# $H$-Decomposition of $r$-graphs when $H$ is an $r$-graph with exactly $k$ independent edges * 

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#### Abstract

Let $\phi_{H}^{r}(n)$ be the smallest integer such that, for all $r$-graphs $G$ on $n$ vertices, the edge set $E(G)$ can be partitioned into at most $\phi_{H}^{r}(n)$ parts, of which every part either is a single edge or forms an $r$-graph isomorphic to $H$. The function $\phi_{H}^{2}(n)$ has been well studied in literature, but for the case $r \geq 3$, the problem that determining the value of $\phi_{H}^{r}(n)$ is widely open. Sousa (2010) gave an asymptotic value of $\phi_{H}^{r}(n)$ when $H$ is an $r$-graph with exactly 2 edges, and determined the exact value of $\phi_{H}^{r}(n)$ in some special cases. In this paper, we first give the exact value of $\phi_{H}^{r}(n)$ when $H$ is an $r$-graph with exactly 2 edges, which improves Sousa's result. Second we determine the exact value of $\phi_{H}^{r}(n)$ when $H$ is an $r$-graph consisting of exactly $k$ independent edges.


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## 1 Introduction

Given two $r$-graphs $G$ and $H$, an $H$-decomposition of $G$ is a partition of the edge set of $G$ such that each part is either a single edge or forms an $r$-graph isomorphic to $H$. The minimum number of parts in an $H$-decomposition of $G$ is denoted by $\phi_{H}^{r}(G)$. The $H$-decomposition number $\phi_{H}^{r}(n)$ is defined as

$$
\phi_{H}^{r}(n)=\max \left\{\phi_{H}^{r}(G): G \text { is an } r \text {-graph with }|V(G)|=n\right\} .
$$

An $r$-graph $G$ with $\phi_{H}^{r}(G)=\phi_{H}^{r}(n)$ is called an extremal graph of $H$.
For the case $r=2$, we omit the index 2 for short in the paper, for example we write graph for 2-graph, and $\phi_{H}(n)$ for $\phi_{H}^{2}(n)$. The function $\phi_{H}(n)$ has been well studied in literature by many researchers. The first exact value of $\phi_{H}(n)$ when $H=K_{3}$ was given by Erdős, Goodman and Pósa [3] in 1966, where $K_{k}$ is the complete graph on $k$ vertices. Ten years later, Bollobás [2] generalized the result to $H=K_{k}, k \geq 3$. Much more exact values of $\phi_{H}(n)$ can be found in the survey by Sousa [11] in 2015. Recently, Hou, Qiu and Liu determined the exact values of $\phi_{H}(n)$ when $H$ is a graph consisting of $k$ complete graphs of order at least 3 which intersect in exactly one common vertex [7] and when $H$ is a graph consisting of $k$ cycles of odd lengths which intersect in exactly one common vertex [8]. An asymptotic value of the function $\phi_{H}(n)$ was given by Pikhurko and Sousa [9, and lately the value was improved by Allen, Böttcher, and Person [1].

For the case $r \geq 3$, the study of the function $\phi_{H}^{r}(n)$ is widely open. Sousa 10 ] gave an asymptotic value of $\phi_{H}^{r}(n)$ when $H$ is an $r$-graph consisting of 2 edges, and determined the exact value of $\phi_{H}^{r}(n)$ in the special cases where the two edges of $H$ intersect exactly 1,2 and $r-1$ vertices. In this paper, we first generalize Sousa's result in [10], that is we obtain the exact value of $\phi_{H}^{r}(n)$ when $H$ is an $r$-graph consisting of exactly 2 edges. Second we focus on the case that $H$ is the $r$-graph consisting of exactly $k$ independent edges, and we determine the exact value of $\phi_{H}^{r}(n)$ in this case.

Given positive integer $n, r$ and $k$ with $n \geq r \geq 2$, let $K_{n}^{r}$ be a complete $r$-graph on $n$ vertices and let $K_{n}^{r}-\ell e$ be a graph obtained from $K_{n}^{r}$ by deleting $\ell$ edges from $K_{n}^{r}$, where $\ell$ is an integer so that $0 \leq \ell \leq k-1$ and $e\left(K_{n}^{r}-\ell e\right) \equiv k-1(\bmod k)$. Note that $\ell$ is determined uniquely by $n, r$ and $k$. Let $\mathcal{K}_{n, k}^{r}$ be the family of $r$-graphs $K_{n}^{r}-\ell e$. The followings are our main results.

Theorem 1. Given integers $r, k$ satisfying $0 \leq k \leq r-1$. Let $H$ be an r-graph consisting of exactly 2 edges which intersect $k$ vertices. If $n \geq 2 r-k$, then $\phi_{H}^{r}(n)=$
$\left\lceil\frac{1}{2}\binom{n}{r}\right\rceil$. Moreover, graphs $G \in \mathcal{K}_{n, 2}^{r}$ and $G=K_{n}^{r}$ if $\binom{n}{r} \equiv 0(\bmod 2)$ are extremal graphs of $H$.

Theorem 1 improves Sousa's result in [10].
Theorem 2. Given integers $k \geq 1, r \geq 2$ and $n_{0}=k r(k+r-2)+2 r-1$, let $H$ be an $r$-graph on $n$ vertices consisting of exactly $k$ independent edges. If $n \geq n_{0}$ then

$$
\phi_{H}^{r}(n)= \begin{cases}\left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor+k-1, & \text { if }\binom{n}{r} \equiv k-1 \quad(\bmod k) ; \\ \left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor+k-2, & \text { otherwise } .\end{cases}
$$

Furthermore, $G$ is an extremal graph of $H$ if and only if $G \in \mathcal{K}_{n, k}^{r}$ or $G=K_{n}^{r}$ if $\binom{n}{r} \equiv k-2(\bmod k)$.

The proofs of Theorems 1 and 2 will be given in Sections 2 and 3, respectively. Before giving the proofs, we first introduce some definitions and notation. Let $H$ be an $r$-graph with vertex set $V(H)$ and $E(H)$. For a vertex $v \in V(H)$, the degree of $v$, denoted by $d_{H}(v)$, is the number of edges of $H$ containing $v$, and the minimum degree of $H$ is denoted by $\delta(H)$. The matching number of $H$ is the maximum number of independent edges in $H$. We write $e(H)$ for the number of edges of $H$, that is $e(H)=|E(H)|$.

## 2 Proof of Theorem 1

We need some basic facts in algebraic graph theory. A graph $G$ is called vertextransitive if its automorphism group acts transitively on $V(G)$. Given nonnegative integers $n, r$ and $k$, let $J(n, r, k)$ be the graph with vertex set $E\left(K_{n}^{r}\right)$, where two vertices are adjacent if and only if their intersection has size $k$. For $n \geq r$, the graphs $J(n, r, r-1)$ and $J(n, r, 0)$ are known as the Johnson graphs and the Kneser graphs, respectively.

Fact 1 (See page 9 and page 35 in [5]). (1) $J(n, r, k)$ has $\binom{n}{r}$ vertices, and each vertex has degree $\binom{r}{k}\binom{n-r}{r-k}$.
(2) The graphs $J(n, r, k)$ are vertex-transitive.
(3) If $n \geq r \geq k, J(n, r, k) \cong J(n, n-r, n-2 r+k)$.

Lemma 3 (Theorem 3.5.1 in [5). If $G$ is a connected vertex-transitive graph, then $G$ has a matching that misses at most one vertex.

Lemma 4 (Theorem 2.3 in [10). Let $H$ be a fixed $r$-graph with 2 edges and $G$ an $r$-graph with $n$ vertices. Then $\phi_{H}^{r}(G) \leq \phi_{H}^{r}\left(K_{n}^{r}\right)$.

Proof of Theorem 1: Lemma 4 implies that $\phi_{H}^{r}(n)=\phi_{H}^{r}\left(K_{n}^{r}\right)$. So, to prove the result, it is sufficient to show that $\phi_{H}^{r}\left(K_{n}^{r}\right)=\left\lceil\frac{1}{2}\binom{n}{r}\right\rceil$. Clearly, $\phi_{H}^{r}\left(K_{n}^{r}\right) \geq\left\lceil\frac{1}{2}\binom{n}{r}\right\rceil$ as $e(H)=2$. To prove $\phi_{H}^{r}\left(K_{n}^{r}\right) \leq\left\lceil\frac{1}{2}\binom{n}{r}\right\rceil$, it is sufficient to find an $H$-decomposition of $K_{n}^{r}$ with $\left\lceil\frac{1}{2}\binom{n}{r}\right\rceil$ parts.

By the definition of $J(n, r, k), K_{n}^{r}$ has an $H$-decomposition with $\left\lceil\frac{1}{2}\binom{n}{r}\right\rceil$ parts is equivalent to the statement that $J(n, r, k)$ has a matching missing at most one vertex. By (3) of Fact 1, we may assume $n \geq 2 r$. If $k=0$ and $n=2 r$, then $J(2 r, r, 0)$ consists of exactly $\frac{1}{2}\binom{n}{r}$ independent edges, so the statement holds.

Now we assume $k>0$ or $n>2 r$. By (2) of Fact [1, $J(n, r, k)$ is vertex-transitive. Hence, by Lemma 3, to show $J(n, r, k)$ has a matching missing at most one vertex, it is sufficient to show that $J(n, r, k)$ is connected. That is, we need to show that any pair of vertices $e, f$ of $J(n, r, k)$ are connected. Suppose $|e \cap f|=i$. We prove $e$ and $f$ are connected by induction on $i$. If $i=r$, then statement is trivial. If $i=r-1$, assume $e=\{1,2, \ldots, r-1, r\}$ and $f=\{1,2, \ldots, r-1, r+1\}$. Then ehf with $h=\{1,2, \ldots, k, r+2, \ldots, 2 r+1-k\}$ is a walk connecting $e$ and $f$ in $J(n, r, k)$. So the result is true for the base case. Now suppose $i<r-1$ and the statement is true for any large $i$. By symmetry, one could assume that $e=\{1,2, \ldots, r-1, r\}$ and $f=\{1,2, \ldots, i, r+1, \ldots, 2 r-i\}$. Let $h=\{1, \ldots, i, i+1, r+1, \ldots, 2 r-i-1\}$. Then $|e \cap h|=i+1>i$ and $|f \cap h|=r-1>i$. By induction hypothesis, $e$ and $h$ (resp. $f$ and $h$ ) are connected in $J(n, r, k)$. By the transitivity of connectivity, $e$ and $f$ are connected in $J(n, r, k)$.

## 3 Proof of Theorem [2

We need two known theorems to prove our result. Given graphs $G$ and $H$, we say $G$ has an $H$-factor if $G$ contains $\left\lfloor\frac{|V(G)|}{|V(H)|}\right\rfloor$ vertex-disjoint copies of $H$.

Theorem 5 (Hajnal, Szemerédi [6]). Let $k$ be a positive integer. If $G$ is a graph on $n$ vertices with minimum degree

$$
\delta(G) \geq\left(1-\frac{1}{k}\right) n
$$

then $G$ has a $K_{k}$-factor.

Theorem 6 (Frankl [4]). If $H$ is an r-graph on $n$ vertices with matching number of size $k$ and $n \geq(2 k+1) r-k$, then

$$
e(H) \leq\binom{ n}{r}-\binom{n-k}{r}
$$

Proof of Theorem 2: Let $G$ be an $r$-graph on $n \geq n_{0}$ vertices with $\phi_{H}(G)=\phi_{H}(n)$. Let $p_{H}(G)$ denote the maximum number of pairwise edge-disjoint copies of $H$ in $G$. Then we have

$$
\begin{equation*}
\phi_{H}(G)=e(G)-(k-1) p_{H}(G) . \tag{1}
\end{equation*}
$$

If we remove the edges of $p_{H}(G)$ pairwise edge-disjoint copies of $H$ from $G$, then we obtain an $H$-free graph, that is a graph with matching number at most $k-1$. Hence by Theorem 6, we have

$$
\begin{equation*}
\binom{n}{r}-\binom{n-k+1}{r} \geq e(G)-k p_{H}(G) . \tag{2}
\end{equation*}
$$

On the other hand,

$$
\begin{equation*}
\phi_{H}(G) \geq \phi_{H}\left(K_{n}^{r}\right) \geq \frac{1}{k}\binom{n}{r} . \tag{3}
\end{equation*}
$$

From (1), (2) and (3), we have

$$
\begin{equation*}
e(G) \geq\binom{ n}{r}-(k-1)\left[\binom{n}{r}-\binom{n-k+1}{r}\right] . \tag{4}
\end{equation*}
$$

Now we define an auxiliary graph $L_{G}$ as follows: let $V\left(L_{G}\right)=E(G)$ and two vertices $e_{1}, e_{2} \in V\left(L_{G}\right)$ is adjacent if and only if $e_{1} \cap e_{2}=\emptyset$ in $G$. Hence the edge set of a copy of $H$ in $E(G)$ induces a clique of order $k$ in $L_{G}$. Therefore, a collection of edge-disjoint copies of $H$ in $G$ corresponds to a collection of vertex-disjoint $K_{k}$ in $L_{G}$.

Claim 1. $L_{G}$ has a $K_{k}$-factor. In particular, $L_{K_{n}^{r}-\ell e}$ has a $K_{k}$-factor for every $\ell$ with $0 \leq \ell \leq k-1$.

Proof of the claim: By definition of $L_{G}$, we have

$$
\delta\left(L_{G}\right) \geq\binom{ n-r}{r}-\left[\binom{n}{r}-e(G)\right] .
$$

By Theorem [5] it suffices to show that

$$
\binom{n-r}{r}-\left[\binom{n}{r}-e(G)\right] \geq\left(1-\frac{1}{k}\right) e(G)
$$

that is

$$
\begin{equation*}
e(G) \geq k\left[\binom{n}{r}-\binom{n-r}{r}\right] . \tag{5}
\end{equation*}
$$

To show (5), by (4) it suffices to show

$$
\binom{n}{r}-(k-1)\left[\binom{n}{r}-\binom{n-k+1}{r}\right] \geq k\left[\binom{n}{r}-\binom{n-r}{r}\right]
$$

that is, we need to show

$$
\begin{equation*}
k\binom{n-r}{r}+(k-1)\binom{n-k+1}{r} \geq(2 k-2)\binom{n}{r} . \tag{6}
\end{equation*}
$$

By the inequality

$$
\frac{\binom{n-t}{r}}{\binom{n}{r}} \geq\left(\frac{n-t-r+1}{n-r+1}\right)^{r} \geq 1-\frac{r t}{n-r+1},(r \geq t \geq 0)
$$

and $n \geq n_{0}=k r(k+r-2)+2 r-1$, it can be easily check that (6) holds. This completes the proof of the claim.

Now suppose $e(G) \equiv i(\bmod k)$ and $e(G)=t k+i(0 \leq i \leq k-1)$ for some $t \leq\left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor$. By Claim 11, $p_{H}(G)=t$ and hence by (1), we have $\phi_{H}(G)=t+i$. In particular,

$$
\phi_{H}^{r}\left(K_{n}^{r}-\ell e\right)= \begin{cases}\left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor+k-1, & \text { if }\binom{n}{r} \equiv k-1 \quad(\bmod k) \\ \left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor+k-2, & \text { otherwise }\end{cases}
$$

If $\binom{n}{r} \equiv k-1(\bmod k)$, then $\phi_{H}^{r}(n)=\phi_{H}^{r}(G)=t+i \leq\left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor+k-1$, and the equality holds if and only if $G=K_{n}^{r} \in \mathcal{K}_{n, k}^{r}$. Otherwise, $\phi_{H}^{r}(n)=\phi_{H}^{r}(G)=t+i \leq$ $\left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor+k-2$, the equality holds if and only if $t=\left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor-1$ and $i=k-1$ or $t=\left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor$ and $i=k-2$, in the former case $G \in \mathcal{K}_{n, k}^{r}$ and in the latter case it happens if and only if $G=K_{n}^{r}$ and $\binom{n}{r} \equiv k-2(\bmod k)$.

## 4 Concluding Remarks

In this paper we determine the exact value of of $\phi_{H}^{r}(n)$ when $H$ is an $r$-graph consisting of exactly 2 edges or consisting of exactly $k$ INDEPENDENT edges. We believe that Theorem 2 still holds when $H$ consists of exactly $k$ edges which intersect the same set of size $i(0 \leq i \leq r-1)$, we leave this as an open problem.

Question 7. Is the following statement true? Given integers $k \geq 1, r \geq 2$, let $H$ be an r-graph consisting of exactly $k$ edges which intersect the same set of size $i$ ( $0 \leq i \leq r-1$ ). If $n$ is sufficiently large, then

$$
\phi_{H}^{r}(n)= \begin{cases}\left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor+k-1, & \text { if }\binom{n}{r} \equiv k-1 \quad(\bmod k) ; \\ \left\lfloor\frac{1}{k}\binom{n}{r}\right\rfloor+k-2, & \text { otherwise } .\end{cases}
$$

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