# EHRHART POLYNOMIALS WITH NEGATIVE COEFFICIENTS 

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#### Abstract

It is shown that, for each $d \geq 4$, there exists an integral convex polytope $\mathcal{P}$ of dimension $d$ such that each of the coefficients of $n, n^{2}, \ldots, n^{d-2}$ of its Ehrhart polynomial $i(\mathcal{P}, n)$ is negative.


In his talk of the Clifford Lectures at Tulane University, 25-27 March 2010, Richard Stanley gave an Ehrhart polynomial with a negative coefficient. More precisely, the polynomial $\frac{13}{6} n^{3}+n^{2}-\frac{1}{6} n+1$ is the Ehrhart polynomial of the tetrahedron in $\mathbb{R}^{3}$ with vertices $(0,0,0),(1,0,0),(0,1,0)$ and $(1,1,13)$. See [1, Example 3.22]. His talk naturally inspired us to find integral convex polytopes of dimension $\geq 4$ whose Ehrhart polynomials possess negative coefficients. Consult [3, Part II] and [4, pp. 235-241] for fundamental materials on Ehrhart polynomials.

A convex polytope is called integral if any of its vertices has integer coordinates. Let $\mathcal{P} \subset \mathbb{R}^{N}$ be an integral convex polytope of dimension $d$ and $\partial \mathcal{P}$ the boundary of $\mathcal{P}$. We introduce the function $i(\mathcal{P}, n)$ by setting

$$
i(\mathcal{P}, n)=\sharp\left(n \mathcal{P} \cap \mathbb{Z}^{N}\right), \quad \text { for } \quad n=1,2, \ldots,
$$

where $n \mathcal{P}=\{n \alpha: \alpha \in \mathcal{P}\}$ and where $\sharp(X)$ is the cardinality of a finite set $X$. The study on $i(\mathcal{P}, n)$ originated in Ehrhart [2] who showed that $i(\mathcal{P}, n)$ is a polynomial in $n$ of degree $d$ with $i(\mathcal{P}, 0)=1$. Furthermore, the coefficients of $n^{d}$ and $n^{d-1}$ of $i(\mathcal{P}, n)$ are always positive ([1, Corollary 3.20 and Theorem 5.6]). We say that $i(\mathcal{P}, n)$ is the Ehrhart polynomial of $\mathcal{P}$.

The purpose of the present paper is, for each $d \geq 4$, to show the existence of an integral convex polytope of dimension $d$ such that each of the coefficients of $n, n^{2}, \ldots, n^{d-2}$ of its Ehrhart polynomial $i(\mathcal{P}, n)$ is negative. In fact,

Theorem 1. Given an arbitrary integer $d \geq 4$, there exists an integral convex polytope $\mathcal{P}$ of dimension $d$ such that each of the coefficients of $n, n^{2}, \ldots, n^{d-2}$ of the Ehrhart polynomial $i(\mathcal{P}, n)$ of $\mathcal{P}$ is negative.

Our proof of Theorem 1 will be given after preparing Lemmata 2 and 3.
Lemma 2. Let $\mathcal{P} \subset \mathbb{R}^{N}$ be an integral convex polytope of dimension $d$ and $i(\mathcal{P}, n)$ its Ehrhart polynomial. Then, given an arbitrary integer $k>0$, there exists an

[^0]integral convex polytope $\mathcal{P}_{k}^{\prime} \subset \mathbb{R}^{N+1}$ of dimension $d+1$ whose Ehrhart polynomial is equal to $(k n+1) i(\mathcal{P}, n)$.

Proof. It follows immediately that the Ehrhart polynomial $i\left(\mathcal{P}_{k}^{\prime}, n\right)$ of the integral convex polytope $\mathcal{P}_{k}^{\prime}=\mathcal{P} \times[0, k] \subset \mathbb{R}^{N+1}$ coincides with $(k n+1) i(\mathcal{P}, n)$.

Lemma 3. Let $d$ and $j$ be integers with $d \geq 5$ and $3 \leq j \leq d-2$, and

$$
g(d, j)=(d-3)^{2}\binom{d-3}{j-1}-\binom{d-3}{j-3}
$$

Then one has $g(d, j)>0$.
Proof. Since $d \geq 5$, one has $g(d, 3)=(d-3)^{2}\binom{d-3}{2}-1>0$ and $g(d, d-2)=$ $(d-3)^{2}-\binom{d-3}{2}>0$. Thus $g(d, j)>0$ for $j=3$ and $j=d-2$. Especially the assertion is true for $d=5$ and $d=6$.

We now work with induction on $d$. Let $d \geq 7$ and $4 \leq j \leq d-3$. Then

$$
\begin{aligned}
g(d, j) & =\left((d-4)^{2}+2 d-7\right)\left(\binom{d-4}{j-1}+\binom{d-4}{j-2}\right)-\left(\binom{d-4}{j-3}+\binom{d-4}{j-4}\right) \\
& =g(d-1, j)+g(d-1, j-1)+(2 d-7)\binom{d-3}{j-1} .
\end{aligned}
$$

It follows from the assumption of induction that $g(d-1, j)+g(d-1, j-1)>0$. Hence $g(d, j)>0$, as desired.

Proof of Theorem [1. It is known [1, Example 3.22] that, given an arbitrary integer $m \geq 1$, there exists an integral convex polytope $\mathcal{Q}_{m}$ of dimension 3 with

$$
i\left(\mathcal{Q}_{m}, n\right)=\frac{m}{6} n^{3}+n^{2}+\frac{-m+12}{6} n+1
$$

Given an arbitrary integer $d \geq 4$, applying Lemma 2 with $k=d-3$ repeatedly yields an integral convex polytope $\mathcal{P}_{m}^{(d)}$ of dimension $d$ such that

$$
\begin{aligned}
i\left(\mathcal{P}_{m}^{(d)}, n\right) & =((d-3) n+1)^{d-3} i\left(\mathcal{Q}_{m}, n\right) \\
& =((d-3) n+1)^{d-3}\left(\frac{m}{6} n^{3}+n^{2}+\frac{-m+12}{6} n+1\right)
\end{aligned}
$$

Let $i\left(\mathcal{P}_{m}^{(d)}, n\right)=\sum_{i=0}^{d} c_{i}^{(d, m)} n^{i}$ with each $c_{i}^{(d, m)} \in \mathbb{Q}$. Then

$$
c_{1}^{(d, m)}=\frac{-m+12}{6}+A_{1}, \quad c_{2}^{(d, m)}=1+\frac{-m+12}{6} A_{1}+A_{2}
$$

and

$$
c_{j}^{(d, m)}=\frac{m}{6} A_{j-3}+A_{j-2}+\frac{-m+12}{6} A_{j-1}+A_{j}, \quad 3 \leq j \leq d-2
$$

where

$$
A_{i}=(d-3)^{i}\binom{d-3}{i}, \quad 0 \leq i \leq d-2 .
$$

Now, since each $A_{j}$ is independent of $m$, it follows that each of $c_{1}^{(d, m)}$ and $c_{2}^{(d, m)}$ is negative for $m$ sufficiently large. Let $3 \leq j \leq d-2$. One has

$$
\begin{aligned}
c_{j}^{(d, m)} & =-\frac{A_{j-1}-A_{j-3}}{6} m+\left(A_{j-2}+2 A_{j-1}+A_{j}\right) \\
& =-(d-3)^{j-3} \frac{g(d, j)}{6} m+\left(A_{j-2}+2 A_{j-1}+A_{j}\right),
\end{aligned}
$$

where $g(d, j)$ is the same function as in Lemma 3. Since $g(d, j)>0$, it follows that $c_{j}^{(d, m)}$ can be negative for $m$ sufficiently large. Hence, for $m$ sufficiently large, the integral convex polytope $\mathcal{P}_{m}^{(d)}$ of dimension $d$ enjoys the required property.

We conclude this paper with
Remark 4. The polynomial

$$
\begin{aligned}
i\left(\mathcal{Q}_{m}, n\right) & =\frac{m}{6} n^{3}+n^{2}+\frac{-m+12}{6} n+1 \\
& =\frac{1}{6}(n+1)\left(m n^{2}+(6-m) n+6\right)
\end{aligned}
$$

has a real positive zero for $m$ sufficient large. Hence $i\left(\mathcal{P}_{m}^{(d)}, n\right)$ has a real positive zero for $m$ sufficient large.

Thus in particular, for $m$ sufficient large and for an arbitrary integral convex polytope $\mathcal{Q}$, the Ehrhart polynomial $i\left(\mathcal{P}_{m}^{(d)} \times \mathcal{Q}, n\right)$ of $\mathcal{P}_{m}^{(d)} \times \mathcal{Q}$ also possesses a negative coefficient.

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## References

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