# Matroid basis graph: Counting Hamiltonian cycles * 

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#### Abstract

We present exponential and super factorial lower bounds on the number of Hamiltonian cycles passing through any edge of the basis graphs of a graphic, generalized Catalan and uniform matroids. All lower bounds were obtained by a common general strategy based on counting appropriated cycles of length four in the corresponding matroid basis graph.


## 1 Introduction

For general background in matroid theory, we refer the reader to Oxley [14 and Welsh [17. A matroid $M=(E, \mathcal{B})$ of rank $r=r(M)$ is a finite set $E$ together with a nonempty collection $\mathcal{B}=\mathcal{B}(M)$ of $r$-subsets of $E$, called the bases of $M$, satisfying the following basis exchange axiom:
(BEA) If $B_{1}$ and $B_{2}$ are members of $\mathcal{B}$ and $e \in B_{1} \backslash B_{2}$, then there is an element $g \in B_{2} \backslash B_{1}$ such that $\left(B_{1}-e\right)+g \in \mathcal{B}$.

The basis graph $\mathrm{BG}(M)$ of a matroid $M$ is the graph having as vertex set the bases of $M$ and two vertices (bases) $B_{1}$ and $B_{2}$ are adjacent if and only if the symmetric difference $B_{1} \Delta B_{2}$ of $B_{1}$ and $B_{2}$ has cardinality two. A graph is a basis graph if it can be labeled to become the basis graph of some matroid. We make no distinction between a basis of $M$ and a vertex of $\mathrm{BG}(M)$.

Basis graphs have been extensively studied. Maurer 13 gave a complete characterization of those graphs that are basis graphs. Liu [10, 12, (11] investigated the connectivity of $\mathrm{BG}(M)$ and Donald, Holzmann, and Tobey 8 gave a

[^0]characterization of basis graphs of uniform matroids. Basis graphs are closely related to matroid basis polytopes. Indeed, Gel'fand and Serganova 9 proved that $\mathrm{BG}(M)$ is the 1 -skeleton of the basis polytope of $M$. We refer the reader to the work developed by Chatelain and Ramírez Alfonsín [5, 6] for further discussion and applications on this direction.

A graph $G$ is edge Hamiltonian if $G$ has order at least three and every edge is in a Hamiltonian cycle. According to Bondy and Ingleton [1], Haff (unpublished) showed that the basis graph $\mathrm{BG}(M)$ of every matroid $M$ is edge Hamiltonian, unless $\mathrm{BG}(M)$ is $K_{1}$ or $K_{2}$, generalizing a result due to Cummins [7] and Shank [15] for graphic matroids. So, if $\mathrm{BG}(M)$ has at least three vertices, then $\mathrm{BG}(M)$ is edge Hamiltonian. In fact, the work of Bondy and Ingleton [1, Theorem 1 and Theorem 2] about pancyclic graphs implies the edge Hamiltonicity proved by Haff.

In this paper, we investigate further the edge Hamiltonicity of $\mathrm{BG}(M)$ by defining the following function. For a given matroid $M$, we let

$$
\operatorname{HC}^{*}(M)=\min \left\{\operatorname{HC}_{e}(M): e \in E(\operatorname{BG}(M))\right\}
$$

where $\mathrm{HC}_{e}(M)$ denotes the number of different Hamiltonian cycles in $\mathrm{BG}(M)$ containing edge $e \in E(\mathrm{BG}(M))$. The function $\mathrm{HC}^{*}(M)$ naturally extends the edge Hamiltonicity. Bondy and Ingleton state that $\mathrm{HC}^{*}(M) \geq 1$ for every matroid $M$.

Along this paper, when we refer that an edge $e$ is in $t$ Hamiltonian cycles, we mean that $e$ is in at least $t$ different Hamiltonian cycles.

In Section 2, we give lower bounds on $\operatorname{HC}^{*}\left(M_{G}\right)$ where $M_{G}$ is the cycle matroid obtained from a $k$-edge-connected graph $G$. The lower bound for $k=$ 2,3 is exponential on the number of vertices of $G$ (Theorems 9 and 14). For $k \geq 4$, the lower bound is superfactorial on $k$ and is exponetial on the number of vertices (Theorem 151). In Section 3, we investigate $\mathrm{HC}^{*}(M)$ when $M$ is in the class of lattice path matroids. We present a lower bound on $\mathrm{HC}_{e}(M)$ when $M$ is a generalized Catalan matroid (Theorem 20). In particular, the derived lower bound for the $k$-Catalan matroid is superfactorial on $k$. Finally, we present a lower bound on $\mathrm{HC}^{*}(M)$ when $M$ is a uniform matroid (Theorem 22).

### 1.1 General strategy

In order to give a lower bound on $\mathrm{HC}^{*}\left(M_{G}\right)$, we follow the strategy described below, which has the same spirit as the one used by Bondy and Ingleton [1].

Let $M$ be a matroid and $\mathrm{BG}(M)$ be its basis graph. Let $B_{1}$ and $B_{2}$ be adjacent vertices (bases) in $\mathrm{BG}(M)$. By (BEA), there exist elements $e$ and $g$ of $M$, with $e \in B_{1} \backslash B_{2}$ and $g \in B_{2} \backslash B_{1}$, such that $B_{2}=B_{1}-e+g$. We define an ( $X, Y$ )-bipartition (determined by $e$ ) of the bases of $M$, with $X=\{B \in \mathcal{B}(M): e \in B\}$ and $Y=\{B \in \mathcal{B}(M): e \notin B\}$. The bases in $X(Y$, respectively) correspond exactly to the bases of the matroid $M^{\prime}=M / e$ obtained by contracting e $\left(M^{\prime \prime}=M \backslash e\right.$, obtained by deleting e, respectively). Moreover, $\mathrm{BG}\left(M^{\prime}\right)\left(\mathrm{BG}\left(M^{\prime \prime}\right)\right.$, respectively $)$ is $\mathrm{BG}(M)[X](\mathrm{BG}(M)[Y]$, respectively), which
is the subgraph of $\mathrm{BG}(M)$ induced by $X$ ( $Y$, respectively). Therefore, there is a $1-1$ correspondence between Hamiltonian cycles of $\mathrm{BG}\left(M^{\prime}\right)\left(\mathrm{BG}\left(M^{\prime \prime}\right)\right.$, respectively) and Hamiltonian cycles of $\mathrm{BG}(M)[X](\mathrm{BG}(M)[Y]$, respectively). For readability, we do not distinguish between $\mathrm{BG}\left(M^{\prime}\right)\left(\mathrm{BG}\left(M^{\prime \prime}\right)\right.$, respectively) and $\mathrm{BG}(M)[X](\mathrm{BG}(M)[Y]$, respectively).

A basis sequence $B_{1} B_{2} B_{3} B_{4}$ is a good cycle for $B_{1} B_{2}$ if it is a cycle (of length four) in $\mathrm{BG}(M)$, each of $B_{1}$ and $B_{4}$ contains $e$, and none of $B_{2}$ and $B_{3}$ contains $e$; that is, $B_{1}$ and $B_{4}$ are adjacent bases of $\mathrm{BG}\left(M^{\prime}\right)$ and $B_{2}$ and $B_{3}$ are adjacent bases of $\operatorname{BG}\left(M^{\prime \prime}\right)$ (Figure 11).


Figure 1: A good cycle $C_{e}=B_{1} B_{2} B_{3} B_{4}$ for $B_{1} B_{2}$.
If $C_{e}=B_{1} B_{2} B_{3} B_{4}$ is good, then the symmetric difference of a Hamiltonian cycle of $\mathrm{BG}\left(M^{\prime}\right)$ passing through the edge $B_{1} B_{4}$, the good $C_{e}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M^{\prime \prime}\right)$ passing through the edge $B_{2} B_{3}$ is a Hamiltonian cycle of $\mathrm{BG}(M)$.

So, if $\mathcal{C}\left(B_{1}, B_{2}\right)$ is the set of good cycles for $B_{1} B_{2}$, then

$$
\operatorname{HC}_{B_{1} B_{2}}(M) \geq \mathrm{HC}^{*}\left(M^{\prime}\right) \times\left|\mathcal{C}\left(B_{1}, B_{2}\right)\right| \times \mathrm{HC}^{*}\left(M^{\prime \prime}\right)
$$

This inequality suggests an inductive way to achieve a lower bound on $\mathrm{HC}^{*}(M)$. A key part in this approach involves proving a lower bound on the number of good cycles for any edge of $\mathrm{BG}(M)$.

## 2 Graphic matroids

In this section, we consider a graphic matroid $M_{G}$ where $G$ is a $k$-edge-connected graph of order $n$; that is, the elements of the ground set of $M_{G}$ are the edges of $G$ and a basis of $M_{G}$ corresponds to a spanning tree of $G$, thus a basis of $M_{G}$ contains exactly $n-1$ edges of $G$. Since loops of $G$ are in no basis of $M_{G}$, we always consider graphs with no loops. For readability, we do not distinguish between a basis of $M_{G}$ and a spanning tree of $G$. If $B$ is a basis of $M_{G}$ and $g$ is an edge of $G$ not in $B$, then $B+g$ induces a unique cycle (circuit) $C(g, B)$ in $G$ (in $M_{G}$, respectively) called the fundamental cycle (circuit, respectively) with respect to $g$ and $B$ [14].

First, note that, by Haff's result, if $G$ is a $k$-edge-connected graph of order $n \geq 3$, for $k \geq 2$, then the graph $\mathrm{BG}\left(M_{G}\right)$ has at least three vertices and is edge Hamiltonian.

Let $G^{\prime}=G / e$ be the graph resulting from contracting the edge $e$ of $G$ and then removing loops and let $G^{\prime \prime}=G \backslash e$ be the graph resulting from deleting the edge $e$.

Let $X$ and $Y$ be disjoint subsets of the vertex set $V(G)$. We denote by $E[X, Y](=E[Y, X])$ the set of edges of $G$ with one end in $X$ and the other end in $Y$, and by $e(X, Y)$ their number.

### 2.1 General structure of good cycles

Now, we fix the structure that we will use in the rest of Section 2 and, unless otherwise stated, we will follow this notation. The facts presented ahead show types of good cycles that this structure induces.

Let $G$ be a graph and $B_{1}$ and $B_{2}$ be bases of $M_{G}$ such that $B_{2}=B_{1}-e+g$. Let $f$ be an edge of $B_{1}-e$. Let $X$ be the vertex set of the component of $B_{1}-e$ that contains no end of $f$. Let $Z$ be the vertex set of the component of $B_{1}-f$ that contains no end of $e$. Let $Y=V(G) \backslash(X \cup Z)$.

Let $\mathcal{C}_{e}=\mathcal{C}\left(B_{1}, B_{2}\right)$ be the set of good cycles for $B_{1} B_{2}$. An arbitrary element of $\mathcal{C}_{e}$ is denoted by $C_{e}$, and is represented as $B_{1} B_{2} B_{3} B_{4}$. For $f \in B_{1}-e=B_{2}-g$, let $\mathcal{C}_{e}(f)=\left\{C_{e} \in \mathcal{C}_{e}: f \notin B_{4}\right\}$. An arbitrary element of $\mathcal{C}_{e}(f)$ is denoted by $C_{e}(f)$. For every $f^{\prime} \in B_{1}-e$ with $f^{\prime} \neq f$, since $f^{\prime}$ belongs to both $B_{3}$ and $B_{4}$ for every cycle $C_{e}(f)$, we have that $\mathcal{C}_{e}(f) \cap \mathcal{C}_{e}\left(f^{\prime}\right)=\emptyset$. Thus $\mathcal{C}_{e}=\bigcup\left\{\mathcal{C}_{e}(f): f \in B_{1}-e\right\}$. For every $w \notin B_{1}+g=B_{2}+e$, we denote by $\mathcal{C}_{e}(f, w)$ the set of cycles in $\mathcal{C}_{e}(f)$ such that $w \in B_{3}$. Similarly, $\mathcal{C}_{e}(f, w) \cap \mathcal{C}_{e}\left(f, w^{\prime}\right)=\emptyset$ for every $w^{\prime} \notin B_{1}+g$ with $w^{\prime} \neq w$. Therefore $\mathcal{C}_{e}(f)=\bigcup\left\{\mathcal{C}_{e}(f, w): w \notin B_{1}+g\right\}$. Summarizing, the following holds.

Remark 1. $\mathcal{C}_{e}(f) \cap \mathcal{C}_{e}\left(f^{\prime}\right)=\emptyset$ and $\mathcal{C}_{e}(f, w) \cap \mathcal{C}_{e}\left(f, w^{\prime}\right)=\emptyset$ for every $f, f^{\prime} \in$ $B_{1}-e$ with $f \neq f^{\prime}$ and every $w, w^{\prime} \notin B_{1}+g$ with $w \neq w^{\prime}$.

Fact 1. If $f$ is not in $C\left(g, B_{1}\right)$ and $w$ is an edge in $E[X \cup Y, Z]$ other than $f$, then there exists a good cycle $C_{e}(f, w)$ by defining

- $B_{4}=B_{1}-f+w$ and $B_{3}=B_{2}-f+w$.

Note that $B_{3}=B_{4}-e+g$. (See Figure (2)

Fact 2. If $f$ is in $C\left(g, B_{1}\right)$ and $\ell$ is an edge in $E[Y, Z]$ other than $f$, then there are two good cycles $C_{e}(f, \ell)$ by defining

- $B_{4}=B_{1}-f+\ell$ and $B_{3}=B_{2}-f+\ell$.
- $B_{4}=B_{1}-f+g$ and $B_{3}=B_{2}-f+\ell$.

Note that, in the first case, $B_{3}=B_{4}-e+g$ and, in the second case, $B_{3}=$ $B_{4}-e+\ell$. (See Figure 3)


| $B_{1}$ | $\{e, f, \ldots\}$ | $\{g, f, \ldots\}$ | $B_{2}$ |
| :--- | :--- | :--- | :--- |
| $B_{4}$ | $\{e, w, \ldots\}$ | $\{g, w, \ldots\}$ | $B_{3}$ |

Figure 2: Edge $f$ is in $B_{1}$. There are edges $w_{1}$ and $w_{2}$ between $X \cup Y$ and $Z$. The table shows a good $C_{e}(f, w)$ 's containing $B_{1} B_{2}$.


| $B_{1}$ | $\{e, f, \ldots\}$ | $\{g, f, \ldots\}$ | $B_{2}$ |
| :---: | :---: | :---: | :---: |
| $B_{4}$ | $\{e, \ell, \ldots\}$ | $\{g, \ell, \ldots\}$ | $B_{3}$ |
| $B_{1}$ | $\{e, f, \ldots\}$ | $\{g, f, \ldots\}$ | $B_{2}$ |
| $B_{4}$ | $\{e, g, \ldots\}$ | $\{g, \ell, \ldots\}$ | $B_{3}$ |

Figure 3: The bold edges are in $B_{1}$. There is an edge $\ell$ between $Y$ and $Z$. The table shows two good cycles $C_{e}(f, \ell)$ containing $B_{1} B_{2}$.

Fact 3. If $f$ is in $C\left(g, B_{1}\right)$ and $h$ is an edge in $E[X, Y]$ other than $e$, then there exists a good cycle $C_{e}(f, h)$ by defining

- $B_{4}=B_{1}-f+g$ and $B_{3}=B_{2}-f+h$.

Note that $B_{3}=B_{4}-e+h$. (See Figure 4.)
Fact 4. If $f$ is in $C\left(g, B_{1}\right)$ and $j$ is an edge in $E[X, Z]$ other than $g$, then there exists a good cycle $C_{e}(f, j)$ by defining

- $B_{4}=B_{1}-f+j$ and $B_{3}=B_{2}-g+j$.

Note that $B_{3}=B_{4}-e+f$.(See Figure 4)


| $B_{1}$ | $\{e, f, \ldots\}$ | $\{g, f, \ldots\}$ | $B_{2}$ |
| :--- | :--- | :--- | :--- |
| $B_{4}$ | $\{e, g, \ldots\}$ | $\{g, h, \ldots\}$ | $B_{3}$ |
| $B_{1}$ | $\{e, f, \ldots\}$ | $\{g, f, \ldots\}$ | $B_{2}$ |
| $B_{4}$ | $\{e, j, \ldots\}$ | $\{j, f, \ldots\}$ | $B_{3}$ |

Figure 4: The bold edges are in $B_{1}$. There is an edge $h$ between $X$ and $Y$, and an edge $j$ between $X$ and $Z$. The table shows the two good cycles $C_{e}$ containing $B_{1} B_{2}$.

### 2.2 2-edge-connected graphs

We start by giving a lower bound on $\mathrm{HC}^{*}\left(M_{G}\right)$ where $M_{G}$ is the cycle matroid obtained from a 2-edge-connected graph $G$. In what follows we shall use the notation introduced in the beginning of this section. In particular, we use extensively the facts and the structure of the vertex sets $X, Y, Z$ provided by adjacent bases $B_{1}, B_{2}$ and a convenient edge $f$ in $B_{1} \cap B_{2}$.

Proposition 2. Let $G$ be a 2-edge-connected graph. Let $B_{1}$ and $B_{2}$ be adjacent bases of $\operatorname{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$. Each edge $f$ of $B_{1}$ with at most one end in $C\left(g, B_{1}\right)$ provides a good cycle $C_{e}(f)$.

Proof. It follows from Fact 1 that, given an edge $f$ in $B_{1}$ with at most one end in $C\left(g, B_{1}\right)$, for every edge $w \in E[X \cup Y, Z]$ (one such $w$ exists because $G$ is 2-edge-connected), there exists a good cycle $C_{e}(f, w)$.

Proposition 3. Let $G$ be a connected graph. Let $B_{1}$ and $B_{2}$ be adjacent bases of $\mathrm{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$. For each edge $w$ not in $B_{1}$ with at most one end in $C\left(g, B_{1}\right)$, there exists an edge $f_{w} \in B_{1}-e$ that provides a good cycle $C_{e}\left(f_{w}, w\right)$.

Proof. Let $w$ be an edge not in $B_{1}$ with at most one end in $C\left(g, B_{1}\right)$. As at most one end of $w$ is in $C\left(g, B_{1}\right)$, there exists an edge $f_{w}$ of $B_{1}-C\left(g, B_{1}\right) \subseteq B_{1}-e$ in the fundamental cycle $C\left(w, B_{1}\right)$. It follows from Fact 1 that there exists a good cycle $C_{e}\left(f_{w}, w\right)$.

Proposition 4. Let $G$ be 2-edge-connected graph. Let $B_{1}$ and $B_{2}$ be adjacent bases of $\operatorname{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$. Suppose that $C\left(g, B_{1}\right)$ has length at least three. For each edge $w$ not in $B_{1}+g$ with both ends in $C\left(g, B_{1}\right)$, there exists an edge $f_{w} \in B_{1}-e$ that provides a good cycle $C_{e}\left(f_{w}, w\right)$.

Proof. As $C\left(g, B_{1}\right)$ has length at least three, $e$ and $g$ are not parallel edges. Let $w$ be an edge not in $B_{1}+g$ with both ends in $C\left(g, B_{1}\right)$.
Case 1. The edge $w$ is parallel to $g$.
Let $f_{w}$ be an edge of $C\left(g, B_{1}\right)-e-g$. In this case $w$ is as $j$ in Fact 4.
Case 2. The edge $w$ is not parallel to $g$ and the fundamental cycle $C\left(w, B_{1}\right)$ contains the edge $e$.

Let $f_{w}$ be an edge of $C\left(g, B_{1}\right)-e-g$ and not in $C\left(w, B_{1}\right)$. In this case $w$ is as $h$ in Fact 3

Case 3. The edge $w$ is not parallel to $g$ and the fundamental cycle $C\left(w, B_{1}\right)$ does not contain the edge e.

Let $f_{w}$ be an edge of $C\left(w, B_{1}\right)-w \subseteq B_{1}-e$. In this case $w$ is as $\ell$ in Fact 2 So, each case leads to one of the previously stated facts where we obtain an $f_{w}$ and a good cycle $C_{e}\left(f_{w}, w\right)$.

Lemma 5. If $G$ is a 2-edge-connected graph of order $n \geq 4$ and size at least $n+2$, then every edge of $\mathrm{BG}\left(M_{G}\right)$ is in two good cycles.

Proof. Let $B_{1}$ and $B_{2}$ be adjacent bases of $\mathrm{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$.
Suppose that $e$ and $g$ are parallel edges; that is, $C\left(g, B_{1}\right)$ is the 2 -cycle eg. Since $G$ has order $n \geq 4$, there are two edges in $B_{1}-e$. By Proposition 2, each one of them gives a good cycle, and they are distinct by Remark 1 .

Now, suppose that $e$ and $g$ are not parallel edges in $G$. Thus, $C\left(g, B_{1}\right)$ has length at least three. If there are two edges in $B_{1}$ with at most one end in $C\left(g, B_{1}\right)$ or two edges not in $B_{1}$ with at most one end in $C\left(g, B_{1}\right)$, by Propositions 2 and 3, respectively, we have two good cycles, distinct by Remark 11 so we are done. Also, if there are two edges not in $B_{1}+g$ with both ends in $C\left(g, B_{1}\right)$, then we are done by Proposition 4 and Remark [1.

Finally, as $G$ has size at least $n+2$, we may assume there exist an edge in $B_{1}$ with at most one end in $C\left(g, B_{1}\right)$ and an edge not in $B_{1}+g$ with both ends in $C\left(g, B_{1}\right)$. Therefore, by Propositions 2 and 4, respectively, and Remark 1 , the lemma follows.

The 1-sum $H \oplus_{1} H^{\prime}$ of two graphs $H$ and $H^{\prime}$ is the graph obtained from identifying a vertex of $H$ with a vertex of $H^{\prime}$.

Lemma 6. Let $G$ be a 2-edge-connected graph of order $n \geq 4$. There exists an edge in $\operatorname{BG}\left(M_{G}\right)$ not in two good cycles if and only if $G$ is either $C_{n}$ or $C_{2} \oplus_{1}$ $C_{n-1}$.

Proof. Let $m$ denote the number of edges of $G$. Since $G$ is 2-edge-connected, every edge is in a cycle, so $m \geq n$. If $m=n$, then $G$ is the $n$-cycle $C_{n}$ and no edge of $\mathrm{BG}\left(M_{G}\right)$ is in a good cycle. For $m \geq n+2$, Lemma 5 implies that every edge of $\operatorname{BG}\left(M_{G}\right)$ is in two good cycles. So, we may assume that $m=n+1$. Because every 2-edge-connected graph has a closed ear-decomposition [2], and $G$ has exactly $m+1$ edges, the closed ear-decomposition of $G$ consist of exactly two ears. Thus, $G$ is either
i) The 1-sum of two cycles, or
ii) The union of three internally disjoint paths that have the same two end vertices.

First, suppose that $G$ is the 1 -sum of two cycle, say $C^{1} \oplus_{1} C^{2}$. Since we only consider graphs with no loops, the length of both $C^{1}$ and $C^{2}$ is at least two. When the length of both $C^{1}$ and $C^{2}$ is at least three, Proposition 2 provides two good cycles for every edge of $\mathrm{BG}\left(M_{G}\right)$. Therefore, $G$ is $C_{2} \oplus_{1} C_{n-1}$ and it can be verified that there are adjacent bases $B_{1}$ and $B_{2}$ in $\operatorname{BG}\left(M_{G}\right)$ for which there is only one good cycle (Figure 5).

Now, suppose that $G$ is the union of three internally disjoint paths, say $P_{1}$, $P_{2}, P_{3}$, that have the same two end vertices. In this case we shall show that every edge of $\mathrm{BG}\left(M_{G}\right)$ is in two good cycles.

Let $B_{1} B_{2}$ be an edge of $\operatorname{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$. First, suppose that $e$ and $g$ are in the same path, say $P_{1}$. Thus, without loss of generality, all edges
of $P_{2}$ are in $B_{1}$, and there exists an edge $w$ in $P_{3}$ not in $B_{1}$. Let $f$ be an edge of $P_{2}$ (and therefore of $B_{1}$ ). Keeping our notation defined in Section [2.1, $w$ is in $E[Y, Z]$ and Fact 2 provides two good cycles $C_{e}(f, w)$.

Finally, suppose that $e$ and $g$ are in different paths; say $e$ belongs to $P_{1}$ and $g$ belongs to $P_{2}$. Thus, all edges of $P_{1}$ are in $B_{1}$, and there exists an edge $w$ in $P_{3}$ not in $B_{1}$. If there exists an edge $f \in B_{1}-e$ in $P_{1}$, then $w$ is in $E[X, Z]$ as $j$ in Fact 4. If there exists an edge $f \in B_{1}$ in $P_{2}$, then $w$ is in $E[X, Y]$ as $h$ in Fact 3. If there exists an edge $f \in B_{1}$ in $P_{3}$, then $w$ is in $E[X \cup Y, Z]$ as in Fact 1. In any case we get a good cycle $C_{e}(f, w)$. Since $G$ has order at least four, there are two edges $f, f^{\prime} \in B_{1}$ other than $e$. Therefore, by Remark 1 every edge $B_{1} B_{2}$ is in two good cycles, named $C_{e}(f, w)$ and $C_{e}\left(f^{\prime}, w\right)$.


Figure 5: A 2-edge-connected graph $G$ whose basis graph $\mathrm{BG}\left(M_{G}\right)$ has an edge, $B_{1} B_{2}$, with no two good cycles. The basis $B_{1}$ is the spanning tree in thick edges and $B_{2}=B_{1}-e+g$.

Proposition 7. For $n \geq 3$, every edge of $K_{n}$ is in $(n-2)$ ! Hamiltonian cycles.
Proof. Let $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ be the vertex set of $K_{n}$. Consider the edge $v_{i} v_{j}$. Without loss of generality, we may assume that $v_{i} v_{j}=v_{1} v_{2}$. Note that $v_{1} v_{2} v_{\sigma(3)} \ldots v_{\sigma(n)}$ is a Hamiltonian cycle for any permutation $\sigma$ of $\{3, \ldots, n\}$. Therefore, the number of Hamiltonian cycles containing the edge $v_{1} v_{2}$ is $(n-2)$ ! and the lemma follows.

Proposition 8. For $n \geq 3$, every edge of $K_{2} \square K_{n-1}$ is in $(n-2)$ ! $(n-3)$ ! Hamiltonian cycles.

Proof. If $n=3$, then $(n-2)!(n-3)!=1$ and $K_{2} \square K_{2}$ is $C_{4}$. So, we may assume that $n \geq 4$.

Let $\left\{u_{1}, u_{2}\right\}$ be the vertex set of $K_{2}$, and $\left\{v_{1}, \ldots, v_{n-1}\right\}$ be the vertex set of $K_{n-1}$. Let $K_{n-1}^{1}=\left\{\left(u_{1}, v_{1}\right), \ldots,\left(u_{1}, v_{n-1}\right)\right\}$ and $K_{n-1}^{2}=\left\{\left(u_{2}, v_{1}\right), \ldots,\left(u_{2}, v_{n-1}\right)\right\}$. There exists a natural partition $\left(K_{n-1}^{1}, K_{n-1}^{2}\right)$ of the vertex set of $K_{2} \square K_{n-1}$. For every permutation $\sigma$ of $\{1, \ldots, n-1\}$, let $C_{\sigma}^{\ell}$ be the Hamiltonian cycle $\left(u_{\ell}, v_{\sigma(1)}\right) \cdots\left(u_{\ell}, v_{\sigma(n-1)}\right)$ of $K_{n-1}^{\ell}$. Consider an edge $\left(u_{\ell}, v_{i}\right)\left(u_{\ell}, v_{j}\right)$ of $K_{2} \square K_{n-1}$. Without loss of generality we may assume that $\ell=i=1$ and $j=2$. For a permutation $\sigma_{1}$ with $\sigma_{1}(1)=1$ and $\sigma_{1}(2)=2$, let $\left(u_{1}, v_{x}\right)\left(u_{1}, v_{y}\right)$ be an edge of $C_{\sigma_{1}}^{1}$ other than $\left(u_{1}, v_{1}\right)\left(u_{1}, v_{2}\right)$. For every permutation $\sigma_{2}$ such that $\sigma_{2}(1)=x$ and $\sigma_{2}(2)=y$, the Hamiltonian cycle $C_{\sigma_{2}}^{2}$ of $K_{n-1}^{2}$ uses the edge $\left(u_{2}, v_{x}\right)\left(u_{2}, v_{y}\right)$. Let $S$ denote the cycle $\left(u_{1}, v_{x}\right)\left(u_{2}, v_{x}\right)\left(u_{2}, v_{y}\right)\left(u_{1}, v_{y}\right)$. The
symmetric difference $C_{\sigma_{1}}^{1} \Delta S \Delta C_{\sigma_{2}}^{2}$ is a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$ containing the edge $\left(u_{1}, v_{1}\right)\left(u_{1}, v_{2}\right)$. Since the edge $\left(u_{1}, v_{x}\right)\left(u_{1}, v_{y}\right)$ can be chosen in $n-2$ different ways, and the number of permutations $\sigma_{1}$ as well as the number of permutations $\sigma_{2}$ is $(n-3)$ !, we obtain $(n-2)((n-3)!)^{2}$ Hamiltonian cycles passing through the edge $\left(u_{1}, v_{1}\right)\left(u_{1}, v_{2}\right)$.

Now consider an edge $\left(u_{1}, v_{i}\right)\left(u_{2}, v_{i}\right)$ of $K_{2} \square K_{n-1}$. Without loss of generality we may assume that $i=1$. Consider the edge $\left(u_{1}, v_{1}\right)\left(u_{1}, v_{j}\right)$. Such edge is in $(n-3)$ ! Hamiltonian cycles $C^{1}$ of $K_{n-1}^{1}$. On the other hand, there are $(n-3)$ ! Hamiltonian cycles $C^{2}$ of $K_{n-1}^{2}$ passing through the edge $\left(u_{2}, v_{1}\right)\left(u_{2}, v_{j}\right)$. Let $S$ denote the cycle $\left(u_{1}, v_{1}\right)\left(u_{1}, v_{j}\right)\left(u_{2}, v_{j}\right)\left(u_{2}, v_{1}\right)$. The symmetric difference $C^{1} \Delta S \Delta C^{2}$ is a Hamiltonian cycle of $\operatorname{BG}\left(M_{G}\right)$ containing the edge $\left(u_{1}, v_{1}\right)\left(u_{2}, v_{1}\right)$. Since there are $(n-3)!$ cycles $C^{1}$, as well as cycles $C^{2}$, and the edge $\left(u_{1}, v_{1}\right)\left(u_{1}, v_{j}\right)$ can be chosen in $n-2$ different ways, we obtain $(n-2)((n-3)!)^{2}$ Hamiltonian cycles containing the edge $\left(u_{1}, v_{1}\right)\left(u_{2}, v_{1}\right)$. Note that $(n-2)((n-3)!)^{2}=$ $(n-2)!(n-3)$ !, hence the proposition follows.

Theorem 9. If $G$ is a 2-edge-connected graph of order $n \geq 3$, then every edge of $\mathrm{BG}\left(M_{G}\right)$ is in $2^{n-3}$ Hamiltonian cycles.

Proof. The proof is by induction on $n$. If $n=3$, then $2^{n-3}=1$ and the theorem follows from the edge Hamiltonicity of $\mathrm{BG}\left(M_{G}\right)$. So we may assume that $n \geq 4$.

By Lemma 6, if there exists an edge in $\mathrm{BG}\left(M_{G}\right)$ not in two good cycles, then $G$ is either $C_{n}$ or $C_{2} \oplus_{1} C_{n-1}$. If $G=C_{n}$, then $\mathrm{BG}\left(M_{G}\right)=K_{n}$ and by Proposition 7 every edge of $\mathrm{BG}\left(M_{G}\right)$ is in $(n-2)!\geq 2^{n-3}$ Hamiltonian cycles. If $G=C_{2} \oplus_{1} C_{n-1}$, then $\operatorname{BG}\left(M_{G}\right)=K_{2} \square K_{n-1}$ and by Proposition 8 every edge of $\operatorname{BG}\left(M_{G}\right)$ is in $(n-2)!(n-3)!\geq 2^{n-3}$ Hamiltonian cycles. Therefore, we may assume that $G$ has at least $n+1$ edges and every edge of $\operatorname{BG}\left(M_{G}\right)$ is in two good cycles.

Let $B_{1} B_{2}$ be an edge of $\operatorname{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$. Let $G^{\prime}=G / e$ and $G^{\prime \prime}=G \backslash e$. As $G^{\prime}$ is 2-edge-connected of order $n-1 \geq 3$, by the induction hypothesis, every edge of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ is in $2^{n-4}$ Hamiltonian cycles in $\mathrm{BG}\left(M_{G^{\prime}}\right)$. As $G^{\prime \prime}$ has $n \geq 4$ vertices and at least $n$ edges, $G^{\prime \prime}$ has at least two spanning trees, and therefore $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ is either $K_{2}$ or edge Hamiltonian.

Let $C_{e}=B_{1} B_{2} B_{3} B_{4}$ be a good cycle. If $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ is $K_{2}$, then the symmetric difference of $C_{e}$ and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ containing $B_{1} B_{4}$ is a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$. On the other hand, if $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ is edge Hamiltonian, then the symmetric difference of $C_{e}$, a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ containing $B_{1} B_{4}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ containing $B_{2} B_{3}$ is a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$. As every edge of $\mathrm{BG}\left(M_{G}\right)$ is in two good cycles, in either case we conclude that every edge of $\mathrm{BG}\left(M_{G}\right)$ is in $2^{n-4} \cdot 2 \cdot 1=2^{n-3}$ Hamiltonian cycles.

## $2.3 k$-edge-connected graphs

Now, we turn our attention to counting Hamiltonian cycles in the basis graph of the cycle matroid of $k$-edge-connected graphs for $k \geq 3$.

Lemma 10. If $G$ is a $k$-edge-connected graph of order $n \geq 3$ for $k \geq 3$, then there are $(n-2)(k-1)$ good cycles for every edge of $\mathrm{BG}\left(M_{G}\right)$.

Proof. Let $B_{1} B_{2}$ be an edge of $\mathrm{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$, and let $f \in B_{1}-e$. First we show that there are $k-1$ good cycles in $\mathcal{C}_{e}(f)$.

As before, let $X$ be the vertex set of the component of $B_{1}-e$ that contains no end of $f$, let $Z$ be the vertex set of the component of $B_{1}-f$ that contains no end of $e$, and let $Y=V(G) \backslash(X \cup Z)$.

Suppose that $f$ is not in $C\left(g, B_{1}\right)$. Since $G$ is $k$-edge-connected, the edge set $E[X \cup Y, Z]$ contains $k-1$ edges distinct from $f$. It follows from Fact $\mathbb{1}$ that there is a good cycle in $\mathcal{C}_{e}(f)$ for each of these edges. Besides, for different edges in $E[X \cup Y, Z]$, the corresponding bases $B_{3}$ are different and so are their corresponding good cycles in $\mathcal{C}_{e}(f)$.

Now, suppose that $f$ is in $C\left(g, B_{1}\right)$. It follows from Fact 2 that there are two good cycles in $\mathcal{C}_{e}(f)$ for every edge in $E[Y, Z] \backslash\{f\}$, from Fact 3 that there is one good cycle in $\mathcal{C}_{e}(f)$ for every edge in $E[X, Y] \backslash\{e\}$, and from Fact团that there is one good cycle in $\mathcal{C}_{e}(f)$ for every edge in $E[X, Z] \backslash\{g\}$. Thus, as these good cycles are distinct, if $(e(X, Y)-1)+(e(X, Z)-1)+2(e(Y, Z)-1) \geq k-1$, we would indeed have $k-1$ good cycles in $\mathcal{C}_{e}(f)$.

By the $k$-edge-connectivity of $G$, we get that

$$
\begin{array}{lll}
|E[Y, X \cup Z] \backslash\{e, f\}| & =e(X, Y)+e(Y, Z)-2 & \geq k-2, \\
|E[Z, X \cup Y] \backslash\{f, g\}| & =e(X, Z)+e(Y, Z)-2 & \geq k-2 . \tag{2}
\end{array}
$$

Hence, summing (11) and (22), we get that $e(X, Y)+e(X, Z)+2 e(Y, Z)-4 \geq 2 k-4$, and $2 k-4 \geq k-1$ as $k \geq 3$. So we have $k-1$ good cycles in $\mathcal{C}_{e}(f)$.

By Remark $\mathbb{1}$ and as there are $n-2$ choices for $f$, there are $(n-2)(k-1)$ good cycles for every edge of $\mathrm{BG}\left(M_{G}\right)$.

Lemma 11. Let $G$ be a 3 -edge-connected graph of order $n \geq 3$ and let $e$ be an edge of $G$. Then $\mathrm{HC}^{*}\left(M_{G}\right) \geq \mathrm{HC}^{*}\left(M_{G / e}\right)$ and $\mathrm{HC}^{*}\left(M_{G}\right) \geq \mathrm{HC}^{*}\left(M_{G \backslash e}\right)$.
Proof. Let $X=\left\{B \in \mathcal{B}\left(M_{G}\right): e \in B\right\}$ and $Y=\left\{B \in \mathcal{B}\left(M_{G}\right): e \notin B\right\}$. Note that $(X, Y)$ is a bipartition of the vertices (bases) of $\mathrm{BG}\left(M_{G}\right)$. Let $G^{\prime}=G / e$ and $G^{\prime \prime}=G \backslash e$. As $G^{\prime}$ is 3 -edge-connected of order $n-1 \geq 2$, the basis graph $\mathrm{BG}\left(M_{G^{\prime}}\right)$ has at least three vertices and is edge Hamiltonian. Similarly, $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ also has at least three vertices and thus is edge Hamiltonian. There is a one-to-one correspondence between the bases in $X$ and the bases of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ and between the bases in $Y$ and the bases of $\operatorname{BG}\left(M_{G^{\prime \prime}}\right)$.

The edge set of $\mathrm{BG}\left(M_{G}\right)$ can be partitioned into: (i) edges with both ends in $\operatorname{BG}\left(M_{G^{\prime}}\right)$, called yellow edges, (ii) edges with one end in $\operatorname{BG}\left(M_{G^{\prime}}\right)$ and the other one in $\operatorname{BG}\left(M_{G^{\prime \prime}}\right)$, called pink edges, and (iii) edges with both ends in $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$, called orange edges.

Case 1. Hamiltonian cycles passing through a pink edge.
Let $B_{1} B_{2}$ be a pink edge. For every good cycle $C_{e}=B_{1} B_{2} B_{3} B_{4}$, the edge $B_{1} B_{4}$ is an edge of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ and $B_{2} B_{3}$ is an edge of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$. The symmetric difference of a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ containing the edge $B_{1} B_{4}$, the good cycle $C_{e}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ containing the edge $B_{2} B_{3}$ is a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$ containing the edge $B_{1} B_{2}$. By Lemma 10 there exists at least one such good cycle $C_{e}$. Thus $\mathrm{HC}_{B_{1} B_{2}}\left(M_{G}\right) \geq \mathrm{HC}^{*}\left(M_{G^{\prime}}\right)$ and $\mathrm{HC}_{B_{1} B_{2}}\left(M_{G}\right) \geq \mathrm{HC}^{*}\left(M_{G^{\prime \prime}}\right)$.
Case 2. Hamiltonian cycles passing through a yellow edge.
First we prove that every yellow edge belongs to a good cycle $C_{e}$. Let $B_{1} B_{4}$ be a yellow edge, say $B_{4}=B_{1}-f+w$. The subgraph $B_{4}-e$ has exactly two components. Since $G$ is 3 -edge-connected, there exists an edge $g$ other than $e$ and $f$ connecting the two components of $B_{4}-e$. Therefore $e \in C\left(g, B_{4}\right)$ and $B_{3}=B_{4}-e+g$ is a basis. If $B_{1}-e+g$ is a basis, we set $B_{2}=B_{1}-e+g$ and we have a good cycle $C_{e}=B_{1} B_{2} B_{3} B_{4}$. So, we may assume that $B_{1}-e+g$ is not a basis. Thus $e \notin C\left(g, B_{1}\right)$. Since $e \in C\left(g, B_{4}\right)$, we have that $C\left(g, B_{1}\right) \neq C\left(g, B_{4}\right)$. This implies that $f \in C\left(g, B_{1}\right)$ and $w \in C\left(g, B_{4}\right)$ since $B_{1}$ and $B_{4}$ only differ by $f$ and $w$. Hence, we know that $C\left(g, B_{1}\right) \Delta C\left(g, B_{4}\right)$, the symmetric difference of $C\left(g, B_{1}\right)$ and $C\left(g, B_{4}\right),(i)$ contains $e, f, w$ but not $g$; (ii) is contained in $B_{1} \cup B_{4}=B_{1}+w$; and (iii) contains a cycle. Therefore, the only cycle contained in $C\left(g, B_{1}\right) \Delta C\left(g, B_{4}\right)$ is $C\left(w, B_{1}\right)$. In this case, $B_{2}=$ $B_{1}-e+w$ is a basis and there is also a good cycle $C_{e}=B_{1} B_{2} B_{3} B_{4}$.

Now, every yellow edge $B_{1} B_{4}$ is in a Hamiltonian cycle $C$ of $\mathrm{BG}\left(M_{G^{\prime}}\right)$. Let us extend $C$ to a Hamiltonian cycle of $\mathrm{BG}(M)$ as follows. Consider an edge $B_{1}^{\prime} B_{4}^{\prime}$ in $C$ other than $B_{1} B_{4}$. The symmetric difference of $C$, a good cycle containing the yellow edge $B_{1}^{\prime} B_{4}^{\prime}$, say $B_{1}^{\prime} B_{2}^{\prime} B_{3}^{\prime} B_{4}^{\prime}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ containing the edge $B_{2}^{\prime} B_{3}^{\prime}$ is an extension of $C$ to a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$ containing the edge $B_{1} B_{4}$. Thus $\mathrm{HC}_{B_{1} B_{4}}\left(M_{G}\right) \geq \mathrm{HC}^{*}\left(M_{G^{\prime \prime}}\right)$ because $B_{2}^{\prime} B_{3}^{\prime}$ is in $\mathrm{HC}^{*}\left(M_{G^{\prime \prime}}\right)$ Hamiltonian cycles of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$.

On the other hand, since $\mathrm{BG}\left(M_{G^{\prime}}\right)$ is a simple graph, two Hamiltonian cycles of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ passing through the edge $B_{1} B_{4}$, say $C$ and $C^{\prime}$, differ in at least two edges. Therefore $C$ and $C^{\prime}$ are extended to different Hamiltonian cycles of $\mathrm{BG}\left(M_{G}\right)$. Hence $\mathrm{HC}_{B_{1} B_{4}}\left(M_{G}\right) \geq \mathrm{HC}^{*}\left(M_{G^{\prime}}\right)$.

Case 3. Hamiltonian cycles passing through an orange edge.
First we prove that every Hamiltonian cycle $C$ in $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ contains two edges, say $B_{2} B_{3}$ and $B_{2}^{\prime} B_{3}^{\prime}$, each of which is in a good cycle in $\mathcal{C}_{e}$. As $G$ has at least three vertices and is 3-edge-connected, there exist an edge $f$ in $G$ not parallel to $e$ and a basis of $M_{G^{\prime \prime}}$ not containing $f$. By traversing $C$, we pass through edges $B_{2} B_{3}$ and $B_{3}^{\prime} B_{2}^{\prime}$ such that $B_{3}=B_{2}-f+w$ and $B_{2}^{\prime}=B_{3}^{\prime}+f-w^{\prime}$ for some elements $w$ and $w^{\prime}$. We shall prove that there exists a $C_{e}$ containing $B_{2} B_{3}$ and a $C_{e}^{\prime}$ containing $B_{2}^{\prime} B_{3}^{\prime}$.

As $f$ is not parallel to $e$, there exists an edge $g \in C\left(e, B_{2}\right)$ other than $f$. Therefore $B_{1}=B_{2}-g+e$ is a basis. If $B_{3}-g+e$ is also a basis, we set
$B_{4}=B_{3}-g+e$ and obtain a good cycle $C_{e}$ with $B_{2} B_{3}$. So, we may assume that $B_{3}-g+e$ is not a basis. Thus $g \notin C\left(e, B_{3}\right)$. Since $g \in C\left(e, B_{2}\right)$, we have that $C\left(e, B_{2}\right) \neq C\left(e, B_{3}\right)$. This implies that $f \in C\left(e, B_{2}\right)$ and $w \in C\left(e, B_{3}\right)$ since $B_{2}$ and $B_{3}$ only differ by $f$ and $w$. In this case $B_{4}=B_{3}-w+e$ is a basis and we obtain a good cycle $C_{e}$ with $B_{2} B_{3}$. This completes the proof for $B_{2} B_{3}$. The proof for $B_{2}^{\prime} B_{3}^{\prime}$ is analogous.

Every orange edge $B_{2} B_{3}$ is in a Hamiltonian cycle $C$ of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$. Since $C$ contains two edges, each in a good cycle, there exists an edge $B_{2}^{\prime} B_{3}^{\prime}$, distinct from $B_{2} B_{3}$, in a good cycle $C_{e}^{\prime}$, say $C_{e}^{\prime}=B_{1}^{\prime} B_{2}^{\prime} B_{3}^{\prime} B_{4}^{\prime}$. The symmetric difference of $C$, the good cycle $C_{e}^{\prime}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ passing through the edge $B_{1}^{\prime} B_{4}^{\prime}$ is a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$ containing the edge $B_{2} B_{3}$. Because there are $\mathrm{HC}^{*}\left(M_{G^{\prime}}\right)$ Hamiltonian cycles passing through the edge $B_{1}^{\prime} B_{4}^{\prime}$, we have that $\mathrm{HC}_{B_{2} B_{3}}\left(M_{G}\right) \geq \mathrm{HC}^{*}\left(M_{G^{\prime}}\right)$. As $B_{2} B_{3}$ is in $\mathrm{HC}^{*}\left(M_{G^{\prime \prime}}\right)$ Hamiltonian cycles of $\operatorname{BG}\left(M_{G^{\prime \prime}}\right)$, and two distinct Hamiltonian cycles differ in at least two edges, we have that $\operatorname{HC}_{B_{2} B_{3}}\left(M_{G}\right) \geq \mathrm{HC}^{*}\left(M_{G^{\prime \prime}}\right)$.

In order to give a bound on $\mathrm{HC}^{*}\left(M_{G}\right)$, we define the function

$$
\operatorname{hc}(n, k)=\min \left\{\mathrm{HC}^{*}\left(M_{G}\right): G \text { is a } k \text {-edge-connected graph of order } n\right\}
$$

Proposition 12. For $k, n \geq 3$, $\mathrm{hc}(n, k) \geq(n-2)(k-1) \mathrm{hc}(n-1, k) \mathrm{hc}(n, k-1)$.
Proof. Let $G$ be a $k$-edge-connected graph of order $n$ such that $\operatorname{HC}^{*}\left(M_{G}\right)=$ hc $(n, k)$. By Lemma [10, there are $(n-2)(k-1)$ good cycles for every edge of $\mathrm{BG}\left(M_{G}\right)$. Let $B_{1} B_{2}$ be an edge of $\mathrm{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$, and let $G^{\prime}=G / e$ and $G^{\prime \prime}=G \backslash e$. The symmetric difference of a good cycle $C_{e}=$ $B_{1} B_{2} B_{3} B_{4}$, a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ containing $B_{1} B_{4}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ containing $B_{2} B_{3}$ is a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$ containing $B_{1} B_{2}$. Hence, $B_{1} B_{2}$ is in $(n-2)(k-1) \mathrm{HC}^{*}\left(M_{G^{\prime}}\right) \mathrm{HC}^{*}\left(M_{G^{\prime \prime}}\right)$ Hamiltonian cycles of $\operatorname{BG}\left(M_{G}\right)$. Now, as $G^{\prime}$ is $k$-edge-connected of order $n-1$, we have that $\operatorname{HC}^{*}\left(M_{G^{\prime}}\right) \geq \operatorname{hc}(n-1, k)$ and, as $G^{\prime \prime}$ is $(k-1)$-edge-connected of order $n$, we have that $\operatorname{HC}^{*}\left(M_{G^{\prime \prime}}\right) \geq \mathrm{hc}(n, k-1)$. Therefore we conclude that $\operatorname{hc}(n, k)=\operatorname{HC}^{*}\left(M_{G}\right) \geq(n-2)(k-1) \operatorname{hc}(n-1, k) \operatorname{hc}(n, k-1)$.

The superfactorial $\operatorname{sf}(x)$ of a positive integer $x$ is the number $x!(x-1)!\cdots 0$ !
Theorem 13. For $k \geq 3, \operatorname{hc}(3, k) \geq \operatorname{sf}(k-1)$.
Proof. We use induction on $k$. Let $G$ be a $k$-edge-connected graph of order three such that $\mathrm{HC}^{*}\left(M_{G}\right)=\mathrm{hc}(3, k)$. Let $B_{1} B_{2}$ be an edge of $\mathrm{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$. Let $G^{\prime}=G / e$ and $G^{\prime \prime}=G \backslash e$. The graph $G^{\prime}$ is $k$-edge-connected of order two.

If $k=3$, then $\mathrm{BG}\left(M_{G^{\prime}}\right)$ has three vertices and is edge Hamiltonian. The graph $G^{\prime \prime}$ has three vertices and is 2-edge-connected, therefore $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ has three vertices and is edge Hamiltonian. By Lemma 10, the edge $B_{1} B_{2}$ is in two good cycles in $\mathcal{C}_{e}$. The symmetric difference of a good cycle $C_{e}=B_{1} B_{2} B_{3} B_{4}$, a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ containing $B_{1} B_{4}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ containing $B_{2} B_{3}$ is a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$ containing $B_{1} B_{2}$.

Hence, every edge of $\mathrm{BG}\left(M_{G}\right)$ is in two Hamiltonian cycles and hc $(3, k)=$ $\mathrm{HC}^{*}\left(M_{G}\right) \geq 2$.

Suppose $k>3$. By the induction hypothesis, $\mathrm{hc}(3, k-1) \geq \operatorname{sf}(k-2)$. The basis graph $\mathrm{BG}\left(M_{G^{\prime}}\right)$ is a complete graph on at least $k$ vertices, thus $\mathrm{HC}^{*}\left(M_{G^{\prime}}\right) \geq(k-2)$ ! Since $G^{\prime \prime}$ is a $(k-1)$-edge-connected graph of order three, every edge of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ is in hc $(3, k-1)$ Hamiltonian cycles. By Lemma 10 the edge $B_{1} B_{2}$ is in $k-1$ good cycles in $\mathcal{C}_{e}$. The symmetric difference of a good cycle $C_{e}=B_{1} B_{2} B_{3} B_{4}$, a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ containing $B_{1} B_{2}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ containing $B_{2} B_{3}$ is a Hamiltonian cycle of $\operatorname{BG}\left(M_{G}\right)$ containing $B_{1} B_{2}$. Hence, every edge of $\operatorname{BG}\left(M_{G}\right)$ is in $(k-$ $2)!(k-1) \mathrm{hc}(3, k-1) \geq(k-1)!\operatorname{sf}(k-2)=\operatorname{sf}(k-1)$ Hamiltonian cycles. Thus, $\mathrm{HC}^{*}\left(M_{G}\right)=\mathrm{hc}(3, k) \geq \operatorname{sf}(k-1)$.

Theorem 14. For $n \geq 3, \operatorname{hc}(n, 3) \geq(n-2)!2^{\binom{n-1}{2}}$.
Proof. We use induction on $n$. Let $G$ be a 3-edge-connected graph of order $n \geq 3$ such that $\mathrm{HC}^{*}\left(M_{G}\right)=\operatorname{hc}(n, 3)$. Let $B_{1} B_{2}$ be an edge of $\mathrm{BG}\left(M_{G}\right)$, say $B_{2}=B_{1}-e+g$. Let $G^{\prime}=G / e$ and $G^{\prime \prime}=G \backslash e$. Note that $G^{\prime}$ is a 3 -edge-connected graph of order $n-1$.

If $n=3$, then $G^{\prime}$ is a 3-edge-connected graph of order two. So $\mathrm{BG}\left(M_{G^{\prime}}\right)$ is the complete graph $K_{m}$, where $m \geq 3$ is the number of edges of $G^{\prime}$. (Remember that we remove loops of $G^{\prime}$ if any.) The graph $G^{\prime \prime}$ has three vertices and is 2-edge-connected, therefore $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ has at least three vertices and is edge Hamiltonian. By Lemma 10, the edge $B_{1} B_{2}$ is in two good cycles in $\mathcal{C}_{e}$. The symmetric difference of a good cycle $C_{e}=B_{1} B_{2} B_{3} B_{4}$, a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ containing $B_{1} B_{4}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ containing $B_{2} B_{3}$ is a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$ containing $B_{1} B_{2}$. Hence, every edge of $\mathrm{BG}\left(M_{G}\right)$ is in two Hamiltonian cycles, and so $\mathrm{HC}^{*}\left(M_{G}\right)=\mathrm{hc}(3,3) \geq 2$.

Suppose $n>3$. By the induction hypothesis, hc $(n-1,3) \geq(n-3)!2^{\binom{n-2}{2}}$. Since $G^{\prime}$ is a 3 -edge-connected graph of order $n-1$, every edge of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ is in hc $(n-1,3)$ Hamiltonian cycles. By Theorem 9, as $G^{\prime \prime}$ is 2-edge-connected of order $n$, every edge of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ is in $2^{n-3}$ Hamiltonian cycles. By Lemma 10 , the edge $B_{1} B_{2}$ is in $2(n-2)$ good cycles in $\mathcal{C}_{e}$. The symmetric difference of a good cycle $C_{e}=B_{1} B_{2} B_{3} B_{4}$, a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime}}\right)$ containing $B_{1} B_{4}$, and a Hamiltonian cycle of $\mathrm{BG}\left(M_{G^{\prime \prime}}\right)$ containing $B_{2} B_{3}$ is a Hamiltonian cycle of $\mathrm{BG}\left(M_{G}\right)$ containing $B_{1} B_{2}$. Therefore, every edge of $\mathrm{BG}\left(M_{G}\right)$ is in $(n-$ 3)! $\left.2^{\left(n_{2}^{2}\right)} 2(n-2) 2^{n-3}=(n-2)!2^{(n-1} 2\right)$ Hamiltonian cycles. Thus, $\operatorname{HC}^{*}\left(M_{G}\right)=$ $\operatorname{hc}(n, 3) \geq(n-2)!2^{\binom{n-1}{2}}$.

The next theorem gives a bound on $\operatorname{hc}(n, k)$ for $n \geq 4$ and $k \geq 4$.
Theorem 15. For $n, k \geq 4$,

$$
\operatorname{hc}(n, k) \geq \frac{2^{\binom{n+k-4}{n-3} \cdot 3^{\binom{n+k-7}{k-3}}} \prod_{r=4}^{k}(r \operatorname{sf}(r-1))^{\binom{n+k-4-r}{n-4}} \cdot \prod_{s=4}^{n}(s-1)!\binom{n+k-4-s}{k-4} . . . ~}{n-1)}
$$

Proof. The proof is by induction on $n+k$ and uses repeatedly Proposition 12
For $n=k=4$, we apply Theorem 13 to hc $(3,4)$ and Theorem 14 to hc $(4,3)$ :

$$
\begin{aligned}
\operatorname{hc}(4,4) & \geq 2 \cdot 3 \mathrm{hc}(3,4) \mathrm{hc}(4,3) \geq 6[\operatorname{sf}(3)] \cdot\left[2!2^{\binom{3}{2}}\right]=3!\cdot \operatorname{sf}(3) 2 \cdot 2^{3} \\
& =2^{\binom{4}{1}} \cdot \frac{3 \cdot 4}{3 \cdot 4} \operatorname{sf}(3) \cdot 3!=\frac{2^{\binom{4}{1}} \cdot 3^{\binom{1}{1}}}{3 \cdot 4}(4 \operatorname{sf}(3))^{\binom{0}{0}} \cdot 3!\binom{0}{0} .
\end{aligned}
$$

The bound on hc $(4, k)$ for $k \geq 5$ comes from applying Theorem 13 to hc $(3, k)$ and the induction hypothesis on hc $(4, k-1)$ :

$$
\begin{aligned}
\operatorname{hc}(4, k) & \geq 2(k-1)[\operatorname{sf}(k-1)] \cdot\left[\frac{2^{\binom{k-1}{1}} \cdot 3^{\binom{k-4}{k-4}}}{3(k-1)} \cdot\left(\prod_{r=4}^{k-1}(r \operatorname{sf}(r-1))^{\binom{k-1-r}{0}}\right) \cdot 3!^{\binom{k-5}{k-5}}\right] \\
& =\frac{2}{3} \operatorname{sf}(k-1) \cdot 2^{k-1} \cdot 3^{\binom{k-3}{k-3}} \cdot\left(\prod_{r=4}^{k-1}(r \operatorname{sf}(r-1))^{\binom{k-r}{0}}\right) \cdot 3!\binom{k-4}{k-4} \\
& =\frac{2^{\binom{k}{1}} \cdot 3^{\binom{k-3}{k-3}}}{3 k} \cdot\left(\prod_{r=4}^{k}(r \operatorname{sf}(r-1))^{\binom{k-r}{0}}\right) \cdot 3!^{\binom{k-4}{k-4}}
\end{aligned}
$$

Similarly, the bound on $\mathrm{hc}(n, 4)$ for $n \geq 5$ comes from applying the induction hypothesis on $\mathrm{hc}(n-1,4)$ and Theorem 14 to $\mathrm{hc}(n, 3)$ :

$$
\begin{aligned}
\operatorname{hc}(n, 4) & \geq(n-2) 3\left[\frac{2^{\binom{n-1}{n-4}} \cdot 3^{\binom{n-4}{1}}}{(n-2) 4} \cdot(4 \operatorname{sf}(3))^{\binom{n-5}{n-5}} \cdot \prod_{s=4}^{n-1}(s-1)!^{\binom{n-1-s}{0}}\right] \cdot\left[(n-2)!2^{\binom{n-1}{2}}\right] \\
& =\frac{3}{4} 2^{\binom{n-1}{n-4}} \cdot 3^{n-4} \cdot(4 \operatorname{sf}(3))^{\binom{n-4}{n-4}} \cdot\left(\prod_{s=4}^{n-1}(s-1)!^{\binom{n-s}{0}}\right) \cdot(n-2)!2^{\binom{n-1}{n-3}} \\
& =\frac{2^{\binom{n-1}{n-4}+\binom{n-1}{n-3}} \cdot 3^{n-3}}{(n-1) 4}(n-1) \cdot(4 \operatorname{sf}(3))^{\binom{n-4}{n-4}} \cdot\left(\prod_{s=4}^{n-1}(s-1)!^{\binom{n-s}{0}}\right) \cdot(n-2)! \\
& =\frac{2^{\binom{n}{n-3}} \cdot 3^{\binom{n-3}{1}}}{(n-1) 4} \cdot(4 \operatorname{sf}(3))^{\binom{n-4}{n-4}} \cdot\left(\prod_{s=4}^{n}(s-1)!^{\binom{n-s}{0}}\right) .
\end{aligned}
$$

Finally, the bound on $\mathrm{hc}(n, k)$ for $n, k \geq 5$ comes from applying the induction hypothesis on both $\operatorname{hc}(n-1, k)$ and $\operatorname{hc}(n, k-1)$ :

$$
\begin{aligned}
\mathrm{hc}(n, k) \geq & \left.(n-2)(k-1) \frac{2^{\left.2^{n+k-5} \begin{array}{c}
n-4
\end{array}\right) \cdot 3^{\binom{n+k-8}{k-3}}}}{(n-2) k} \prod_{r=4}^{k}(r \operatorname{sf}(r-1))^{\binom{n+k-5-r}{n-5}} \cdot \prod_{s=4}^{n-1}(s-1)\right)^{\binom{n+k-5-s}{k-4}} \\
& \cdot \frac{2^{\binom{n+k-5}{n-3}} \cdot 3^{\binom{n+k-8}{k-4}}}{(n-1)(k-1)} \prod_{r=4}^{k-1}(r \operatorname{sf}(r-1))^{\binom{n+k-5-r}{n-4}} \cdot \prod_{s=4}^{n}(s-1)!^{\binom{n+k-5-s}{k-5}} \\
= & \frac{2^{\binom{n+k-4}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1) k}\left(\prod_{r=4}^{k-1}(r \operatorname{sf}(r-1))^{\binom{n+k-4-r}{n-4}}\right) \cdot(k \operatorname{sf}(k-1))^{\binom{n-5}{n-5}}
\end{aligned}
$$

$$
\begin{aligned}
& \cdot\left(\prod_{s=4}^{n-1}(s-1)!!_{\binom{n+k-4-s}{k-4}}^{\substack{\left.k-5 \\
!^{k-5} \\
k-5\\
\right)}}\right. \\
& =\frac{2^{\binom{n+k-4}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1) k} \prod_{r=4}^{k}(r \operatorname{sf}(r-1))^{\binom{n+k-4-r}{n-4}} \cdot \prod_{s=4}^{n}(s-1)!\binom{n+k-4-s}{k-4} \text {. }
\end{aligned}
$$

This completes the proof of the theorem.
The following corollary follows from mathematical manipulations on the right side of the inequality given by Theorem 15 and it gives a more explicit and concise expression.

Corollary 16. For $n>k \geq 5$,

$$
\operatorname{hc}(n, k)>\prod_{r=3}^{n} \operatorname{sf}(r-1)\binom{n+k-5-r}{n-6}+\binom{n+k-4-r}{n-4}+\binom{n+k-5-r}{k-5} .
$$

Proof. We start proving two auxiliary equalities that shall be used to prove the corollary. Firstly,

$$
\begin{align*}
2\binom{n+k-4-2}{n-4} \cdot 3^{\binom{n+k-4-3}{k-3}} \cdot \prod_{r=4}^{k} r\binom{n+k-4-r}{n-4} & =\prod_{r=2}^{k} r\binom{n+k-4-r}{n-4} \\
& =\prod_{r=2}^{k} r!\binom{n+k-4-r}{n-4}-\binom{n+k-4-(r+1)}{n-4}  \tag{3}\\
& =\prod_{r=2}^{k} r!\binom{n+k-5-r}{n-5} \\
& =\prod_{r=2}^{k} \operatorname{sf}(r)\left(\begin{array}{c}
\substack{n+k-5-r \\
n-5}
\end{array}\right)-\binom{n+k-5-(r+1)}{n-5}  \tag{4}\\
& =\prod_{r=2}^{k} \operatorname{sf}(r)^{\binom{n+k-6-r}{n-6}} \\
& \left.=\prod_{r=3}^{k+1} \operatorname{sf}(r-1)^{\substack{n+k-6-(r-1) \\
n-6}}\right) \\
& =\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-5-r}{n-6}}
\end{align*}
$$

Equalities (3) and (4) follow from the hypothesis that $n \geq 6$.
Secondly,

$$
2^{\binom{n+k-4-3}{k-4}} \cdot \prod_{s=4}^{n}(s-1)!\binom{n+k-4-s}{k-4}=\prod_{s=3}^{n}(s-1)!\binom{n+k-4-s}{k-4}
$$

$$
\begin{align*}
& =\prod_{s=3}^{n} \operatorname{sf}(s-1)^{\binom{n+k-4-s}{k-4}-\binom{n+k-4-(s+1)}{k-4}}  \tag{5}\\
& =\prod_{s=3}^{n} \operatorname{sf}(s-1)^{\binom{n+k-5-s}{k-5}}
\end{align*}
$$

Equality (5) follows from the hypothesis that $k \geq 5$.
Thus, by Theorem 15, we have that

$$
\begin{align*}
& \mathrm{hc}(n, k) \geq \frac{2^{\binom{n+k-4}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1) k} \cdot \prod_{r=4}^{k}(r \operatorname{sf}(r-1))^{\binom{n+k-4-r}{n-4}} \cdot \prod_{s=4}^{n}(s-1)!^{\binom{n+k-4-s}{k-4}} \\
& =\frac{2^{\binom{n+k-4}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1) k} \cdot \prod_{r=4}^{k} r^{\binom{n+k-4-4}{n-4}} \\
& \cdot \prod_{r=4}^{k} \operatorname{sf}(r-1)\left(\begin{array}{c}
\binom{n+k-4-r}{n-4}
\end{array}\right. \\
& \cdot \prod_{s=4}^{n}(s-1)!\left(\begin{array}{c}
\binom{n k-4-s}{k-4}
\end{array}\right. \\
& =\frac{2^{\binom{n-k-5}{n-4}}}{(n-1) k} \cdot\left(2^{\binom{n+k-6}{n-4}} \cdot 3^{\binom{n+k-7}{k-3}} \prod_{r=4}^{k} r^{\binom{n+k-4-4}{n-4}}\right)  \tag{6}\\
& \cdot\left(2^{\binom{n+k-7}{n-4}} \prod_{r=4}^{k} \operatorname{sf}(r-1)^{\binom{n+k-4-r}{n-4}}\right) \\
& \cdot\left(2^{\binom{n-k-7}{n-3}} \cdot \prod_{s=4}^{n}(s-1)!^{\binom{n+k-4-s}{k-4}}\right) \\
& =\frac{2^{\binom{n+k-5}{n-4}}}{(n-1) k} \cdot\left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\left(\begin{array}{c}
\binom{n+5-5-r}{n-6}
\end{array}\right), ~}\right.  \tag{7}\\
& \cdot\left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-4-r}{n-4}}\right) \\
& \left.\cdot\left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\substack{n+k-5-r \\
k-5}}\right)\right) \\
& \left.=\frac{2^{\binom{n k-5}{n-4}}}{(n-1) k} \cdot \prod_{r=3}^{n} \operatorname{sf}(r-1)\right)^{\binom{n+k-5-r}{n-6}+\binom{n+k-4-r}{n-4}+\binom{n+k-5-r}{k-5}} \\
& >\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-5-r}{n-6}+\binom{n+k-4-r}{n-4}+\binom{n+k-5-r}{k-5}} .
\end{align*}
$$

Equality (6) holds because $\binom{n+k-4}{n-3}=\binom{n+k-7}{n-3}+\binom{n+k-7}{n-4}+\binom{n+k-6}{n-4}+\binom{n+k-5}{n-4}$, and (7) follows from the two previous equalities.

## 3 Generalized Catalan matroids

In this section we address a special class of transversal matroids introduced by Bonin, de Mier, and Noy [4]. We follow the description of Bonin and de Mier [3] and Stanley [16].

Let $S$ be a subset of $\mathbb{Z}^{d}$. A lattice path $L$ in $\mathbb{Z}^{d}$ of length $k$ with steps in $S$ is a sequence $v_{0}, \ldots, v_{k} \in \mathbb{Z}^{d}$ such that each consecutive difference $s_{j}=v_{j}-v_{j-1}$ lies in $S$. We call $s_{j}$ the $j$ th step of the lattice path $L$. We say that $L$ starts at $v_{0}$ and ends at $v_{k}$, or simply that $L$ goes from $v_{0}$ to $v_{k}$.

All lattice paths we consider are in $\mathbb{Z}^{2}$, start at $(0,0)$ and end at $(m, r)$, and use steps in $S=\{(1,0),(0,1)\}$. We call the steps $(1,0)$ and $(0,1)$ as East $(E)$ and North $(N)$, respectively. Sometimes it is convenient to represent a lattice path $L$ as a sequence of steps; that is, as a word of length $m+r$ on the alphabet $\{E, N\}$; other times, as a subset of $\{1, \ldots, m+r\}=[m+r]$, say $\{j: j$ th step of $L$ is $N\}$.

Let $P$ and $Q$ be lattice paths from $(0,0)$ to $(m, r)$ with $P$ never going above $Q$. Let $\mathcal{P}$ be the set of all lattice paths from $(0,0)$ to $(m, r)$ that go neither below $P$ nor above $Q$. For each $i$ with $1 \leq i \leq r$, let $A_{i}$ be the set

$$
A_{i}=\{j: j \text { th step is the } i \text { th North for some path in } \mathcal{P}\} .
$$

Observe that $A_{1}, \ldots, A_{r}$ are intervals $A_{i}=\left[a_{i}, b_{i}\right]$ in $[m+r]$. Moreover $a_{1}<$ $\cdots<a_{r}$ and $b_{1}<\cdots<b_{r}$; and $a_{i}$ and $b_{i}$ correspond to the positions of the $i$ th North step of $Q$ and $P$, respectively. An example is shown in Figure 6.


Figure 6: Lattice paths $P$ and $Q$ from $(0,0)$ to $(10,8)$ and the corresponding sets $A_{1}, \ldots, A_{8}$. Representations of $P$ and $Q$ as words of length $10+8$ in the alphabet $\{E, N\}$ and as subsets of $[10+8]$. Lattice path $B$ goes neither below $P$ nor above $Q$ and its representations as a word and as a subset.

Let $M[P, Q]$ be the transversal matroid on the ground set $[m+r]$ and $\left(A_{1}, \ldots, A_{r}\right)$ its presentation. We call $\left(A_{1}, \ldots, A_{r}\right)$ the standard presentation of $M[P, Q]$. Note that $M[P, Q]$ has rank $r$ and corank (or nullity) $m$. A transversal matroid is a lattice path matroid if it is a matroid of the type $M[P, Q]$. Each basis of $M[P, Q]$ corresponds to a lattice path from $(0,0)$ to $(m, r)$ that goes neither below $P$ nor above $Q$. Figure 6 shows an illustration of a matroid $M[P, Q]$ and a basis $B$.

Let $M[P, Q]$ be a lattice path matroid. Let $P=y_{1} \cdots y_{i} \cdots y_{m+r}\left(=y^{[m+r]}\right)$ and $Q=x_{1} \cdots x_{i} \cdots x_{m+r}\left(=x^{[m+r]}\right)$, with $x_{i}, y_{i} \in\{N, E\}$ for $i \in[m+r]$.

A generalized Catalan matroid is a lattice path matroid $M[P, Q]$, where $P=$ $E^{m} N^{r}$. We simply write $M[Q]$ for generalized Catalan matroids. The class of generalized Catalan matroid is minor-closed [3, Theorem 4.2]. The k-Catalan matroid is the generalized Catalan matroid $M\left[(N E)^{k}\right]$; that is, $Q=(N E)^{k}$.

Lemma 17. Let $M[Q]$ be a generalized Catalan matroid of rank $r$ and corank $m$, for $m \geq r \geq 2$, with neither a loop nor an isthmus. Then, every edge of $\mathrm{BG}(M[Q])$ is in $r-1$ good cycles.

Proof. As $M[Q]$ has neither a loop nor an isthmus, the first step of $Q$ is North and the last one is East. By convenience, we consider the bases of $M[Q]$ as words of length $m+r$ in the alphabet $\{N, E\}$.

Let $B_{1} B_{2}$ be an edge of $\operatorname{BG}(M[Q])$, say $B_{2}=B_{1}-e+g$. Thus, $B_{1}=$ $x^{[m+r]}, B_{2}=y^{[m+r]}$, and there exist indices $e$ and $g$ such that $x_{e}=y_{g}=N$, $x_{g}=y_{e}=E$, and $x_{\ell}=y_{\ell}$ for $\ell \neq e, g$. Without loss of generality we may assume that $e<g$.
Case 1. There exists an index $\ell$ less than $e$ (and therefore less than $g$ ) such that $x_{\ell}=y_{\ell}=N$ (Figure 7).

Let $f$ be the least index such that $x_{f}=y_{f}=N$. For every index $w$ such that $x_{w}=y_{w}=E$, basis $B_{4}$ rises by switching $x_{w}$ for $N$ and $x_{f}$ for $E$ in $B_{1}$ and basis $B_{3}$ rises by switching $y_{w}$ for $N$ and $y_{f}$ for $E$ in $B_{2}$; that is, $B_{4}=B_{1}-f+w$ and $B_{3}=B_{2}-f+w$. Since the first step of $Q$ is North and the last one is East, the paths corresponding to the words $B_{3}$ and $B_{4}$, respectively, are in $M[Q]$. Thus, for every common $E$ step of $B_{1}$ and $B_{2}$, we obtain a good cycle $C_{e}$. Therefore, there are $m-1$ good $C_{e}$ passing through the edge $B_{1} B_{2}$.


Figure 7: Illustration of lattice paths corresponding to Case 1.

Case 2. There exists an index $\ell$ greater that $g$ (and therefore greater than $e$ ) such that $x_{\ell}=y_{\ell}=E$.

Let $w$ be the last index such that $x_{w}=y_{w}=E$. For every index $f$ such that $x_{f}=y_{f}=N$, basis $B_{4}$ rises by switching $x_{f}$ for $E$ and $x_{w}$ for $N$ in $B_{1}$ and basis $B_{3}$ rises by switching $y_{f}$ for $E$ and $y_{w}$ for $N$ in $B_{2}$; that is, $B_{4}=B_{1}-f+w$ and $B_{3}=B_{2}-f+w$. Since the first step of $Q$ is North and the last one is East, the paths corresponding to the words $B_{3}$ and $B_{4}$, respectively, are in $M[Q]$. Thus, for every common $N$ step of $B_{1}$ and $B_{2}$, we obtain a good cycle $C_{e}$. Therefore, there are $r-1$ good $C_{e}$ passing through the edge $B_{1} B_{2}$.
Case 3. There exist no indices $\ell$ and $\ell^{\prime}$ with $\ell<e$ and $\ell^{\prime}>g$ such that $x_{\ell}=$ $y_{\ell}=N$ and $x_{\ell^{\prime}}=y_{\ell^{\prime}}=E$.

Thus, $x_{e}$ is the first $N$ in $B_{1}$ and $x_{g}$ is the last $E$. Let $x_{h}$ be the penultimate $E$ in $B_{1}$. Such $x_{h}$ exists because $m \geq r \geq 2$. As $y_{g}$ is $N$ in $B_{2}, y_{h}$ is the last $E$ in $B_{2}$.

In order to count the number of good cycles, we partition the $N$ 's in the words corresponding to the bases $B_{1}$ and $B_{2}$ in maximal blocks, and for each $N$ we shall show a good cycle associated with it.

Block of Type I. Consider the block $x_{i} \cdots x_{w-1} x_{w}$ such that $x_{i}=\cdots=$ $x_{w-1}=N$ and $x_{w}=E$ with $e<i<w<g$.

Also, we have that $y_{i}=\cdots=y_{w-1}=N$ and $y_{w}=E$. For every $f \in$ $\{i, \ldots, w-1\}$, basis $B_{4}$ rises by switching $x_{f}$ for $E$ and $x_{w}$ for $N$ in $B_{1}$ and basis $B_{3}$ rises by switching $y_{f}$ for $E$ and $y_{w}$ for $N$ in $B_{2}$; that is, $B_{4}=B_{1}-f+w$ and $B_{3}=B_{2}-f+w$.

Block of Type II. Consider the block $x_{i} \cdots x_{w-1} x_{w}$ such that $x_{i}=\cdots x_{w-1}=$ $N$ and $x_{w}=E$ with $e<i<w=g$.

Also, we have that $y_{i}=\cdots=y_{w-1}=N$. Let $x_{h}$ the penultimate $E$ in $B_{1}$. As $y_{w}=y_{g}$ is $N$ in $B_{2}, y_{h}$ is the last $E$ in $B_{2}$. For every $f \in\{i, \ldots, g-1\}$, basis $B_{4}$ rises by switching $x_{f}$ for $E$ and $x_{g}$ for $N$ in $B_{1}$, and basis $B_{3}$ rises by switching $y_{f}$ for $E$ and $y_{h}$ for $N$ in $B_{2}$; that is, $B_{4}=B_{1}-f+g$ and $B_{3}=B_{2}-f+h$.

Block of Type III. Consider a block $x_{g+1} \cdots x_{m+r}$ of $N$ 's in $B_{1}$.
Also, we have that $y_{g+1} \cdots y_{m+r}$ is a block of $N$ 's in $B_{2}$. For every element $f \in\{g+1, \ldots, m+r\}$, basis $B_{4}$ rises by switching $x_{f}$ for $E$ and $x_{g}$ for $N$ in $B_{1}$, and basis $B_{3}$ rises by switching $y_{f}$ for $E$ and $y_{h}$ for $N$ in $B_{2}$; that is, $B_{4}=B_{1}-f+g$ and $B_{3}=B_{2}-f+h$.

Since every $N$ distinct of $x_{e}$ belongs to some type of block, we get $r-1$ good $C_{e}$ passing through the edge $B_{1} B_{2}$.

Bonin and de Mier [3] observed that the class of all generalized Catalan matroids is closed under duals. Moreover, a basis $B^{*}$ of the dual of $M[P, Q]$ corresponds to the $E$ steps of the basis $B$ in $M[P, Q]$. Therefore, the following is a consequence of this fact and Lemma 17

Corollary 18. For $r, m \geq 2$, let $M[Q]$ be a generalized Catalan matroid of rank $r$ and corank $m$, with neither a loop nor an isthmus. Then every edge of $\mathrm{BG}(M[Q])$ is in $\min \{r-1, m-1\}$ good cycles.

Let $M[P, Q]$ be a lattice path matroid. Let $P=y^{[m+r]}$ and $Q=x^{[m+r]}$, with $x_{i}, y_{i} \in\{N, E\}$ for $i \in[m+r]$. Assume $e$ is neither a loop nor an isthmus. In [3] was observed that:
(1) $M[P, Q] \backslash e$ is the lattice path matroid $M\left[P^{\prime}, Q^{\prime}\right]$ where the upper bounding path $Q^{\prime}$ is formed by deleting from $Q$ the first $E$ step that is at or after step $e$; the lower bounding path $P^{\prime}$ is formed by deleting from $P$ the last $E$ step that is at or before step $x$.
(2) $M[P, Q] / e$ is the lattice path matroid $M\left[P^{\prime \prime}, Q^{\prime \prime}\right]$ where the upper bounding path $Q^{\prime \prime}$ is formed by deleting from $Q$ the last $N$ step that is at or before step $e$; the lower bounding path $P^{\prime \prime}$ is formed by deleting from $P$ the first $N$ step that is at or after step $e$.

Observation 1. If the $k$-Catalan matroid is a minor of the generalized Catalan matroid $M[Q]$, then for every element e of $M[Q]$, the $(k-1)$-Catalan matroid is a minor of both the generalized Catalan matroid $M[Q] \backslash e$ and $M[Q] / e$.

In fact, Observation 1 also holds if we replace generalized Catalan matroid for lattice path matroid.

For the class of generalized Catalan matroids, we define the function
$\operatorname{hc}_{L}(k)=\min \left\{\mathrm{HC}^{*}(M[Q]): M[Q]\right.$ has a $k$-Catalan matroid as a minor $\}$.
Proposition 19. For $k \geq 2, \mathrm{hc}_{L}(k) \geq(k-1) \mathrm{hc}_{L}(k-1)^{2}$.
Proof. Let $M[Q]=M_{Q}$ be a generalized Catalan matroid such that $\mathrm{HC}^{*}\left(M_{Q}\right)=$ $\mathrm{hc}_{L}(k)$. We may assume that $M_{Q}$ has neither a loop nor an isthmus. Thus, both the rank and corank of $M$ are at least $k$. By Corollary [18, there are $\min \{r-1, m-1\} \geq k-1$ good cycles for every edge of $\operatorname{BG}\left(M_{Q}\right)$. Let $B_{1} B_{2}$ be an edge of $\operatorname{BG}\left(M_{Q}\right)$, say $B_{1}=B_{2}-e+g$, and let $M^{\prime}=M_{Q} \backslash e$ and $M^{\prime \prime}=M_{Q} / e$. It follows from Observation 1 that both $M^{\prime}$ and $M^{\prime \prime}$ contain a $(k-1)$-Catalan matroid as a minor. Thus $\mathrm{HC}^{*}\left(M^{\prime}\right) \geq \mathrm{hc}_{L}(k-1)$ and $\mathrm{HC}^{*}\left(M^{\prime \prime}\right) \geq \mathrm{hc}_{L}(k-1)$. Therefore we conclude that hc $c_{L}(k) \geq(k-1) \mathrm{hc}_{L}(k-1)^{2}$.

Theorem 20. For $k \geq 2, \operatorname{hc}_{L}(k) \geq \operatorname{sf}(k-1) \operatorname{sf}(k-2)$.
Proof. The proof is by induction on $k$. We write simply $M_{Q}$ instead $M[Q]$. Let $M_{Q}$ be a generalized Catalan matroid such that $\mathrm{HC}^{*}\left(M_{Q}\right)=\mathrm{hc}_{L}(k)$. We may assume that $M_{Q}$ has neither a loop nor a isthmus. In particular, $M_{Q}$ has both rank and corank at least $k$. Let $k=2$. So $\mathrm{BG}\left(M_{Q}\right)$ has at least three vertices and is edge Hamiltonian. Therefore $\mathrm{hc}_{L}(2) \geq 1=\operatorname{sf}(1) \operatorname{sf}(0)$.

Now let $k \geq 3$. Let $B_{1} B_{2}$ be an edge of $\mathrm{BG}\left(M_{Q}\right)$, say $B_{2}=B_{1}-e+g$. By Corollary 18, the edge $B_{1} B_{2}$ is in $\min \{r-1, m-1\} \geq k-1$ good cycles.

Table 1: The three types of good cycles for $U_{r, n}$.


Consider $M^{\prime}=M_{Q} \backslash e$ and $M^{\prime \prime}=M_{Q} / e$. By Observation 1 the $(k-1)$-Catalan matroid is a minor of both $M^{\prime}$ and $M^{\prime \prime}$. Thus, by the induction hypothesis, $\mathrm{HC}^{*}\left(M^{\prime}\right), \mathrm{HC}^{*}\left(M^{\prime \prime}\right) \geq \mathrm{hc}_{L}(k-1) \geq \mathrm{sf}(k-2) \operatorname{sf}(k-3)$. Hence, every edge of $\mathrm{BG}\left(M_{Q}\right)$ is in $(k-1)(\operatorname{sf}(k-2) \operatorname{sf}(k-3))^{2} \geq \operatorname{sf}(k-1) \operatorname{sf}(k-2)$ Hamiltonian cycles.

### 3.1 Uniform matroids

Recall that the set of bases of the uniform matroid of rank $r$ on $n$ elements, denoted by $U_{r, n}$, consists of all $r$-subsets of $[n]$. Also, $U_{r, n}$ can be considered as the lattice path matroid $M[P, Q]$ where $Q=N^{r} E^{n-r}$ and $P=E^{n-r} N^{r}$.

Let $B_{1} B_{2}$ be an edge of $\mathrm{BG}\left(U_{r, n}\right)$, say $B_{1}=B_{2}-e+g$. So, we have that $B_{1}=\left\{e, f_{2}, \ldots, f_{r}\right\}$ and $B_{2}=\left\{g, f_{2}, \ldots, f_{r}\right\}$, with $f_{i} \in[n] \backslash\{e, g\}$ for $i \in\{2, \ldots, r\}$. For every $w$ in $[n] \backslash\left\{e, g, f_{2}, \ldots, f_{r}\right\}$, we can obtain three types of good cycles in $C_{e}$ by replacing an $f_{i}$ by $w$ as shown in Table 1 We thus have the following result.

Proposition 21. Let $n>r \geq 1$ be integers. Then every edge of $\mathrm{BG}\left(U_{r, n}\right)$ is in $3(n-r-1)(r-1)$ good cycles.

Finally, the next theorem can be proved by induction on the number of elements of the matroid, applying Proposition 21 and following the same strategy as above.

Theorem 22. Let $n>r \geq 1$ be integers. Then every edge of $\mathrm{BG}\left(U_{r, n}\right)$ is in $((n-r-1)!(r-1)!)^{\min \{n-r-1, r-1\}}$ Hamiltonian cycles.

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