

On the maximum number of maximum independent sets

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Abstract

We give a very short and simple proof of Zykov's generalization of Turán's theorem, which implies that the number of maximum independent sets of a graph of order n and independence number α with $\alpha < n$ is at most $\lceil \frac{n}{\alpha} \rceil^{n \bmod \alpha} \lfloor \frac{n}{\alpha} \rfloor^{\alpha - (n \bmod \alpha)}$. Generalizing a result of Zito, we show that the number of maximum independent sets of a tree of order n and independence number α is at most $2^{n-\alpha-1} + 1$, if $2\alpha = n$, and, $2^{n-\alpha-1}$, if $2\alpha > n$, and we also characterize the extremal graphs. Finally, we show that the number of maximum independent sets of a subcubic tree of order n and independence number α is at most $\left(\frac{1+\sqrt{5}}{2}\right)^{2n-3\alpha+1}$, and we provide more precise results for extremal values of α .

Keywords: Turán graph; tree; independence number; maximum independent set; Fibonacci number

1 Introduction

We consider only finite, simple, and undirected graphs, and use standard terminology and notation. An *independent set* in a graph G is a set of pairwise non-adjacent vertices of G . The *independence number* $\alpha(G)$ of G is the maximum cardinality of an independent set in G . An independent set in G is *maximal* if no proper superset is an independent set in G , and *maximum* if it has cardinality $\alpha(G)$. For a graph G , let $\sharp\alpha(G)$ be the number of maximum independent sets in G .

In the present paper we study the maximum number of maximum independent sets as a function of the order and the independence number in general graphs, trees, and subcubic trees. Before we come to our results, we mention some related research.

For a tree T of order $n > 1$, Zito [12] showed

$$\sharp\alpha(T) \leq \begin{cases} 2^{\frac{n-2}{2}} + 1 & , \text{ if } n \text{ is even, and} \\ 2^{\frac{n-3}{2}} & , \text{ if } n \text{ is odd.} \end{cases} \quad (1) \quad \{\mathbf{e1}\}$$

Since $\alpha(T) \geq n/2$, it is not difficult to show that (1) implies

$$\sharp\alpha(T) \leq 2^{\alpha(T)-1} + 1, \quad (2) \quad \{\mathbf{e2}\}$$

cf. [2] for a simple independent proof. For similar results concerning the maximum number of maximal independent sets see [6, 11].

Jou and Chang [5] observed that Moon and Moser's [7] result on the maximum number of maximal independent sets implies

$$\sharp\alpha(G) \leq \begin{cases} 3^{\frac{n}{3}} & , \text{ if } n \bmod 3 = 0, \\ 4 \cdot 3^{\frac{n-4}{3}} & , \text{ if } n \bmod 3 = 1, \text{ and} \\ 2 \cdot 3^{\frac{n-2}{3}} & , \text{ if } n \bmod 3 = 2, \end{cases}$$

for every graph G of order n . This is actually an immediate consequence of Zykov's generalization [13] of Turán's theorem [10]; independently shown also by Roman [9]. For positive integers n and p , let $T_p(n)$ be the complete p -partite graph with $n \bmod p$ partite sets of order $\left\lceil \frac{n}{p} \right\rceil$ and $p - (n \bmod p)$ partite sets of order $\left\lfloor \frac{n}{p} \right\rfloor$, that is, $T_p(n)$ is the *Turán graph*. A *clique* in a graph G is a set of pairwise adjacent vertices of G . For a graph G and a positive integer q , let $\sharp\omega^{(q)}(G)$ be the number of cliques of order q in G .

{theorem}

Theorem 1 (Zykov [13]). *Let n , q , and p be integers with $2 \leq q < p \leq n$. If G is a graph of order n with no clique of order p , then $\sharp\omega^{(q)}(G) \leq \sharp\omega^{(q)}(T_{p-1}(n))$ with equality if and only if $G = T_{p-1}(n)$.*

As our first contribution, we give a very short and simple proof of Theorem 1 inspired by the 5th proof from *The Book* [1] of Turán's theorem. Applying the special case $q = p - 1$ of Theorem 1 to the complement \bar{G} of a graph G immediately implies the following.

{corollar}

Corollary 2. *If G is a graph of order n and independence number α with $\alpha < n$, then*

$$\sharp\alpha(G) \leq \left\lceil \frac{n}{\alpha} \right\rceil^{n \bmod \alpha} \left\lfloor \frac{n}{\alpha} \right\rfloor^{\alpha - (n \bmod \alpha)}. \quad (3) \quad \{\mathbf{ecor}\}$$

Furthermore, equality holds in (3) if and only if G is the complement of $T_\alpha(n)$.

Corollary 2 also follows from a result of Nielsen [8] who showed that the right hand side of (3) is a tight upper bound on the number of maximal independent sets of cardinality exactly α for every graph G of order n regardless of the independence number of G .

Our further results concern trees and subcubic trees.

The next result is a common generalization of (1) and (2).

Theorem 3. *If T is a tree of order n and independence number α , then*

{theorem1}

$$\sharp\alpha(T) \leq \begin{cases} 2^{n-\alpha-1} + 1 & , \text{ if } 2\alpha = n, \text{ and} \\ 2^{n-\alpha-1} & , \text{ if } 2\alpha > n. \end{cases} \quad (4) \quad \{\mathbf{e4}\}$$

Furthermore, equality holds in (4) if and only if T arises by subdividing $n - \alpha - 1$ edges of $K_{1,\alpha}$ once.

As it turns out, the maximum number of maximum independent sets in subcubic trees is closely related to the famous Fibonacci numbers. Let $f(n)$ denote the n -th Fibonacci number, that is,

$$f(n) = \begin{cases} 0 & , \text{ if } n = 0, \\ 1 & , \text{ if } n = 1, \\ f(n-1) + f(n-2) & , \text{ if } n \geq 2. \end{cases}$$

Our first result for subcubic trees concerns the smallest possible value of the independence number in (subcubic) trees. For a positive integer k , let $T(k)$ arise by attaching a new endvertex to every vertex of a path of order k . Since

$$\begin{aligned} \sharp\alpha(T(1)) &= 2, \\ \sharp\alpha(T(2)) &= 3, \text{ and} \\ \sharp\alpha(T(k)) &= \sharp\alpha(T(k-1)) + \sharp\alpha(T(k-2)) \text{ for every } k \geq 3, \end{aligned}$$

we obtain

$$\sharp\alpha(T(k)) = f(k+2)$$

for every positive integer k .

Theorem 4. *If T is a subcubic tree of order n and independence number $\alpha = \frac{n}{2}$, then*

{theorem2}

$$\sharp\alpha(T) \leq f(\alpha + 2) \quad (5) \quad \{\mathbf{e9}\}$$

Furthermore, equality holds in (5) if and only if $T = T(\alpha)$.

Our second result for subcubic trees concerns the largest possible value of the independence number in subcubic trees. If T is a tree, then T' arises from T by attaching a P_3 if $V(T)$ is the disjoint union of $V(T)$ and $\{x, y, z\}$, and $E(T) = E(T') \cup \{uy, xy, yz\}$, where u is some vertex of T .

{theorem3}

Theorem 5. *If T is a subcubic tree of order n and independence number α , then*

$$\alpha(T) \leq \frac{2n+1}{3}. \quad (6) \quad \{\mathbf{e11}\}$$

Furthermore, equality holds in (6) if and only if T arises from K_1 by iteratively attaching P_3 s, in which case $\sharp\alpha(T) = 1$.

For given positive integers n and α with $\alpha \leq \frac{2n+1}{3}$, suitably combining the extremal trees from Theorem 4 and Theorem 5 allows to construct subcubic trees with order n and independence number α that satisfy

$$\sharp\alpha(T) = \Omega\left(f(2n - 3\alpha + 1)\right).$$

This implies that our last result for subcubic trees is best possible up to small constant factors and additive terms.

{theorem4}

Theorem 6. *If T is a subcubic tree of order n and independence number α , then*

$$\sharp\alpha(T) \leq \left(\frac{1+\sqrt{5}}{2}\right)^{2n-3\alpha+1}.$$

All proofs are give in the next section.

2 Proofs

Proof of Theorem 1. Let G be a graph of order n with no clique of order p that maximizes $\sharp\omega^{(q)}(G)$. Let G_0 arise from G by removing all edges that do not belong to a clique of order q in G . Clearly, G_0 has no clique of order p , and $\sharp\omega^{(q)}(G_0) = \sharp\omega^{(q)}(G)$. {claim1}

Claim 1. G_0 is a complete multipartite graph.

Proof of Claim 1. Suppose, for a contradiction, that the claim fails. This implies the existence of three vertices u, v , and w such that u is not adjacent to v or w , but v and w are adjacent. Let $d^{(q)}(u)$ be the number of cliques of order q in G_0 that contain u , that is, $d^{(q)}(u) = \sharp\omega^{(q-1)}(G_0[N_{G_0}(u)])$. Let $d^{(q)}(v)$ and $d^{(q)}(w)$ be defined analogously. If $d^{(q)}(u) < d^{(q)}(v)$, then the graph that arises from G_0 by removing u and duplicating v has no clique of order p but $\sharp\omega^{(q)}(G_0) - d^{(q)}(u) + d^{(q)}(v) > \sharp\omega^{(q)}(G)$ cliques of order q , contradicting the choice of G . Hence, by symmetry, we may assume that $d^{(q)}(u) \geq d^{(q)}(v), d^{(q)}(w)$. Now, since the edge vw belongs to some clique of order q in G_0 , the graph that arises from G_0 by removing v and w , and triplicating u has no clique of order p but $\sharp\omega^{(q)}(G_0) + 2d^{(q)}(u) - d^{(q)}(v) - d^{(q)}(w) + 1 > \sharp\omega^{(q)}(G)$ cliques of order q , contradicting the choice of G . \square

Since G_0 has no clique of order p , the multipartite graph G_0 has $p-1$ (possibly empty) partite sets V_1, \dots, V_{p-1} , of orders $n_1 \geq \dots \geq n_{p-1}$, respectively. Since $\sharp\omega^{(q)}(G_0) > 0$, the graph $G'_0 = G_0 - (V_1 \cup V_{p-1})$ has a clique of order $q-2$, that is, $\sharp\omega^{(q-2)}(G'_0) > 0$. If $n_1 \geq n_{p-1} + 2$, then G_0 has

$$n_1 n_{p-1} \sharp\omega^{(q-2)}(G'_0) + (n_1 + n_{p-1}) \sharp\omega^{(q-1)}(G'_0) + \sharp\omega^{(q)}(G'_0)$$

cliques of order q , while the graph that arises from G_0 by moving one vertex from V_i to V_j has

$$(n_1 - 1)(n_{p-1} + 1) \sharp\omega^{(q-2)}(G'_0) + (n_1 - 1 + n_{p-1} + 1) \sharp\omega^{(q-1)}(G'_0) + \sharp\omega^{(q)}(G'_0)$$

cliques of order q . Since $\sharp\omega^{(q-2)}(G'_0) > 0$ and $(n_1 - 1)(n_{p-1} + 1) > n_1 n_{p-1}$, this contradicts the choice of G . Hence, we obtain $|n_i - n_j| \leq 1$ for every $1 \leq i \leq j \leq p-1$, which implies $G_0 = T_{p-1}(n)$. Since $n \geq p$, all $p-1$ partite sets of G_0 are non-empty. Therefore, adding any non-edge of G_0 to G_0 results in a graph that has a clique of order p , which implies $G = G_0$, and completes the proof. \square

A vertex of degree at most 1 is an *endvertex*, and a neighbor of an endvertex is a *support vertex*.

Proof of Theorem 3. Within this proof, we call a tree *special* if it arises by subdividing $n - \alpha - 1$ edges of $K_{1,\alpha}$ once. Suppose, for a contradiction, that the theorem is false, and let n be the smallest order for which it fails. Let T be a tree of order n and independence number α such that

- either $\sharp\alpha(T)$ does not satisfy (4),
- or $\sharp\alpha(T)$ satisfies (4) with equality but T is not special.

It is easy to see that T is not special and has diameter at least 3, which implies $\frac{n}{2} \leq \alpha \leq n-2$. We root T at an endvertex of a longest path in T . Let y be the parent of an endvertex of maximum depth in T , let x_1, \dots, x_k be the children of y , and let z be the parent of y .

The tree $T' = T - \{x_1, \dots, x_k, y\}$ has order $n' = n - k - 1$ and independence number $\alpha' = \alpha - k$.

First, we assume that $k \geq 2$. In this case, every maximum independent set in T contains $\{x_1, \dots, x_k\}$, and the choice of n implies

$$\begin{aligned} \sharp\alpha(T) &= \sharp\alpha(T') \\ &\stackrel{(4)}{\leq} 2^{n'-\alpha'-1} + 1 \\ &= 2^{n-\alpha-2} + 1 \end{aligned} \tag{7} \quad \{\text{e5}\}$$

$$\stackrel{\alpha \leq n-2}{\leq} 2^{n-\alpha-1}. \tag{8} \quad \{\text{e6}\}$$

Now, if $\sharp\alpha(T) = 2^{n-\alpha-1}$, then

- equality holds in (7), which implies $2(\alpha - k) = 2\alpha' = n' = n - k - 1$, and
- equality holds in (8), which implies $\alpha = n - 2$.

These equations imply $k = n - 3$, $\alpha' = 1$, and $n' = 2$, that is, T' is K_2 . We obtain the contradiction, that T arises by subdividing one edge of $K_{1,\alpha}$, that is, T is special. Hence, we may assume that $k = 1$.

Since the number of maximum independent sets in T that contain y is less or equal than the number of maximum independent sets in T that contain x , we obtain $\sharp\alpha(T) \leq 2\sharp\alpha(T')$, and $\sharp\alpha(T) < 2\sharp\alpha(T')$ if some maximum independent set in T' that contain z .

First, we assume that $2\alpha = n$ and that T' is not special. Since $2\alpha' = 2\alpha - 2 = n - 2 = n'$, the tree T' is a bipartite graph whose partite sets both have order exactly α' . This implies that some maximum independent set in T' contains z , and the choice of n implies the contradiction

$$\sharp\alpha(T) < 2\sharp\alpha(T') \stackrel{(4)}{\leq} 2 \cdot 2^{n'-\alpha'-1} = 2^{n-\alpha-1}.$$

Next, we assume that $2\alpha = n$ and that T' is special. There are only three possibilities for the structure of T illustrated in Figure 1 together with the resulting values of $\sharp\alpha$.

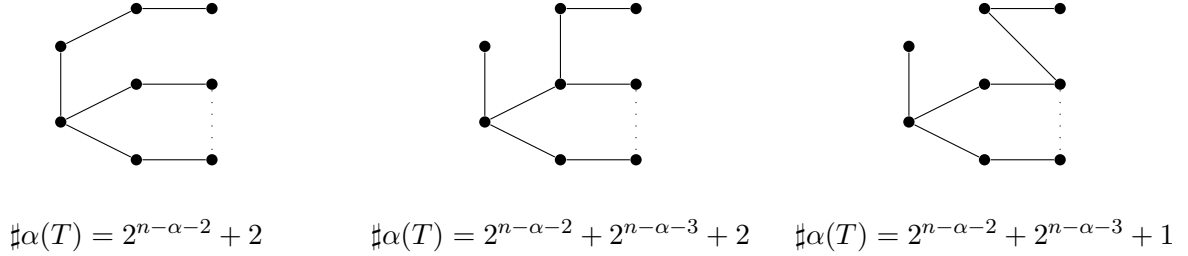


Figure 1: Three possibilities for the structure of T .

{fig:3cas

In all three cases, we have $n - \alpha - 2 \geq 1$, because otherwise either T would be special or the configuration would not be possible. In the first and third case, this already implies a contradiction, because $2^{n-\alpha-2} + 2 \leq 2^{n-\alpha-2} + 2^{n-\alpha-3} + 1 \leq 2^{n-\alpha-1}$. In the second case, we obtain $n - \alpha - 2 \geq 2$, because T is not special. Thus, also in this case, we obtain a contradiction, because $2^{n-\alpha-2} + 2^{n-\alpha-3} + 2 \leq 2^{n-\alpha-1}$.

Finally, we assume that $2\alpha > n$. Since $2\alpha' > n'$, the choice of n implies

$$\sharp\alpha(T) \leq 2\sharp\alpha(T') \tag{9} \quad \{\text{e7}\}$$

$$\stackrel{(4)}{\leq} 2 \cdot 2^{n'-\alpha'-1} \tag{10} \quad \{\text{e8}\}$$

$$= 2^{n-\alpha-1}.$$

Now, if $\sharp\alpha(T) = 2^{n-\alpha-1}$, then

- equality holds in (9), which implies that no maximum independent set in T' contains z , and
- equality holds in (10), which implies that T' is special.

Since the only vertex of T' that does not belong to some maximum independent set in T' is the unique vertex of degree more than 2 in T' , we obtain the contradiction that T is special, which completes the proof. \square

Proof of Theorem 4. Suppose, for a contradiction, that the theorem is false, and let n be the smallest order for which it fails. Let T be a subcubic tree of order n and independence number $\alpha = \frac{n}{2}$ such that $\sharp\alpha(T)$ is as large as possible. Note that n is necessarily even.

If A and B are the two partite sets of the bipartite graph T , then $\alpha = \frac{n}{2}$ implies $|A| = |B| = \frac{n}{2}$. Furthermore, since A and B are both maximum independent sets in T , the neighborhood $N_T(S)$ of

every subset S of A is at least as large as S , which, by Hall's theorem [4], implies that T has a perfect matching M . If $n \in \{2, 4\}$, then $T = T(\alpha)$ follows immediately. Hence, we may assume that $n \geq 6$.

Let the tree \tilde{T} arise from T by contracting all edges in M . Let $e_1 \dots e_p$ be a longest path in \tilde{T} . Since $n \geq 6$, we have $p \geq 3$. Let $e_i = u_i v_i$ for $i \in [3]$. By symmetry, we may assume that $u_2 u_3$ is the (unique) edge between e_2 and e_3 . By the choice of P , all neighbors of e_2 in \tilde{T} that are distinct from e_3 are endvertices of \tilde{T} . Since T has maximum degree at most 3, the set $N_{\tilde{T}}(e_2) \setminus \{e_3\}$ contains

- $d_1 \leq 1$ edges e of T such that u_2 has a neighbor in e , and
- $d_2 \leq 2$ edges e of T such that v_2 has a neighbor in e .

Since e_1 is one of the edges counted by $d_1 + d_2$, we obtain

$$(d_1, d_2) \in \{(0, 1), (0, 2), (1, 1), (1, 2), (1, 0)\}.$$

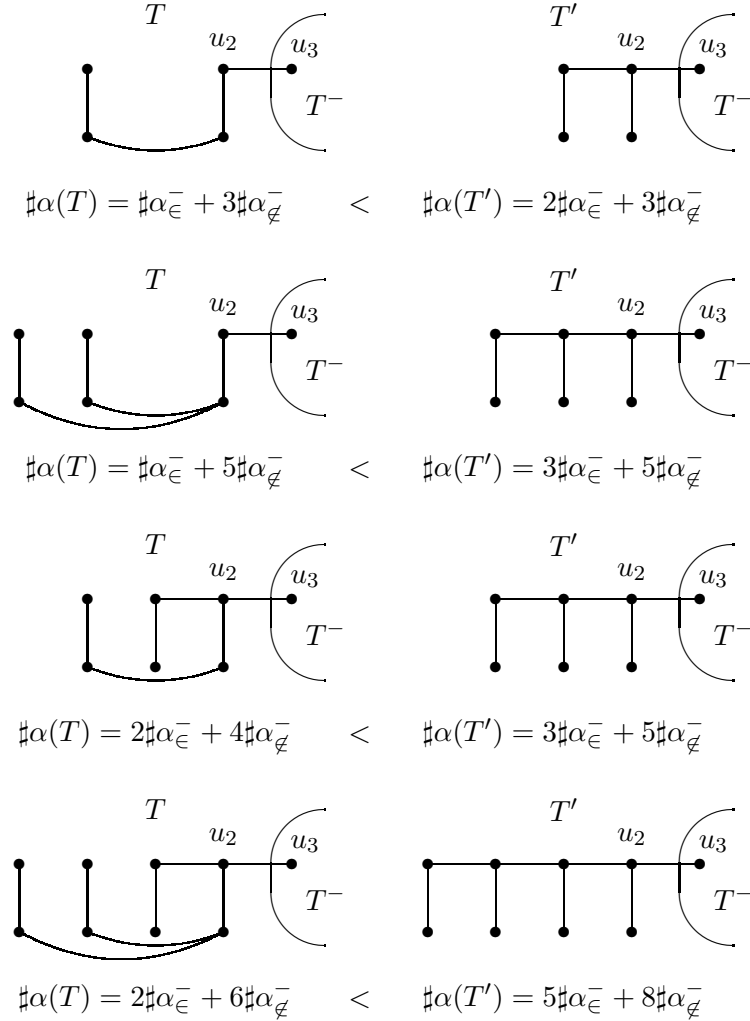


Figure 2: $(d_1, d_2) \notin \{(0, 1), (0, 2), (1, 1), (1, 2)\}$.

{figexcl}

Our next goal is to exclude the first four of these possible values of (d_1, d_2) . In each case, we construct a subcubic tree T' of order n and independence number $\alpha = \frac{n}{2}$ such that $\sharp\alpha(T') > \sharp\alpha(T)$, contradicting the choice of T . Let $T^- = T - \bigcup_{e \in N_{\tilde{T}}(e_2) \setminus \{e_3\}} e$. By construction, the tree T^- still has a perfect matching, which implies $\alpha(T^-) = \frac{n(T^-)}{2}$.

Let

- $\sharp\alpha_{\bar{e}}^-$ be the number of maximum independent sets in T^- that contain u_3 , and let

- $\sharp\alpha_{\bar{\epsilon}}^-$ be the number of maximum independent sets in T^- that do not contain u_3 .

Since $\alpha(T^-) = \frac{n(T^-)}{2}$, arguing as above implies that both partite sets of the bipartite graph T^- are maximum independent sets in T^- , which implies $\sharp\alpha_{\bar{\epsilon}}^-, \sharp\alpha_{\bar{\epsilon}}^- > 0$. Figure 2 illustrates the construction of T' in each case, together with the values of $\sharp\alpha(T)$ and $\sharp\alpha(T')$.

We conclude that $(d_1, d_2) = (1, 0)$, which implies that the subcubic tree T' has order $n - 4$ and independence number $\alpha - 2 = \frac{n-4}{2}$. Let $T'' = T - \{u_1, v_1\}$. The subcubic tree T'' has order $n - 2$ and independence number $\alpha - 1 = \frac{n-2}{2}$. Therefore, by the choice of n , we obtain

$$\begin{aligned}
\sharp\alpha(T) &= 2\sharp\alpha_{\bar{\epsilon}}^- + 3\sharp\alpha_{\bar{\epsilon}}^- \\
&= \left(\sharp\alpha_{\bar{\epsilon}}^- + 2\sharp\alpha_{\bar{\epsilon}}^-\right) + \left(\sharp\alpha_{\bar{\epsilon}}^- + \sharp\alpha_{\bar{\epsilon}}^-\right) \\
&= \sharp\alpha(T'') + \sharp\alpha(T') \\
&\leq f(\alpha - 1 + 2) + f(\alpha - 2 + 2) \\
&= f(\alpha + 2),
\end{aligned} \tag{11} \quad \{\text{e10}\}$$

that is, $\sharp\alpha(T) \leq f(\alpha + 2)$. Furthermore, if $\sharp\alpha(T) = f(\alpha + 2)$, then equality holds in (11), which, by the choice of n , implies $T' = T(\alpha - 2)$ and $T'' = T(\alpha - 1)$, and, hence, $T = T(\alpha)$. This contradiction completes the proof. \square

Proof of Theorem 5. Suppose, for a contradiction, that the theorem is false, and let n be the smallest order for which it fails. Let T be a subcubic tree of order n and independence number α . Let u be an endvertex of a longest path P in T . By the choice of n , the path P has order at least 3. Let v be the neighbor of u , and let w be the neighbor of v on P that is distinct from u . The subcubic tree $T' = T - (N_T[v] \setminus \{w\})$ has order $n - d_T(v)$ and independence number $\alpha - (d_T(v) - 1)$. By the choice of n , we obtain

$$\begin{aligned}
\alpha &= \alpha(T') + (d_T(v) - 1) \\
&\leq \frac{2n(T') + 1}{3} + (d_T(v) - 1) \\
&= \frac{2(n - d_T(v)) + 1}{3} + (d_T(v) - 1)
\end{aligned} \tag{12} \quad \{\text{e12}\}$$

$$\begin{aligned}
&= \frac{2n + 1}{3} - \frac{3 - d_T(v)}{3} \\
&\leq \frac{2n + 1}{3},
\end{aligned} \tag{13} \quad \{\text{e13}\}$$

which implies (6). Now, equality in (6) implies equality in (12) and (13). By the choice of n , the tree T' arises from K_1 by iteratively attaching P_3 s, and that v has degree 3. Hence, also T arises from K_1 by iteratively attaching P_3 s. The uniqueness of the maximum independent set follows easily by an inductive argument exploiting the constructive characterization of T . This completes the proof. \square

Proof of Theorem 6. Suppose, for a contradiction, that the theorem is false, and let n be the smallest order for which it fails. Let T be a subcubic tree of order n and independence number α such that $\sharp\alpha(T)$ is as large as possible.

$\{\text{claim2}\}$

Claim 1. *The tree T contains a path of length at least 3.*

Proof of Claim 1. Suppose, for a contradiction, that T is a star $K_{1,n-1}$.

If $n = 1$, then

$$\sharp\alpha(T) = 1 = \left(\frac{1 + \sqrt{5}}{2}\right)^{2-3+1},$$

if $n = 2$, then

$$\sharp\alpha(T) = 2 < 2.618 \approx \left(\frac{1 + \sqrt{5}}{2}\right)^{4-3+1},$$

if $n = 3$, then

$$\sharp\alpha(T) = 1 < 1.618 \approx \left(\frac{1+\sqrt{5}}{2}\right)^{6-6+1},$$

and, if $n = 4$, then

$$\sharp\alpha(T) = 1 = \left(\frac{1+\sqrt{5}}{2}\right)^{8-9+1}.$$

In each case, we obtain a contradiction to the choice of n and T . \square

Let $uvw \dots r$ be a longest path in T , and consider T as rooted in r . For a vertex z of T , let V_z be the set that contains z and all its descendants. {claim3}

Claim 2. $d_T(v) = 2$

Proof of Claim 2. Suppose, for a contradiction, that $d_T(v) = 3$. Note that every maximum independent set in T contains both children of v but not v . Hence, the subcubic tree $T' = T - V(T_v)$ has order $n - 3$ and independence number $\alpha - 2$, and satisfies $\sharp\alpha(T) = \sharp\alpha(T')$. By the choice of n , we obtain

$$\sharp\alpha(T) = \sharp\alpha(T') \leq \left(\frac{1+\sqrt{5}}{2}\right)^{2 \cdot (n-3) - 3 \cdot (\alpha-2) + 1} = \left(\frac{1+\sqrt{5}}{2}\right)^{2n-3\alpha+1},$$

which contradicts the choice of T . \square

Claim 3. w is not a support vertex. {claim4}

Proof of Claim 3. Suppose, for a contradiction, that w is a support vertex. The subcubic tree $T' = T - V(T_v)$ has order $n - 2$ and independence number $\alpha - 1$, while the subcubic tree $T'' = T - V(T_w)$ has order $n - 4$ and independence number $\alpha - 2$. Since there are $\sharp\alpha(T')$ maximum independent sets in T that contain u , and $\sharp\alpha(T'')$ maximum independent sets in T that do not contain u , the choice of n implies

$$\begin{aligned} \sharp\alpha(T) &= \sharp\alpha(T') + \sharp\alpha(T'') \\ &\leq \left(\frac{1+\sqrt{5}}{2}\right)^{2 \cdot (n-2) - 3 \cdot (\alpha-1) + 1} + \left(\frac{1+\sqrt{5}}{2}\right)^{2 \cdot (n-4) - 3 \cdot (\alpha-2) + 1} \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{2n-3\alpha+1} \left(\left(\frac{1+\sqrt{5}}{2}\right)^{-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{-2} \right) \\ &= \left(\frac{1+\sqrt{5}}{2}\right)^{2n-3\alpha+1}, \end{aligned}$$

which contradicts the choice of T . \square

Claim 4. $d_T(w) = 2$. {claim5}

Proof of Claim 4. Suppose, for a contradiction, that w has a child v' distinct from v . By Claims 2 and 3, the vertex v' has exactly one child u' , which is an endvertex. The subcubic tree $T' = T - \{u, v, u', v'\}$ has order $n - 4$ and independence number $\alpha - 2$. Since for every maximum independent set I' of T' that does not contain w , we have $x \in I'$, and $(I' \setminus \{x\}) \cup \{w\}$ is a maximum independent set in T' that contains w , there are at most $\frac{\sharp\alpha(T')}{2}$ maximum independent sets in T' that do not contain w , and at least $\frac{\sharp\alpha(T')}{2}$ maximum independent sets in T' that contain w . A maximum independent set in T' that contains w can only be extended in a unique way to a maximum independent set in T , while a maximum independent set in T' that does not contain w can be extended in four different ways to a

maximum independent set in T . Since all maximum independent sets in T are of one of these types, the choice of n implies

$$\begin{aligned}
\sharp\alpha(T) &\leq 4 \cdot \frac{\sharp\alpha(T')}{2} + \frac{\sharp\alpha(T')}{2} \\
&\leq \frac{5}{2} \cdot \left(\frac{1+\sqrt{5}}{2} \right)^{2 \cdot (n-4) - 3 \cdot (\alpha-2) + 1} \\
&= \frac{5}{2} \cdot \left(\frac{1+\sqrt{5}}{2} \right)^{-2} \left(\frac{1+\sqrt{5}}{2} \right)^{2n-3\alpha+1} \\
&< \left(\frac{1+\sqrt{5}}{2} \right)^{2n-3\alpha+1},
\end{aligned}$$

using $\frac{5}{2} < \left(\frac{1+\sqrt{5}}{2} \right)^2$, which contradicts the choice of T . \square

Since $\sharp\alpha(P_4) = 3 < \left(\frac{1+\sqrt{5}}{2} \right)^{2 \cdot 4 - 3 \cdot 2 + 1}$ we may assume that x has a parent y .

Claim 5. x is not a support vertex. {claim6}

Proof of Claim 5. Suppose, for a contradiction, that x has a child w' that is an endvertex. The subcubic tree $T' = T - \{u, v, w\}$ has order $n - 3$ and independence number $\alpha - 2$. Every maximum independent set I of T contains u , w , and w' , and $I \setminus \{u, w\}$ is a maximum independent set in T' . By the choice of n , this implies

$$\sharp\alpha(T) \leq \sharp\alpha(T') \leq \left(\frac{1+\sqrt{5}}{2} \right)^{2 \cdot (n-3) - 3 \cdot (\alpha-2) + 1} = \left(\frac{1+\sqrt{5}}{2} \right)^{2n-3\alpha+1},$$

which contradicts the choice of T . \square

Claim 6. x has no child that is a support vertex. {claim7}

Proof of Claim 6. Suppose, for a contradiction, that x has a child w' that is a support vertex. If w' has two children that are endvertices, then arguing as in the proof of Claim 2 yields a contradiction. If w' has a child that is not an endvertex, then $d_T(w') = 3$, which leads to a similar contradiction as in the proof of Claim 4. Hence, w' has a unique child v' , which is an endvertex. The subcubic tree $T' = T - V(T_x)$ has order $n - 6$ and independence number $\alpha - 3$. A maximum independent set I' of T' can be extended in at most four different ways to a maximum independent set in T : $I' \cup \{u, v', x\}$, $I' \cup \{v, v', x\}$, $I' \cup \{u, w, w'\}$ and $I' \cup \{u, v', w\}$. Since all maximum independent sets in T are of such a form, the choice of n implies

$$\sharp\alpha(T) \leq 4\sharp\alpha(T') \leq 4 \left(\frac{1+\sqrt{5}}{2} \right)^{2 \cdot (n-6) - 3 \cdot (\alpha-3) + 1} < \left(\frac{1+\sqrt{5}}{2} \right)^{2n-3\alpha+1},$$

using $4 < \left(\frac{1+\sqrt{5}}{2} \right)^3$, which contradicts the choice of T . \square

Claim 7. $d_T(x) = 2$. {claim8}

Proof of Claim 7. Suppose, for a contradiction, that x has a child w' distinct from w . By Claims 5 and 6, w' has a child v' that has a child u' . By Claims 2 and 4, $d_T(w') = d_T(v') = 2$. The subcubic tree $T' = T - V(T_x)$ has order $n - 7$ and independence number $\alpha - 4$. Note that every maximum

independent set in T' can be extended in a unique way to a maximum independent set in T , and that the maximum independent sets in T are exactly those sets. Hence, by the choice of n , we obtain

$$\sharp\alpha(T) \leq \sharp\alpha(T') \leq \left(\frac{1+\sqrt{5}}{2}\right)^{2\cdot(n-7)-3\cdot(\alpha-4)+1} < \left(\frac{1+\sqrt{5}}{2}\right)^{2n-3\alpha+1}.$$

□

By the above claims, we know that $d_T(v) = d_T(w) = d_T(x) = 2$. Let $T' = T - V(T_x)$, $T_1 = T - \{vu\} + \{xu\}$, and $T'' = T_1 - \{v, w\}$. Clearly, all these trees are subcubic.

A maximum independent set in T' that contains y can only be extended in a unique way to a maximum independent set in T , and all maximum independent set in T that contain y are of that form. A maximum independent set I' of T' that does not contain y can be extended to a maximum independent set I of T in three ways, $I' \cup \{u, w\}$, $I' \cup \{u, x\}$ and, $I' \cup \{v, x\}$, and every maximum independent set in T that does not contain y is of that form.

Similarly, a maximum independent set in T' that contains y can be extended to a maximum independent set in T_1 in two different ways, and all maximum independent set in T_1 that contain y are of that form. A maximum independent set I' of T' that does not contain y can be extended to a maximum independent set I_1 of T_1 in three ways, $I' \cup \{u, w\}$, $I' \cup \{u, v\}$ and, $I' \cup \{v, x\}$, and every maximum independent set in T_1 that does not contain y is of that form. Arguing as in the proof of Claim 3, we obtain

$$\sharp\alpha(T) \leq \sharp\alpha(T_1) = \sharp\alpha(T') + \sharp\alpha(T'') \leq \left(\frac{1+\sqrt{5}}{2}\right)^{2n-3\alpha+1}.$$

This final contradiction completes the proof. □

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