

Chromatic symmetric functions and H -free graphs

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Abstract

Two celebrated conjectures in chromatic symmetric function theory concern the positivity chromatics symmetric functions of claw-free graphs. Here we extend the claw-free idea to general graphs and consider the e-positivity question for H -free graphs where $H = \{\text{claw}, F\}$ and $H = \{\text{claw}, F, \text{co-}F\}$, where F is a four-vertex graph. We settle the question for all cases except $H = \{\text{claw}, \text{co-diamond}\}$, and we provide some partial results in that case.

1 Introduction

A key area of investigation in symmetric function theory concerns the e-positivity, and/or Schur positivity, of a particular class of symmetric functions. In chromatic symmetric function theory there are two celebrated conjectures in this regard that focus on claw-free graphs, where the claw is the four-vertex bipartite graph $K_{1,3}$ (see Figure 3). One conjecture, due to Stanley and Stembridge [12], hypothesizes that the chromatic symmetric function of a claw-free incomparability graph is e-positive. The second conjecture, due to Stanley [10] with credit to Gasharov, hypothesizes that the chromatic symmetric function of a claw-free graph is Schur positive (for definitions see Section 2).

In parallel to this, in graph theory, much effort has been spent in characterizing the chromatic characteristics of graphs that are H -free, where H is some set of induced subgraphs. A key question in this domain is, can the chromatic number of a H -free graph be determined in polynomial time? This answer is known for large classes of graphs, but, interestingly, the classification for all combinations of subgraphs with four vertices (of which the claw is obviously one) is not yet complete [7]. Thus, combining these two areas, it is a natural question to generalize the claw-free conjectures and ask about other H -free graphs: “For which H -free graphs are their chromatic symmetric functions e-positive?” and to ask in particular, “For which H -free graphs are their chromatic symmetric functions e-positive, where H is the claw along with one other four-vertex graphs?” Obviously the corresponding Schur positivity questions would be subcases of the Gasharov-Stanley conjecture.

Set H	Positivity	Reference
claw, P_4	e-positive	[13]
claw, paw	e-positive	Theorem 3.2
claw, co-paw	e-positive	Theorem 3.3
claw, co-diamond	unknown	unknown
claw, diamond	not necessarily e-positive	Lemma 3.1
claw, co-claw	not necessarily e-positive	Lemma 3.1
claw, K_4	not necessarily e-positive	Lemma 3.1
claw, $4K_1$	not necessarily e-positive	Lemma 3.1
claw, C_4	not necessarily e-positive	Lemma 3.1
claw, $2K_2$	not necessarily e-positive	Lemma 3.1

Figure 1: Table of e-positivity of pairs of four-vertex graphs

Formally, let H be a set of four-vertex graphs. By abuse of notation we say a graph is e-positive if its chromatic symmetric function is e-positive. Then we want to know for which sets H of four-vertex graphs, an H -free graph is e-positive (or not). There are eleven graphs on four vertices, see Figure 3. The claw is not e-positive, since $X_{K_{1,3}} = e_4 + 5e_{3,1} - 2e_{2,2} + e_{2,1,1}$, and we include it always as part of H (as an aside, note that some intuition as to why the claw is not e-positive can be gained by considering partition dominance and the concept of *nice* graphs, as defined in Stanley [9]). We consider two different directions.

First, $H = \{\text{claw}, F\}$ for F a single four-vertex graph. By using an example of Stanley, given here in Figure 4, we see that this example does not contain diamond, co-claw, K_4 , $4K_1$, C_4 , or $2K_2$, and we can thus show that a graph that is free of claw paired with each of these is not necessarily e-positive. The remaining four-vertex graphs are P_4 , paw, co-paw, and co-diamond. By a result of Tsujie [13], if H contains a claw and a P_4 , then H -free graphs are e-positive. In this paper we show that if H contains a paw, or if H contains a co-paw, then H -free graphs are e-positive. Thus the only outstanding case of $H = \{\text{claw}, F\}$ is $H = \{\text{claw}, \text{co-diamond}\}$. In Section 3 we present some partial results on this case. We summarize the $H = \{\text{claw}, F\}$ results in Table 1.

Second, $H = \{\text{claw}, F, \text{co-}F\}$, where F is a single four-vertex graph and $\text{co-}F$ is the graph complement of F . Again, the Stanley example, Figure 4, takes care of a number of cases and our theorems in Section 3 handle all but one of the rest. The last case, $H = \{\text{claw}, \text{diamond}, \text{co-diamond}\}$ is handled by a case-by-case argument.

2 Background and Notation

The original interest in claw-free graphs in the context of chromatic symmetric functions stems from Conjecture 5.1 of Stanley and Stembridge [12]. This interest can be traced farther back to another historically significant conjecture, the Goulden-Jackson immanant conjecture [3], that hypothesizes that the expansion of the immanant of the Jacobi-Trudi matrix has nonnegative coefficients (the immanant is similar to a determinant but with

Set H	Positivity	Reference
claw, P_4	e-positive	[13]
claw, paw, co-paw	e-positive	Theorem 5.2
claw, diamond, co-diamond	e-positive	Theorem 5.1
claw, co-claw	not necessarily e-positive	Lemma 3.1
claw, $C_4, 2K_2$	not necessarily e-positive	Theorem 5.3
claw, $K_4, 4K_1$	not necessarily e-positive	Theorem 5.3

Figure 2: Table of e-positivity of claw plus F and co- F .

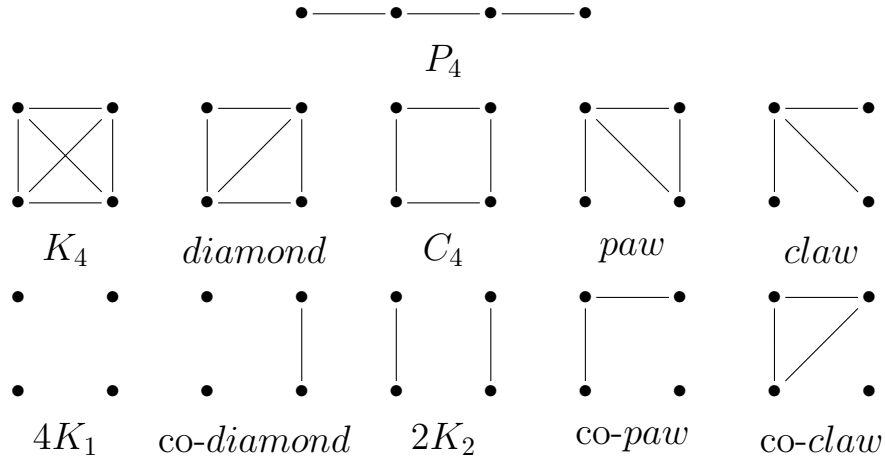


Figure 3: All four-vertex graphs

the sign replaced by another symmetric group character). This conjecture was proved soon after by Curtis Greene [4]. Related to this, Stanley and Stembridge [12] investigated conjectures involving the positivity of coefficients in immanants and conjectured that if a poset is $(3 + 1)$ -free, then the chromatic symmetric function of its incomparability graph is e -positive (see Stanley [9] or Stanley and Stembridge [12] for definitions).

Gasharov [2] proved that the chromatic symmetric functions of the incomparability graphs of $(3 + 1)$ -free posets are Schur positive. Note that if a symmetric function is e -positive it is necessarily Schur positive, but not the other way around, so this is a weaker result than the one asked for. The e -positivity of the chromatic symmetric function of the incomparability graph of a $(3 + 1)$ -free poset remains open, although there has been much progress on this question, see Guay-Paquet [5].

We now turn to some definitions. Further details on definitions and notation can be found in Stanley [11]. A coloring of the set of vertices V of some graph $G = (V, E)$ is a function κ from V to the positive integers \mathbb{Z}^+ : $\kappa : V \rightarrow \mathbb{Z}^+$. A coloring κ is proper if $\kappa(u) \neq \kappa(v)$ whenever vertex u is not adjacent to vertex v . Chromatic symmetric functions were defined by Stanley [9] as a generalization of the chromatic polynomial. Indeed, if $x_1 = x_2 = x_3 = \dots = 1$, this expression reduces to the chromatic polynomial for a graph.

Definition 2.1. *For a graph G with vertex set $V = \{v_1, v_2, \dots, v_N\}$ and edge set E , the chromatic symmetric function is defined to be*

$$X_G = \sum_{\kappa} x_{\kappa(v_1)} x_{\kappa(v_2)} \cdots x_{\kappa(v_N)}$$

where the sum is over all proper colorings κ of G .

A partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_\ell)$ of a positive integer n is a nondecreasing sequence of positive integers: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_\ell$, where λ_i is called the i th part of λ , $1 \leq i \leq \ell$. The transpose, λ' , of λ , is defined by its parts: $\lambda'_i = |\{j : \lambda_j \geq i\}|$. The elementary symmetric function, $e_i(x)$, is defined as

$$e_i(x) = \sum_{j_1 < j_2 < \dots < j_i} x_{j_1} \cdots x_{j_i}$$

and the elementary symmetric function, $e_\lambda(x)$, is defined as $e_\lambda(x) = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_\ell}$.

The Schur function is another common symmetric functions. There are a number of useful definitions of the Schur functions, but the easiest in this context is one in terms of the elementary symmetric functions: $s_\lambda(x) = \det(e_{\lambda'_i - i + j}(x))_{1 \leq i, j \leq \lambda_1}$ where if $\lambda'_i - i + j < 0$ then $e_{\lambda'_i - i + j}(x) = 0$.

Both the set of elementary symmetric functions and the set of Schur functions each form a basis for the algebra of symmetric functions (for details, see Stanley[11]). If a given symmetric function can be written as a nonnegative linear combination of elementary (resp. Schur) symmetric functions we say the symmetric function is e -positive (resp. Schur positive).

In graph theory we often discuss the idea of an H -free graph. Define H -free to be the class of graphs that do not contain any graph is H as an induced subgraph. Let P_k

be the chordless path on k vertices and C_k be the chordless cycle on k vertices. The complete graph K_n is the graph on n vertices such that there is an edge between all pairs of vertices. The graph K_3 is called the *triangle*, and its complement, which is $3K_1$, three vertices with no edges at all, is called the *co-triangle*. A *stable set* is a set S of vertices of a graph such that there are no edges between any of the vertices in S , e.g. a co-triangle is a stable set of size 3. The notation $\alpha(G)$ denotes the number of vertices in the largest stable set of G .

We will make use of a number of known results concerning chromatic symmetric functions and e -positivity. These lemmas are as follows:

Lemma 2.2. [9, Proposition 2.3] *If a finite simple graph G is a disjoint union of subgraphs $G_1 \cup G_2$, then $X_G = X_{G_1}X_{G_2}$.*

Lemma 2.3. [1, Theorem 8] $X_{K_n} = n!e_n$.

Lemma 2.4. [9, Proposition 5.3] X_{P_k} is e -positive.

Lemma 2.5. [9, Proposition 5.4] X_{C_k} is e -positive.

Lemma 2.6. [11, Exercise 7.47j] *If G is co-triangle-free, G is e -positive.*

Moreover, we also employ a number of results from the graph theory literature:

Theorem 2.7. [13, Theorem 1.5] *If G is (claw, P_4)-free, then X_G is e -positive.*

Theorem 2.8. [8] *If G is paw-free, then every connected component of G is triangle-free or is a complete multipartite graph.*

We will also need the following statement of the structure of (claw,triangle)-free graphs.

Lemma 2.9. *If G is (claw, triangle)-free, then each component of G is a chordless path or cycle, i.e. $G = P_k$ or $G = C_k$.*

Proof of Lemma 2.9. Let T be a component of G . Suppose T contains a chordless cycle C . Since G is triangle-free, we may assume C has at least four vertices. If every vertex of T lies on C , then we are done. So we may assume there is a vertex $v \in T - C$ that has a neighbor c in C . If v is adjacent to a neighbor c' of c on C , then T contains a triangle with vertices v, c, c' ; otherwise c is the center of a claw in T . Now, we may assume T contains no cycle, that is, T is a tree, but a claw-free tree has maximum degree at most two and so is necessarily a chordless path. \square

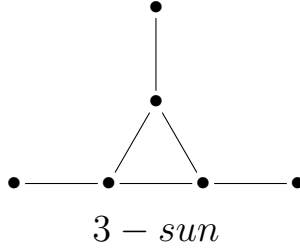


Figure 4: Example from Stanley [9] of a graph that is not e -positive.

3 $H = \{\text{claw}, F\}$

We are now ready to show which (claw, F) -free graphs G , for F a single four-vertex graph, can guarantee that X_G is e -positive; however, we begin with a negative result that will eliminate a number of cases:

Lemma 3.1. *A graph that is H -free for H equal to $\{\text{claw}, \text{diamond}\}$, $\{\text{claw}, K_4\}$, $\{\text{claw}, 4K_1\}$, $\{\text{claw}, C_4\}$, $\{\text{claw}, 2K_2\}$, or $\{\text{claw}, \text{co-claw}\}$, is not necessarily e -positive.*

Proof. Consider the graph G which is the 3-sun graph (Figure 4). Note that G is (claw, K_4) -free, $(\text{claw}, \text{diamond})$ -free, (claw, C_4) -free, $(\text{claw}, 4K_1)$ -free, $(\text{claw}, 2K_2)$ -free, and $(\text{claw}, \text{co-claw})$ -free; however, recall from Stanley [9] that the chromatic symmetric function for the 3-sun is

$$X_G = 6e_{3,2,1} - 6e_{3,3} + 6e_{4,1,1} + 12e_{4,2} + 18e_{5,1} + 12e_6,$$

implying X_G is not e -positive. □

Theorem 3.2. *If G is $(\text{claw}, \text{paw})$ -free, then X_G is e -positive.*

Proof. We prove by induction on the number of vertices. Let G be $(\text{claw}, \text{paw})$ -free. If G is disconnected, then by the induction hypothesis, each component of G is e -positive and so by Lemma 2.2, G is e -positive. So we may assume G is connected. By Theorem 2.8 (Olariu's theorem), G is triangle-free or is a complete multipartite graph. By Lemma 2.9, G is either P_k or C_k , or G is a complete multipartite graph.

If G is P_k or C_k , then by Lemma 2.4 or Lemma 2.5, G is e -positive. If G is a complete multipartite graph, since G is also claw-free we have $\alpha(G) \leq 2$. It then follows from Lemma 2.6 that G is e -positive. Therefore, $(\text{claw}, \text{paw})$ -free graphs are e -positive. □

Theorem 3.3. *If G is $(\text{claw}, \text{co-paw})$ -free, then X_G is e -positive.*

Proof. We prove by induction on the number of vertices. Let G be $(\text{claw}, \text{co-paw})$ -free. If \overline{G} is disconnected, then G is the join of two vertex sets V_A and V_B . Since G is claw-free, we have $\alpha(V_A) \leq 2$ and $\alpha(V_B) \leq 2$, and thus $\alpha(G) \leq 2$, and so G is e -positive by Lemma 2.6. Thus, we may assume \overline{G} is connected.

By Theorem 2.8, G is the graph $\overline{P_k}$ or $\overline{C_k}$, or G is the complement of a complete bipartite graph. If G is $\overline{P_k}$ or $\overline{C_k}$, then $\alpha(G) \leq 2$, which implies G is e -positive by Lemma 2.6. If G is the complement of a complete bipartite graph, then G is a disjoint union of cliques. So, G is e -positive by Lemma 2.2 and Lemma 2.3. \square

The last four-vertex graph F to consider is the co-diamond. However, there is no straightforward decomposition that allows us to handle this case. To this end, we will make some observations about the structure of $(\text{claw}, \text{co-diamond})$ -free graphs in the next section.

4 The structure of $(\text{claw}, \text{co-diamond})$ -free graphs

In this section, we study the structure of $(\text{claw}, \text{co-diamond})$ -free graphs.

Lemma 4.1. *Let G be a co-diamond-free graph that is disconnected. Then*

- *Each component of G is a clique, or*
- *G is the union of a complete multipartite graph and the graph K_1 .*

Proof. Let G be a co-diamond-free graph that is disconnected. We may assume some component C of G is not a clique, for otherwise we are done. So C contains a P_3 as induced subgraph. If G has at least three components, then $G - C$ contains a $2K_1$ and this $2K_1$ together with an edge of C induce a co-diamond, a contradiction. So G has exactly two components. Let C' be the component different from C . If C' contains an edge, then this edge and the two end-points of the P_3 in C form a co-diamond. So C' is the graph K_1 . The component C cannot contain a co- P_3 as induced subgraph, for otherwise this co- P_3 and C' form a co-diamond. Since a connected graph without co- P_3 must be a complete multipartite graph, the Lemma follows. \square

Lemma 4.2. *If a $(\text{claw}, \text{co-diamond})$ -free graph G is disconnected, then G is e -positive.*

Proof. Let G be a $(\text{claw}, \text{co-diamond})$ -free graph that is disconnected. We will rely on Lemma 4.1. If each component of G is a clique, the G is e -positive by Lemma 2.2 and Lemma 2.3. So we may assume G is the union of a complete multipartite graph M and the graph K_1 . Since G is claw-free, we have $\alpha(M) \leq 2$, and so M is e -positive by Lemma 2.6. It follows from Lemma 2.2 that G is e -positive. \square

Lemma 4.3. *If a $(\text{claw}, \text{co-diamond})$ -free graph G has $\alpha(G) \geq 4$, then G is e -positive.*

Proof. We use a result of Lozin and Malyshev (Lemma 9 in [7]) that shows a $(\text{claw}, \text{co-diamond})$ -free graph G with $\alpha(G) \geq 4$ must be a stable set. For the sake of completeness, we are going to provide a proof of this statement. Let G be a $(\text{claw}, \text{co-diamond})$ -free graph G with $\alpha(G) \geq 4$. Consider a stable set S that has at least four vertices and is maximal (that is, not belonging to a larger stable set). Consider any vertex $v \in G - S$, if such a vertex exists. Vertex v must have a neighbor in S by the maximality of S . If v has at least three neighbors in S , then G contains a claw, otherwise G contains a co-diamond. So G must be S . Clearly, a stable set is e -positive. \square

So we only need to consider $(\text{claw}, \text{co-diamond})$ -free graphs G that are connected and have $\alpha(G) \leq 3$ ($4K_1$ -free). If G is triangle-free, then by Lemma 2.9, G is e -positive. Similarly, if G is co-triangle-free, then G is e -positive. So, we have $\alpha(G) = 3$. Suppose G contains a triangle and co-triangle. Let us name the vertices of the co-triangle a, b, c . Since G is co-diamond-free, any vertex not in the co-triangle must be joined to two vertices in the co-triangle (if it is joined to none we get a $4K_1$, if it is joined to one then we get a co-diamond, and if it is joined to three then we get a claw). Let the set $S_{x,y}$ be the vertices adjacent to the two vertices x, y of the co-triangle. This means that $(\text{claw}, \text{co-diamond})$ -free graphs which are not known to be e -positive from the previous theorems have the structure depicted in Figure 5 (we will call such graphs *peculiar*).

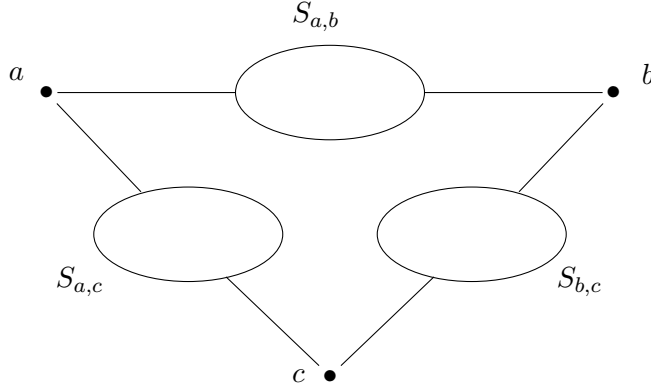


Figure 5: The structure of a connected $(\text{claw}, \text{co-diamond})$ -free graph that is not known to be e -positive. The three black vertices are the co-triangle. Each oval represents a subgraph, with each vertex in subgraph being joined to the two corresponding vertices of the co-triangle. At least two ovals are non-empty.

Now that we understand the structure of $(\text{claw}, \text{co-diamond})$ -free graphs, we can consider adding additional restrictions to the graph. In the following theorems we explore the e -positivity question for $(\text{claw}, \text{co-diamond}, F)$ -free graphs for F a four-vertex graph.

Theorem 4.4. *If G is $(\text{claw}, \text{co-diamond}, P_4)$ -free, $(\text{claw}, \text{co-diamond}, \text{paw})$ -free, or $(\text{claw}, \text{co-diamond}, \text{co-paw})$ -free, then X_G is e -positive.*

Proof. This follows directly from Theorems 2.7, 3.2, and 3.3. \square

Theorem 4.5. *If G is $(\text{claw}, \text{co-diamond}, \text{diamond})$ -free or $(\text{claw}, \text{co-diamond}, \text{co-claw})$ -free, then X_G is e -positive.*

Proof. Let G have a structure defined in Figure 5. If G is $(\text{claw}, \text{co-diamond}, \text{diamond})$ -free, then the vertices in each oval is a stable set (if $S_{x,y}$ contains an edge, then this edge and $\{x, y\}$ form a diamond). Similarly, if G is $(\text{claw}, \text{co-diamond}, \text{co-claw})$ -free, then each oval is a stable set (if $S_{x,y}$ contains an edge, then this edge form a co-claw with $\{x, z\}$ where z is the vertex of the co-triangle different from x and y). However, if there are three or more vertices in a oval then there exists an induced claw. Therefore each oval can have a maximum of two vertices.

Since G is connected, there must be two non-empty ovals. There are seven base cases to consider when placing vertices in each oval. For each of these base graphs, the number of possible edges (between the ovals) each graph can have are also noted:

- one vertex in two ovals (one possible edge)
- two vertices in an oval, one vertex in another oval (two possible edges)
- two vertices in two ovals (four possible edges)
- one vertex in all three ovals (three possible edges)
- two vertices in an oval, one vertex in the remaining two ovals (five possible edges)
- two vertices in two ovals, one vertex in the remaining oval (eight possible edges)
- two vertices in all three ovals (twelve possible edges)

For each of the cases, there are a finite number of possible graphs. For each of these graphs, we check if the graph is $(\textit{claw}, \textit{co-diamond}, \textit{diamond})$ -free/ $(\textit{claw}, \textit{co-diamond}, \textit{co-claw})$ -free and if it has an induced triangle. If it meets these requirements, we then check whether the X_G is e -positive.

Since the number of total cases is relatively small, they are easily verifiable through the use of a computer program. Combining every possible case listed above, there are only 36 graphs (including isomorphic cases) which are $(\textit{claw}, \textit{co-diamond}, \textit{diamond})$ -free or $(\textit{claw}, \textit{co-diamond}, \textit{co-claw})$ -free and have an induced triangle, and all of them are e -positive. \square

Thus, for $(\textit{claw}, \textit{co-diamond}, H)$ -free graphs, there are only three remaining cases to consider: $(\textit{claw}, \textit{co-diamond}, C_4)$ -free, $(\textit{claw}, \textit{co-diamond}, 2K_2)$ -free, and $(\textit{claw}, \textit{co-diamond}, K_4)$ -free.

Observation 4.6. *If a peculiar graph G is $(\textit{claw}, \textit{co-diamond}, C_4)$ -free, then each oval induces a clique.*

Proof. Suppose G is $(\textit{claw}, \textit{co-diamond}, C_4)$ -free. If $S_{x,y}$ contains two non-adjacent vertices u, v , then the set $\{x, y, u, v\}$ induces a C_4 . \square

Consider the graph G of Figure 5. We will call G the *generalized pyramid* if every oval is a clique and there are all edges between any two ovals, see Figure 6.

Lemma 4.7. *If a peculiar graph G is $(\textit{claw}, \textit{co-diamond}, 2K_2)$ -free, then G is the generalized pyramid.*

Proof. Let G be a peculiar $(\textit{claw}, \textit{co-diamond}, 2K_2)$ -free graph. Suppose there are non-adjacent vertices u, v with $u \in S_{a,b}$ and $v \in S_{b,c}$, then the set $\{a, u, c, v\}$ induces a $2K_2$. Thus, there are all edges between any two ovals. Suppose $S_{a,b}$ contains two non-adjacent vertices u, v . Assume without loss of generality that $S_{b,c}$ is non-empty. Then, for any vertex $x \in S_{b,c}$, the set $\{x, a, b, c\}$ induces a claw. So G must be the generalized pyramid. \square

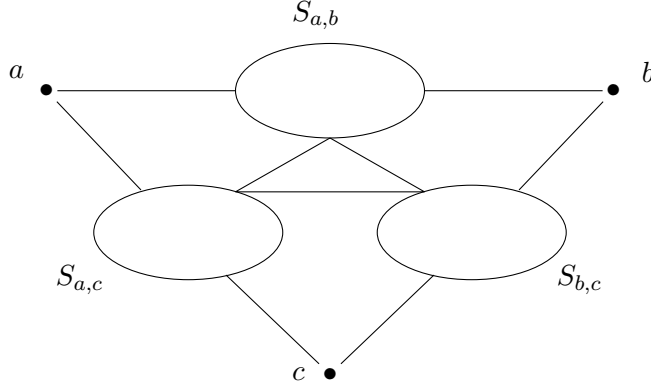


Figure 6: The generalized pyramid graph.

Lemma 4.8. *If a peculiar graph G is $(\text{claw}, \text{co-diamond}, K_4)$ -free, then G has at most 18 vertices.*

Proof. Consider a peculiar graph G is $(\text{claw}, \text{co-diamond}, K_4)$ -free. Consider the oval $S_{a,b}$. Now, let T be a set with three vertices in $S_{a,b}$. If T is a stable set, the T and a induce a claw. If T is a clique, then T and a induce a K_4 . So $S_{a,b}$ contains no triangle, and no co-triangle. It follows from a folklore (Ramsey's theorem) that $S_{a,b}$ has at most five vertices. So G has at most $5 + 5 + 5 + 3 = 18$ vertices. \square

By Lemma 4.8, there are a finite number of graphs to verify. Brute force checking is theoretically possible, but it is infeasible unless further properties of $(\text{claw}, \text{co-diamond}, K_4)$ -free graphs are discovered.

5 $H = \{\text{claw}, F, \text{co-}F\}$

We can also consider which graphs are e -positive which are $(\text{claw}, H, \text{co-}H)$ -free. Referring back to Figure 3, we see that:

- K_4 is complementary to $4K_1$
- diamond is complementary to co-diamond
- C_4 is complementary to $2K_2$
- paw is complementary to co-paw
- claw is complementary to co-claw
- P_4 is self-complementary.

The case of the (claw, P_4) -free graphs was settled by Tsujie [13]. The remaining five cases are settled by the results of this paper.

Theorem 5.1. *If G is $(\text{claw}, \text{diamond}, \text{co-diamond})$ -free, then X_G is e -positive.*

Proof. By Theorem 4.5, X_G is e -positive. □

Theorem 5.2. *If G is $(\text{claw}, \text{paw}, \text{co-paw})$ -free, then X_G is e -positive.*

Proof. By Theorem 3.2 and Theorem 3.3, X_G is e -positive. □

Theorem 5.3. *If G is $(\text{claw}, K_4, 4K_1)$ -free, $(\text{claw}, C_4, 2K_2)$ -free, or $(\text{claw}, \text{co-claw})$ -free, then X_G may not be e -positive.*

Proof. Use the 3-sun graph in Figure 4 and the same reasoning as in Lemma 3.1. □

6 Concluding Remarks

We have considered the e -positivity question for the chromatic symmetric functions for two different classes of H -free graphs: $H = \{\text{claw}, F\}$ and $H = \{\text{claw}, F, \text{co-}F\}$, where F is a four-vertex graph. In the case of $H = \{\text{claw}, F\}$ there are ten different cases to consider, of which one of these, the case $H = \{\text{claw}, P_4\}$, was previously proved e -positive by Tsujie [13]. Here we settle eight of the remaining cases, leaving just the case of $H = \{\text{claw}, \text{co-diamond}\}$. Based on our results and our investigations towards a result for $H = \{\text{claw}, \text{co-diamond}\}$, we propose the following two open problems:

Open Problem 6.1. *Are generalized pyramids e -positive?*

Open Problem 6.2. *Are $(\text{claw}, \text{co-diamond})$ -free graphs e -positive?*

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References

- [1] S. Cho and S. van Willigenburg, Chromatic classical symmetric functions, *J. Comb.*, to appear. Also arXiv:1508.07670.
- [2] V. Gasharov, Incomparability graphs of $(3+1)$ -free posets are s -positive, *Disc. Math.* 157 (1996), 193–197.
- [3] I. P. Goulden and D. M. Jackson, Immanants of combinatorial matrices, *J. Algebra* 148 (1992), 305–324.

- [4] C. Greene, Proof of a conjecture on immanants of the Jacobi-Trudi matrix, *Linear Algebra Appl.* 171 (1992), 65–79.
- [5] M. Guay-Paquet, A modular relation for the chromatic symmetric function of $(3+1)$ -posets, arXiv:1306.2400, 2013.
- [6] C.T. Hoang and B. Reed, On the co- P_3 -structure of perfect graphs, *SIAM J. Disc. Math.* 18 (2005), 571–576.
- [7] V.V. Lozin and D.S. Malyshev, Vertex coloring of graphs with few obstructions, *Disc. Appl. Math.*, (2015), <http://dx.doi.org/10.1016/j.dam.2015.02.015>.
- [8] Olariu, Paw-free graphs, *Information Proc. Letters* 28 (1988), 53–54.
- [9] R.P. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.* 111 (1995), 166–194.
- [10] R. P. Stanley, Graph colorings and related symmetric functions: ideas and applications, *Disc. Math.* 193 (1998), 267–286.
- [11] R.P. Stanley, *Enumerative Combinatorics, Vol 2*, Cambridge: Cambridge University Press, 1999.
- [12] R. P. Stanley and J. R. Stembridge, On immanants of Jacobi-Trudi Matrices and permutations with restricted position, *JCTA* 62 (1993), 261–279.
- [13] S. Tsujie, The chromatic symmetric functions of trivially perfect graphs and cographs, arXiv: 1707.0458, 2017.