Matroid basis graph: Counting Hamiltonian cycles *

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Abstract

We present exponential and super factorial lower bounds on the number of Hamiltonian cycles passing through any edge of the basis graphs of a graphic, generalized Catalan and uniform matroids. All lower bounds were obtained by a common general strategy based on counting appropriated cycles of length four in the corresponding matroid basis graph.

1 Introduction

For general background in matroid theory, we refer the reader to Oxley [14] and Welsh [17]. A matroid $M = (E, \mathcal{B})$ of rank r = r(M) is a finite set E together with a nonempty collection $\mathcal{B} = \mathcal{B}(M)$ of r-subsets of E, called the bases of M, satisfying the following basis exchange axiom:

(*BEA*) If B_1 and B_2 are members of \mathcal{B} and $e \in B_1 \setminus B_2$,

then there is an element $g \in B_2 \setminus B_1$ such that $(B_1 - e) + g \in \mathcal{B}$.

The basis graph BG(M) of a matroid M is the graph having as vertex set the bases of M and two vertices (bases) B_1 and B_2 are adjacent if and only if the symmetric difference $B_1\Delta B_2$ of B_1 and B_2 has cardinality two. A graph is a basis graph if it can be labeled to become the basis graph of some matroid. We make no distinction between a basis of M and a vertex of BG(M).

Basis graphs have been extensively studied. Maurer [13] gave a complete characterization of those graphs that are basis graphs. Liu [10, 12, 11] investigated the connectivity of BG(M) and Donald, Holzmann, and Tobey [8] gave a

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characterization of basis graphs of uniform matroids. Basis graphs are closely related to *matroid basis polytopes*. Indeed, Gel'fand and Serganova [9] proved that BG(M) is the 1-skeleton of the *basis polytope* of M. We refer the reader to the work developed by Chatelain and Ramírez Alfonsín [5, 6] for further discussion and applications on this direction.

A graph G is edge Hamiltonian if G has order at least three and every edge is in a Hamiltonian cycle. According to Bondy and Ingleton [1], Haff (unpublished) showed that the basis graph BG(M) of every matroid M is edge Hamiltonian, unless BG(M) is K_1 or K_2 , generalizing a result due to Cummins [7] and Shank [15] for graphic matroids. So, if BG(M) has at least three vertices, then BG(M) is edge Hamiltonian. In fact, the work of Bondy and Ingleton [1, Theorem 1 and Theorem 2] about pancyclic graphs implies the edge Hamiltonicity proved by Haff.

In this paper, we investigate further the edge Hamiltonicity of BG(M) by defining the following function. For a given matroid M, we let

$$\operatorname{HC}^*(M) = \min\{\operatorname{HC}_e(M) : e \in E(\operatorname{BG}(M))\}$$

where $\operatorname{HC}_e(M)$ denotes the number of different Hamiltonian cycles in $\operatorname{BG}(M)$ containing edge $e \in E(\operatorname{BG}(M))$. The function $\operatorname{HC}^*(M)$ naturally extends the edge Hamiltonicity. Bondy and Ingleton state that $\operatorname{HC}^*(M) \geq 1$ for every matroid M.

Along this paper, when we refer that an edge e is in t Hamiltonian cycles, we mean that e is in at least t different Hamiltonian cycles.

In Section 2, we give lower bounds on $\mathrm{HC}^*(M_G)$ where M_G is the cycle matroid obtained from a k-edge-connected graph G. The lower bound for k =2,3 is exponential on the number of vertices of G (Theorems 9 and 14). For $k \geq 4$, the lower bound is superfactorial on k and is exponential on the number of vertices (Theorem 15). In Section 3, we investigate $\mathrm{HC}^*(M)$ when M is in the class of lattice path matroids. We present a lower bound on $\mathrm{HC}_e(M)$ when M is a generalized Catalan matroid (Theorem 20). In particular, the derived lower bound for the k-Catalan matroid is superfactorial on k. Finally, we present a lower bound on $\mathrm{HC}^*(M)$ when M is a uniform matroid (Theorem 22).

1.1 General strategy

In order to give a lower bound on $HC^*(M_G)$, we follow the strategy described below, which has the same spirit as the one used by Bondy and Ingleton [1].

Let M be a matroid and BG(M) be its basis graph. Let B_1 and B_2 be adjacent vertices (bases) in BG(M). By (*BEA*), there exist elements e and g of M, with $e \in B_1 \setminus B_2$ and $g \in B_2 \setminus B_1$, such that $B_2 = B_1 - e + g$. We define an (X, Y)-bipartition (determined by e) of the bases of M, with $X = \{B \in \mathcal{B}(M): e \in B\}$ and $Y = \{B \in \mathcal{B}(M): e \notin B\}$. The bases in X (Y, respectively) correspond exactly to the bases of the matroid M' = M/e obtained by contracting e ($M'' = M \setminus e$, obtained by deleting e, respectively). Moreover, BG(M') (BG(M''), respectively) is BG(M)[X] (BG(M)[Y], respectively), which is the subgraph of BG(M) induced by X (Y, respectively). Therefore, there is a 1–1 correspondence between Hamiltonian cycles of BG(M') (BG(M''), respectively) and Hamiltonian cycles of BG(M)[X] (BG(M)[Y], respectively). For readability, we do not distinguish between BG(M') (BG(M''), respectively) and BG(M)[X] (BG(M)[Y], respectively).

A basis sequence $B_1B_2B_3B_4$ is a good cycle for B_1B_2 if it is a cycle (of length four) in BG(M), each of B_1 and B_4 contains e, and none of B_2 and B_3 contains e; that is, B_1 and B_4 are adjacent bases of BG(M') and B_2 and B_3 are adjacent bases of BG(M'') (Figure 1).

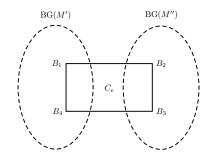


Figure 1: A good cycle $C_e = B_1 B_2 B_3 B_4$ for $B_1 B_2$.

If $C_e = B_1 B_2 B_3 B_4$ is good, then the symmetric difference of a Hamiltonian cycle of BG(M') passing through the edge $B_1 B_4$, the good C_e , and a Hamiltonian cycle of BG(M'') passing through the edge $B_2 B_3$ is a Hamiltonian cycle of BG(M).

So, if $\mathcal{C}(B_1, B_2)$ is the set of good cycles for B_1B_2 , then

 $\operatorname{HC}_{B_1B_2}(M) \ge \operatorname{HC}^*(M') \times |\mathcal{C}(B_1, B_2)| \times \operatorname{HC}^*(M'').$

This inequality suggests an inductive way to achieve a lower bound on $\mathrm{HC}^*(M)$. A key part in this approach involves proving a lower bound on the number of good cycles for any edge of $\mathrm{BG}(M)$.

2 Graphic matroids

In this section, we consider a graphic matroid M_G where G is a k-edge-connected graph of order n; that is, the elements of the ground set of M_G are the edges of G and a basis of M_G corresponds to a spanning tree of G, thus a basis of M_G contains exactly n - 1 edges of G. Since loops of G are in no basis of M_G , we always consider graphs with no loops. For readability, we do not distinguish between a basis of M_G and a spanning tree of G. If B is a basis of M_G and gis an edge of G not in B, then B + g induces a unique cycle (circuit) C(g, B)in G (in M_G , respectively) called the fundamental cycle (circuit, respectively) with respect to g and B [14]. First, note that, by Haff's result, if G is a k-edge-connected graph of order $n \geq 3$, for $k \geq 2$, then the graph $BG(M_G)$ has at least three vertices and is edge Hamiltonian.

Let G' = G/e be the graph resulting from contracting the edge e of G and then removing loops and let $G'' = G \setminus e$ be the graph resulting from deleting the edge e.

Let X and Y be disjoint subsets of the vertex set V(G). We denote by E[X, Y] (= E[Y, X]) the set of edges of G with one end in X and the other end in Y, and by e(X, Y) their number.

2.1 General structure of good cycles

Now, we fix the structure that we will use in the rest of Section 2 and, unless otherwise stated, we will follow this notation. The facts presented ahead show types of good cycles that this structure induces.

Let G be a graph and B_1 and B_2 be bases of M_G such that $B_2 = B_1 - e + g$. Let f be an edge of $B_1 - e$. Let X be the vertex set of the component of $B_1 - e$ that contains no end of f. Let Z be the vertex set of the component of $B_1 - f$ that contains no end of e. Let $Y = V(G) \setminus (X \cup Z)$.

Let $C_e = C(B_1, B_2)$ be the set of good cycles for B_1B_2 . An arbitrary element of C_e is denoted by C_e , and is represented as $B_1B_2B_3B_4$. For $f \in B_1 - e = B_2 - g$, let $C_e(f) = \{C_e \in C_e : f \notin B_4\}$. An arbitrary element of $C_e(f)$ is denoted by $C_e(f)$. For every $f' \in B_1 - e$ with $f' \neq f$, since f' belongs to both B_3 and B_4 for every cycle $C_e(f)$, we have that $C_e(f) \cap C_e(f') = \emptyset$. Thus $C_e = \bigcup \{C_e(f) : f \in B_1 - e\}$. For every $w \notin B_1 + g = B_2 + e$, we denote by $C_e(f, w)$ the set of cycles in $C_e(f)$ such that $w \in B_3$. Similarly, $C_e(f, w) \cap C_e(f, w') = \emptyset$ for every $w' \notin B_1 + g$ with $w' \neq w$. Therefore $C_e(f) = \bigcup \{C_e(f, w) : w \notin B_1 + g\}$. Summarizing, the following holds.

Remark 1. $C_e(f) \cap C_e(f') = \emptyset$ and $C_e(f, w) \cap C_e(f, w') = \emptyset$ for every $f, f' \in B_1 - e$ with $f \neq f'$ and every $w, w' \notin B_1 + g$ with $w \neq w'$.

Fact 1. If f is not in $C(g, B_1)$ and w is an edge in $E[X \cup Y, Z]$ other than f, then there exists a good cycle $C_e(f, w)$ by defining

• $B_4 = B_1 - f + w$ and $B_3 = B_2 - f + w$.

Note that $B_3 = B_4 - e + g$. (See Figure 2.)

Fact 2. If f is in $C(g, B_1)$ and ℓ is an edge in E[Y, Z] other than f, then there are two good cycles $C_e(f, \ell)$ by defining

- $B_4 = B_1 f + \ell$ and $B_3 = B_2 f + \ell$.
- $B_4 = B_1 f + g$ and $B_3 = B_2 f + \ell$.

Note that, in the first case, $B_3 = B_4 - e + g$ and, in the second case, $B_3 = B_4 - e + \ell$. (See Figure 3.)

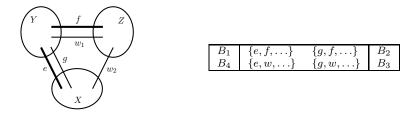


Figure 2: Edge f is in B_1 . There are edges w_1 and w_2 between $X \cup Y$ and Z. The table shows a good $C_e(f, w)$'s containing B_1B_2 .

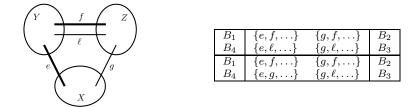


Figure 3: The bold edges are in B_1 . There is an edge ℓ between Y and Z. The table shows two good cycles $C_e(f, \ell)$ containing B_1B_2 .

Fact 3. If f is in $C(g, B_1)$ and h is an edge in E[X, Y] other than e, then there exists a good cycle $C_e(f, h)$ by defining

• $B_4 = B_1 - f + g$ and $B_3 = B_2 - f + h$.

Note that $B_3 = B_4 - e + h$. (See Figure 4.)

Fact 4. If f is in $C(g, B_1)$ and j is an edge in E[X, Z] other than g, then there exists a good cycle $C_e(f, j)$ by defining

• $B_4 = B_1 - f + j$ and $B_3 = B_2 - g + j$.

Note that $B_3 = B_4 - e + f$. (See Figure 4.)

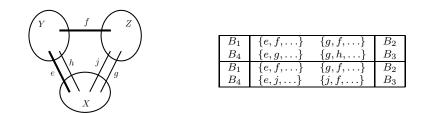


Figure 4: The bold edges are in B_1 . There is an edge h between X and Y, and an edge j between X and Z. The table shows the two good cycles C_e containing B_1B_2 .

2.2 2-edge-connected graphs

We start by giving a lower bound on $\mathrm{HC}^*(M_G)$ where M_G is the cycle matroid obtained from a 2-edge-connected graph G. In what follows we shall use the notation introduced in the beginning of this section. In particular, we use extensively the facts and the structure of the vertex sets X, Y, Z provided by adjacent bases B_1, B_2 and a convenient edge f in $B_1 \cap B_2$.

Proposition 2. Let G be a 2-edge-connected graph. Let B_1 and B_2 be adjacent bases of $BG(M_G)$, say $B_2 = B_1 - e + g$. Each edge f of B_1 with at most one end in $C(g, B_1)$ provides a good cycle $C_e(f)$.

Proof. It follows from Fact 1 that, given an edge f in B_1 with at most one end in $C(g, B_1)$, for every edge $w \in E[X \cup Y, Z]$ (one such w exists because G is 2-edge-connected), there exists a good cycle $C_e(f, w)$.

Proposition 3. Let G be a connected graph. Let B_1 and B_2 be adjacent bases of $BG(M_G)$, say $B_2 = B_1 - e + g$. For each edge w not in B_1 with at most one end in $C(g, B_1)$, there exists an edge $f_w \in B_1 - e$ that provides a good cycle $C_e(f_w, w)$.

Proof. Let w be an edge not in B_1 with at most one end in $C(g, B_1)$. As at most one end of w is in $C(g, B_1)$, there exists an edge f_w of $B_1 - C(g, B_1) \subseteq B_1 - e$ in the fundamental cycle $C(w, B_1)$. It follows from Fact 1 that there exists a good cycle $C_e(f_w, w)$.

Proposition 4. Let G be 2-edge-connected graph. Let B_1 and B_2 be adjacent bases of $BG(M_G)$, say $B_2 = B_1 - e + g$. Suppose that $C(g, B_1)$ has length at least three. For each edge w not in $B_1 + g$ with both ends in $C(g, B_1)$, there exists an edge $f_w \in B_1 - e$ that provides a good cycle $C_e(f_w, w)$.

Proof. As $C(g, B_1)$ has length at least three, e and g are not parallel edges. Let w be an edge not in $B_1 + g$ with both ends in $C(g, B_1)$.

Case 1. The edge w is parallel to g.

Let f_w be an edge of $C(g, B_1) - e - g$. In this case w is as j in Fact 4.

Case 2. The edge w is not parallel to g and the fundamental cycle $C(w, B_1)$ contains the edge e.

Let f_w be an edge of $C(g, B_1) - e - g$ and not in $C(w, B_1)$. In this case w is as h in Fact 3.

Case 3. The edge w is not parallel to g and the fundamental cycle $C(w, B_1)$ does not contain the edge e.

Let f_w be an edge of $C(w, B_1) - w \subseteq B_1 - e$. In this case w is as ℓ in Fact 2. So, each case leads to one of the previously stated facts where we obtain an f_w and a good cycle $C_e(f_w, w)$. **Lemma 5.** If G is a 2-edge-connected graph of order $n \ge 4$ and size at least n+2, then every edge of BG(M_G) is in two good cycles.

Proof. Let B_1 and B_2 be adjacent bases of BG(M_G), say $B_2 = B_1 - e + g$.

Suppose that e and g are parallel edges; that is, $C(g, B_1)$ is the 2-cycle eg. Since G has order $n \ge 4$, there are two edges in $B_1 - e$. By Proposition 2, each one of them gives a good cycle, and they are distinct by Remark 1.

Now, suppose that e and g are not parallel edges in G. Thus, $C(g, B_1)$ has length at least three. If there are two edges in B_1 with at most one end in $C(g, B_1)$ or two edges not in B_1 with at most one end in $C(g, B_1)$, by Propositions 2 and 3, respectively, we have two good cycles, distinct by Remark 1, so we are done. Also, if there are two edges not in $B_1 + g$ with both ends in $C(g, B_1)$, then we are done by Proposition 4 and Remark 1.

Finally, as G has size at least n+2, we may assume there exist an edge in B_1 with at most one end in $C(g, B_1)$ and an edge not in $B_1 + g$ with both ends in $C(g, B_1)$. Therefore, by Propositions 2 and 4, respectively, and Remark 1, the lemma follows.

The 1-sum $H \oplus_1 H'$ of two graphs H and H' is the graph obtained from identifying a vertex of H with a vertex of H'.

Lemma 6. Let G be a 2-edge-connected graph of order $n \ge 4$. There exists an edge in BG(M_G) not in two good cycles if and only if G is either C_n or $C_2 \oplus_1 C_{n-1}$.

Proof. Let m denote the number of edges of G. Since G is 2-edge-connected, every edge is in a cycle, so $m \ge n$. If m = n, then G is the *n*-cycle C_n and no edge of BG(M_G) is in a good cycle. For $m \ge n+2$, Lemma 5 implies that every edge of BG(M_G) is in two good cycles. So, we may assume that m = n + 1. Because every 2-edge-connected graph has a closed ear-decomposition [2], and G has exactly m + 1 edges, the closed ear-decomposition of G consist of exactly two ears. Thus, G is either

- i) The 1-sum of two cycles, or
- ii) The union of three internally disjoint paths that have the same two end vertices.

First, suppose that G is the 1-sum of two cycle, say $C^1 \oplus_1 C^2$. Since we only consider graphs with no loops, the length of both C^1 and C^2 is at least two. When the length of both C^1 and C^2 is at least three, Proposition 2 provides two good cycles for every edge of BG (M_G) . Therefore, G is $C_2 \oplus_1 C_{n-1}$ and it can be verified that there are adjacent bases B_1 and B_2 in BG (M_G) for which there is only one good cycle (Figure 5).

Now, suppose that G is the union of three internally disjoint paths, say P_1 , P_2 , P_3 , that have the same two end vertices. In this case we shall show that every edge of BG(M_G) is in two good cycles.

Let B_1B_2 be an edge of BG (M_G) , say $B_2 = B_1 - e + g$. First, suppose that e and g are in the same path, say P_1 . Thus, without loss of generality, all edges

of P_2 are in B_1 , and there exists an edge w in P_3 not in B_1 . Let f be an edge of P_2 (and therefore of B_1). Keeping our notation defined in Section 2.1, w is in E[Y, Z] and Fact 2 provides two good cycles $C_e(f, w)$.

Finally, suppose that e and g are in different paths; say e belongs to P_1 and g belongs to P_2 . Thus, all edges of P_1 are in B_1 , and there exists an edge w in P_3 not in B_1 . If there exists an edge $f \in B_1 - e$ in P_1 , then w is in E[X, Z] as j in Fact 4. If there exists an edge $f \in B_1$ in P_2 , then w is in E[X, Y] as h in Fact 3. If there exists an edge $f \in B_1$ in P_3 , then w is in $E[X \cup Y, Z]$ as in Fact 1. In any case we get a good cycle $C_e(f, w)$. Since G has order at least four, there are two edges $f, f' \in B_1$ other than e. Therefore, by Remark 1, every edge B_1B_2 is in two good cycles, named $C_e(f, w)$ and $C_e(f', w)$.



Figure 5: A 2-edge-connected graph G whose basis graph BG(M_G) has an edge, B_1B_2 , with no two good cycles. The basis B_1 is the spanning tree in thick edges and $B_2 = B_1 - e + g$.

Proposition 7. For $n \ge 3$, every edge of K_n is in (n-2)! Hamiltonian cycles.

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of K_n . Consider the edge $v_i v_j$. Without loss of generality, we may assume that $v_i v_j = v_1 v_2$. Note that $v_1 v_2 v_{\sigma(3)} \ldots v_{\sigma(n)}$ is a Hamiltonian cycle for any permutation σ of $\{3, \ldots, n\}$. Therefore, the number of Hamiltonian cycles containing the edge $v_1 v_2$ is (n-2)! and the lemma follows.

Proposition 8. For $n \ge 3$, every edge of $K_2 \Box K_{n-1}$ is in (n-2)!(n-3)!Hamiltonian cycles.

Proof. If n = 3, then (n-2)!(n-3)! = 1 and $K_2 \Box K_2$ is C_4 . So, we may assume that $n \ge 4$.

Let $\{u_1, u_2\}$ be the vertex set of K_2 , and $\{v_1, \ldots, v_{n-1}\}$ be the vertex set of K_{n-1} . Let $K_{n-1}^1 = \{(u_1, v_1), \ldots, (u_1, v_{n-1})\}$ and $K_{n-1}^2 = \{(u_2, v_1), \ldots, (u_2, v_{n-1})\}$. There exists a natural partition (K_{n-1}^1, K_{n-1}^2) of the vertex set of $K_2 \Box K_{n-1}$. For every permutation σ of $\{1, \ldots, n-1\}$, let C_{σ}^{ℓ} be the Hamiltonian cycle $(u_{\ell}, v_{\sigma(1)}) \cdots (u_{\ell}, v_{\sigma(n-1)})$ of K_{n-1}^{ℓ} . Consider an edge $(u_{\ell}, v_i)(u_{\ell}, v_j)$ of $K_2 \Box K_{n-1}$. Without loss of generality we may assume that $\ell = i = 1$ and j = 2. For a permutation σ_1 with $\sigma_1(1) = 1$ and $\sigma_1(2) = 2$, let $(u_1, v_x)(u_1, v_y)$ be an edge of $C_{\sigma_1}^1$ other than $(u_1, v_1)(u_1, v_2)$. For every permutation σ_2 such that $\sigma_2(1) = x$ and $\sigma_2(2) = y$, the Hamiltonian cycle $C_{\sigma_2}^2$ of K_{n-1}^2 uses the edge $(u_2, v_x)(u_2, v_y)$. Let S denote the cycle $(u_1, v_x)(u_2, v_x)(u_2, v_y)(u_1, v_y)$. The symmetric difference $C_{\sigma_1}^1 \Delta S \Delta C_{\sigma_2}^2$ is a Hamiltonian cycle of BG(M_G) containing the edge $(u_1, v_1)(u_1, v_2)$. Since the edge $(u_1, v_x)(u_1, v_y)$ can be chosen in n-2 different ways, and the number of permutations σ_1 as well as the number of permutations σ_2 is (n-3)!, we obtain $(n-2)((n-3)!)^2$ Hamiltonian cycles passing through the edge $(u_1, v_1)(u_1, v_2)$.

Now consider an edge $(u_1, v_i)(u_2, v_i)$ of $K_2 \Box K_{n-1}$. Without loss of generality we may assume that i = 1. Consider the edge $(u_1, v_1)(u_1, v_j)$. Such edge is in (n-3)! Hamiltonian cycles C^1 of K_{n-1}^1 . On the other hand, there are (n-3)! Hamiltonian cycles C^2 of K_{n-1}^2 passing through the edge $(u_2, v_1)(u_2, v_j)$. Let S denote the cycle $(u_1, v_1)(u_1, v_j)(u_2, v_j)(u_2, v_1)$. The symmetric difference $C^1 \Delta S \Delta C^2$ is a Hamiltonian cycle of BG (M_G) containing the edge $(u_1, v_1)(u_1, v_j)(u_2, v_1)$. Since there are (n-3)! cycles C^1 , as well as cycles C^2 , and the edge $(u_1, v_1)(u_1, v_j)$ can be chosen in n-2 different ways, we obtain $(n-2)((n-3)!)^2$ Hamiltonian cycles containing the edge $(u_1, v_1)(u_2, v_1)$. Note that $(n-2)((n-3)!)^2 = (n-2)!(n-3)!$, hence the proposition follows. \Box

Theorem 9. If G is a 2-edge-connected graph of order $n \ge 3$, then every edge of BG(M_G) is in 2^{n-3} Hamiltonian cycles.

Proof. The proof is by induction on n. If n = 3, then $2^{n-3} = 1$ and the theorem follows from the edge Hamiltonicity of BG(M_G). So we may assume that $n \ge 4$.

By Lemma 6, if there exists an edge in $BG(M_G)$ not in two good cycles, then G is either C_n or $C_2 \oplus_1 C_{n-1}$. If $G = C_n$, then $BG(M_G) = K_n$ and by Proposition 7 every edge of $BG(M_G)$ is in $(n-2)! \ge 2^{n-3}$ Hamiltonian cycles. If $G = C_2 \oplus_1 C_{n-1}$, then $BG(M_G) = K_2 \Box K_{n-1}$ and by Proposition 8 every edge of $BG(M_G)$ is in $(n-2)!(n-3)! \ge 2^{n-3}$ Hamiltonian cycles. Therefore, we may assume that G has at least n+1 edges and every edge of $BG(M_G)$ is in two good cycles.

Let B_1B_2 be an edge of $BG(M_G)$, say $B_2 = B_1 - e + g$. Let G' = G/eand $G'' = G \setminus e$. As G' is 2-edge-connected of order $n - 1 \ge 3$, by the induction hypothesis, every edge of $BG(M_{G'})$ is in 2^{n-4} Hamiltonian cycles in $BG(M_{G'})$. As G'' has $n \ge 4$ vertices and at least n edges, G'' has at least two spanning trees, and therefore $BG(M_{G''})$ is either K_2 or edge Hamiltonian.

Let $C_e = B_1 B_2 B_3 B_4$ be a good cycle. If $BG(M_{G''})$ is K_2 , then the symmetric difference of C_e and a Hamiltonian cycle of $BG(M_{G'})$ containing $B_1 B_4$ is a Hamiltonian cycle of $BG(M_G)$. On the other hand, if $BG(M_{G''})$ is edge Hamiltonian, then the symmetric difference of C_e , a Hamiltonian cycle of $BG(M_{G'})$ containing $B_1 B_4$, and a Hamiltonian cycle of $BG(M_{G''})$ containing $B_2 B_3$ is a Hamiltonian cycle of $BG(M_G)$. As every edge of $BG(M_G)$ is in two good cycles, in either case we conclude that every edge of $BG(M_G)$ is in $2^{n-4} \cdot 2 \cdot 1 = 2^{n-3}$ Hamiltonian cycles.

2.3 *k*-edge-connected graphs

Now, we turn our attention to counting Hamiltonian cycles in the basis graph of the cycle matroid of k-edge-connected graphs for $k \ge 3$.

Lemma 10. If G is a k-edge-connected graph of order $n \ge 3$ for $k \ge 3$, then there are (n-2)(k-1) good cycles for every edge of BG(M_G).

Proof. Let B_1B_2 be an edge of BG (M_G) , say $B_2 = B_1 - e + g$, and let $f \in B_1 - e$. First we show that there are k - 1 good cycles in $C_e(f)$.

As before, let X be the vertex set of the component of $B_1 - e$ that contains no end of f, let Z be the vertex set of the component of $B_1 - f$ that contains no end of e, and let $Y = V(G) \setminus (X \cup Z)$.

Suppose that f is not in $C(g, B_1)$. Since G is k-edge-connected, the edge set $E[X \cup Y, Z]$ contains k - 1 edges distinct from f. It follows from Fact 1 that there is a good cycle in $\mathcal{C}_e(f)$ for each of these edges. Besides, for different edges in $E[X \cup Y, Z]$, the corresponding bases B_3 are different and so are their corresponding good cycles in $\mathcal{C}_e(f)$.

Now, suppose that f is in $C(g, B_1)$. It follows from Fact 2 that there are two good cycles in $C_e(f)$ for every edge in $E[Y, Z] \setminus \{f\}$, from Fact 3 that there is one good cycle in $C_e(f)$ for every edge in $E[X, Y] \setminus \{e\}$, and from Fact 4 that there is one good cycle in $C_e(f)$ for every edge in $E[X, Z] \setminus \{g\}$. Thus, as these good cycles are distinct, if $(e(X, Y) - 1) + (e(X, Z) - 1) + 2(e(Y, Z) - 1) \ge k - 1$, we would indeed have k - 1 good cycles in $C_e(f)$.

By the k-edge-connectivity of G, we get that

$$|E[Y, X \cup Z] \setminus \{e, f\}| = e(X, Y) + e(Y, Z) - 2 \ge k - 2, \tag{1}$$

$$|E[Z, X \cup Y] \setminus \{f, g\}| = e(X, Z) + e(Y, Z) - 2 \ge k - 2.$$
(2)

Hence, summing (1) and (2), we get that $e(X, Y) + e(X, Z) + 2e(Y, Z) - 4 \ge 2k - 4$, and $2k - 4 \ge k - 1$ as $k \ge 3$. So we have k - 1 good cycles in $\mathcal{C}_e(f)$.

By Remark 1 and as there are n-2 choices for f, there are (n-2)(k-1) good cycles for every edge of BG (M_G) .

Lemma 11. Let G be a 3-edge-connected graph of order $n \ge 3$ and let e be an edge of G. Then $\operatorname{HC}^*(M_G) \ge \operatorname{HC}^*(M_{G/e})$ and $\operatorname{HC}^*(M_G) \ge \operatorname{HC}^*(M_{G\backslash e})$.

Proof. Let $X = \{B \in \mathcal{B}(M_G): e \in B\}$ and $Y = \{B \in \mathcal{B}(M_G): e \notin B\}$. Note that (X, Y) is a bipartition of the vertices (bases) of BG (M_G) . Let G' = G/e and $G'' = G \setminus e$. As G' is 3-edge-connected of order $n - 1 \geq 2$, the basis graph BG $(M_{G'})$ has at least three vertices and is edge Hamiltonian. Similarly, BG $(M_{G''})$ also has at least three vertices and thus is edge Hamiltonian. There is a one-to-one correspondence between the bases in X and the bases of BG $(M_{G'})$ and between the bases in Y and the bases of BG $(M_{G''})$.

The edge set of $BG(M_G)$ can be partitioned into: (i) edges with both ends in $BG(M_{G'})$, called *yellow edges*, (ii) edges with one end in $BG(M_{G'})$ and the other one in $BG(M_{G''})$, called *pink edges*, and (iii) edges with both ends in $BG(M_{G''})$, called *orange edges*. Case 1. Hamiltonian cycles passing through a pink edge.

Let B_1B_2 be a pink edge. For every good cycle $C_e = B_1B_2B_3B_4$, the edge B_1B_4 is an edge of BG $(M_{G'})$ and B_2B_3 is an edge of BG $(M_{G''})$. The symmetric difference of a Hamiltonian cycle of BG $(M_{G'})$ containing the edge B_1B_4 , the good cycle C_e , and a Hamiltonian cycle of BG $(M_{G''})$ containing the edge B_2B_3 is a Hamiltonian cycle of BG (M_G) containing the edge B_1B_2 . By Lemma 10, there exists at least one such good cycle C_e . Thus HC $_{B_1B_2}(M_G) \ge \text{HC}^*(M_{G''})$ and HC $_{B_1B_2}(M_G) \ge \text{HC}^*(M_{G''})$.

Case 2. Hamiltonian cycles passing through a yellow edge.

First we prove that every yellow edge belongs to a good cycle C_e . Let B_1B_4 be a yellow edge, say $B_4 = B_1 - f + w$. The subgraph $B_4 - e$ has exactly two components. Since G is 3-edge-connected, there exists an edge g other than e and f connecting the two components of $B_4 - e$. Therefore $e \in C(g, B_4)$ and $B_3 = B_4 - e + g$ is a basis. If $B_1 - e + g$ is a basis, we set $B_2 = B_1 - e + g$ and we have a good cycle $C_e = B_1B_2B_3B_4$. So, we may assume that $B_1 - e + g$ is not a basis. Thus $e \notin C(g, B_1)$. Since $e \in C(g, B_4)$, we have that $C(g, B_1) \neq C(g, B_4)$. This implies that $f \in C(g, B_1)$ and $w \in C(g, B_4)$ since B_1 and B_4 only differ by f and w. Hence, we know that $C(g, B_1)\Delta C(g, B_4)$, the symmetric difference of $C(g, B_1)$ and $C(g, B_4)$, (*i*) contains e, f, w but not g; (*ii*) is contained in $B_1 \cup B_4 = B_1 + w$; and (*iii*) contains a cycle. Therefore, the only cycle contained in $C(g, B_1)\Delta C(g, B_4)$ is $C(w, B_1)$. In this case, $B_2 = B_1 - e + w$ is a basis and there is also a good cycle $C_e = B_1B_2B_3B_4$.

Now, every yellow edge B_1B_4 is in a Hamiltonian cycle C of $BG(M_{G'})$. Let us extend C to a Hamiltonian cycle of BG(M) as follows. Consider an edge $B'_1B'_4$ in C other than B_1B_4 . The symmetric difference of C, a good cycle containing the yellow edge $B'_1B'_4$, say $B'_1B'_2B'_3B'_4$, and a Hamiltonian cycle of $BG(M_{G''})$ containing the edge $B'_2B'_3$ is an extension of C to a Hamiltonian cycle of $BG(M_G)$ containing the edge B_1B_4 . Thus $HC_{B_1B_4}(M_G) \ge HC^*(M_{G''})$ because $B'_2B'_3$ is in $HC^*(M_{G''})$ Hamiltonian cycles of $BG(M_{G''})$.

On the other hand, since $BG(M_{G'})$ is a simple graph, two Hamiltonian cycles of $BG(M_{G'})$ passing through the edge B_1B_4 , say C and C', differ in at least two edges. Therefore C and C' are extended to different Hamiltonian cycles of $BG(M_G)$. Hence $HC_{B_1B_4}(M_G) \ge HC^*(M_{G'})$.

Case 3. Hamiltonian cycles passing through an orange edge.

First we prove that every Hamiltonian cycle C in $BG(M_{G''})$ contains two edges, say B_2B_3 and $B'_2B'_3$, each of which is in a good cycle in C_e . As G has at least three vertices and is 3-edge-connected, there exist an edge f in G not parallel to e and a basis of $M_{G''}$ not containing f. By traversing C, we pass through edges B_2B_3 and $B'_3B'_2$ such that $B_3 = B_2 - f + w$ and $B'_2 = B'_3 + f - w'$ for some elements w and w'. We shall prove that there exists a C_e containing B_2B_3 and a C'_e containing $B'_2B'_3$.

As f is not parallel to e, there exists an edge $g \in C(e, B_2)$ other than f. Therefore $B_1 = B_2 - g + e$ is a basis. If $B_3 - g + e$ is also a basis, we set $B_4 = B_3 - g + e$ and obtain a good cycle C_e with B_2B_3 . So, we may assume that $B_3 - g + e$ is not a basis. Thus $g \notin C(e, B_3)$. Since $g \in C(e, B_2)$, we have that $C(e, B_2) \neq C(e, B_3)$. This implies that $f \in C(e, B_2)$ and $w \in C(e, B_3)$ since B_2 and B_3 only differ by f and w. In this case $B_4 = B_3 - w + e$ is a basis and we obtain a good cycle C_e with B_2B_3 . This completes the proof for B_2B_3 . The proof for $B'_2B'_3$ is analogous.

Every orange edge B_2B_3 is in a Hamiltonian cycle C of $BG(M_{G''})$. Since C contains two edges, each in a good cycle, there exists an edge $B'_2B'_3$, distinct from B_2B_3 , in a good cycle C'_e , say $C'_e = B'_1B'_2B'_3B'_4$. The symmetric difference of C, the good cycle C'_e , and a Hamiltonian cycle of $BG(M_{G'})$ passing through the edge $B'_1B'_4$ is a Hamiltonian cycle of $BG(M_G)$ containing the edge B_2B_3 . Because there are $HC^*(M_{G'})$ Hamiltonian cycles passing through the edge $B'_1B'_4$, we have that $HC_{B_2B_3}(M_G) \ge HC^*(M_{G'})$. As B_2B_3 is in $HC^*(M_{G''})$ Hamiltonian cycles of $BG(M_{G''})$ containing the edge $B'_1B'_4$.

In order to give a bound on $HC^*(M_G)$, we define the function

 $hc(n,k) = min\{HC^*(M_G): G \text{ is a } k\text{-edge-connected graph of order } n\}.$

Proposition 12. For $k, n \ge 3$, $hc(n, k) \ge (n-2)(k-1)hc(n-1, k)hc(n, k-1)$.

Proof. Let G be a k-edge-connected graph of order n such that $\operatorname{HC}^*(M_G) = \operatorname{hc}(n,k)$. By Lemma 10, there are (n-2)(k-1) good cycles for every edge of $\operatorname{BG}(M_G)$. Let B_1B_2 be an edge of $\operatorname{BG}(M_G)$, say $B_2 = B_1 - e + g$, and let G' = G/e and $G'' = G \setminus e$. The symmetric difference of a good cycle $C_e = B_1B_2B_3B_4$, a Hamiltonian cycle of $\operatorname{BG}(M_{G'})$ containing B_1B_4 , and a Hamiltonian cycle of $\operatorname{BG}(M_{G'})$ containing B_1B_2 . Hence, B_1B_2 is in $(n-2)(k-1)\operatorname{HC}^*(M_{G'})\operatorname{HC}^*(M_{G''})$ Hamiltonian cycles of $\operatorname{BG}(M_G)$. Now, as G' is k-edge-connected of order n-1, we have that $\operatorname{HC}^*(M_{G'}) \geq \operatorname{hc}(n-1,k)$ and, as G'' is (k-1)-edge-connected of order n, we have that $\operatorname{HC}^*(M_{G''}) \geq \operatorname{hc}(n,k-1)$. Therefore we conclude that $\operatorname{hc}(n,k) = \operatorname{HC}^*(M_G) \geq (n-2)(k-1)\operatorname{hc}(n-1,k)\operatorname{hc}(n,k-1)$. □

The superfactorial sf(x) of a positive integer x is the number $x!(x-1)!\cdots 0!$

Theorem 13. For $k \ge 3$, $hc(3, k) \ge sf(k - 1)$.

Proof. We use induction on k. Let G be a k-edge-connected graph of order three such that $\mathrm{HC}^*(M_G) = \mathrm{hc}(3,k)$. Let B_1B_2 be an edge of $\mathrm{BG}(M_G)$, say $B_2 = B_1 - e + g$. Let G' = G/e and $G'' = G \setminus e$. The graph G' is k-edge-connected of order two.

If k = 3, then BG $(M_{G'})$ has three vertices and is edge Hamiltonian. The graph G'' has three vertices and is 2-edge-connected, therefore BG $(M_{G''})$ has three vertices and is edge Hamiltonian. By Lemma 10, the edge B_1B_2 is in two good cycles in C_e . The symmetric difference of a good cycle $C_e = B_1B_2B_3B_4$, a Hamiltonian cycle of BG $(M_{G'})$ containing B_1B_4 , and a Hamiltonian cycle of BG $(M_{G''})$ containing B_2B_3 is a Hamiltonian cycle of BG (M_G) containing B_1B_2 .

Hence, every edge of $BG(M_G)$ is in two Hamiltonian cycles and $hc(3, k) = HC^*(M_G) \ge 2$.

Suppose k > 3. By the induction hypothesis, $hc(3, k - 1) \ge sf(k - 2)$. The basis graph $BG(M_{G'})$ is a complete graph on at least k vertices, thus $HC^*(M_{G'}) \ge (k-2)!$ Since G'' is a (k-1)-edge-connected graph of order three, every edge of $BG(M_{G''})$ is in hc(3, k - 1) Hamiltonian cycles. By Lemma 10, the edge B_1B_2 is in k - 1 good cycles in C_e . The symmetric difference of a good cycle $C_e = B_1B_2B_3B_4$, a Hamiltonian cycle of $BG(M_{G'})$ containing B_1B_2 , and a Hamiltonian cycle of $BG(M_{G''})$ containing B_1B_2 . Hence, every edge of $BG(M_G)$ is in $(k - 2)!(k - 1)hc(3, k - 1) \ge (k - 1)! sf(k - 2) = sf(k - 1)$ Hamiltonian cycles. Thus, $HC^*(M_G) = hc(3, k) \ge sf(k - 1)$.

Theorem 14. For $n \ge 3$, $hc(n,3) \ge (n-2)! 2^{\binom{n-1}{2}}$.

Proof. We use induction on n. Let G be a 3-edge-connected graph of order $n \geq 3$ such that $\operatorname{HC}^*(M_G) = \operatorname{hc}(n,3)$. Let B_1B_2 be an edge of $\operatorname{BG}(M_G)$, say $B_2 = B_1 - e + g$. Let G' = G/e and $G'' = G \setminus e$. Note that G' is a 3-edge-connected graph of order n-1.

If n = 3, then G' is a 3-edge-connected graph of order two. So $BG(M_{G'})$ is the complete graph K_m , where $m \geq 3$ is the number of edges of G'. (Remember that we remove loops of G' if any.) The graph G'' has three vertices and is 2-edge-connected, therefore $BG(M_{G''})$ has at least three vertices and is edge Hamiltonian. By Lemma 10, the edge B_1B_2 is in two good cycles in C_e . The symmetric difference of a good cycle $C_e = B_1B_2B_3B_4$, a Hamiltonian cycle of $BG(M_{G'})$ containing B_1B_4 , and a Hamiltonian cycle of $BG(M_{G''})$ containing B_1B_4 , and a Hamiltonian B_1B_2 . Hence, every edge of $BG(M_G)$ is in two Hamiltonian cycles, and so $HC^*(M_G) = hc(3,3) \geq 2$.

Suppose n > 3. By the induction hypothesis, $hc(n-1,3) \ge (n-3)! 2^{\binom{n-2}{2}}$. Since G' is a 3-edge-connected graph of order n-1, every edge of BG $(M_{G'})$ is in hc(n-1,3) Hamiltonian cycles. By Theorem 9, as G'' is 2-edge-connected of order n, every edge of BG $(M_{G''})$ is in 2^{n-3} Hamiltonian cycles. By Lemma 10, the edge B_1B_2 is in 2(n-2) good cycles in \mathcal{C}_e . The symmetric difference of a good cycle $C_e = B_1B_2B_3B_4$, a Hamiltonian cycle of BG $(M_{G'})$ containing B_1B_4 , and a Hamiltonian cycle of BG $(M_{G''})$ containing B_2B_3 is a Hamiltonian cycle of BG (M_G) containing B_1B_2 . Therefore, every edge of BG (M_G) is in $(n-3)! 2^{\binom{n-2}{2}} 2(n-2)2^{n-3} = (n-2)! 2^{\binom{n-1}{2}}$ Hamiltonian cycles. Thus, HC* $(M_G) =$ $hc(n,3) \ge (n-2)! 2^{\binom{n-1}{2}}$.

The next theorem gives a bound on hc(n, k) for $n \ge 4$ and $k \ge 4$.

Theorem 15. For $n, k \geq 4$,

$$hc(n,k) \geq \frac{2^{\binom{n+k-4}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1)k} \prod_{r=4}^{k} (r \operatorname{sf}(r-1))^{\binom{n+k-4-r}{n-4}} \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-4-s}{k-4}}.$$

Proof. The proof is by induction on n + k and uses repeatedly Proposition 12. For n = k = 4, we apply Theorem 13 to hc(3, 4) and Theorem 14 to hc(4, 3):

$$\begin{aligned} \operatorname{hc}(4,4) &\geq 2 \cdot 3 \, \operatorname{hc}(3,4) \, \operatorname{hc}(4,3) \geq 6 \left[\operatorname{sf}(3)\right] \cdot \left[2! \, 2^{\binom{3}{2}}\right] &= 3! \cdot \operatorname{sf}(3) \, 2 \cdot 2^3 \\ &= 2^{\binom{4}{1}} \cdot \frac{3 \cdot 4}{3 \cdot 4} \, \operatorname{sf}(3) \cdot 3! \, = \, \frac{2^{\binom{4}{1}} \cdot 3^{\binom{1}{1}}}{3 \cdot 4} \big(4 \operatorname{sf}(3)\big)^{\binom{0}{0}} \cdot 3!^{\binom{0}{0}}. \end{aligned}$$

The bound on hc(4, k) for $k \ge 5$ comes from applying Theorem 13 to hc(3, k) and the induction hypothesis on hc(4, k - 1):

$$\begin{split} \operatorname{hc}(4,k) &\geq 2(k-1) \left[\operatorname{sf}(k-1) \right] \cdot \left[\frac{2^{\binom{k-1}{1}} \cdot 3^{\binom{k-4}{k-4}}}{3(k-1)} \cdot \left(\prod_{r=4}^{k-1} \left(r \operatorname{sf}(r-1) \right)^{\binom{k-1-r}{0}} \right) \cdot 3!^{\binom{k-5}{k-5}} \right] \\ &= \frac{2}{3} \operatorname{sf}(k-1) \cdot 2^{k-1} \cdot 3^{\binom{k-3}{k-3}} \cdot \left(\prod_{r=4}^{k-1} \left(r \operatorname{sf}(r-1) \right)^{\binom{k-r}{0}} \right) \cdot 3!^{\binom{k-4}{k-4}} \\ &= \frac{2^{\binom{k}{1}} \cdot 3^{\binom{k-3}{k-3}}}{3k} \cdot \left(\prod_{r=4}^{k} \left(r \operatorname{sf}(r-1) \right)^{\binom{k-r}{0}} \right) \cdot 3!^{\binom{k-4}{k-4}}. \end{split}$$

Similarly, the bound on hc(n, 4) for $n \ge 5$ comes from applying the induction hypothesis on hc(n - 1, 4) and Theorem 14 to hc(n, 3):

$$\begin{aligned} \operatorname{hc}(n,4) &\geq (n-2)3 \left[\frac{2^{\binom{n-1}{n-4}} \cdot 3^{\binom{n-4}{1}}}{(n-2)4} \cdot \left(4\operatorname{sf}(3)\right)^{\binom{n-5}{n-5}} \cdot \prod_{s=4}^{n-1} (s-1)!^{\binom{n-1-s}{0}} \right] \cdot \left[(n-2)! \, 2^{\binom{n-1}{2}} \right] \\ &= \frac{3}{4} \, 2^{\binom{n-1}{n-4}} \cdot 3^{n-4} \cdot \left(4\operatorname{sf}(3)\right)^{\binom{n-4}{n-4}} \cdot \left(\prod_{s=4}^{n-1} (s-1)!^{\binom{n-s}{0}}\right) \cdot (n-2)! \, 2^{\binom{n-1}{n-3}} \\ &= \frac{2^{\binom{n-1}{n-4} + \binom{n-1}{n-3}} \cdot 3^{n-3}}{(n-1)4} \, (n-1) \cdot \left(4\operatorname{sf}(3)\right)^{\binom{n-4}{n-4}} \cdot \left(\prod_{s=4}^{n-1} (s-1)!^{\binom{n-s}{0}}\right) \cdot (n-2)! \\ &= \frac{2^{\binom{n}{n-3}} \cdot 3^{\binom{n-3}{1}}}{(n-1)4} \cdot \left(4\operatorname{sf}(3)\right)^{\binom{n-4}{n-4}} \cdot \left(\prod_{s=4}^{n} (s-1)!^{\binom{n-s}{0}}\right). \end{aligned}$$

Finally, the bound on hc(n, k) for $n, k \ge 5$ comes from applying the induction hypothesis on both hc(n - 1, k) and hc(n, k - 1):

$$\begin{split} \operatorname{hc}(n,k) &\geq (n-2)(k-1) \frac{2^{\binom{n+k-5}{n-4}} \cdot 3^{\binom{n+k-8}{k-3}}}{(n-2)k} \prod_{r=4}^{k} (r\operatorname{sf}(r-1))^{\binom{n+k-5-r}{n-5}} \cdot \prod_{s=4}^{n-1} (s-1)!^{\binom{n+k-5-s}{k-4}} \\ &\cdot \frac{2^{\binom{n+k-5}{n-3}} \cdot 3^{\binom{n+k-8}{k-4}}}{(n-1)(k-1)} \prod_{r=4}^{k-1} (r\operatorname{sf}(r-1))^{\binom{n+k-5-r}{n-4}} \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-5-s}{k-5}} \\ &= \frac{2^{\binom{n+k-4}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1)k} \left(\prod_{r=4}^{k-1} (r\operatorname{sf}(r-1))^{\binom{n+k-4-r}{n-4}} \right) \cdot \left(k\operatorname{sf}(k-1)\right)^{\binom{n-5}{n-5}} \end{split}$$

$$\cdot \left(\prod_{s=4}^{n-1} (s-1)!^{\binom{n+k-4-s}{k-4}}\right) \cdot (n-1)!^{\binom{k-5}{k-5}}$$

$$= \frac{2^{\binom{n+k-4}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1)k} \prod_{r=4}^{k} (r \operatorname{sf}(r-1))^{\binom{n+k-4-r}{n-4}} \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-4-s}{k-4}}.$$

This completes the proof of the theorem.

The following corollary follows from mathematical manipulations on the right side of the inequality given by Theorem 15 and it gives a more explicit and concise expression.

Corollary 16. For $n > k \ge 5$,

$$hc(n,k) > \prod_{r=3}^{n} sf(r-1)^{\binom{n+k-5-r}{n-6} + \binom{n+k-4-r}{n-4} + \binom{n+k-5-r}{k-5}}.$$

Proof. We start proving two auxiliary equalities that shall be used to prove the corollary. Firstly,

$$2^{\binom{n+k-4-2}{n-4}} \cdot 3^{\binom{n+k-4-3}{k-3}} \cdot \prod_{r=4}^{k} r^{\binom{n+k-4-r}{n-4}} = \prod_{r=2}^{k} r^{\binom{n+k-4-r}{n-4}}$$

$$= \prod_{r=2}^{k} r!^{\binom{n+k-4-r}{n-4} - \binom{n+k-4-(r+1)}{n-4}} \qquad (3)$$

$$= \prod_{r=2}^{k} r!^{\binom{n+k-5-r}{n-5}}$$

$$= \prod_{r=2}^{k} \mathrm{sf}(r)^{\binom{n+k-5-r}{n-5} - \binom{n+k-5-(r+1)}{n-5}} \qquad (4)$$

$$= \prod_{r=3}^{k} \mathrm{sf}(r-1)^{\binom{n+k-6-r}{n-6}}$$

$$= \prod_{r=3}^{k+1} \mathrm{sf}(r-1)^{\binom{n+k-6-(r-1)}{n-6}}$$

$$= \prod_{r=3}^{n} \mathrm{sf}(r-1)^{\binom{n+k-5-r}{n-6}}.$$

Equalities (3) and (4) follow from the hypothesis that $n \ge 6$. Secondly,

$$2^{\binom{n+k-4-3}{k-4}} \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-4-s}{k-4}} = \prod_{s=3}^{n} (s-1)!^{\binom{n+k-4-s}{k-4}}$$

$$= \prod_{s=3}^{n} \mathrm{sf}(s-1)^{\binom{n+k-4-s}{k-4} - \binom{n+k-4-(s+1)}{k-4}}$$
(5)
$$= \prod_{s=3}^{n} \mathrm{sf}(s-1)^{\binom{n+k-5-s}{k-5}}.$$

Equality (5) follows from the hypothesis that $k \geq 5$. Thus, by Theorem 15, we have that

$$\begin{split} \operatorname{hc}(n,k) &\geq \frac{2^{\binom{n+k-3}{n-3}} \cdot 3^{\binom{n+k-7}{n-4}}}{(n-1)k} \cdot \prod_{r=4}^{k} \left(r \operatorname{sf}(r-1) \right)^{\binom{n+k-4-r}{n-4}} \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-4-s}{k-4}} \\ &= \frac{2^{\binom{n+k-3}{n-3}} \cdot 3^{\binom{n+k-7}{k-3}}}{(n-1)k} \cdot \prod_{r=4}^{k} r^{\binom{n+k-4-r}{n-4}} \\ &\quad \cdot \prod_{r=4}^{k} \operatorname{sf}(r-1)^{\binom{n+k-4-r}{n-4}} \\ &\quad \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-4-r}{n-4}} \\ &\quad = \frac{2^{\binom{n+k-5}{n-4}}}{(n-1)k} \cdot \left(2^{\binom{n+k-6}{n-4}} \cdot 3^{\binom{n+k-7}{k-3}} \prod_{r=4}^{k} r^{\binom{n+k-4-r}{n-4}} \right) \\ &\quad \left(2^{\binom{n+k-7}{n-4}} \cdot \prod_{r=4}^{k} \operatorname{sf}(r-1)^{\binom{n+k-4-r}{n-4}} \right) \\ &\quad \left(2^{\binom{n+k-7}{n-4}} \cdot \prod_{r=4}^{n} (s-1)!^{\binom{n+k-4-r}{n-4}} \right) \\ &\quad \left(2^{\binom{n+k-7}{n-4}} \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-4-r}{n-4}} \right) \\ &\quad \left(2^{\binom{n+k-7}{n-4}} \cdot \prod_{s=4}^{n} (s-1)!^{\binom{n+k-4-r}{n-4}} \right) \\ &\quad \left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-4-r}{n-4}} \right) \\ &\quad \left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-5-r}{n-4}} \right) \\ &\quad \left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-5-r}{n-4}} + \binom{\binom{n+k-4-r}{k-5}}{\binom{n-1}{k-5}} \right) \\ &= \frac{2^{\binom{(n+k-5)}{n-1}}} \cdot \prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-5-r}{n-4}} + \binom{\binom{n+k-4-r}{k-5}}{\binom{n-1}{k-5}} \\ &\quad \left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-5-r}{n-4}} + \binom{\binom{n+k-5-r}{k-5}}{\binom{n-1}{k-5}} \right) \\ &= \frac{2^{\binom{(n+k-5)}{n-1}}} \cdot \prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-5-r}{n-4}} + \binom{\binom{n+k-5-r}{k-5}}{\binom{n-1}{k-5}} \\ &\quad \left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-5-r}{n-4}} + \binom{\binom{n+k-5-r}{k-5}}{\binom{n-1}{k-5}} \right) \\ &\quad \left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{\binom{n+k-5-r}{n-4}} + \binom{\binom{n+k-5-r}{k-5}}{\binom{n-1}{k-5}} + \binom{n-1}{k-5}} + \binom{n-1}{k-5}} \\ &\quad \left(\prod_{r=3}^{n} \operatorname{sf}(r-1)^{$$

Equality (6) holds because $\binom{n+k-4}{n-3} = \binom{n+k-7}{n-3} + \binom{n+k-7}{n-4} + \binom{n+k-6}{n-4} + \binom{n+k-5}{n-4}$, and (7) follows from the two previous equalities.

3 Generalized Catalan matroids

In this section we address a special class of transversal matroids introduced by Bonin, de Mier, and Noy [4]. We follow the description of Bonin and de Mier [3] and Stanley [16].

Let S be a subset of \mathbb{Z}^d . A *lattice path* L in \mathbb{Z}^d of length k with steps in S is a sequence $v_0, \ldots, v_k \in \mathbb{Z}^d$ such that each consecutive difference $s_j = v_j - v_{j-1}$ lies in S. We call s_j the *j*th step of the lattice path L. We say that L starts at v_0 and ends at v_k , or simply that L goes from v_0 to v_k .

All lattice paths we consider are in \mathbb{Z}^2 , start at (0,0) and end at (m,r), and use steps in $S = \{(1,0), (0,1)\}$. We call the steps (1,0) and (0,1) as *East* (E) and *North* (N), respectively. Sometimes it is convenient to represent a lattice path L as a sequence of steps; that is, as a word of length m + r on the alphabet $\{E, N\}$; other times, as a subset of $\{1, \ldots, m + r\} = [m+r]$, say $\{j: j$ th step of L is $N\}$.

Let P and Q be lattice paths from (0,0) to (m,r) with P never going above Q. Let \mathcal{P} be the set of all lattice paths from (0,0) to (m,r) that go neither below P nor above Q. For each i with $1 \leq i \leq r$, let A_i be the set

 $A_i = \{j: j \text{th step is the } i \text{th North for some path in } \mathcal{P}\}.$

Observe that A_1, \ldots, A_r are intervals $A_i = [a_i, b_i]$ in [m+r]. Moreover $a_1 < \cdots < a_r$ and $b_1 < \cdots < b_r$; and a_i and b_i correspond to the positions of the *i*th North step of Q and P, respectively. An example is shown in Figure 6.

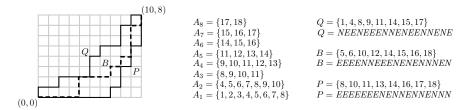


Figure 6: Lattice paths P and Q from (0,0) to (10,8) and the corresponding sets A_1, \ldots, A_8 . Representations of P and Q as words of length 10 + 8 in the alphabet $\{E, N\}$ and as subsets of [10 + 8]. Lattice path B goes neither below P nor above Q and its representations as a word and as a subset.

Let M[P,Q] be the transversal matroid on the ground set [m+r] and (A_1, \ldots, A_r) its presentation. We call (A_1, \ldots, A_r) the standard presentation of M[P,Q]. Note that M[P,Q] has rank r and corank (or nullity) m. A transversal matroid is a *lattice path matroid* if it is a matroid of the type M[P,Q]. Each basis of M[P,Q] corresponds to a lattice path from (0,0) to (m,r) that goes neither below P nor above Q. Figure 6 shows an illustration of a matroid M[P,Q] and a basis B.

Let M[P,Q] be a lattice path matroid. Let $P = y_1 \cdots y_i \cdots y_{m+r}$ $(= y^{[m+r]})$ and $Q = x_1 \cdots x_i \cdots x_{m+r}$ $(= x^{[m+r]})$, with $x_i, y_i \in \{N, E\}$ for $i \in [m+r]$. A generalized Catalan matroid is a lattice path matroid M[P,Q], where $P = E^m N^r$. We simply write M[Q] for generalized Catalan matroids. The class of generalized Catalan matroid is minor-closed [3, Theorem 4.2]. The k-Catalan matroid is the generalized Catalan matroid $M[(NE)^k]$; that is, $Q = (NE)^k$.

Lemma 17. Let M[Q] be a generalized Catalan matroid of rank r and corank m, for $m \ge r \ge 2$, with neither a loop nor an isthmus. Then, every edge of BG(M[Q]) is in r-1 good cycles.

Proof. As M[Q] has neither a loop nor an isthmus, the first step of Q is North and the last one is East. By convenience, we consider the bases of M[Q] as words of length m + r in the alphabet $\{N, E\}$.

Let B_1B_2 be an edge of BG(M[Q]), say $B_2 = B_1 - e + g$. Thus, $B_1 = x^{[m+r]}$, $B_2 = y^{[m+r]}$, and there exist indices e and g such that $x_e = y_g = N$, $x_g = y_e = E$, and $x_\ell = y_\ell$ for $\ell \neq e, g$. Without loss of generality we may assume that e < g.

Case 1. There exists an index ℓ less than e (and therefore less than g) such that $x_{\ell} = y_{\ell} = N$ (Figure 7).

Let f be the least index such that $x_f = y_f = N$. For every index w such that $x_w = y_w = E$, basis B_4 rises by switching x_w for N and x_f for E in B_1 and basis B_3 rises by switching y_w for N and y_f for E in B_2 ; that is, $B_4 = B_1 - f + w$ and $B_3 = B_2 - f + w$. Since the first step of Q is North and the last one is East, the paths corresponding to the words B_3 and B_4 , respectively, are in M[Q]. Thus, for every common E step of B_1 and B_2 , we obtain a good cycle C_e . Therefore, there are m - 1 good C_e passing through the edge B_1B_2 .

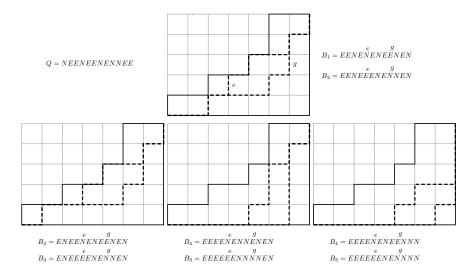


Figure 7: Illustration of lattice paths corresponding to Case 1.

Case 2. There exists an index ℓ greater that g (and therefore greater than e) such that $x_{\ell} = y_{\ell} = E$.

Let w be the last index such that $x_w = y_w = E$. For every index f such that $x_f = y_f = N$, basis B_4 rises by switching x_f for E and x_w for N in B_1 and basis B_3 rises by switching y_f for E and y_w for N in B_2 ; that is, $B_4 = B_1 - f + w$ and $B_3 = B_2 - f + w$. Since the first step of Q is North and the last one is East, the paths corresponding to the words B_3 and B_4 , respectively, are in M[Q]. Thus, for every common N step of B_1 and B_2 , we obtain a good cycle C_e . Therefore, there are r - 1 good C_e passing through the edge B_1B_2 .

Case 3. There exist no indices ℓ and ℓ' with $\ell < e$ and $\ell' > g$ such that $x_{\ell} = y_{\ell} = N$ and $x_{\ell'} = y_{\ell'} = E$.

Thus, x_e is the first N in B_1 and x_g is the last E. Let x_h be the penultimate E in B_1 . Such x_h exists because $m \ge r \ge 2$. As y_g is N in B_2 , y_h is the last E in B_2 .

In order to count the number of good cycles, we partition the N's in the words corresponding to the bases B_1 and B_2 in maximal *blocks*, and for each N we shall show a good cycle associated with it.

Block of Type I. Consider the block $x_i \cdots x_{w-1} x_w$ such that $x_i = \cdots = x_{w-1} = N$ and $x_w = E$ with e < i < w < g.

Also, we have that $y_i = \cdots = y_{w-1} = N$ and $y_w = E$. For every $f \in \{i, \ldots, w-1\}$, basis B_4 rises by switching x_f for E and x_w for N in B_1 and basis B_3 rises by switching y_f for E and y_w for N in B_2 ; that is, $B_4 = B_1 - f + w$ and $B_3 = B_2 - f + w$.

Block of Type II. Consider the block $x_i \cdots x_{w-1} x_w$ such that $x_i = \cdots x_{w-1} = N$ and $x_w = E$ with e < i < w = g.

Also, we have that $y_i = \cdots = y_{w-1} = N$. Let x_h the penultimate E in B_1 . As $y_w = y_g$ is N in B_2 , y_h is the last E in B_2 . For every $f \in \{i, \ldots, g-1\}$, basis B_4 rises by switching x_f for E and x_g for N in B_1 , and basis B_3 rises by switching y_f for E and y_h for N in B_2 ; that is, $B_4 = B_1 - f + g$ and $B_3 = B_2 - f + h$.

Block of Type III. Consider a block $x_{g+1} \cdots x_{m+r}$ of N's in B_1 .

Also, we have that $y_{g+1} \cdots y_{m+r}$ is a block of N's in B_2 . For every element $f \in \{g+1, \ldots, m+r\}$, basis B_4 rises by switching x_f for E and x_g for N in B_1 , and basis B_3 rises by switching y_f for E and y_h for N in B_2 ; that is, $B_4 = B_1 - f + g$ and $B_3 = B_2 - f + h$.

Since every N distinct of x_e belongs to some type of block, we get r-1 good C_e passing through the edge B_1B_2 .

Bonin and de Mier [3] observed that the class of all generalized Catalan matroids is closed under duals. Moreover, a basis B^* of the dual of M[P,Q] corresponds to the *E* steps of the basis *B* in M[P,Q]. Therefore, the following is a consequence of this fact and Lemma 17.

Corollary 18. For $r, m \ge 2$, let M[Q] be a generalized Catalan matroid of rank r and corank m, with neither a loop nor an isthmus. Then every edge of BG(M[Q]) is in min $\{r-1, m-1\}$ good cycles.

Let M[P,Q] be a lattice path matroid. Let $P = y^{[m+r]}$ and $Q = x^{[m+r]}$, with $x_i, y_i \in \{N, E\}$ for $i \in [m+r]$. Assume e is neither a loop nor an isthmus. In [3] was observed that:

- M[P,Q]\e is the lattice path matroid M[P',Q'] where the upper bounding path Q' is formed by deleting from Q the first E step that is at or after step e; the lower bounding path P' is formed by deleting from P the last E step that is at or before step x.
- (2) M[P,Q]/e is the lattice path matroid M[P",Q"] where the upper bounding path Q" is formed by deleting from Q the last N step that is at or before step e; the lower bounding path P" is formed by deleting from P the first N step that is at or after step e.

Observation 1. If the k-Catalan matroid is a minor of the generalized Catalan matroid M[Q], then for every element e of M[Q], the (k-1)-Catalan matroid is a minor of both the generalized Catalan matroid $M[Q] \setminus e$ and M[Q] / e.

In fact, Observation 1 also holds if we replace generalized Catalan matroid for lattice path matroid.

For the class of generalized Catalan matroids, we define the function

 $hc_L(k) = min\{HC^*(M[Q]): M[Q] \text{ has a } k\text{-Catalan matroid as a minor}\}.$

Proposition 19. For $k \ge 2$, $hc_L(k) \ge (k-1)hc_L(k-1)^2$.

Proof. Let $M[Q] = M_Q$ be a generalized Catalan matroid such that $\operatorname{HC}^*(M_Q) = \operatorname{hc}_L(k)$. We may assume that M_Q has neither a loop nor an isthmus. Thus, both the rank and corank of M are at least k. By Corollary 18, there are $\min\{r-1, m-1\} \ge k-1$ good cycles for every edge of BG (M_Q) . Let B_1B_2 be an edge of BG (M_Q) , say $B_1 = B_2 - e + g$, and let $M' = M_Q \setminus e$ and $M'' = M_Q/e$. It follows from Observation 1 that both M' and M'' contain a (k-1)-Catalan matroid as a minor. Thus $\operatorname{HC}^*(M') \ge \operatorname{hc}_L(k-1)$ and $\operatorname{HC}^*(M'') \ge \operatorname{hc}_L(k-1)$. Therefore we conclude that $\operatorname{hc}_L(k) \ge (k-1)\operatorname{hc}_L(k-1)^2$. □

Theorem 20. For $k \ge 2$, $hc_L(k) \ge sf(k-1)sf(k-2)$.

Proof. The proof is by induction on k. We write simply M_Q instead M[Q]. Let M_Q be a generalized Catalan matroid such that $\mathrm{HC}^*(M_Q) = \mathrm{hc}_L(k)$. We may assume that M_Q has neither a loop nor a isthmus. In particular, M_Q has both rank and corank at least k. Let k = 2. So $\mathrm{BG}(M_Q)$ has at least three vertices and is edge Hamiltonian. Therefore $\mathrm{hc}_L(2) \geq 1 = \mathrm{sf}(1) \mathrm{sf}(0)$.

Now let $k \ge 3$. Let B_1B_2 be an edge of BG (M_Q) , say $B_2 = B_1 - e + g$. By Corollary 18, the edge B_1B_2 is in min $\{r-1, m-1\} \ge k-1$ good cycles.

Table 1: The three types of good cycles for $U_{r,n}$.

B_1	$\{e, f_i, \ldots\}$	$\{g, f_i, \ldots\}$	B_2	B_1	$\{e, f_i\}$	$_i,\ldots\}$	$\{g, f_i, \ldots\}$	B_2
B_4	$\{e, w, \ldots\}$	$\{g, w, \ldots\}$	B_3	B_4	$\{e, g\}$	$,\ldots \}$	$\{g, f_i, \ldots\} \\ \{g, w, \ldots\}$	B_3
		$\begin{array}{c c} B_1 & \{e, f_2 \\ B_4 & \{e, u\} \end{array}$	$i,\ldots\}$ $v,\ldots\}$	$\begin{cases} g, f_i, \\ \{w, f_i\} \end{cases}$	$\ldots \}$	B_2 B_3		

Consider $M' = M_Q \setminus e$ and $M'' = M_Q/e$. By Observation 1, the (k-1)-Catalan matroid is a minor of both M' and M''. Thus, by the induction hypothesis, $\operatorname{HC}^*(M'), \operatorname{HC}^*(M'') \ge \operatorname{hc}_L(k-1) \ge \operatorname{sf}(k-2)\operatorname{sf}(k-3)$. Hence, every edge of $\operatorname{BG}(M_Q)$ is in $(k-1)(\operatorname{sf}(k-2)\operatorname{sf}(k-3))^2 \ge \operatorname{sf}(k-1)\operatorname{sf}(k-2)$ Hamiltonian cycles.

3.1 Uniform matroids

Recall that the set of bases of the uniform matroid of rank r on n elements, denoted by $U_{r,n}$, consists of all r-subsets of [n]. Also, $U_{r,n}$ can be considered as the lattice path matroid M[P,Q] where $Q = N^r E^{n-r}$ and $P = E^{n-r}N^r$.

Let B_1B_2 be an edge of $BG(U_{r,n})$, say $B_1 = B_2 - e + g$. So, we have that $B_1 = \{e, f_2, \ldots, f_r\}$ and $B_2 = \{g, f_2, \ldots, f_r\}$, with $f_i \in [n] \setminus \{e, g\}$ for $i \in \{2, \ldots, r\}$. For every w in $[n] \setminus \{e, g, f_2, \ldots, f_r\}$, we can obtain three types of good cycles in C_e by replacing an f_i by w as shown in Table 1. We thus have the following result.

Proposition 21. Let $n > r \ge 1$ be integers. Then every edge of BG $(U_{r,n})$ is in 3(n-r-1)(r-1) good cycles.

Finally, the next theorem can be proved by induction on the number of elements of the matroid, applying Proposition 21, and following the same strategy as above.

Theorem 22. Let $n > r \ge 1$ be integers. Then every edge of BG $(U_{r,n})$ is in $((n-r-1)!(r-1)!)^{\min\{n-r-1,r-1\}}$ Hamiltonian cycles.

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