A characterization of domination weak bicritical graphs with large diameter

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Abstract

The domination number of a graph G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. A vertex of a graph is called critical if its deletion decreases the domination number, and a graph is called critical if its all vertices are critical. A graph G is called weak bicritical if for every non-critical vertex $x \in V(G)$, $G - x$ is a critical graph with $\gamma(G - x) = \gamma(G)$. In this paper, we characterize the connected weak bicritical graphs G whose diameter is exactly $2\gamma(G) - 2$. This is a generalization of some known results concerning the diameter of graphs with a domination-criticality.

Key words and phrases. weak bicritical graph, critical graph, bicritical graph, diameter

AMS 2010 Mathematics Subject Classification. 05C69.

1 Introduction

All graphs considered in this paper are finite, simple, and undirected.

Let G be a graph. We let $V(G)$ and $E(G)$ denote the vertex set and the edge set of G, respectively. For $x \in V(G)$, we let $N_G(x)$ and $N_G[x]$ denote the *open neighborhood* and the *closed neighborhood* of x, respectively; thus $N_G(x) = \{y \in V(G) : xy \in E(G)\}$ and $N_G[x] = N_G(x) \cup \{x\}$. For $x, y \in V(G)$, we let $d_G(x, y)$ denote the *distance* between x and y in G. For $x \in V(G)$ and a non-negative integer i, let $N_G^{(i)}(x) = \{y \in V(G) : d_G(x, y) = i\}$; thus $N_G^{(0)}(x) = \{x\}$ and $N_G^{(1)}(x) = N_G(x)$. The *diameter* of G, denoted by $\mathrm{diam}(G)$, is defined to be the maximum of $d_G(x, y)$

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as x, y range over $V(G)$. A vertex $x \in V(G)$ is *diametrical* if $\max\{d_G(x, y) : y \in V(G)\} = \mathrm{diam}(G)$.

We let $\overline{G}$ denote the *complement* of G. For two graphs H_1 and H_2, we let $H_1 \cup H_2$ denote the *union* of H_1 and H_2. For a graph H and a non-negative integer s, sH denote the disjoint union of s copies of H. We let K_n and P_n denote the *complete graph* and the *path* of order n, respectively.

For two subsets X, Y of $V(G)$, we say that X *dominates* Y if $Y \subseteq \bigcup_{x \in X} N_G[x]$. A subset of $V(G)$ which dominates $V(G)$ is called a *dominating set* of G. The minimum cardinality of a dominating set of G, denoted by $\gamma(G)$, is called the *domination number* of G. A dominating set of G with the cardinality $\gamma(G)$ is called a γ*-set* of G.

For terms and symbols not defined here, we refer the reader to [7].

1.1 Motivations

For a given graph G, we can divide the set $V(G)$ into the following three subsets;

$$\begin{aligned} V^0(G) &= \{x \in V(G) : \gamma(G - x) = \gamma(G)\}, \\ V^+(G) &= \{x \in V(G) : \gamma(G - x) > \gamma(G)\}, \text{ and} \\ V^-(G) &= \{x \in V(G) : \gamma(G - x) < \gamma(G)\}. \end{aligned}$$

A vertex in $V^-(G)$ is said to be *critical*. A graph G is *critical* if every vertex of G is critical (i.e., $V(G) = V^-(G)$), and G is k*-critical* if G is critical and $\gamma(G) = k$. Many researchers have studied critical vertices or critical graphs (for example, see [1, 2, 11, 12, 13]). Among them, we focus on the following theorem which was conjectured by Brigham, Chinn and Dutton [4].

Theorem A (Fulman, Hanson and MacGillivray [8]) *Let $k \geq 2$ be an integer, and let G be a connected k-critical graph. Then diam$(G) \leq 2k - 2$.*

After that, Ao [3] characterized the connected k-critical graphs G with $\mathrm{diam}(G) = 2k - 2$ (see Theorem E in Subsection 1.2).

Now we introduce other criticality for the domination. A graph G is *bicritical* if $\gamma(G - \{x, y\}) < \gamma(G)$ for any pair of distinct vertices $x, y \in V(G)$, and G is k*-bicritical* if G is bicritical and $\gamma(G) = k$. It is known that for $k \leq 2$, the order of a k-bicritical graph is at most 3 (see [5]), and hence we are interested in k-bicritical graphs with $k \geq 3$. Brigham, Haynes, Henning and Rall [5] gave a conjecture concerning the diameter of bicritical graphs: For $k \geq 3$, every connected k-bicritical graph G satisfies $\mathrm{diam}(G) \leq k - 1$. However, the conjecture was disproved by the following theorem.

Theorem B (Furuya [9, 10]) *Let $k \geq 3$ be an integer. Then there exist infinitely many connected k-bicritical graphs G with*

$$diam(G) = \begin{cases} 3 & (k = 3) \\ 6 & (k = 5) \\ \frac{3k-1}{2} & (k \text{ is odd and } k \geq 7) \\ \frac{3k-2}{2} & (k \text{ is even}). \end{cases}$$

Thus one might be interested in an upper bound of the diameter of bicritical graphs. In [10], the author proved the following theorem. (However, it is open to find a sharp upper bound of the diameter of bicritical graphs.)

Theorem C (Furuya [10]) *Let $k \geq 3$ be an integer, and let G be a connected k-bicritical graph. Then $diam(G) \leq 2k - 3$.*

For convenience, let $\mathcal{C}$ and $\mathcal{C}_B$ denote the family of connected critical graphs and the family of connected bicritical graphs, respectively. Here we compare Theorem A with Theorem C. Although the inequalities in the theorems are similar, the two theorems are essentially different because $\mathcal{C}$ is different from $\mathcal{C}_B$:

- We can easily check that the graphs in $\mathcal{F}_k$ defined in Subsection 1.2 are critical and not bicritical.
- It is known that there exist infinitely many connected critical and bicritical graphs (see [5, 9]), and Brigham et al. [5] proved that a graph obtained from a critical and bicritical graph by expanding one vertex is bicritical and not critical. On the other hand, there exist infinitely many connected 4-bicritical graphs which is not critical and not obtained by the above operation (see the graph L_s in [10]).

In particular, $\mathcal{C}$ and $\mathcal{C}_B$ seems to be remotely related.

To treat the criticality and the bicriticality simultaneously, a new critical concept was defined in [10]. A graph G is *weak bicritical* if $V^+(G) = \emptyset$ and $G - x$ is critical for every $x \in V^0(G)$, and G is *weak k-bicritical* if G is weak bicritical and $\gamma(G) = k$. Since all critical graphs and all bicritical graphs are weak bicritical, the weak bicriticality is a unification of the criticality and the bicriticality. In [10], the author showed the following theorem which is a generalization of Theorem A.

Theorem D (Furuya [10]) *Let $k \geq 2$ be an integer, and let G be a connected weak k-bicritical graph. Then $diam(G) \leq 2k - 2$.*

However, Theorem C cannot directly follow from Theorem D. In this paper, our main aim is to give a common generalization of Theorems A and C by characterizing the connected weak k-bicritical graphs G with $\mathrm{diam}(G) = 2k - 2$.

1.2 Main result

Before we state our main result, we introduce Ao's characterization.

Let $k \geq 2$ be an integer. We define the family $\mathcal{F}_k$ of graphs as follows: Let $m_i \geq 2$ $(1 \leq i \leq k-1)$ be integers. For each $1 \leq i \leq k-1$, let G_i be a graph isomorphic to $\overline{m_i K_2}$ (i.e., G_i is a graph obtained from the complete graph of order $2m_i$ by deleting a perfect matching), and take two vertices $u_i, v_i \in V(G_i)$ with $u_i v_i \notin E(G_i)$. Let $G(m_1, \dots, m_{k-1})$ be the graph obtained from $G_1, \dots, G_{k-1}$ by identifying v_i and u_{i+1} for each $1 \leq i \leq k-2$, and set

$$\mathcal{F}_k = \{G(m_1, \dots, m_{k-1}) : m_i \geq 2,\ 1 \leq i \leq k-1\}.$$

By the definition of $\mathcal{F}_k$, we see the following observation.

Observation 1.1 *Let $k \geq 3$, $k_1 \geq 2$ and $k_2 \geq 2$ be integers with $k_1 + k_2 - 1 = k$. Then a graph G belongs to $\mathcal{F}_k$ if and only if G is obtained from two graphs $H_1 \in \mathcal{F}_{k_1}$ and $H_2 \in \mathcal{F}_{k_2}$ by identifying diametrical vertices u_i of H_i $(i \in \{1, 2\})$.*

Ao [3] proved the following theorem. (By using lemmas for our main result, the following theorem can be easily proved. Hence we will give its proof in Section 4).

Theorem E (Ao [3]) *Let $k \geq 2$ be an integer, and let G be a connected k-critical graph. Then $diam(G) \leq 2k-2$, with the equality if and only if $G \in \mathcal{F}_k$.*

Now we recursively define the family $\mathcal{F}_k^*$ $(k \geq 2)$ of graphs and the identifiable vertices of graphs in $\mathcal{F}_k^*$. Let

$$\mathcal{F}_2^* = \{\overline{(m+1)K_2},\ \overline{mK_2 \cup K_3},\ \overline{mK_2 \cup P_3} : m \geq 1\}.$$

Note that $\mathcal{F}_2^*$ is equal to the family of connected weak 2-bicritical graphs (see Lemma 1.5 in Subsection 1.3). For each $G \in \mathcal{F}_2^*$, a vertex $x \in V(G)$ is *identifiable* if $x \in V^-(G)$. Note that if $G = \overline{(m+1)K_2}$, then all vertices of G are identifiable; if $G = \overline{mK_2 \cup K_3}$, then G has exactly three non-identifiable vertices; if $G = \overline{mK_2 \cup P_3}$, then G has exactly two non-identifiable vertices. We assume that $k \geq 3$, and for $2 \leq k' \leq k-1$, the family $\mathcal{F}_{k'}^*$ and the identifiable vertices of graphs in $\mathcal{F}_{k'}^*$ has been defined. Let $\mathcal{F}_k'$ be the family of graphs obtained from two graphs $H_1 \in \mathcal{F}_{k_1}$ and $H_2 \in \mathcal{F}_{k_2}^*$ with $k_1 \geq 2$, $k_2 \geq 2$ and $k_1 + k_2 - 1 = k$ by identifying a diametrical vertex of H_1 and an identifiable vertex of H_2. Let $m_i \geq 2$ $(i \in \{1, 2\})$, and let u be the unique cut vertex of the graph $G(m_1, m_2)$ $(\in \mathcal{F}_3)$. Let $G^1(m_1, m_2)$ be the graph obtained from $G(m_1, m_2)$ by adding a new vertex u' and joining u' to all vertices in $N_{G(m_1,m_2)}(u)$, and let $G^2(m_1, m_2) = G^1(m_1, m_2) + uu'$. Let

$$\mathcal{F}_3'' = \{G^1(m_1, m_2),\ G^2(m_1, m_2) : m_i \geq 2,\ i \in \{1, 2\}\},$$

and let $\mathcal{F}''_k = \emptyset$ for $k \geq 4$. Then by tedious argument, we see that every graph in $\mathcal{F}''_3$ is weak 3-bicritical (but we omit detail). Let $\mathcal{F}^*_k = \mathcal{F}'_k \cup \mathcal{F}''_k$ for $k \geq 3$. For each $G \in \mathcal{F}^*_k$, a vertex $x \in V(G)$ is *identifiable* if $x \in V^-(G)$ and x is a diametrical vertex of G. By induction and Lemma 1.6(ii) in Subsection 1.3, we see that every graph $G \in \mathcal{F}^*_k$ has at least one identifiable vertex, and hence $\mathcal{F}^*_k$ is well-defined. Furthermore, by the definition of $\mathcal{F}_k$ and $\mathcal{F}^*_k$ and Observation 1.1, we also see that $\mathcal{F}_k \subseteq \mathcal{F}^*_k$ and the diameter of graphs in $\mathcal{F}^*_k$ is exactly $2k-2$.

Our main result is the following.

Theorem 1.2 *Let $k \geq 2$ be an integer, and let G be a connected weak k-bicritical graph. Then $diam(G) \leq 2k-2$, with the equality if and only if $G \in \mathcal{F}^*_k$.*

Theorem 1.2 clearly leads to Theorems A and D. Furthermore, it is not hard to check that no graph in $\mathcal{F}^*_k$ is bicritical and no graph in $\mathcal{F}^*_k - \mathcal{F}_k$ is critical, and so Theorem 1.2 leads to Theorems C and E. Therefore, Theorem 1.2 is a common generalization of some known results.

1.3 Preliminaries

In this subsection, we enumerate some fundamental or preliminary results.

The following has been known property which will be used in our argument.

Lemma 1.3 *Let G be a graph, and let $u, v \in V(G)$. If $N_G[u] \subseteq N_G[v]$, then v is not critical.*

In [10], the author showed that the minimum degree of a connected weak bicritical graph of order at least 3 is at least 2. Now we let G be a disconnected weak bicritical graph. Then we can verify that each component of G is weak bicritical. (Indeed, all components of G are critical with at most one exception.) Thus the following lemma holds.

Lemma 1.4 *Let G be a weak bicritical graph, and let G_1 be a component of G with $|V(G_1)| \geq 3$. Then the minimum degree of G_1 is at least 2.*

Since the weak 1-bicritical graphs are only K_1 and K_2, we are interested in weak k-bicritical graphs for $k \geq 2$. The following lemma gives a characterization of weak 2-bicritical graphs (or 2-critical graphs).

Lemma 1.5 (Furuya [10]) *A graph G is weak 2-bicritical if and only if*

$$G \in \{\overline{mK_2},\ \overline{mK_2 \cup K_3},\ \overline{(m-1)K_2 \cup P_3} : m \geq 1\}.$$

In particular, a graph G is 2-critical if and only if $G \in \{\overline{mK_2} : m \geq 1\}$.

We next focus on the coalescence of graphs. Let H_1 and H_2 be two vertex-disjoint graphs, and let $x_1 \in V(H_1)$ and $x_2 \in V(H_2)$. Under this notation, we let $(H_1 \bullet H_2)(x_1, x_2; x)$ denote the graph obtained from H_1 and H_2 by identifying vertices x_1 and x_2 into a vertex labeled x. We call $(H_1 \bullet H_2)(x_1, x_2; x)$ the *coalescence* of H_1 and H_2 via x_1 and x_2.

Lemma 1.6 ([4, 5, 6, 9]) *Let H_1 and H_2 be graphs, and for each $i \in \{1, 2\}$, let x_i be a non-isolated vertex of H_i. Let $G = (H_1 \bullet H_2)(x_1, x_2; x)$. Then the following hold.*

(i) *We have $\gamma(H_1) + \gamma(H_2) - 1 \leq \gamma(G) \leq \gamma(H_1) + \gamma(H_2)$. If x_i is a critical vertex of H_i for some $i \in \{1, 2\}$, then $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.*

(ii) *If x_i is a critical vertex of H_i for each $i \in \{1, 2\}$, then*

$$V^-(G) = (V^-(H_1) - \{x_1\}) \cup (V^-(H_2) - \{x_2\}) \cup \{x\}.$$

In particular, the graph G is critical if and only if both H_1 and H_2 are critical.

2 Coalescences

In this section, we prove the following theorem.

Theorem 2.1 *Let H_1 and H_2 be graphs, and for each $i \in \{1, 2\}$, let x_i be a non-isolated vertex of H_i. Let $G = (H_1 \bullet H_2)(x_1, x_2; x)$. Then G is weak bicritical if and only if for some $i \in \{1, 2\}$,*

(1) *H_i is critical,*

(2) *H_{3-i} is weak bicritical, and*

(3) *x_{3-i} is a critical vertex of H_{3-i}.*

Furthermore, if G is weak bicritical, then $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$.

Proof. We first assume that G is weak bicritical, and show that $\gamma(G) = \gamma(H_1) + \gamma(H_2) - 1$ and (1)–(3) hold.

Claim 2.1 *The vertex x belongs to $V^-(G)$.*

Proof. Suppose that $x \notin V^-(G)$. Then $x \in V^0(G)$ and $G - x$ is critical. Since $G - x$ is the union of $H_1 - x_1$ and $H_2 - x_2$, $\gamma(G) = \gamma(H_1 - x_1) + \gamma(H_2 - x_2)$ and $H_i - x_i$ is critical for each $i \in \{1, 2\}$. For $i \in \{1, 2\}$, let $y_i \in N_{H_i}(x_i)$, and let S_i be a γ-set of $H_i - \{x_i, y_i\}$. Then $\gamma(H_i - \{x_i, y_i\}) \leq \gamma(H_i - x_i) - 1$. Since $S_1 \cup S_2 \cup \{x\}$ is a dominating set of G, we have $\gamma(H_1-\{x_1,y_1\})+\gamma(H_2-\{x_2,y_2\})+1=|S_1|+|S_2|+|\{x\}|\geq\gamma(G)$. Consequently,

$$\begin{aligned}\gamma(G)&=\gamma(G-x)\\&=\gamma(H_1-x_1)+\gamma(H_2-x_2)\\&\geq\gamma(H_1-\{x_1,y_1\})+\gamma(H_2-\{x_2,y_2\})+2\\&\geq\gamma(G)+1,\end{aligned}$$

which is a contradiction. $\square$

Claim 2.2 *For $i\in\{1,2\}$, x_i is a critical vertex of H_i.*

Proof. Let S be a γ-set of $G-x$. Then by Claim 2.1 and Lemma 1.6(i), $|S|\leq\gamma(G)-1\leq\gamma(H_1)+\gamma(H_2)-1$. Since $\{S\cap V(H_1),S\cap V(H_2)\}$ is a partition of S, we have $|S\cap V(H_i)|\leq\gamma(H_i)-1$ for some $i\in\{1,2\}$. Without loss of generality, we may assume that $|S\cap V(H_1)|\leq\gamma(H_1)-1$. Since removing a vertex can decrease the domination number at most by one and $S\cap V(H_1)$ is a dominating set of H_1-x_1, this implies that $\gamma(H_1-x_1)=|S\cap V(H_1)|=\gamma(H_1)-1$ and x_1 is a critical vertex of H_1. Again by Lemma 1.6(i), $\gamma(G)=\gamma(H_1)+\gamma(H_2)-1$, and hence $|S|\leq\gamma(G)-1=\gamma(H_1)+\gamma(H_2)-2$. Consequently

$$\begin{aligned}|S\cap V(H_2)|&=|S|-|S\cap V(H_1)|\\&\leq(\gamma(H_1)+\gamma(H_2)-2)-(\gamma(H_1)-1)\\&=\gamma(H_2)-1.\end{aligned}$$

Since $S\cap V(H_2)$ is a dominating set of H_2-x_2, $\gamma(H_2-x_2)\leq|S\cap V(H_2)|\leq\gamma(H_2)-1$ and x_2 is a critical vertex of H_2. $\square$

By Lemma 1.6 and Claim 2.2,

$$\gamma(G)=\gamma(H_1)+\gamma(H_2)-1 \tag{2.1}$$

and

$$V^-(G)=(V^-(H_1)-\{x_1\})\cup(V^-(H_2)-\{x_2\})\cup\{x\}. \tag{2.2}$$

If H_1 and H_2 are critical, then (1)–(3) hold. Thus, without loss of generality, we may assume that H_1 is not critical (i.e., $V(H_1)-V^-(H_1)\neq\emptyset$). Let $y\in V(H_1)-V^-(H_1)$. By (2.2), $y\notin V^-(G)$, and hence $G-y$ is critical.

Claim 2.3 *We have $y\in V^0(H_1)$.*

Proof. Note that $\gamma(G-\{x,y\})<\gamma(G)$, and $\gamma(H_2-x_2)=\gamma(H_2)-1$ because x_2 is a critical vertex of H_2 and removing a vertex can decrease the domination number at most by one. Since $G-\{x,y\}$ is the union of $H_1-\{x_1,y\}$ and H_2-x_2, this together with (2.1) leads to

$$\begin{aligned}\gamma(H_1)+\gamma(H_2)-2 &= \gamma(G)-1\\ &\geq \gamma(G-\{x,y\})\\ &= \gamma(H_1-\{x_1,y\})+\gamma(H_2-x_2)\\ &= \gamma(H_1-\{x_1,y\})+\gamma(H_2)-1,\end{aligned}$$

and so $\gamma(H_1-\{x_1,y\})\leq\gamma(H_1)-1$. Since $S_1\cup\{x_1\}$ is a dominating set of H_1-y for a γ-set S_1 of $H_1-\{x_1,y\}$, we have

$$\gamma(H_1-y)\leq\gamma(H_1-\{x_1,y\})+1\leq\gamma(H_1).$$

Since $y\notin V^-(H_1)$, the desired conclusion holds. $\square$

Since y is an arbitrary vertex in $V(H_1)-V^-(H_1)$, it suffices to show that both H_1-y and H_2 are critical. Note that $y\neq x_1$. Now we show that

$$x_1 \text{ is a non-isolated vertex of } H_1-y. \tag{2.3}$$

By way of contradiction, we suppose that x_1 is an isolated vertex of H_1-y. Since x_1 is a non-isolated vertex of H_1, $N_{H_1}(x_1)=\{y\}$. Since G is weak bicritical and x_2 is a non-isolated vertex of H_2, the component of G containing y has at least three vertices. This together with Lemma 1.4 implies $N_{H_1}(y)-\{x_1\}\neq\emptyset$. Let $y'\in N_{H_1}(y)-\{x_1\}$. Since $G-y$ is critical, $\gamma(G-\{y,y'\})\leq\gamma(G)-1=\gamma(H_1)+\gamma(H_2)-2$. Let S be a γ-set of $G-\{y,y'\}$. If $x\in S$, let $S'=((S-\{x\})\cap V(H_2))\cup\{x_2\}$; if $x\notin S$, let $S'=S\cap V(H_2)$. In either case, S' is a dominating set of H_2, and hence $|(S-\{x\})\cap V(H_1)|=|S|-|S'|\leq(\gamma(H_1)+\gamma(H_2)-2)-\gamma(H_2)=\gamma(H_1)-2$. Since $(S-\{x\})\cap V(H_1)$ is a dominating set of $H_1-\{x,y,y'\}$, $S''=((S-\{x\})\cap V(H_1))\cup\{y\}$ is a dominating set of H_1 with $|S''|\leq\gamma(H_1)-1$, which is a contradiction. Thus (2.3) holds.

Recall that $G-y$ is critical. Since $G-y=((H_1-y)\bullet H_2)(x_1,x_2;x)$, it follows from Lemma 1.6(ii) and (2.3) that H_1-y and H_2 are critical.

We next assume that (1)–(3) hold, and show that G is weak bicritical. We may assume that $i=1$ (i.e., H_1 is critical, H_2 is weak bicritical, and x_2 is a critical vertex of H_2). By Lemma 1.6(i), $\gamma(G)=\gamma(H_1)+\gamma(H_2)-1$. If G is critical, then the desired conclusion holds. Thus $V(G)-V^-(G)\neq\emptyset$. Let $y\in V(G)-V^-(G)$. By Lemma 1.6(ii), $y\in V^0(H_2)$, and hence H_2-y is critical.

Claim 2.4 *We have $y \in V^0(G)$.*

Proof. Let S_1 be a γ-set of H_1, and let S_2 be a γ-set of $H_2 - \{x_2, y\}$. If $x_1 \in S_1$, let $S = (S_1 - \{x_1\}) \cup S_2 \cup \{x\}$; if $x_1 \notin S_1$, let $S = S_1 \cup S_2$. In either case, S is a dominating set of $G - y$. Since $|S| = \gamma(H_1) + \gamma(H_2 - \{x_2, y\}) \leq \gamma(H_1) + (\gamma(H_2) - 1) = \gamma(G)$, we have $\gamma(G - y) \leq \gamma(G)$. Since $y \notin V^-(G)$, the desired conclusion holds. $\square$

Since y is an arbitrary vertex in $V(G) - V^-(G)$, it suffices to show that $G - y$ is critical. Note that $y \neq x$. Now we show that

$$x_2 \text{ is a non-isolated vertex of } H_2 - y. \tag{2.4}$$

Recall that x_2 is a non-isolated vertex of H_2. Furthermore, since x_2 is a critical vertex of H_2, the component of H_2 containing x_2 is not isomorphic to K_2, and hence the component of H_2 containing x_2 has at least three vertices. This together with Lemma 1.4 implies that the degree of x_2 in H_2 is at least 2, and so the degree of x_2 in $H_2 - y$ is at least 2. Thus (2.4) holds.

Recall that both H_1 and $H_2 - y$ are critical. Since $G - y = (H_1 \bullet (H_2 - y))(x_1, x_2; x)$, it follows from Lemma 1.6(ii) and (2.4) that $G - y$ is critical.

This completes the proof of Theorem 2.1. $\square$

3 Sufficient pairs

Let $l \geq 3$ be an integer, and let G be a connected graph. A pair (x, j) of a vertex $x \in V(G)$ and an integer $j \geq 2$ is *l-sufficient* if x is a diametrical vertex of G and there exists a γ-set S of G with $|S \cap (\bigcup_{0 \leq i \leq j} N_G^{(i)}(x))| \geq (j + l)/2$.

Lemma 3.1 (Furuya [10]) *Let $k \geq 3$ and $l \geq 3$ be integers, and let G be a connected weak k-bicritical graph having an l-sufficient pair. Then diam$(G) \leq 2k - l + 1$.*

Theorem 3.2 *Let $k \geq 3$ be an integer, and let G be a connected weak k-bicritical graph. If G has a diametrical vertex x such that $\bigcup_{1 \leq i \leq 3} N_G^{(i)}(x) \subseteq V^-(G)$ and $|N_G^{(2)}(x)| \geq 2$, then diam$(G) \leq 2k - 3$.*

Proof. We show that diam$(G) \leq 3$ or G has a 4-sufficient pair. By way of contradiction, we suppose that diam$(G) \geq 4$ and G has no 4-sufficient pair. For each $i \geq 0$, let $X_i = N_G^{(i)}(x)$ and $U_i = X_0 \cup X_1 \cup \cdots \cup X_i$.

Claim 3.1 *If a set $S \subseteq V(G)$ dominates $N_G[x]$ and $|S \cap U_2| \leq 1$, then x is the unique vertex of $S \cap U_2$.*

Proof. By the assumption of the claim, there exists a vertex $z \in N_G[x]$ dominating $N_G[x]$ in G. Since $N_G[x] \subseteq N_G[z]$, if $z \neq x$, then $z \in N_G^{(1)}(x)$ and z is not a critical vertex of G by Lemma 1.3, which contradicts the assumption of the theorem. □

Let $w_2, w_2' \in X_2$ be distinct vertices, and let S_1 be a γ-set of $G - w_2$. Note that $S_1 \cup \{w_2\}$ is a γ-set of G. Since G has no 4-sufficient pair, $|(S_1 \cup \{w_2\}) \cap U_2| < (2+4)/2 = 3$, and so $|S_1 \cap U_2| \leq 1$. Since S_1 dominates $N_G[x]$ in G, it follows from Claim 3.1 that x is the unique vertex in $S_1 \cap U_2$. Since G has no 4-sufficient pair, $|(S_1 \cup \{w_2\}) \cap U_4| < (4+4)/2 = 4$, and so $|S_1 \cap U_4| \leq 2$. Since $|X_2| \geq 2$ and S_2 dominates $(X_2 \cup X_3) - \{w_2\}$, there exists a vertex $w_3 \in X_3$ dominating $(X_2 \cup X_3) - \{w_2\}$ in $G - w_2$.

Let S_2 be a γ-set of $G - w_3$. Note that $S_2 \cup \{w_2'\}$ is a γ-set of G because $w_3 w_2' \in E(G)$. Since G has no 4-sufficient pair, $|(S_2 \cup \{w_2'\}) \cap U_2| < (2+4)/2 = 3$, and so $|S_2 \cap U_2| \leq 1$. Since S_2 dominates $N_G[x]$ in G, it follows from Claim 3.1 that x is the unique vertex in $S_2 \cap U_2$. Since G has no 4-sufficient pair, $|(S_2 \cup \{w_2'\}) \cap U_4| < (4+4)/2 = 4$, and so $|S_2 \cap U_4| \leq 2$. Since S_2 dominates $(X_2 \cup X_3) - \{w_3\}$, there exists a vertex $w_3' \in X_3$ dominating $(X_2 \cup X_3) - \{w_3\}$ in $G - w_3$. Recall that w_3 dominates X_3 in $G - w_2$. Thus $w_3 w_3' \in E(G)$, and hence S_2 is a dominating set of G, which is a contradiction.

Consequently $\mathrm{diam}(G) \leq 3$ or G has a 4-sufficient pair. In either case, it follows from Lemma 3.1 that the desired conclusion holds. □

4 Proof of Theorems E and 1.2

In this section, we prove Theorems E and 1.2. As we mentioned in Subsection 1.2, $\mathcal{F}_k \subseteq \mathcal{F}_k^*$ and the diameter of graphs in $\mathcal{F}_k^*$ is exactly $2k-2$. By Lemma 1.5, $\mathcal{F}_2$ is equal to the family of connected 2-critical graphs. Thus by induction and Lemma 1.6(ii), we see that all graphs in $\mathcal{F}_k$ are k-critical, and so

$$\text{if a graph } G \text{ belongs to } \mathcal{F}_k, \text{ then } G \text{ is } k\text{-critical and } \mathrm{diam}(G) = 2k-2. \tag{4.1}$$

Recall that every graph in $\mathcal{F}_2^*$ is weak 2-bicritical and every graph in $\mathcal{F}_3''$ is weak 3-bicritical. This together with induction and Theorem 2.1 implies that all graphs in $\mathcal{F}_k^*$ are weak k-bicritical, and so

$$\text{if a graph } G \text{ belongs to } \mathcal{F}_k^*, \text{ then } G \text{ is weak } k\text{-bicritical and } \mathrm{diam}(G) = 2k-2. \tag{4.2}$$

Proof of Theorem E. Let k and G be as in Theorem E. By (4.1), it suffices to show that

$$\text{if } \mathrm{diam}(G) \geq 2k-2, \text{ then } G \in \mathcal{F}_k. \tag{4.3}$$

We proceed by induction on k.

If $k = 2$, then Lemma 1.5 leads to (4.3). Thus we may assume that $k \geq 3$. Suppose that $\mathrm{diam}(G) \geq 2k-2$. Let w be a diametrical vertex of G. If $|N_G^{(2)}(w)| \geq 2$, then $\mathrm{diam}(G) \leq 2k-3$ by Theorem 3.2, which is a contradiction. Thus $|N_G^{(2)}(w)| = 1$. In particular, G has a cut vertex x. Hence we can write G as $G = (H_1 \bullet H_2)(x_1, x_2; x)$ for two graphs H_1 and H_2 and vertices $x_i \in V(H_i)$ ($i \in \{1,2\}$). For each $i \in \{1,2\}$, set $k_i = \gamma(H_i)$. By Lemma 1.6, H_1 and H_2 are critical and $k_1 + k_2 - 1 = \gamma(H_1) + \gamma(H_2) - 1 = \gamma(G) = k$. Furthermore, we have $\mathrm{diam}(G) \leq \mathrm{diam}(H_1) + \mathrm{diam}(H_2)$. By induction hypothesis, $\mathrm{diam}(H_i) \leq 2k_i - 2$, with the equality if and only if $H_i \in \mathcal{F}_{k_i}$. Consequently, we have $2k-2 \leq \mathrm{diam}(G) \leq (2k_1-2) + (2k_2-2) = 2k-2$. This implies that $H_i \in \mathcal{F}_{k_i}$ and x_i is a diametrical vertex of H_i. Then by Observation 1.1, we have $G \in \mathcal{F}_k$.

This completes the proof of Theorem E. $\square$

Proof of Theorem 1.2. Let k and G be as in Theorem 1.2. By (4.2), it suffices to show that

$$\text{if } \mathrm{diam}(G) \geq 2k-2, \text{ then } G \in \mathcal{F}_k^*. \tag{4.4}$$

We proceed by induction on k.

If $k = 2$, then Lemma 1.5 leads to (4.4). Thus we may assume that $k \geq 3$. Suppose that $\mathrm{diam}(G) \geq 2k-2$. If G is critical, then it follows from Theorem E that $G \in \mathcal{F}_k$ ($\subseteq \mathcal{F}_k^*$), as desired. Thus we may assume that G is not critical (i.e., $V^0(G) \neq \emptyset$). Let $w, w' \in V(G)$ be vertices with $d_G(w,w') = \mathrm{diam}(G)$.

Claim 4.1 *If G has no cut vertex, then $G \in \mathcal{F}_k^*$.*

Proof. Note that $|N^{(2)}(w)| \geq 2$. If $V^0(G) \subseteq \{w, w'\}$ (i.e., $V(G) - \{w,w'\} \subseteq V^-(G)$), then by Theorem 3.2, we have $\mathrm{diam}(G) \leq 2k-3$, which is a contradiction. Thus $V^0(G) - \{w,w'\} \neq \emptyset$. Let $z \in V^0(G) - \{w,w'\}$. Then $G - z$ is a connected critical graph and

$$\mathrm{diam}(G-z) \geq d_{G-z}(w,w') \geq d_G(w,w') = \mathrm{diam}(G) \geq 2k-2.$$

This together with Theorem E forces $G - z \in \mathcal{F}_k$ and $\mathrm{diam}(G-z) = d_{G-z}(w,w') = \mathrm{diam}(G) = 2k-2$. By the definition of $\mathcal{F}_k$, we have $|N_{G-z}^{(2)}(w)| = |N_{G-z}^{(4)}(w)| = 1$. Write $N_{G-z}^{(2)}(w) = \{z'\}$. Since G has no cut vertex, the following hold:

- $k = 3$,
- z is adjacent to a vertex in $N^{(1)}_{G-z}(w)$ and a vertex in $N^{(3)}_{G}(w)$, and
- $N_G(z) \subseteq \bigcup_{1\le i\le 3} N^{(i)}_{G-z}(w)$.

Suppose that z' is a critical vertex of G, and let S be a γ-set of $G - z'$. Since $N_G(z) \subseteq N_G[z']$ and S is not a dominating set of G, this forces $zz' \notin E(G)$ and $z \in S$. Since S dominates w, $S \cap N_G[w] \neq \emptyset$. In particular, $|(S \cup \{z'\}) \cap (\bigcup_{0\le i\le 2} N^{(i)}_{G}(w))| \geq 3$. Since $S \cup \{z'\}$ is a γ-set, $(w, 2)$ is a 4-sufficient pair. This together with Lemma 3.1 implies that $\mathrm{diam}(G) \leq 2k - 3$, which is a contradiction. Thus z' is not a critical vertex of G (i.e., $z' \in V^0(G)$).

Replacing the role of z and z', we have $G - z' \in \mathcal{F}_k$ and $N_{G-z'}(z) = N^{(1)}_{G-z'}(w) \cup N^{(3)}_{G-z'}(w)$. Hence G is isomorphic to a graph in $\mathcal{F}''_3$ $(\subseteq \mathcal{F}^*_3)$. $\square$

By Claim 4.1, we may assume that G has a cut vertex x. Then we can write G as $G = (H_1 \bullet H_2)(x_1, x_2; x)$ for two graphs H_1 and H_2 and vertices $x_i \in V(H_i)$ $(i \in \{1, 2\})$. For each $i \in \{1, 2\}$, set $k_i = \gamma(H_i)$. Having Theorem 2.1 in mind, we may assume that H_1 is critical, H_2 is weak bicritical and x_2 is a critical vertex of H_2. Furthermore, $k_1 + k_2 - 1 = \gamma(H_1) + \gamma(H_2) - 1 = \gamma(G) = k$. By induction hypothesis, $\mathrm{diam}(H_1) \leq 2k_1 - 2$, with the equality if and only if $H_1 \in \mathcal{F}_{k_1}$. By Theorem E, $\mathrm{diam}(H_2) \leq 2k_2 - 2$, with the equality if and only if $H_2 \in \mathcal{F}^*_{k_2}$. Since $\mathrm{diam}(G) \leq \mathrm{diam}(H_1) + \mathrm{diam}(H_2)$, we have $2k - 2 \leq \mathrm{diam}(G) \leq (2k_1 - 2) + (2k_2 - 2) = 2k - 2$. This implies that $H_1 \in \mathcal{F}_{k_1}$, $H_2 \in \mathcal{F}^*_{k_2}$ and x_i is a diametrical vertex of H_i. Since x_2 is a critical vertex of H_2, it follows from the definition of $\mathcal{F}^*_k$, we have $G \in \mathcal{F}^*_k$.

This completes the proof of Theorem 1.2. $\square$

Acknowledgment

This work was supported by JSPS KAKENHI Grant number 26800086.

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