## EHRHART POLYNOMIALS WITH NEGATIVE COEFFICIENTS

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ABSTRACT. It is shown that, for each  $d \ge 4$ , there exists an integral convex polytope  $\mathcal{P}$  of dimension d such that each of the coefficients of  $n, n^2, \ldots, n^{d-2}$  of its Ehrhart polynomial  $i(\mathcal{P}, n)$  is negative.

In his talk of the Clifford Lectures at Tulane University, 25–27 March 2010, Richard Stanley gave an Ehrhart polynomial with a negative coefficient. More precisely, the polynomial  $\frac{13}{6}n^3 + n^2 - \frac{1}{6}n + 1$  is the Ehrhart polynomial of the tetrahedron in  $\mathbb{R}^3$  with vertices (0,0,0), (1,0,0), (0,1,0) and (1,1,13). See [1, Example 3.22]. His talk naturally inspired us to find integral convex polytopes of dimension  $\geq 4$  whose Ehrhart polynomials possess negative coefficients. Consult [3, Part II] and [4, pp. 235–241] for fundamental materials on Ehrhart polynomials.

A convex polytope is called *integral* if any of its vertices has integer coordinates. Let  $\mathcal{P} \subset \mathbb{R}^N$  be an integral convex polytope of dimension d and  $\partial \mathcal{P}$  the boundary of  $\mathcal{P}$ . We introduce the function  $i(\mathcal{P}, n)$  by setting

$$i(\mathcal{P}, n) = \sharp(n\mathcal{P} \cap \mathbb{Z}^N), \text{ for } n = 1, 2, \dots,$$

where  $n\mathcal{P} = \{n\alpha : \alpha \in \mathcal{P}\}\)$  and where  $\sharp(X)$  is the cardinality of a finite set X. The study on  $i(\mathcal{P}, n)$  originated in Ehrhart [2] who showed that  $i(\mathcal{P}, n)$  is a polynomial in n of degree d with  $i(\mathcal{P}, 0) = 1$ . Furthermore, the coefficients of  $n^d$  and  $n^{d-1}$  of  $i(\mathcal{P}, n)$  are always positive ([1, Corollary 3.20 and Theorem 5.6]). We say that  $i(\mathcal{P}, n)$  is the Ehrhart polynomial of  $\mathcal{P}$ .

The purpose of the present paper is, for each  $d \ge 4$ , to show the existence of an integral convex polytope of dimension d such that each of the coefficients of  $n, n^2, \ldots, n^{d-2}$  of its Ehrhart polynomial  $i(\mathcal{P}, n)$  is negative. In fact,

**Theorem 1.** Given an arbitrary integer  $d \ge 4$ , there exists an integral convex polytope  $\mathcal{P}$  of dimension d such that each of the coefficients of  $n, n^2, \ldots, n^{d-2}$  of the Ehrhart polynomial  $i(\mathcal{P}, n)$  of  $\mathcal{P}$  is negative.

Our proof of Theorem 1 will be given after preparing Lemmata 2 and 3.

**Lemma 2.** Let  $\mathcal{P} \subset \mathbb{R}^N$  be an integral convex polytope of dimension d and  $i(\mathcal{P}, n)$  its Ehrhart polynomial. Then, given an arbitrary integer k > 0, there exists an

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integral convex polytope  $\mathcal{P}'_k \subset \mathbb{R}^{N+1}$  of dimension d+1 whose Ehrhart polynomial is equal to  $(kn+1)i(\mathcal{P},n)$ .

*Proof.* It follows immediately that the Ehrhart polynomial  $i(\mathcal{P}'_k, n)$  of the integral convex polytope  $\mathcal{P}'_k = \mathcal{P} \times [0, k] \subset \mathbb{R}^{N+1}$  coincides with  $(kn+1)i(\mathcal{P}, n)$ .  $\Box$ 

**Lemma 3.** Let d and j be integers with  $d \ge 5$  and  $3 \le j \le d-2$ , and

$$g(d,j) = (d-3)^2 {d-3 \choose j-1} - {d-3 \choose j-3}.$$

Then one has g(d, j) > 0.

Proof. Since  $d \ge 5$ , one has  $g(d,3) = (d-3)^2 \binom{d-3}{2} - 1 > 0$  and  $g(d,d-2) = (d-3)^2 - \binom{d-3}{2} > 0$ . Thus g(d,j) > 0 for j = 3 and j = d-2. Especially the assertion is true for d = 5 and d = 6.

We now work with induction on d. Let  $d \ge 7$  and  $4 \le j \le d - 3$ . Then

$$g(d,j) = ((d-4)^2 + 2d - 7) \left( \binom{d-4}{j-1} + \binom{d-4}{j-2} \right) - \left( \binom{d-4}{j-3} + \binom{d-4}{j-4} \right)$$
$$= g(d-1,j) + g(d-1,j-1) + (2d-7)\binom{d-3}{j-1}.$$

It follows from the assumption of induction that g(d-1,j) + g(d-1,j-1) > 0. Hence g(d,j) > 0, as desired.

Proof of Theorem 1. It is known [1, Example 3.22] that, given an arbitrary integer  $m \ge 1$ , there exists an integral convex polytope  $\mathcal{Q}_m$  of dimension 3 with

$$i(\mathcal{Q}_m, n) = \frac{m}{6}n^3 + n^2 + \frac{-m+12}{6}n + 1.$$

Given an arbitrary integer  $d \ge 4$ , applying Lemma 2 with k = d - 3 repeatedly yields an integral convex polytope  $\mathcal{P}_m^{(d)}$  of dimension d such that

$$i(\mathcal{P}_m^{(d)}, n) = ((d-3)n+1)^{d-3}i(\mathcal{Q}_m, n)$$
$$= ((d-3)n+1)^{d-3} \left(\frac{m}{6}n^3 + n^2 + \frac{-m+12}{6}n + 1\right)$$

Let  $i(\mathcal{P}_m^{(d)}, n) = \sum_{i=0}^d c_i^{(d,m)} n^i$  with each  $c_i^{(d,m)} \in \mathbb{Q}$ . Then  $c_1^{(d,m)} = \frac{-m+12}{6} + A_1, \qquad c_2^{(d,m)} = 1 + \frac{-m+12}{6} A_1 + A_2$ 

$$c_j^{(d,m)} = \frac{m}{6}A_{j-3} + A_{j-2} + \frac{-m+12}{6}A_{j-1} + A_j, \qquad 3 \le j \le d-2,$$

where

$$A_{i} = (d-3)^{i} \binom{d-3}{i}_{2}, \qquad 0 \le i \le d-2.$$

Now, since each  $A_j$  is independent of m, it follows that each of  $c_1^{(d,m)}$  and  $c_2^{(d,m)}$  is negative for m sufficiently large. Let  $3 \leq j \leq d-2$ . One has

$$c_{j}^{(d,m)} = -\frac{A_{j-1} - A_{j-3}}{6}m + (A_{j-2} + 2A_{j-1} + A_{j})$$
$$= -(d-3)^{j-3}\frac{g(d,j)}{6}m + (A_{j-2} + 2A_{j-1} + A_{j}),$$

where g(d, j) is the same function as in Lemma 3. Since g(d, j) > 0, it follows that  $c_j^{(d,m)}$  can be negative for m sufficiently large. Hence, for m sufficiently large, the integral convex polytope  $\mathcal{P}_m^{(d)}$  of dimension d enjoys the required property.  $\Box$ 

We conclude this paper with

**Remark 4.** The polynomial

$$i(\mathcal{Q}_m, n) = \frac{m}{6}n^3 + n^2 + \frac{-m + 12}{6}n + 1$$
$$= \frac{1}{6}(n+1)(mn^2 + (6-m)n + 6)$$

has a real positive zero for m sufficient large. Hence  $i(\mathcal{P}_m^{(d)}, n)$  has a real positive zero for m sufficient large.

Thus in particular, for m sufficient large and for an arbitrary integral convex polytope  $\mathcal{Q}$ , the Ehrhart polynomial  $i(\mathcal{P}_m^{(d)} \times \mathcal{Q}, n)$  of  $\mathcal{P}_m^{(d)} \times \mathcal{Q}$  also possesses a negative coefficient.

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