On the integrability of strongly regular graphs

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March 23, 2022

Abstract

Koolen et al. showed that if a connected graph with smallest eigenvalue at least -3 has large minimal valency, then it is 2-integrable. In this paper, we will prove that a lower bound for the minimal valency is 166.

Keywords and phrases: *strongly regular graph, lattice, s-integrability*
Mathematics Subject Classification: *05C50, 05E30, 11H99*

1 Introduction

(For undefined notations, we refer to next section.) In 1976, Cameron, Goethals, Seidel and Shult [6] showed that:

Theorem 1.1. *If G is a connected graph with smallest eigenvalue at least -2, then either G has at most 36 vertices, or G is integrable.*

Note that for a connected graph G, the graph G is complete if and only if $\theta_{\min}(G) \geq -1$; for $-1 > \theta_{\min}(G) \geq -2$, Cameron et al. showed that the lattice $\Lambda(G)$ is one of the root lattices A_p, D_q, E_6, E_7, E_8 with $p \geq 1$ and $q \geq 4$, and note that A_p and D_q are integrable, and E_6, E_7, E_8 are 2-integrable, but not integrable, see [10].

Thus as a corollary of (the proof of) the result of Cameron et al. [6], we have

Corollary 1.2. *Any connected graph G with smallest eigenvalue at least -2 is 2-integrable. Moreover, if G is not integrable, then G has at most 36 vertices.*

For the case of graphs with smallest eigenvalue at least -3, Koolen, Yang and Yang [24] generalized a result of Hoffman (1977) [21] and showed the following.

Theorem 1.3. *There exists a positive integer κ such that if a connected graph G has smallest eigenvalue at least -3 and minimal valency at least κ, then G is 2-integrable.*

In this paper, we will show that the number κ in Theorem 1.3 is at least 166. Our main result is as follows, which immediately follows from Theorems 4.4, 5.2, 6.3, 7.2, 8.3 and Proposition 8.1 (ii).

Theorem 1.4. (i) *If G is an integrable strongly regular graph with parameters (v, k, λ, μ) and smallest eigenvalue $\theta_{\min}(G) \in [-3, -2)$, then either $\mu \leq 9$ or G is the strongly regular graph $K_{\frac{v}{3} \times 3}$.*

  (ii) *The strongly regular graph $\overline{SG}$, which is the complement of the Sims-Gewirtz graph, with parameters $(56, 45, 36, 36)$ and smallest eigenvalue -3 is not integrable, but 2-integrable.*

  (iii) *The Hoffman-Singleton graph $HoSi$ with parameters $(50, 7, 0, 1)$ and smallest eigenvalue -3 is not 2-integrable.*

  (iv) *The strongly regular graph $\overline{GQ}(3, 9)$, which is the complement of the point graph of the generalized quadrangle of order $(3, 9)$, with parameters $(112, 81, 60, 54)$ and smallest eigenvalue -3 is not 2-integrable, but 4-integrable.*

  (v) *The strongly regular graph $\overline{McL}$, which is the complement of the McLaughlin graph, with parameters $(275, 162, 105, 81)$ and smallest eigenvalue -3 is not 2-integrable, but 4-integrable.*

Remark 1.5. (i) *The result $\mu \leq 9$ in Theorem 1.4 (i) can not be improved, as the strongly regular graph with parameters $(35, 18, 9, 9)$ and smallest eigenvalue -3 is integrable.*

  (ii) *Adding three distinct vertices x_1, x_2, x_3 to the graph $\overline{McL}$ such that x_1, x_2, x_3 are adjacent to each other and to all vertices of $\overline{McL}$, we obtain a new graph with smallest eigenvalue -3 and minimal valency 166. This shows that the value of κ in Theorem 1.3 is at least 166.*

2   Definitions and preliminaries

All the graphs considered in this paper are finite, undirected and simple. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $E(G)$. Recall that the *adjacency matrix* $A(G)$ of G is the 01-matrix with rows and columns indexed by the vertices of G, such that the xy-entry of $A(G)$ is 1 if and only if x and y are adjacent. We write $x \sim y$ if the vertices x and y are adjacent and $x \nsim y$ otherwise. The *eigenvalues* of G are the eigenvalues of $A(G)$ and we denote by $\theta_{\min}(G)$ the smallest eigenvalue of G. For a vertex x in G, we write $k(x)$ for the valency of x, that is, the number of neighbors of x. We call the graph G *k-regular* if $k(x) = k$ for any vertex x.

Recall that a *c-clique* (resp. *c-coclique*) is a subgraph of G with c vertices such that every pair of vertices are adjacent (resp. not adjacent). The well-known Ramsey's Theorem is as follows.

Theorem 2.1. *([30]) Let a, b be two positive integers. Then there exists a minimum positive integer $R(a, b)$ such that for any graph G on $m \geq R(a, b)$ vertices, the graph G contains a a-clique or a b-coclique as a subgraph.*

Let $\pi = \{V_1, \ldots, V_p\}$ be a partition of the vertex set $V(G)$ of the graph G. For each vertex x in V_i, write $d_x^{(j)}$ for the number of neighbors of x in V_j. Then we write $q_{i,j} = \frac{1}{|V_i|} \sum_{x \in V_i} d_x^{(j)}$ for the average number of neighbors in V_j of vertices in V_i. The matrix $Q_\pi := (q_{i,j})$ is called the *quotient matrix* of G relative to π and π is called *equitable* if for all i and j, we have $d_x^{(j)} = q_{i,j}$ for each $x \in V_i$.

Lemma 2.2. *([17, Theorem 9.3.3 and Lemma 9.6.1]) If π is a partition of the vertex set of a graph G which has three distinct eigenvalues, then π is equitable if and only if all eigenvalues of the quotient matrix Q_π are the eigenvalues of G.*

Now let us look at a class of highly structured graphs, that is, the class of strongly regular graphs. A graph G with v vertices is said to be a *strongly regular graph* with parameters (v, k, λ, μ) if it is k-regular, every pair of adjacent vertices has λ common neighbors, and every pair of distinct nonadjacent vertices has μ common neighbors.

Note that if G is a strongly regular graph with parameters (v, k, λ, μ), then the complement $\overline{G}$ of the graph G is also strongly regular and its parameters are

$$(v, v - 1 - k, v - 2 - 2k + \mu, v - 2k + \lambda). \tag{2.1}$$

Let G be a strongly regular graph with parameters (v, k, λ, μ). For any vertex x of G, we denote by $N_i(x)$ the set of vertices at distance i from x, where $i = 1, 2$. We call the graph induced by $N_1(x)$ (resp. $N_2(x)$) the *first* (resp. *second*) *subconstituent* of G relative to x. Let $\pi(x) := \{\{x\}, N_1(x), N_2(x)\}$ be a partition of $V(G)$. Note that $\pi(x)$ is equitable and the quotient matrix $Q_{\pi(x)}$ of G relative to π is as follows:

$$Q_{\pi(x)} = \begin{pmatrix} 0 & k & 0 \\ 1 & \lambda & k - 1 - \lambda \\ 0 & \mu & k - \mu \end{pmatrix}. \tag{2.2}$$

By Lemma 2.2, the eigenvalues of the quotient matrix $Q_{\pi(x)}$ are also the eigenvalues of G.

2.1 Graphs and lattices

In this subsection, we will define the s-integrability of graphs. Before that, we will introduce integral lattices and their s-integrability.

Let Λ be a subset of $\mathbb{R}^n$. We say that Λ is a *lattice* if there exist vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m \in \mathbb{R}^n$ (for some m) such that $\Lambda = \left\{ \sum a_i \mathbf{u}_i \mid a_i \in \mathbb{Z} \right\}$ and we call $\{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ a *generator set* of the lattice Λ.

A lattice Λ is called *integral*, if $(\mathbf{v}_i, \mathbf{v}_j) \in \mathbb{Z}$ for all $\mathbf{v}_i, \mathbf{v}_j \in \Lambda$. An integral lattice Λ is called *s-integrable* (for some positive integer s) if $\sqrt{s}\Lambda$ can be embedded in the standard lattice. Note that a 1-integrable lattice is also called *integrable*.

For a positive real number t, a *representation of norm t* of a graph G is a map $\varphi : V(G) \mapsto \mathbb{R}^n$ (for some positive integer n) such that

$$(\varphi(x), \varphi(y)) = \begin{cases} t & \text{if } x = y, \\ 1 & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

Note that G has a representation of norm t if and only if $\theta_{\min}(G) \geq -t$, as $A(G) + tI$ has to be positive semidefinite.

Let φ be a representation of norm $-\lfloor \theta_{\min}(G) \rfloor$ of G. The integral lattice generated by φ is the lattice $\Lambda^\varphi(G) := \left\{ \sum a_x \varphi(x) \mid a_x \in \mathbb{Z}, x \in V(G) \right\}$. As the Gram matrix of its generator set $\{\varphi(x) \mid x \in V(G)\}$ is equal to $A(G) - \lfloor \theta_{\min}(G) \rfloor I$, (the isomorphic class of) the lattice $\Lambda^\varphi(G)$ only depends on the matrix $A(G) - \lfloor \theta_{\min}(G) \rfloor I$, not on the particular representation φ. Thus, we denote this lattice by $\Lambda(G)$ and we say the graph G is *s-integrable* if the lattice $\Lambda(G)$ is s-integrable. If the lattice $\Lambda(G)$ is 1-integrable, we also say that the graph G is *integrable*.

A graph G is s-integrable if and only if there exists an integral matrix N such that the equality $A(G) - \lfloor \theta_{\min}(G) \rfloor I = \frac{1}{s} N^T N$ holds.

Note that if G is s-integrable, it is also τs-integrable for all positive integers τ. The smallest positive integer s such that a graph is s-integrable gives a measure of the complexity of the graphs structure.

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2.2 Designs

Definition 2.3. *A* design *is a pair* $(S, \mathcal{B})$, *where S is a set of elements called* points *and $\mathcal{B}$ is a collection of nonempty subsets of S called* blocks. *Let $\tilde{v}, \tilde{k}, \tilde{\lambda}$ and $\tilde{t}$ be positive integers such that $\tilde{v} > \tilde{k} \geq \tilde{t}$. A $\tilde{t}$-$(\tilde{v}, \tilde{k}, \tilde{\lambda})$ design is a design $(S, \mathcal{B})$ such that the following properties are satisfied:*

(i) $|S| = \tilde{v}$,

(ii) *each block contains exactly $\tilde{k}$ points, and*

(iii) *every set of $\tilde{t}$ distinct points is contained in exactly $\tilde{\lambda}$ blocks.*

Lemma 2.4. *([33, Theorem 9.4]) Suppose $(S, \mathcal{B})$ is a $\tilde{t}$-$(\tilde{v}, \tilde{k}, \tilde{\lambda})$ design and $1 \leq \tilde{s} \leq \tilde{t}$. Then $(S, \mathcal{B})$ is an $\tilde{s}$-$(\tilde{v}, \tilde{k}, \tilde{\lambda}_{\tilde{s}})$ design, where*

$$\tilde{\lambda}_{\tilde{s}} = \frac{\tilde{\lambda}\binom{\tilde{v}-\tilde{s}}{\tilde{t}-\tilde{s}}}{\binom{\tilde{k}-\tilde{s}}{\tilde{t}-\tilde{s}}}. \tag{2.3}$$

If $\tilde{\lambda} = 1$, then the $\tilde{t}$-$(\tilde{v}, \tilde{k}, 1)$ design is called a *Steiner system* and denote by $S(\tilde{t}, \tilde{k}, \tilde{v})$. Note that a Steiner system $S(\tilde{t}, \tilde{k}, \tilde{v})$ with $\tilde{t} \geq 2$ is also a 2-$(\tilde{v}, \tilde{k}, \tilde{\lambda}_2)$ design, where $\tilde{\lambda}_2 = \frac{\binom{\tilde{v}-\tilde{s}}{\tilde{t}-\tilde{s}}}{\binom{\tilde{k}-\tilde{s}}{\tilde{t}-\tilde{s}}}$.

Definition 2.5. *A 2-$(\tilde{v}, \tilde{k}, \tilde{\lambda})$ design is called* quasi-symmetric *if the cardinality of the intersection of two blocks takes only two values.*

Proposition 2.6. *([27, Theorem 3.7]) The Steiner system $S(4, 7, 23)$ is quasi-symmetric with intersection numbers 1 and 3.*

Let $(S, \mathcal{B})$ be a quasi-symmetric design with intersection numbers l_1 and l_2, where $l_1 < l_2$. The *block graph* of $(S, \mathcal{B})$ is the graph whose vertices are the blocks of $(S, \mathcal{B})$, with two vertices adjacent if and only if their intersection has cardinality l_2.

Lemma 2.7. *(cf. [7, Theorem 3.2]) The block graph of a quasi-symmetric design is strongly regular.*

Let $(S, \mathcal{B})$ be a design where $S = \{x_1, \ldots, x_{\tilde{v}}\}$ and $\mathcal{B} = \{B_1, \ldots, B_{\tilde{b}}\}$. The *incidence matrix* of $(S, \mathcal{B})$ is the $\tilde{v} \times \tilde{b}$ 01-matrix $\mathcal{I} = (\mathcal{I}_{x_i, B_j})$ defined by the rule $\mathcal{I}_{x_i, B_j}$ equals 1 if $x_i \in B_j$ and 0 otherwise.

3 Basic results on s-integrable graphs

In this section, we will give several results about the s-integrability of graphs. Some of the results play an important role in the proof of our main theorem. Before stating our results, the following notations are necessary. For a given vector $\mathbf{r}$, we denote by $\operatorname{supp}(\mathbf{r})$ the support of $\mathbf{r}$, that is, $\operatorname{supp}(\mathbf{r}) = \{i \mid \mathbf{r}_i \neq 0\}$. If a vector $\mathbf{r}$ is indexed by the vertex set of a graph G, we denote by $G(\mathbf{r})$ the subgraph of G induced by the set $\operatorname{supp}(\mathbf{r})$. For convenience, we also write

$$\gamma_x^{\mathbf{r}} := \mathbf{r}_x, \ \ \delta_x^{\mathbf{r}} := \sum_{y \sim x} \mathbf{r}_y, \ \ \zeta_x^{\mathbf{r}} := \sum_{y \neq x, y \not\sim x} \mathbf{r}_y,$$

and

$$\sigma^{\mathbf{r}} := \gamma_x^{\mathbf{r}} + \delta_x^{\mathbf{r}} + \delta_x^{\mathbf{r}}.$$

Now we start our work.

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Proposition 3.1. *Assume that G is an s-integrable graph with smallest eigenvalue $\theta_{\min}(G)$ and N is a $\rho \times |V(G)|$ integral matrix satisfying $A(G) + \lceil -\theta_{\min}(G) \rceil I = \frac{1}{s} N^T N$. Then:*

(i) *There exists a row vector $\mathbf{r}$ of N such that*

$$|\mathrm{supp}(\mathbf{r})| \leq \frac{s\lceil -\theta_{\min}(G) \rceil |V(G)|}{\rho} \leq \frac{s\lceil -\theta_{\min}(G) \rceil |V(G)|}{\mathrm{rank}(N)} \tag{3.1}$$

holds, where $\mathrm{rank}(N)$ is the rank of N.

(ii) *Let $\pi := \{V_1, \ldots, V_p\}$ be an equitable partition of the vertex set of G such that the quotient matrix Q_π has $\theta_{\min}(G)$ as its smallest eigenvalue. If $\theta_{\min}(G)$ is an integer, then for any row $\mathbf{r}$ of N and any eigenvector $\mathbf{u}$ of Q_π with eigenvalue $\theta_{\min}(G)$, we have*

$$\sum_{i=1}^{p} \mathbf{u}_i \sum_{x \in V_i} \mathbf{r}_x = 0, \quad \text{where } \mathbf{u}_i \text{ is the } V_i\text{-entry of } \mathbf{u}. \tag{3.2}$$

(iii) *For any row $\mathbf{r}$ of N, we have the following inequality:*

$$\sum_{x \in V(G)} \gamma_x^{\mathbf{r}} \delta_x^{\mathbf{r}} \geq \frac{1}{s}(\mathbf{r}\mathbf{r}^T)^2 + \lfloor \theta_{\min}(G) \rfloor \mathbf{r}\mathbf{r}^T. \tag{3.3}$$

(iv) *If G' is an induced subgraph of G with $\lfloor \theta_{\min}(G') \rfloor = \lfloor \theta_{\min}(G) \rfloor$, then G' is also s-integrable.*

Proof. (i) Suppose N has b non-zero entries. For any column N_x of N, it has at most $s\lceil -\theta_{\min}(G) \rceil$ non-zero entries since N is integral and $(N_x, N_x) = s\lceil -\theta_{\min}(G) \rceil$. Thus, $b \leq s\lceil -\theta_{\min}(G) \rceil |V(G)|$. Choose $\mathbf{r}$ to be a row of N whose support size is the smallest. It is easy to see that $b \geq |\mathrm{supp}(\mathbf{r})|\rho \geq |\mathrm{supp}(\mathbf{r})|\mathrm{rank}(N)$ and hence (i) holds.

(ii) Define a vector $\widetilde{\mathbf{u}}$ such that $\widetilde{\mathbf{u}}_x = \mathbf{u}_i$ if $x \in V_i$. It is easy to check that $A(G)\widetilde{\mathbf{u}} = \theta_{\min}(G)\widetilde{\mathbf{u}}$ holds and thus $N\widetilde{\mathbf{u}} = \mathbf{0}$. Clearly, for the row $\mathbf{r}$, $\mathbf{r}\widetilde{\mathbf{u}} = \sum_{x \in V(G)} \mathbf{r}_x \widetilde{\mathbf{u}}_x = \sum_{i=1}^{p} \mathbf{u}_i \sum_{x \in V_i} \mathbf{r}_x = 0$ and (ii) holds.

(iii) Since $\frac{1}{s}(\mathbf{r}\mathbf{r}^T)^2 \leq \frac{1}{s}\mathbf{r}N^T N\mathbf{r}^T = \mathbf{r}(A(G) + \lceil -\theta_{\min}(G) \rceil I)\mathbf{r}^T = \mathbf{r}A(G)\mathbf{r}^T - \lfloor \theta_{\min}(G) \rfloor \mathbf{r}\mathbf{r}^T$, we have $\frac{1}{s}(\mathbf{r}\mathbf{r}^T)^2 + \lfloor \theta_{\min}(G) \rfloor \mathbf{r}\mathbf{r}^T \leq \mathbf{r}A(G)\mathbf{r}^T = \sum_{x \in V(G)} \mathbf{r}_x \sum_{y \sim x} \mathbf{r}_y = \sum_{x \in V(G)} \gamma_x^{\mathbf{r}} \delta_x^{\mathbf{r}}$ and (iii) holds.

(iv) This is clear. $\qquad\square$

Proposition 3.2. *Let G be a graph with smallest eigenvalue $\theta_{\min}(G) \in [-3, -2)$. If G is 2-integrable, then for any integral matrix N satisfying $A(G) + 3I = \frac{1}{2}N^T N$ and any row $\mathbf{r}$ of N, the order of a coclique in the subgraph $G(\mathbf{r})$ is at most 6.*

Moreover, if $|\mathbf{r}_x| = 2$ for some $x \in V(G)$, then the order of a coclique in the subgraph $G(\mathbf{r})$ is at most 3.

Proof. Assume $\mathbf{r}$ is the rth row of N and $\overline{C}$ is a coclique in $G(\mathbf{r})$ with $V(\overline{C}) = \{x_1, \ldots, x_{\bar{c}}\}$, where $\bar{c}$ is the order of $\overline{C}$. Denote by $\widetilde{N}$ the matrix whose columns are indexed by the vertices of $\overline{C}$ such that its x_i-column equals $\frac{\mathbf{r}_{x_i}}{|\mathbf{r}_{x_i}|} N_{x_i}$, where N_{x_i} is the x_i-column of N. By deleting the rth row from $\widetilde{N}$, we obtain its submatrix $\widetilde{N}'$. Note that $N^T N = 2A + 6I$ and there exists at most one vertex x in $\overline{C}$ such that $|\mathbf{r}_x| = 2$. If $|\mathbf{r}_x| = 2$ for some x, then $(\widetilde{N}')^T \widetilde{N}' = \begin{pmatrix} 2 & -2\mathbf{j}_{\bar{c}-1}^T \\ -2\mathbf{j}_{\bar{c}-1} & 6I_{\bar{c}-1} - J_{\bar{c}-1} \end{pmatrix}$;

otherwise $(\widetilde{N}')^T \widetilde{N}' = 6I_{\bar{c}} - J_{\bar{c}}$, where $\mathbf{j}$ and J are the all-ones vector and all-ones matrix respectively. Since the matrix $(\widetilde{N}')^T \widetilde{N}'$ is positive semidefinite, for the former case we have $\bar{c} \leq 3$, since the matrix $\begin{pmatrix} 2 & -2\mathbf{j}_3^T \\ -2\mathbf{j}_3 & 6I_3 - J_3 \end{pmatrix}$ is not positive semidefinite; for the latter case we have $\bar{c} \leq 6$, since the matrix $6I_7 - J_7$ is not positive semidefinite. This completes the proof. $\square$

4 Integrable strongly regular graphs

4.1 Two classes of integrable strongly regular graphs

In this subsection, we will introduce geometric strongly regular graphs, and prove that they are integrable. Moreover, we will show that the regular complete multipartite graphs are also integrable at the end of this subsection.

Let G be a strongly regular graph with parameters (v, k, λ, μ) and smallest eigenvalue $\theta_{\min}(G)$. In 1973, Delsarte [12] showed that any clique C of G has the order at most $1 - \frac{k}{\theta_{\min}(G)}$ and we call C *Delsarte* if its order is exactly $1 - \frac{k}{\theta_{\min}(G)}$. We say G is *geometric* if there exists a set $\mathcal{C}$ of Delsarte cliques of G such that every pair of adjacent vertices of G lies in a unique $C \in \mathcal{C}$. In 1979, Neumaier [25] showed that

Theorem 4.1. *For any integer $t \geq 2$, there are only finitely many connected non-geometric strongly regular graphs with smallest eigenvalue at least $-t$.*

For geometric strongly regular graphs, we have the following result:

Lemma 4.2. *If G is a geometric strongly regular graph, then G is integrable.*

Proof. Without loss of generality, we may assume that G has valency k and smallest eigenvalue $\lambda_{\min}(G)$. (Note that $\lambda_{\min}(G)$ must be an integer.) Let N be the vertex-clique incidence matrix with respect to the set $\mathcal{C}$ of Delsarte cliques, that is $N_{C,x} = \begin{cases} 1 & \text{if } x \sim V(C) \\ 0 & \text{otherwise} \end{cases}$ for $x \in V(G)$ and $C \in \mathcal{C}$. Then $N^T N = A(G) - \theta_{\min}(G)I$ holds and this completes the proof of this lemma. $\square$

Note that the complete multipartite graph $K_{n \times t} = K_{\underbrace{t, \ldots, t}_{n}}$, which is the strongly regular graph with parameters $(nt, (n-1)t, (n-2)t, (n-1)t)$ and smallest eigenvalue $-t$, is also integrable. The proof is as follows:

Let $\cup_{i=1}^n V_i$ be the partition of $K_{n \times t}$ into its color classes where $V_i = \{x_1^i, \ldots, x_t^i\}$, and let $\{\mathbf{e}, \mathbf{e}_{i,l,j} \mid 1 \leq i \leq n, 1 \leq l < j \leq t\}$ be an integral orthonormal basis for $\mathbb{R}^{1 + \frac{nt(t-1)}{2}}$. Define the x_j^i-column $N_{x_j^i}$ of N as follows:

$$N_{x_j^i} := \mathbf{e} - \sum_{1 \leq l < j} \mathbf{e}_{i,l,j} + \sum_{j < l \leq t} \mathbf{e}_{i,j,l}.$$

We find that $N^T N = A(K_{n \times t}) + tI$ and this completes the proof.

4.2 Integrable strongly regular graphs with smallest eigenvalue at least -2

Now let us look at the known results on the strongly regular graphs with smallest eigenvalue at least -2. In 1968, Seidel [31] showed that

Theorem 4.3. *If G is a connected strongly regular graph with smallest eigenvalue -2, then G is a triangular graph $T(n)$ ($n \geq 5$), a lattice graph $L_2(n)$ ($n \geq 3$), a complete multipartite graph $K_{n \times 2}$ ($n \geq 2$), or one of the graphs of Petersen, Clebsch, Schläfli, Shrikhande, or Chang.*

From ([2, Corollary 3.12.3]), we find that if G is a connected strongly regular graph with v vertices, valency k and smallest eigenvalue $\theta_{\min}(G) > -2$, then either G is complete or G is a pentagon. Therefore, the strongly regular graphs with smallest eigenvalue at least -2, which are not integrable, are the Petersen graph, Clebsch graph, Schläfli graph, Shrikhande graph and the three Chang graphs. Note that all of these seven strongly regular graphs are 2-integrable, see [2, Section 3.11].

4.3 Integrable strongly regular graphs with smallest eigenvalue at least -3

In this subsection, we will show the following theorem.

Theorem 4.4. *Assume that G is an integrable strongly regular graph with parameters (v, k, λ, μ) and smallest eigenvalue $\theta_{\min}(G) \in [-3, -2)$. If $\mu \geq 10$, then G is the strongly regular graph $K_{\frac{v}{3} \times 3}$.*

Proof. Let $\mu \geq 10$. For convenience, we denote by $W_{x,y} := \{w \mid w \nsim x, w \nsim y \text{ and } w \neq x, y\}$ the set of common nonadjacent vertices of x and y. It is easy to see that

$$v = \begin{cases} 2k - \lambda + |W_{x,y}| & \text{if } x \sim y, \\ 2 + 2k - \mu + |W_{x,y}| & \text{if } x \nsim y. \end{cases} \tag{4.1}$$

Since G is integrable, there exists an integral matrix N such that

$$A(G) + 3I = N^T N$$

holds. Note that $N_{i,j} \in \{1, -1, 0\}$ for any i, j. For any vertex x, we denote by N_x the x-column of N and $\operatorname{supp}(N_x) := \{i \mid (N_x)_i \neq 0\}$ the support of N_x.

For any two distinct vertices x and y, we first loot at the case $|\operatorname{supp}(N_x) \cap \operatorname{supp}(N_y)| = 3$. It follows immediately that x and y are adjacent and we may assume $N_x = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ and $N_y = \mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$. For any vertex $z \in V(G) - \{x, y\}$, if z is adjacent to x, then $(N_z, N_x) = 1$. As $0 \leq (N_z, N_y) \leq 1$, we find that $N_z = \mathbf{e}_1 + \mathbf{e}_i + \mathbf{e}_j$ or $N_z = \mathbf{e}_2 + \mathbf{e}_i + \mathbf{e}_j$ for some $3 < i < j$, and thus $z \sim y$. This implies that $N_G(x) - \{y\} = N_G(y) - \{x\}$, that is, $\lambda = k - 1$. Hence, we obtain that G is the strongly regular graph $\frac{v}{k+1} K_{k+1}$ (see [17, Lemma 10.1.1]), which has smallest eigenvalue -2. This gives a contradiction. Therefore, we have

$$|\operatorname{supp}(N_x) \cap \operatorname{supp}(N_y)| = 1, \text{ if } x \sim y. \tag{4.2}$$

From now on, we may assume $x \nsim y$, and we will show that

$$|\operatorname{supp}(N_x) \cap \operatorname{supp}(N_y)| = 2, \text{ if } x \nsim y. \tag{4.3}$$

Suppose not, then $|\operatorname{supp}(N_x) \cap \operatorname{supp}(N_y)| = 0$ and we may assume $N_x = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$, $N_y = \mathbf{e}_4 + \mathbf{e}_5 + \mathbf{e}_6$. Since $\mu \geq 10$, there must exist two vertices $z, z' \in N_G(x) \cap N_G(y)$ such that $|\operatorname{supp}(N_z) \cap \operatorname{supp}(N_{z'})| = 3$. This is not possible and (4.3) holds. Moreover, if there exists an integral unit vector $\mathbf{e}$ such that $(N_x, \mathbf{e}) = (N_y, \mathbf{e}) = 1$, then we have

$$(N_z, \mathbf{e}) \geq 0, \text{ for } z \in V(G). \tag{4.4}$$

Otherwise let $N_x := \mathbf{e} + \mathbf{e}' + \mathbf{e}''$ and $N_y := \mathbf{e} - \mathbf{e}' + \mathbf{e}'''$ be the x-column and y-column of N. It is easy to see that, by (4.2), there exist at least $\mu - 1$ vertices in the set $N_G(x) \cap N_G(y)$ such that the columns of N indexed by these vertices have inner product 1 with $\mathbf{e}$, and 0 with all $\mathbf{e}'$, $\mathbf{e}''$ and $\mathbf{e}'''$. Suppose that there exists a vertex z such that $(N_z, \mathbf{e}) = -1$, then $N_z = -\mathbf{e} + \mathbf{e}'' + \mathbf{e}'''$.

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In this case, N_z has inner product -1 with the column of N indexed by some vertex in the set $N_G(x) \cap N_G(y)$. This is not possible and (4.4) holds.

Given any two nonadjacent vertices u and v, we may assume, by (4.3), that

$$N_u = \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3, \ N_v = \mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_4.$$

Let us look at the set $W_{u,v}$. For any vertex $w \in W_{u,v}$, if $(N_w, \mathbf{e}_2) = 1$, then we have, by (4.4), that $(N_v, \mathbf{e}_2) = -1 \geq 0$ as $w \not\sim u$ and $(N_w, \mathbf{e}_2) = (N_u, \mathbf{e}_2) = 1$. This is not possible. Thus $(N_w, \mathbf{e}_2) \neq 1$. Similarly, $(N_w, -\mathbf{e}_2) \neq 1$. By using (4.4) again, we find that $(N_w, \mathbf{e}_1) \geq 0$ and thus $N_w = \mathbf{e}_1 - \mathbf{e}_3 - \mathbf{e}_4$. This implies that

$$|W_{u,v}| \leq 1.$$

If $|W_{u,v}| = 0$, then $v = 2 + 2k - \mu$ by (4.1). For the number $k - \mu$, which is the cardinality of the set $N_G(u) - N_G(u) \cap N_G(v)$, we have $k - \mu \geq 1$ otherwise $G = K_{\frac{v}{2} \times 2}$ with $\theta_{\min}(G) = -1$. For any vertex $u' \in N_G(u) - N_G(u) \cap N_G(v)$, we have $(N_{u'}, -\mathbf{e}_2) \neq 1$. Otherwise $(N_u, -\mathbf{e}_2) = -1 \geq 0$ by (4.4), as $u' \not\sim v$ and $(N_{u'}, -\mathbf{e}_2) = (N_v, -\mathbf{e}_2) = 1$. This is not possible. Note that $|\mathrm{supp}(N_{u'}) \cap \mathrm{supp}(N_u)| = 1$, $|\mathrm{supp}(N_{u'}) \cap \mathrm{supp}(N_v)| = 2$ and $(N_{u'}, \mathbf{e}_1) \geq 0$ by (4.2)-(4.4). Thus,

$$N_{u'} = \mathbf{e}_1 - \mathbf{e}_4 + \mathbf{e}_i \text{ or } N_{u'} = \mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_i \text{ for some } i \geq 5.$$

If $k - \mu = |N_G(u) - N_G(u) \cap N_G(v)| = |\{u', u''\}| = 2$, then we may assume $N_{u'} = \mathbf{e}_1 - \mathbf{e}_4 + \mathbf{e}_i$ and $N_{u''} = \mathbf{e}_2 + \mathbf{e}_4 + \mathbf{e}_i$ for some $i \geq 5$. As $v \not\sim u''$ and $(N_v, \mathbf{e}_4) = (N_{u''}, \mathbf{e}_4) = 1$, we have, by (4.4), that $(N_{u'}, \mathbf{e}_4) = -1 \geq 0$. This gives the contradiction. Thus, $k - \mu = 1$ and $v = 3 + k$. We conclude that the complement $\overline{G}$ of G is 2-regular and any two adjacent vertices of $\overline{G}$ have no common neighbors. Since $\overline{G}$ is a strongly regular graph with at least 12 vertices, we find that this is not possible. Therefore, $|W_{u,v}| = 1$.

Assume that $W_{u,v} = \{w\}$, then $N_w = \mathbf{e}_1 - \mathbf{e}_3 - \mathbf{e}_4$ and it is easy to check that $N_G(u) = N_G(v) = N_G(w)$, that is, $\mu = k$ and $v = 2 + k + 1$ by (4.1). We conclude that the complement $\overline{G}$ of G is 2-regular and any two adjacent vertices of $\overline{G}$ have exactly one common neighbor, that is, $\overline{G}$ is a disjoint union of K_3. This implies that G is the graph $K_{\frac{v}{3} \times 3}$.

This completes the proof. $\qquad \square$

5 The complement of the Sims-Gewirtz graph

In 1969, A. Gewirtz [16] showed that the strongly regular graph with parameters $(56, 10, 0, 2)$ exists and is unique, named the Sims-Gewirtz graph. Thus, the strongly regular graph, by (2.1), with parameters $(56, 45, 36, 36)$ is also unique, which is the complement of the Sims-Gewirtz graph, and its spectrum is $\{45^{(1)}, 3^{(20)}, -3^{(35)}\}$. By Theorem 4.4, we easily find that the latter strongly regular graph is not integrable. Note that

Lemma 5.1. ([26, Theorem 4]) *The block graph of the quasi-symmetric* 2-(21, 6, 4) *design with intersection numbers* $l_1 = 0$ *and* $l_2 = 2$ *is strongly regular with parameters* $(56, 45, 36, 36)$.

Thus, we have the following result.

Theorem 5.2. *The strongly regular graph* $\overline{SG}$, *which is the complement of the Sims-Gewirtz graph, with parameters* $(56, 45, 36, 36)$ *and smallest eigenvalue* -3 *is* 2-*integrable.*

Proof. By using Lemma 5.1 and the fact that the strongly regular graph with parameters $(56, 45, 36, 36)$ is unique, we conclude that $\overline{SG}$ is the block graph of the quasi-symmetric 2-$(21, 6, 4)$ design with intersection numbers $l_1 = 0$ and $l_2 = 2$. Denote by $\mathcal{I}$ the incidence matrix

of this design. We find that $2A(\overline{SG}) + 6I = \mathcal{I}^T\mathcal{I}$, that is,

$$A(\overline{SG}) + 3I = \frac{1}{2}\mathcal{I}^T\mathcal{I}.$$

This completes the proof. □

6 The Hoffman-Singleton graph

The Hoffman-Singleton graph, constructed by A.J. Hoffman and R.R. Singleton in 1960 [22], is the unique strongly regular graph with parameters $(50, 7, 0, 1)$ and spectrum $\{7^{(1)}, 2^{(28)}, (-3)^{(21)}\}$. For the uniqueness, see [23]. Here we give a construction of the Hoffman-Singleton graph, which is introduced by N.Robertson, see [2, p. 391].
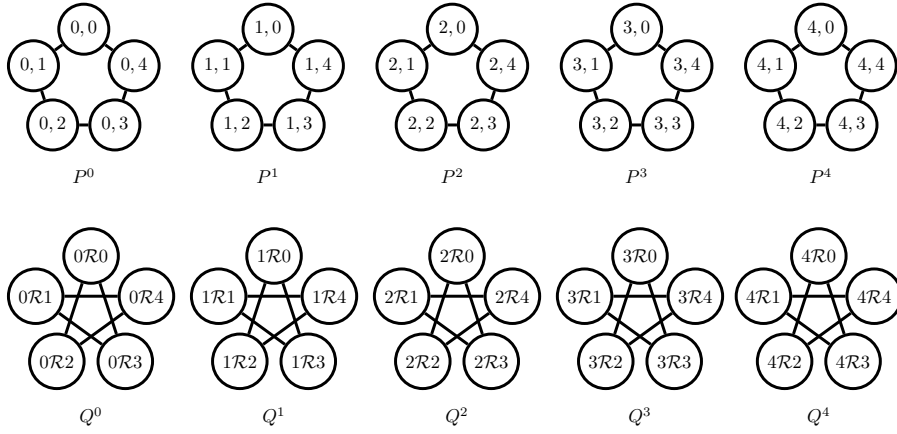
Let P^i and Q^i be the graphs in Figure 1,

Figure 1

where $V(P^i) = \{(i, 0), \ldots, (i, 4)\}$ and $V(Q^i) = \{i\mathcal{R}0, \ldots, i\mathcal{R}4\}$. Now join vertex $i\mathcal{R}l$ of Q^i to vertex $(j, i \cdot j + l)$ of P^j for all $0 \le i, j, l \le 4$. Here $i \cdot j + l$ is calculated modulo 5. This yields the Hoffman-Singleton graph.

For the pentagons in a Hoffman-Singleton graph, C. Fan and A.J. Schwenk [15] showed the following.

Proposition 6.1. *Suppose that HoSi is a Hoffman-Singleton graph with a pentagon H_0 as a subgraph. Denote by $N_i(H_0)$ the set of vertices not in H_0 each of which has distance i from some vertex of H_0, where $i = 1, 2$. Then the subgraph induced by $N_1(H_0)$ is a disjoint union of 5 pentagons $H^0, \ldots, H^4$ and the subgraph induced by $N_2(H_0)$ is a disjoint union of 4 pentagons $H_1, \ldots, H_4$. Moreover, the induced subgraph on $V(H_i) \cup V(H^j)$ is a Petersen graph for $0 \le i, j \le 4$.*

In [20], P.R. Hafner showed that

Proposition 6.2. *In a Hoffman-Singleton graph, there are 1260 pentagons and 126 sets of 10 disjoint pentagons. The automorphism group of the Hoffman-Singleton graph acts transitively on the set of all sets of 10 disjoint pentagons.*

In this section, we will show the following result.

Theorem 6.3. *The Hoffman-Singleton graph $HoSi$ with parameters $(50, 7, 0, 1)$ and smallest eigenvalue -3 is not 2-integrable.*

Proof. Suppose that $HoSi$ is 2-integrable. Then there exists an integral matrix N such that $2A(HoSi) + 6I = N^T N$ holds. Note that for any entry $N_{i,j}$ of N, $N_{i,j} \in \{\pm 1, \pm 2, 0\}$. Without loss of generality, we may assume that for any row $\mathbf{r}$ of N, $|\mathrm{supp}(\mathbf{r})| > 0$ and $\sigma^{\mathbf{r}} \geq 0$.

For any $x \in V(HoSi)$, let $\pi(x) := \{\{x\}, N_1(x), N_2(x)\}$ be an equitable partition of $HoSi$. From (2.2), we find the quotient matrix $Q_{\pi(x)}$ of $HoSi$ relative to $\pi(x)$ is as follows:

$$Q_{\pi(x)} = \begin{pmatrix} 0 & 7 & 0 \\ 1 & 0 & 6 \\ 0 & 1 & 6 \end{pmatrix}.$$

The vector $\mathbf{u} := (21, -9, 1)^T$ is an eigenvector of $Q_{\pi(x)}$ with eigenvalue -3. We have, by (3.2),

$$21\gamma_x^{\mathbf{r}} - 9\delta_x^{\mathbf{r}} + \zeta_x^{\mathbf{r}} = 0 \tag{6.1}$$

holds, for any row $\mathbf{r}$ of N.

First, we claim that each entry of N equals 1, -1 or 0. Suppose not, then there exists a row $\mathbf{r}'$ of N and a vertex x of $HoSi$ such that $|\mathbf{r}'_x| = 2$. It is straightforward to check that there exists no vertex y satisfying $|\mathbf{r}'_y| = 2$ except the vertex x. From Proposition 3.2, we obtain that the graph $HoSi(\mathbf{r}')$ has no 4-coclique. Note that $HoSi(\mathbf{r}')$ also has no 3-clique, as the graph $HoSi$ does not contain 3-clique. Thus, we have

$$|\gamma_x^{\mathbf{r}'}| - 1 + |\delta_x^{\mathbf{r}'}| + |\zeta_x^{\mathbf{r}'}| \leq |\mathrm{supp}(\mathbf{r}')| < R(3, 4) = 9. \tag{6.2}$$

But we can not find the possible solution for $(\gamma_x^{\mathbf{r}'}, \delta_x^{\mathbf{r}'}, \zeta_x^{\mathbf{r}'})$ which satisfies both (6.1) and (6.2). This implies that the claim holds.

Now for any row $\mathbf{r}$ of N, we have that, from Proposition 3.2, the graph $HoSi(\mathbf{r})$ has no 7-coclique. Note that $HoSi(\mathbf{r})$ also has no 3-clique, as the graph $HoSi$ does not contain 3-clique. Thus, we obtain

$$|\gamma_x^{\mathbf{r}}| + |\delta_x^{\mathbf{r}}| + |\zeta_x^{\mathbf{r}}| \leq |\mathrm{supp}(\mathbf{r})| < R(3, 7) = 23, \tag{6.3}$$

$$|\zeta_x^{\mathbf{r}}| < R(3, 6) = 18. \tag{6.4}$$

Thus, we have

$$(\gamma_x^{\mathbf{r}}, \delta_x^{\mathbf{r}}, \zeta_x^{\mathbf{r}}, \sigma^{\mathbf{r}}) \in \{(-1, -1, 12, 10), (1, 3, 6, 10), (1, 4, 15, 20), (\pm 1, \pm 2, \mp 3, 0)\} \tag{6.5}$$

for any x with $\mathbf{r}_x \neq 0$.

First we show $\sigma^{\mathbf{r}} \neq 20$. Otherwise, for any x with $\mathbf{r}_x \neq 0$, we have $\gamma_x^{\mathbf{r}} = \mathbf{r}_x = 1$ and $\delta_x^{\mathbf{r}} = 4$. From (3.3), we also have $4|\mathrm{supp}(\mathbf{r})| \geq \frac{|\mathrm{supp}(\mathbf{r})|^2}{2} - 3|\mathrm{supp}(\mathbf{r})|$, that is, $|\mathrm{supp}(\mathbf{r})| \leq 14$. But $|\mathrm{supp}(\mathbf{r})| \geq 1 + 4 + 15 = 20$. This gives a contradiction.

If $\sigma^{\mathbf{r}} = 10$, then $\delta_x^{\mathbf{r}} = 3$ when $\gamma_x^{\mathbf{r}} = \mathbf{r}_x = 1$ and $\delta_x^{\mathbf{r}} = -1$ when $\gamma_x^{\mathbf{r}} = \mathbf{r}_x = -1$. By using (3.3), we have $3 \cdot \frac{|\mathrm{supp}(\mathbf{r})|+10}{2} + \frac{|\mathrm{supp}(\mathbf{r})|-10}{2} \geq \frac{|\mathrm{supp}(\mathbf{r})|^2}{2} - 3|\mathrm{supp}(\mathbf{r})|$ and thus $|\mathrm{supp}(\mathbf{r})| \leq 10$. This implies that $|\mathrm{supp}(\mathbf{r})| = 10$ and for any x with $\mathbf{r}_x \neq 0$, $(\gamma_x^{\mathbf{r}}, \delta_x^{\mathbf{r}}, \zeta_x^{\mathbf{r}}) = (1, 3, 6)$, that is, the induced subgraph $HoSi(\mathbf{r})$ is the Petersen graph.

If $\sigma^{\mathbf{r}} = 0$, then $\delta_x^{\mathbf{r}} = 2$ when $\gamma_x^{\mathbf{r}} = \mathbf{r}_x = 1$ and $\delta_x^{\mathbf{r}} = -2$ when $\gamma_x^{\mathbf{r}} = \mathbf{r}_x = -1$, that is, for any vertex x with $\mathbf{r}_x \neq 0$, there exist at least two vertices y_1 and y_2 such that $y_i \sim x$ and $\mathbf{r}_{y_i} = \mathbf{r}_x$.

Considering that $HoSi$ has no 3- and 4-cycle, we find that $|\text{supp}(\mathbf{r})| = |V(HoSi(\mathbf{r}))| \geq 10$. Note that in this case, we also have

$$|\text{supp}(\mathbf{r})|^2 = (\mathbf{r}\mathbf{r}^T)^2 \leq \mathbf{r}N^T N\mathbf{r}^T = 2\mathbf{r}A(HoSi)\mathbf{r}^T + 6\mathbf{r}\mathbf{r}^T = 2 \cdot 2 \cdot |\text{supp}(\mathbf{r})| + 6|\text{supp}(\mathbf{r})|.$$

Thus $|\text{supp}(\mathbf{r})| = 10$ and the induced subgraph $HoSi(\mathbf{r})$ is a disjoint union of 2 pentagons. Moreover, the row $\mathbf{r}$ is orthogonal to all of the other rows of N except itself.

Now let us look at the matrix N. It is easy to see that N has exactly 30 rows, since N has 300 $(= 6 \times 50)$ non-zero entries and each row $\mathbf{r}$ of N has 10 non-zero entries. Denote by $\mathbf{r}^0, \ldots, \mathbf{r}^{29}$ all of the rows of N. For convenience, we may assume $\sigma^{\mathbf{r}^i} = 10$ for $0 \leq i \leq s-1$ and $\sigma^{\mathbf{r}^i} = 0$ for $s \leq i \leq 29$. Considering that $\text{rank}(N) = \text{rank}(A(HoSi) + 3I) = 50 - 21 = 29$ and $\mathbf{r}^j(\mathbf{r}^i)^T = 0$ for $s \leq j \leq 29$ and $j \neq i$, there exist constants $f_1, \ldots, f_s$ such that the following holds.

$$\sum_{i=0}^{s-1} f_i \mathbf{r}^i = \mathbf{0}, \text{ where } \sum_{i=0}^{s-1} f_i^2 \neq 0 \tag{6.6}$$

Let H be the graph with vertex set $\{\mathbf{r}^0, \ldots, \mathbf{r}^{s-1}\}$, where $\mathbf{r}^i$ and $\mathbf{r}^j$ are adjacent if and only if there exists a vertex $y \in V(G)$ such that $\mathbf{r}^i_y = \mathbf{r}^j_y = 1$. For any vertex x, assume $l_0(x) := |\{\mathbf{r}^i \mid \mathbf{r}^i_x \neq 0, 0 \leq i \leq s-1\}|$ and $l_1(x) := |\{\mathbf{r}^i \mid \mathbf{r}^i_x \neq 0, s \leq i \leq 29\}|$. We have

$$\sum_{0 \leq i \leq 29, \mathbf{r}^i_x \neq 0} |\gamma^{\mathbf{r}^i}_x| = l_0(x) + l_1(x) = 6,$$

$$\sum_{0 \leq i \leq 29, \mathbf{r}^i_x \neq 0} |\delta^{\mathbf{r}^i}_x| = 3l_0(x) + 2l_1(x) = 2 \times 7 = 14.$$

This shows

$$l_0(x) = 2. \tag{6.7}$$

By using (6.7), we obtain

$$|\{(x, \mathbf{r}^i) \mid \mathbf{r}^i_x \neq 0, 0 \leq i \leq s-1\}| = 2 \cdot 50 = 10s.$$

Thus $|V(H)| = s = 10$.

Now we figure out the minimal valency of H. For any vertex $\mathbf{r}^i$ of H, let $\pi := \{V(HoSi(\mathbf{r}^i)), V(HoSi) - V(HoSi(\mathbf{r}^i))\}$ be a partition of the vertices of $HoSi$. The quotient matrix Q_π of $HoSi$ relative to π is

$$Q_\pi = \begin{pmatrix} 3 & 4 \\ 1 & 6 \end{pmatrix}$$

with 7 and 2 as eigenvalues. Note that π is equitable by Lemma 2.2 and the vector $\mathbf{w} := (4, -1)^T$ is an eigenvector of Q_π with 2 as eigenvalue. Define $\widetilde{\mathbf{w}}$ to be a vector satisfying $\widetilde{\mathbf{w}}_x = 4$ if $x \in V(HoSi(\mathbf{r}^i))$ and $\widetilde{\mathbf{w}}_x = -1$ if $x \in V(HoSi) - V(HoSi(\mathbf{r}^i))$. Then $A(HoSi)\widetilde{\mathbf{w}} = 2\widetilde{\mathbf{w}}$ and $\widetilde{\mathbf{w}}^T(2A(HoSi)+6I)\widetilde{\mathbf{w}} = 10\widetilde{\mathbf{w}}^T\widetilde{\mathbf{w}} = 2000$. Note that for any $\mathbf{r}^j \in V(H)$, $\mathbf{r}^j$ is a 01-vector. Hence, if $\mathbf{r}^i(\mathbf{r}^j)^T = t$, then $(\mathbf{r}^j\widetilde{\mathbf{w}}, \mathbf{r}^j\widetilde{\mathbf{w}}) = (5t - 10)^2$. Define $p^i_t := |\{j \mid j \neq i, \mathbf{r}^j \in V(H), \mathbf{r}^i(\mathbf{r}^j)^T = t\}|$, where $0 \leq t \leq 10$. Thus,

$$\sum_{t=1}^{10} p^i_t = 9, \quad \sum_{t=1}^{10} p^i_t(5t-10)^2 \leq \widetilde{\mathbf{w}}^T N^T N\widetilde{\mathbf{w}} - (\mathbf{r}^i\widetilde{\mathbf{w}}, \mathbf{r}^i\widetilde{\mathbf{w}}) = \widetilde{\mathbf{w}}^T(2A(HoSi)+6I)\widetilde{\mathbf{w}} - 1600 = 400.$$

$$\tag{6.8}$$

From (6.8), we find $p^i_0 \leq 4$. This shows that for any vertex $\mathbf{r}^i$ of H, it has valency at least 5.

Since $\mathbf{r}^i$ is a 01-vector for any $0 \leq i \leq s-1$, we infer that, from (6.6) and (6.7), any two distinct vertices $\mathbf{r}^i$ and $\mathbf{r}^j$ of H are adjacent if and only if $f_i = -f_j$. As the minimal valency of H is at least 5 and $\sum_{i=0}^{s-1} f_i^2 \neq 0$, we conclude that H is the bipartite graph $K_{5,5}$ and thus

$$p_0^i = 4, \ p_2^i = 5, \ \text{and} \ p_t^i = 0 \ \text{for} \ t \neq 0, 2. \tag{6.9}$$

Assume $\{\mathbf{r}_0, \ldots, \mathbf{r}_4\}$ and $\{\mathbf{r}_5, \ldots \mathbf{r}_9\}$ are the color classes of H. The induced subgraphs $HoSi(\mathbf{r}^0)$, ..., $HoSi(\mathbf{r}^9)$ have the following properties.

(i) $HoSi(\mathbf{r}^i)$ is a Petersen graph for any $0 \leq i \leq 9$;

(ii) $|V(HoSi(\mathbf{r}^i)) \cap V(HoSi(\mathbf{r}^j))| = 0$ for $0 \leq i < j \leq 4$;

(iii) $|V(HoSi(\mathbf{r}^i)) \cap V(HoSi(\mathbf{r}^j))| = 0$ for $5 \leq i < j \leq 9$;

(iv) $|V(HoSi(\mathbf{r}^i)) \cap V(HoSi(\mathbf{r}^j))| = 2$ for $0 \leq i \leq 4$ and $5 \leq j \leq 9$.

Without loss of generality, we may assume, by Proposition 6.2 and Proposition 6.1, $V(HoSi(\mathbf{r}^i)) = V(P^i) \cup V(Q^i)$ for $i = 0, \ldots, 4$, where P^i and Q^i are the graphs in Figure 1. But now we can not find the Petersen graph $HoSi(\mathbf{r}^5)$ satisfying $|V(HoSi(\mathbf{r}^5)) \cap V(HoSi(\mathbf{r}^i))| = 2$ for $0 \leq i \leq 4$.

This shows that the graph $HoSi$ is not 2-integrable and we complete the proof. $\square$

7 The complement of $GQ(3,9)$

A *generalized quadrangle* of order (s, t) is an incidence structure of points and lines with the properties that

(i) every point lies on $t+1$ lines and any two points are on at most one line;

(ii) every line contains $s+1$ points;

(iii) for any point P and line L which are not incident, there is a unique point on L collinear with P.

The *point graph* of a generalized quadrangle, also denoted by $GQ(s, t)$, is the graph with the points of the quadrangle as its vertices, with two points adjacent if and only if they are collinear. Note that the graph $GQ(s, t)$ is a strongly regular graph with parameters

$$((st+1)(s+1), s(t+1), s-1, t+1) \tag{7.1}$$

(see [17, Lemma 10.8.1]).

It is well-known that the strongly regular graph $GQ(q, q^2)$ exists when q is a prime power, and is unique for $q = 2, 3$ (see [28]). The complement of $GQ(q, q^2)$, denoted by $\overline{GQ}(q, q^2)$, is a strongly regular graph with parameters, by (2.1) and (7.1), as follows:

$$((q+1)(q^3+1), q^4, q(q-1)(q^2+1), (q-1)q^3).$$

In particular $GQ(3, 9)$ is the unique strongly regular graph with parameters $(112, 30, 2, 10)$ and spectrum $\{30^{(1)}, 2^{(90)}, -10^{(21)}\}$, and its first subconstituent is a disjoint union of 10 3-cliques and second subconstituent is the unique strongly regular graph with parameters $(81, 20, 1, 6)$, called the Brouwer-Haemers graph (see [3]). Therefore, we have

Proposition 7.1. (i) *The graph* $\overline{GQ}(3, 9)$ *is the unique strongly regular graph with parameters* $(112, 81, 60, 54)$ *and spectrum* $\{81^{(1)}, 9^{(21)}, -3^{(90)}\}$.

(ii) *The first subconstituent of $\overline{GQ}(3,9)$ is the strongly regular graph with parameters $(81, 60, 45, 42)$, which is the complement of the Brouwer-Haemers graph.*

(iii) *The maximal order of cocliques in $\overline{GQ}(3,9)$ is 4 and the maximal order of cliques in $\overline{GQ}(3,9)$ is 16.*

In this section, we will show the following result.

Theorem 7.2. *The strongly regular graph $\overline{GQ}(3,9)$, which is the complement of the point graph of the generalized quadrangle of order $(3,9)$, with parameters $(112, 81, 60, 54)$ and smallest eigenvalue -3 is not 2-integrable.*

Proof. Suppose that the graph $\overline{GQ}(3,9)$ is 2-integrable, then there exists an integral matrix N such that $2A(\overline{GQ}(3,9)) + 6I = N^T N$ holds. Note that for any entry $N_{i,j}$, $N_{i,j} \in \{\pm 1, \pm 2, 0\}$. Without loss of generality, we may assume that for any row $\mathbf{r}$ of N, $|\mathrm{supp}(\mathbf{r})| > 0$ and $\sigma^{\mathbf{r}} \geq 0$. Choose $\mathbf{r}'$ to be a row of N whose support has the minimum size.

From Proposition 7.1 (iii), we find that there exists a 4-coclique $\overline{C}$ in $\overline{GQ}(3,9)$. Let $\pi = \{V(\overline{C}), V(G) - V(\overline{C})\}$ be a partition of $\overline{GQ}(3,9)$. The quotient matrix Q_π of $\overline{GQ}(3,9)$ relative to π is $\begin{pmatrix} 0 & 81 \\ 3 & 78 \end{pmatrix}$ with 81 and -3 as eigenvalues, where $(27, -1)^T$ is an eigenvector of Q_π with eigenvalue -3. Thus, π is an equitable partition by Lemma 2.2 and $27 \sum_{x \in V(\overline{C})} \mathbf{r}'_x - \sum_{x \in V(\overline{GQ}(3,9)) - V(\overline{C})} \mathbf{r}'_x = 0$ holds by Propostition 3.1 (ii). This implies

$$28 \mid \sigma^{\mathbf{r}'}, \quad \text{where } \sigma^{\mathbf{r}'} = \sum_{x \in V(\overline{GQ}(3,9))} \mathbf{r}'_x. \tag{7.2}$$

For any $x \in V(\overline{GQ}(3,9))$, the quotient matrix $Q_{\pi(x)}$, by (2.2), of $\overline{GQ}(3,9)$ relative to the equitable partition $\pi(x) := \{\{x\}, N_1(x), N_2(x)\}$ is as follows:

$$Q_{\pi(x)} = \begin{pmatrix} 0 & 81 & 0 \\ 1 & 60 & 20 \\ 0 & 54 & 27 \end{pmatrix},$$

and the vector $\mathbf{u} := (135, -5, 9)^T$ is an eigenvector of $Q_{\pi(x)}$ with eigenvalue -3. We have, by (3.2),

$$135 \gamma_x^{\mathbf{r}'} - 5 \delta_x^{\mathbf{r}'} + 9 \zeta_x^{\mathbf{r}'} = 0. \tag{7.3}$$

If there exists a vertex x such that $\mathbf{r}'_x = \gamma_x^{\mathbf{r}'} = \pm 2$, then for any vertex $y \neq x$, $\mathbf{r}'_y \in \{\pm 1, 0\}$. In this case,

$$|\mathrm{supp}(\mathbf{r}')| \geq |\gamma_x^{\mathbf{r}'}| - 1 + |\delta_x^{\mathbf{r}'}| + |\zeta_x^{\mathbf{r}'}| \geq 31 \tag{7.4}$$

by (7.2) and (7.3).

Now we may assume $\mathbf{r}'_x \in \{\pm 1, 0\}$ for all $x \in V(\overline{GQ}(3,9))$. We will show that in this case $|\mathrm{supp}(\mathbf{r}')| \geq 31$ also holds. By using (7.2) and (7.3) again, only the following cases should be discussed.

$$(\gamma_y^{\mathbf{r}'}, \delta_y^{\mathbf{r}'}, \zeta_y^{\mathbf{r}'}, \sigma^{\mathbf{r}'}) \in \{(1, 27, 0, 28), (-1, 9, 20, 28), (\pm 1, \pm 9, \mp 10, 0)\} \tag{7.5}$$

with $\mathbf{r}'_y \neq 0$.

We claim that if $\delta_y^{\mathbf{r}'} = \pm 9$, there exist 17 vertices $w_1, \ldots, w_{17}$ such that $w_i \sim y$ and $\mathbf{r}'_{w_i} \neq 0$. Define $\mathbf{r}'(y)$ be the vector obtained from $\mathbf{r}'$ be removing the coordinates indexed by $V(\overline{GQ}(3,9)) - \{y\} - N_2(y)$. It is sufficient to show that $|\mathrm{supp}(\mathbf{r}'(y))| \geq 17$. Now let us look at the first subconstituent G_y of $\overline{GQ}(3,9)$, that is, the subgraph induced by the vertex set $N_1(y)$. From Proposition 7.1 (ii), we find that G_y is a strongly regular graph with parameters

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$(81, 60, 45, 42)$ and smallest eigenvalue -3. The quotient matrix $Q_{\pi'(z)}$ of G_y relative to the partition $\pi'(z) = \{\{z\}, N_1(z), N_2(z)\}$ of the vertices of G_y is as follows:

$$Q_{\pi'(z)} = \begin{pmatrix} 0 & 60 & 0 \\ 1 & 45 & 14 \\ 0 & 42 & 18 \end{pmatrix},$$

and the vector $(20, -1, 2)^T$ is an eigenvector of $Q_{\pi'(z)}$ with eigenvalue -3. It is not hard to check that $20\gamma_z^{\mathbf{r}'(y)} - \delta_z^{\mathbf{r}'(y)} + 2\zeta_z^{\mathbf{r}'(y)} = 0$ holds. If $\delta_y^{\mathbf{r}'} = \sigma^{\mathbf{r}'(y)} = \pm 9$, we find $(\gamma_z^{\mathbf{r}'(y)}, \delta_z^{\mathbf{r}'(y)}, \zeta_z^{\mathbf{r}'(y)}) = (\pm 1, \pm 12, \mp 4)$ when $\mathbf{r}'(y)_z \neq 0$ and thus $|\mathrm{supp}(\mathbf{r}'(y))| \geq 17$.

Now we claim that $\sigma^{\mathbf{r}'} \neq 0$. Otherwise $(\gamma_y^{\mathbf{r}'}, \delta_y^{\mathbf{r}'}, \zeta_y^{\mathbf{r}'}) = (\pm 1, \pm 9, \mp 10)$ for any $\mathbf{r}'_y \neq 0$ and $|\mathrm{supp}(\mathbf{r}')| \geq 1 + |\mathrm{supp}(\mathbf{r}'(y))| + 10 \geq 1 + 17 + 10 = 28$. We also have, by (3.3), $9|\mathrm{supp}(\mathbf{r}')| \geq \frac{1}{2}|\mathrm{supp}(\mathbf{r}')|^2 - 3|\mathrm{supp}(\mathbf{r}')|$ and this is not possible.

If $\sigma^{\mathbf{r}'} = 28$, there must be a vertex z^- such that $(\gamma_{z^-}^{\mathbf{r}'}, \delta_{z^-}^{\mathbf{r}'}, \zeta_{z^-}^{\mathbf{r}'}) = (-1, 9, 20)$ and thus $|\mathrm{supp}(\mathbf{r}')| \geq 1 + 17 + 20 \geq 31$, otherwise the induced subgraph $\overline{GQ}(3, 9)(\mathbf{r}')$ is a 28-clique and this contradicts Proposition 7.1 (iii).

But we have, by Proposition 3.1 (i), $|\mathrm{supp}(\mathbf{r}')| \leq \frac{2 \cdot 3 \cdot 112}{\mathrm{rank}(N)} = \frac{672}{\mathrm{rank}(A(\overline{GQ}(3,9))+3I)} = \frac{672}{112-90} < 31$.

This shows that the graph $\overline{GQ}(3, 9)$ is not 2-integrable and the theorem holds. $\square$

8 The complement of the McLaughlin graph

In 1975, J.M. Goethals and J.J. Seidel [18] showed that the McLaughlin graph is the unique strongly regular graph with parameters $(275, 112, 30, 56)$ and spectrum $\{112^{(1)}, 2^{(252)}, -28^{(22)}\}$. For its first and second subconstituents, they are the unique strongly regular graphs with parameters $(112, 30, 2, 10)$ and $(162, 56, 10, 24)$, respectively (see [5] and [4]). Therefore we have

Proposition 8.1.   (i) *The complement of the McLaughlin graph is the unique strongly regular graph with parameters $(275, 162, 105, 81)$ and spectrum $\{162^{(1)}, 27^{(22)}, -3^{(252)}\}$.*

(ii) *The second subconstituent of the complement of the McLaughlin graph is the strongly regular graph $\overline{GQ}(3, 9)$.*

Note that the McLaughlin graph can be constructed from the Steiner system $S(4, 7, 23)$ (cf. [2, Chapter 11]). Thus, we have the following construction for the complement of McLaughlin graph.

Proposition 8.2. *Let $(\Omega, \mathcal{B})$ be the Steiner system $(4, 7, 23)$ with $\Omega = \{0, 1, \ldots, 22\}$. The complement of McLaughlin graph can be constructed as a graph with vertex set $\{1, \ldots, 22\} \cup \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1$ is the set of 77 blocks in $S(4, 7, 23)$ which contains 0 and $\mathcal{B}_2$ is the set of 176 blocks in $S(4, 7, 23)$ which does not contain 0. The edges in this graph are defined as follows:*

(i) *vertices in $\{1, \ldots, 22\}$ are pairwise adjacent;*

(ii) *a vertex x in $\{1, \ldots, 22\}$ is adjacent to a block $B \in \mathcal{B}_1$ if and only if $x \in B$;*

(iii) *a vertex x in $\{1, \ldots, 22\}$ is adjacent to a block $B \in \mathcal{B}_2$ if and only if $x \notin B$;*

(iv) *two blocks $B_1, B_2 \in \mathcal{B}_i$ are adjacent if and only if $|B_1 \cap B_2| = 3$, for $i = 1, 2$;*

(v) *two blocks $B_1 \in \mathcal{B}_1$ and $B_2 \in \mathcal{B}_2$ are adjacent if and only if $|B_1 \cap B_2| = 1$.*

In this section, we will show that

Theorem 8.3. *The strongly regular graph* $\overline{McL}$*, which is the complement of the McLaughlin graph, with parameters* $(275, 162, 105, 81)$ *and smallest eigenvalue* -3 *is* 4*-integrable.*

Proof. In order to do so, we will show that the lattice $\Lambda(\overline{McL})$ is a sublattice of the shorter Leech lattice Λ_{23}.

First we define some vectors in $\mathbb{R}^{24}$. For convenience, the coordinates of the vectors in $\mathbb{R}^{24}$ are indexed by the elements of the set $\{\infty, 0, 1, \ldots, 22\}$. Moreover, the Steiner systems $S(5, 8, 24)$ and $S(4, 7, 23)$ have $\{\infty, 0, 1, \ldots, 22\}$ and $\{0, 1, \ldots, 22\}$ as their point sets, respectively. By following [13], we find that the Leech lattice Λ_{24} is generated by the following $760 \, (= 1 + 759)$ vectors:

- $\frac{1}{\sqrt{8}}(-3, 1^{23})$,

- $\frac{1}{\sqrt{8}}(2^8, 0^{16})$, in which the positions of the eight 2's form a block of $S(5, 8, 24)$.

Note that the vector $\mathbf{a}_0 := \frac{1}{\sqrt{8}}(4, 4, 0, \ldots, 0)$ is a minimal vector in Λ_{24}. We have, from [9, p. 179], that

$$\Lambda_{23} = \left\{ \mathbf{v} - \frac{(\mathbf{v}, \mathbf{a}_0)}{(\mathbf{a}_0, \mathbf{a}_0)} \mathbf{a}_0 \mid \mathbf{v} \in \Lambda_{24}, (\mathbf{v}, \mathbf{a}_0) \text{ is even.} \right\}$$

Let Δ be the set of the following 275 vectors:

(i) $\mathbf{a}_i := \frac{1}{\sqrt{8}}(4, 0, \ldots, 0, 4, 0, \ldots, 0)^T$, where the first 4 is in the first position (∞) and second 4 is in the i-th position, for $1 \le i \le 22$.

(ii) $\mathbf{b}_B := \frac{1}{\sqrt{8}}(2, 2^7, 0^{16})^T$, where the first 2 is in the position (∞) and the positions of the seven other 2's form a block B of $S(4, 7, 23)$ which contains 0.

(iii) $\mathbf{c}_B := \frac{1}{\sqrt{8}}(3, -1^7, 1^{16})^T$, where the 3 is in the position (∞) and the positions of the seven other -1's form a block B of $S(4, 7, 23)$ which does not contain 0.

Note that Δ is a subset of Λ_{24} (see [13]), and for any vector $\mathbf{u}$ in Δ, $(\mathbf{u}, \mathbf{a}_0) = 2$. Hence, the lattice generated by the following set

$$\widetilde{\Delta} := \{\mathbf{u} - \frac{1}{2}\mathbf{a}_0 \mid \mathbf{u} \in \Delta\}$$

is a sublattice of Λ_{23}. Moreover, by Proposition 2.6 and Proposition 8.2, we find that the matrix N with the 275 vectors in $\widetilde{\Delta}$ columns satisfies $N^T N = A(\overline{McL}) + 3I$. This shows that the lattice $\Lambda(\overline{McL})$ is a sublattice of Λ_{23}.

Since that the shorter Leech lattice Λ_{23} is an unimodular lattice with dimension 23, it is 4 integrable (see [10, Theorem 18]). This completes the proof. $\square$

Acknowledgments

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