Constructing 2-Arc-Transitive Covers of Hypercubes

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Abstract

We introduce the notion of a symmetric basis of a vector space equipped with a quadratic form, and provide a sufficient and necessary condition for the existence to such a basis. Symmetric bases are then used to study Cayley graphs of certain extraspecial 2-groups of order 2^{2r+1} ($r \geq 1$), which are further shown to be normal Cayley graphs and 2-arc-transitive covers of $2r$-dimensional hypercubes.

Keywords extraspecial 2-group symmetric basis quadratic form locally-primitive graph normal Cayley graph

1 Introduction

Throughout this paper, all graphs are simple, connected and regular. Let Γ be a graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. An *s-arc* of Γ is a sequence $(v_1, \ldots, v_{s+1})$ of $s+1$ vertices such that for all $1 \leq i \leq s$, $\{v_i, v_{i+1}\}$ is an edge in Γ and $v_i \neq v_{i+2}$, and Γ is said to be *s-arc-transitive* if the automorphism group of Γ is transitive on the set of s-arcs. The study of s-arc-transitive graphs is motivated by a result of Tutte (1949), which says that there are no s-arc-transitive graphs of valency 3 for $s \geq 6$. Later this result was extended by Weiss [16] saying that there are no 8-arc-transitive graphs of valency at least 3. Thus analysing the s-arc-transitive graphs for $2 \leq s \leq 7$ has become one of the central goals in algebraic graph theory, and the classification of some 2-arc-transitive graphs has been obtained. For example, the 2-arc-transitive circulants are classified in [1]; a complete classification of 2-arc-transitive dihedrants is given in [7]; and a class of 2-arc-transitive Cayley graphs of elementary abelian 2-groups is classified in [9].

A natural idea to investigate the 2-arc-transitive graphs is to study their quotient graphs. Let $\mathcal{P}$ be a partition of the vertex set $V(\Gamma)$. Define the *quotient graph* $\Gamma_{\mathcal{P}}$ of Γ to be the graph with vertex set $\mathcal{P}$ and two parts $P, P' \in \mathcal{P}$ form an edge if and only if there is at least one edge in Γ joining a vertex of P

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and a vertex of P'. If $\mathcal{P}$ is G-*invariant* for some group G of automorphisms of Γ, then the action of G on Γ induces an action of G on $\Gamma_{\mathcal{P}}$. Let N be a nontrivial normal subgroup of G and $\mathcal{P}$ be the set of N-orbits in $V(\Gamma)$. The quotient graph $\Gamma_{\mathcal{P}}$ is said to be a *normal quotient* of Γ, denoted Γ_N. In general, the valency of Γ_N divides the valency of Γ. If the valency of Γ equals the valency of Γ_N, then Γ is said to be a *cover* of Γ_N. It has been proved by Praeger [11, Theorem 4.1] that if G is vertex-transitive and 2-arc-transitive on Γ, and N has more than two orbits in $V(\Gamma)$, then

1. G/N is s-arc transitive on Γ_N and G/N is faithful on $V(\Gamma_N)$,
2. Γ is a cover of Γ_N, and
3. N is semiregular on $V(\Gamma)$.

We say that a permutation group is *quasiprimitive* on a set Ω if every nontrivial normal subgroup of the permutation group is transitive on Ω, and *primitive* if it acts transitively on Ω and preserves no nontrivial partition of Ω. A permutation group is said to be *bi-quasiprimitive* on Ω if

(i) each nontrivial normal subgroup of the permutation group has at most two orbits on Ω, and

(ii) there exists a normal subgroup with two orbits on Ω.

The structure of finite quasiprimitive permutation groups was investigated in [11] and the types of quasiprimitive groups that are 2-arc-transitive on a graph were determined. Praeger studied bi-quasiprimitive groups in [12] and one specific class identified was previously studied in [10]. One family of such bipartite bi-quasiprimitive graphs are the affine ones. A 2-arc-transitive graph is said to be *affine*, if there is a vector space N and a group G of automorphisms of the graph such that $N \leq G \leq AGL(N)$ with N regular on the vertices and G acting transitively on the set of 2-arcs. Table 1 in [9] classifies all affine bipartite 2-arc-transitive graphs with the stabilizer of the bipartition of the vertices being primitive on each bipartition.

Another interesting topic is to reconstruct 2-arc-transitive covers of 2-arc-transitive graphs. It is known that every finite regular graph has a 2-arc-transitive cover [2]. In [8] Du, Malnič and Waller investigate the regular covers of complete graphs which are 2-arc-transitive, and they give a complete classification of all graphs whose group of covering transformations is either cyclic or isomorphic to $Z_p \times Z_p$ where p is a prime and whose fibre-preserving subgroup of automorphisms acts 2-arc-transitively. In particular, two families of 2-arc-transitive graphs are obtained. After that, many more results related to the reconstruction of the 2-arc-transitive graphs have been obtained, see [6] for examples.

The main subject of this paper is to construct a 2-arc-transitive cover for one family of affine graphs, namely the hypercubes. Let $V = \mathbb{Z}_2^d$ be a d-dimensional vector space over the field $\mathbb{F}_2$, let $e_1, \ldots, e_d$ be a basis of V and $\mathcal{B} = \{e_1, \ldots, e_d\}$. A *$d$-dimensional hypercube* is a Cayley graph $Q_d = Cay(V, \mathcal{B})$. It is known that Q_d admits a regular group of automorphisms Z_2^d and $Aut(Q_d) = Z_2^d \rtimes S_d$, where $S_d = Aut_{\mathbf{1}}(Q_d)$ and permutes $e_1, \ldots, e_d$ naturally (see [15]). Thus the hypercubes are 2-arc-transitive affine graphs. Furthermore, it has been shown in [9] that Q_d is bi-quasiprimitive if and only if $d = 2$ or d is odd. In this paper, we are interested in the even-dimensional hypercubes, in particular, we construct a 2-arc-transitive cover for even-dimensional hypercubes. We also show that such a cover is a normal Cayley graph.

Let G be a finite group, and S be a subset of G such that S does not contain the identity of G and $S = S^{-1} = \{s^{-1} | s \in S\}$. We say that an element g of G is an *involution* if it has order 2, that is, $g^2 = \mathbf{1}$ and $g \neq \mathbf{1}$. The *Cayley graph* $\Gamma = Cay(G,S)$ is defined to have vertex set $V(\Gamma) = G$, and edge set $E(\Gamma) = \{\{g, sg\} | s \in S\}$. It is well known that a graph is a Cayley graph if and only if its full automorphism group contains a subgroup acting regularly on the vertex set of the graph (see [14]).

Let $\Gamma = Cay(G,S)$ be a Cayley graph for some group G and $Aut(\Gamma)$ be the full automorphism group of Γ. For each $g \in G$, define a map $\hat{g} : G \to G$ by the right multiplication of g on G as below:

$$\hat{g} : x \to xg, \ for \ x \in G.$$

Then $\hat{g}$ is an automorphism of Γ. It follows from the definition that the group $\hat{G} = \{\hat{g} \mid g \in G\}$ is a subgroup of $Aut(\Gamma)$ and acts regularly on $V(\Gamma)$. Following Xu [18], we say that Γ is a *normal Cayley graph for G* (or normal) if $\hat{G} \trianglelefteq Aut(\Gamma)$, otherwise we say that Γ is a *non-normal Cayley graph for G* (or non-normal).

Suppose that V is a d-dimensional vector space with a nondegenerate quadratic form Q where the associated bilinear form B_Q is symmetric. Let $\mathcal{C} = \{v_1, v_2, \ldots, v_d\}$ be a basis of V. We say that $\mathcal{C}$ is *symmetric* if $Q(v_i) = 0$ and $B(v_i, v_j) = 1$ for all i, j with $1 \leq i < j \leq d$. In Section 3, we determine a necessary and sufficient condition for a vector space to have a symmetric basis.

Let G be an extraspecial 2-group of order 2^{2r+1} with $r \geq 1$, that is, $|Z(G)| = 2$ and $G/Z(G) \cong \mathbb{Z}_2^{2r}$. There are two extraspecial 2-groups of each order, for which we will give more details in Section 3. Let $\overline{S} = \{\overline{s_1}, \ldots, \overline{s_{2r}}\}$ be a symmetric basis of $G/Z(G)$, and for each $1 \leq i \leq 2r$, let s_i be a preimage of $\overline{s_i}$ in G. Notice that generally for a basis of $G/Z(G)$, the preimages of the basis elements are not necessary involutions in G. However in Section 3 we show that in the case where it is a symmetric basis the preimages of the basis elements are all involutions in G, which is crucial for the proofs of the main results. Note that $\Sigma = Cay(G/Z(G), \overline{S})$ is a $2r$-dimensional hypercube. We will prove the following result.

Theorem 1.1. *Let G be an extraspecial 2-group of order 2^{2r+1} with $r \geq 1$ such that $G/Z(G)$ has a symmetric basis $\{\overline{s_1}, \ldots, \overline{s_{2r}}\}$. Let $\Gamma = Cay(G,S)$ be a Cayley graph of G with $S = \{s_1, \ldots, s_{2r}\}$. Then Γ is a 2-arc-transitive cover of some $2r$-dimensional hypercube Σ, and Γ is a normal Cayley graph with $Aut(\Gamma) = G \rtimes S_{2r}$.*

2 Preliminaries

Let V be a $2r$-dimensional ($r \geq 1$) vector space over a field $\mathbb{F}_q$, where q is a prime-power. Let B be a bilinear form on V. We say that B is *symmetric* if $B(u,v) = B(v,u)$ for all $u, v \in V$, and B is *alternating* if $B(u,u) = 0$ for all $u \in V$. The *radical* of B is the subspace

$$rad(B) = \{u \in V \mid B(u,v) = 0 \text{ for all } v \in V\},$$

and B is said to be *nondegenerate* if $\mathrm{rad}(B) = \{0\}$. Let W be a subspace of V. Define

$$W^{\perp} = \{v \in V \mid B(w,v) = 0 \text{ for all } w \in W\}$$

to be the *orthogonal complement* of W. It is known that if B is nondegenerate, then $dim(W) + dim(W^{\perp}) = dim(V)$. A map $Q : V \to \mathbb{F}_q$ is a *quadratic form* on V if the following two conditions are satisfied:

(i) $Q(\lambda u) = \lambda^2 Q(u)$ for all $u \in V$ and $\lambda \in \mathbb{F}_q$, and

(ii) the map $B_Q : V \times V \to \mathbb{F}_q$ defined by

$$B_Q(u, v) = Q(u + v) - Q(u) - Q(v)$$

is a bilinear form.

The bilinear form B_Q is called the *associated bilinear form* of Q. A quadratic form Q is said to be *nondegenerate* if and only if its associated bilinear form is nondegenerate.

Let u, v be two distinct vectors of V. We say that $\{u, v\}$ is a *hyperbolic pair* if $Q(u) = Q(v) = 0$ and $B(u, v) = 1$. By [4, Proposition 2.2.7], when Q is nondegenerate and $B = B_Q$ is symmetric , V has the following two types of standard bases, in particular, the basis is *hyperbolic* in case (i), and *elliptic* in case (ii):

(i). $\mathcal{B} = \{e_1, \dots, e_{n/2}, f_1, \dots, f_{n/2}\}$ where

$$Q(e_i) = Q(f_i) = 0,\ B(e_i, f_j) = \delta_{ij} \text{ for all } i, j;$$

(ii). $\mathcal{B} = \{e_1, \dots, e_{n/2-1}, f_1, \dots, f_{n/2-1}, x, y\}$ where $Q(e_i) = Q(f_i) = 0$,

$$B(e_i, x) = B(e_i, y) = B(f_i, x) = B(f_i, y) = 0,\ B(e_i, f_j) = \delta_{ij}$$

for all i, j, $Q(x) = 1$, $B(x, y) = 1$ and $Q(y) = \zeta$ where $x^2 + x + \zeta \in \mathbb{F}_q[x]$ is irreducible,

where

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

If V has a hyperbolic basis, then Q is said to be a *hyperbolic* quadratic form (or hyperbolic in short). Similarly we say that Q is *elliptic* when V has an elliptic basis.

Let U be a subspace of V. We say that U is *totally singular* if $Q(u) = 0$ for all $u \in U$. The next result is a consequence of Witt's Lemma (see [4])

Proposition 2.1 ([5, Page 38]). *Let V be a d-dimensional vector space over the field $\mathbb{F}_q$ equipped with a nondegenerate quadratic form Q, and U be a maximal totally singular subspace of V. Then*

$$dimU = \frac{1}{2} dimV - \delta$$

where $\delta = 0$ if Q is hyperbolic, and $\delta = 1$ if Q is elliptic.

3 Symmetric Basis of a Vector Space

Let V be a $2r$-dimensional vector space over $\mathbb{F}_2$ with a nondegenerate quadratic form $Q : V \to \mathbb{F}_2$ and an associated symmetric bilinear form $B = B_Q$. Suppose that $\mathcal{C} = \{v_1, \dots, v_{2r}\}$ is a symmetric basis of V. Let $\mathcal{C}' = \{c_1, \dots, c_{r-1}\}$ where for $1 \leqslant i \leqslant r - 1$, we have

$$c_i = v_{2i-1} + v_{2i} + v_{2i+1} + v_{2i+2}.$$

Let U be the subspace of V generated by $\mathcal{C}'$. Then $dim(U) = \frac{1}{2}dim(V) - 1$. Also U is totally singular as $Q(c_i) = 0$ and $B(c_i, c_j) = 0$ for all $1 \leqslant i, j \leqslant r - 1$. If U is maximal subject to being totally singular in V, then Q is elliptic. Otherwise Q is hyperbolic, and there is a vector $v \in V \backslash U$ such that $W = \langle v, \mathcal{C}' \rangle$ is a maximal totally singular subspace of V.

Let $\alpha \in V$ be a non-zero vector that is not in $\mathcal{C}'$. We may assume that $\alpha = \mu_1 + \mu_2 + \cdots + \mu_t$, where $\mu_i \in \mathcal{C}$ and $\mu_i \neq \mu_j$ for all $1 \leqslant i \neq j \leqslant t$.

Lemma 3.1. *$Q(\alpha) = 0$ if and only if $t \equiv 0$ or 1 (mod 4).*

Proof. First suppose that $t = 2k$ with $k \geqslant 0$ and let $u_i = \mu_{2i-1} + \mu_{2i}$, for $i = 1, 2, \ldots, k$. So

$$\begin{aligned}
&Q(\mu_1 + \mu_2 + \cdots + \mu_{2k})\\
&= Q(u_1 + u_2 + \cdots + u_k)\\
&= Q(u_1) + Q(u_2 + \cdots + u_k) + \sum_{i=2}^{k} B(u_1, u_i)\\
&= Q(u_1) + Q(u_2) + Q(u_3 + \cdots + u_k) + \sum_{i=2}^{k} B(u_1, u_i) + \sum_{i=3}^{k} B(u_2, u_i)\\
&= Q(u_1) + Q(u_2) + Q(u_3) + Q(u_4 + \cdots + u_k) + \sum_{i=2}^{k} B(u_1, u_i) + \sum_{i=3}^{k} B(u_2, u_i) + \sum_{i=4}^{k} B(u_3, u_i)\\
&= \sum_{i=1}^{k} Q(u_i) + \sum_{i=1}^{k-1} \sum_{j=i+1}^{k} B(u_i, u_j).
\end{aligned}$$

For $1 \leqslant i < j \leqslant k$ we have

$$\begin{aligned}
B(u_i, u_j) &= B(\mu_{2i-1} + \mu_{2i}, \mu_{2j-1} + \mu_{2j})\\
&= B(\mu_{2i-1}, \mu_{2j-1}) + B(\mu_{2i-1}, \mu_{2j}) + B(\mu_{2i}, \mu_{2j-1}) + B(\mu_{2i}, \mu_{2j})\\
&= 0,
\end{aligned}$$

and

$$\begin{aligned}
Q(u_i) =& Q(\mu_{2i-1} + \mu_{2i})\\
&= Q(\mu_{2i-1}) + Q(\mu_{2i}) + B(\mu_{2i-1}, \mu_{2i})\\
&= 1.
\end{aligned}$$

Thus $Q(\alpha) = 0$ if and only if $k \equiv 0$ (mod 2), that is, if and only if $t \equiv 0$ (mod 4).

Next suppose that t is odd, and so $t = 2k + 1$ with $k \geqslant 0$. Then

$$\begin{aligned}
Q(\mu_1 + \mu_2 + \cdots + \mu_{2k} + \mu_{2k+1}) &= Q(\mu_1 + \cdots + \mu_{2k}) + Q(\mu_{2k+1}) + \sum_{i=1}^{2k} B(\mu_i, \mu_{2k+1})\\
&= Q(\mu_1 + \cdots + \mu_{2k}).
\end{aligned}$$

which implies that $Q(\alpha) = 0$ if and only if $t \equiv 1$ (mod 4). ∎

Before we introduce the sufficient and necessary conditions for V to have a symmetric basis, we first give the following lemma.

Lemma 3.2. *Let W be a nontrivial subspace of V with dimension $d \equiv 2 \pmod 4$. Suppose that W has a symmetric basis and there are three pairwise perpendicular hyperbolic pairs $\{a,b\},\{c,d\},\{g,h\}$ in $W^{\perp}\backslash W$. Let*

$$W' = W \perp \langle a,b\rangle \perp \langle c,d\rangle \perp \langle g,h\rangle.$$

Then W' has a symmetric basis.

Proof. Let $\mathcal{C}_W = \{w_1,\dots,w_d\}$ be a symmetric basis of W. Let

$$\begin{aligned} u_1 &= a+c+d+\sum_{i=1}^{d} w_i;\\ u_2 &= b+c+d+\sum_{i=1}^{d} w_i;\\ u_3 &= c+g+h+\sum_{i=1}^{d} w_i;\\ u_4 &= d+g+h+\sum_{i=1}^{d} w_i;\\ u_5 &= g+a+b+\sum_{i=1}^{d} w_i;\\ u_6 &= h+a+b+\sum_{i=1}^{d} w_i. \end{aligned}$$

Since $d \equiv 2 \pmod 4$, Lemma 3.1 implies $Q(\sum_{i=1}^{d} w_i) = 1$. So $Q(u_j) = 0$ for all $j = 1,2,\dots,6$. For any $1 \leqslant j_1 \neq j_2 \leqslant 6$, one can check that

$$B(u_{j_1}, u_{j_2}) = 1.$$

Also we have $B(u_j, w_i) = 1$ for all $1 \leqslant i \leqslant d$ and $j = 1,2,\dots,6$. Therefore $\mathcal{C}_W \cup \{u_1,\dots,u_6\}$ is a symmetric basis of W'. ∎

Lemma 3.3. *Let V be a vector space of dimension $2r$ with nondegenerate quadratic form Q such that $r \geqslant 1$.*

(i). If Q is hyperbolic and $r \equiv 0$ or $1 \pmod 4$, then V has a symmetric basis;

(ii). If Q is elliptic and $r \equiv 2$ or $3 \pmod 4$, then V has a symmetric basis.

Proof. We prove this by using induction on r.

(i) Suppose that Q is hyperbolic with $r \equiv 0$ or $1 \pmod 4$. When $r = 1$, a hyperbolic basis of V is also a symmetric basis of V. Now assume the lemma holds for all $r \leqslant 4\ell + 1$ with $r \equiv 0$ or $1 \pmod 4$, where ℓ is a nonnegative integer. Note that we have seen that the lemma holds when $\ell = 0$. Let $r' = 4\ell$ and suppose that $r = r' + 4 = 4(\ell + 1)$. Let $W \leqslant V$ be a subspace of dimension $2(r'+1)$ such that Q is hyperbolic on W and

$$V = W \perp \langle a,b\rangle \perp \langle c,d\rangle \perp \langle g,h\rangle.$$

where $\{a, b\}, \{c, d\}, \{g, h\}$ are hyperbolic pairs in $V \backslash W$. By induction, W has a symmetric basis $\mathcal{C}_W = \{w_1, w_2, \ldots, w_{2(r'+1)}\}$. Take $d = 2(r'+1)$. Then V has a symmetric basis $\mathcal{C}_W \cup \{u_1, \ldots, u_6\}$ as constructed in Lemma 3.2.

Now suppose that $r = 4(\ell+1)+1$. We may assume that

$$V = W \perp \langle a, b\rangle \perp \langle c, d\rangle \perp \langle g, h\rangle \perp \langle x, y\rangle,$$

where $\{a, b\}, \{c, d\}, \{g, h\}, \{x, y\}$ are hyperbolic. Clearly V contains a subspace U with a symmetric basis $\mathcal{C}_W \cup \{u_1, \ldots, u_6\}$ as above, and let

$$\alpha = x + \sum_{i=1}^{2r'+2} w_i + a + b + c + d + g + h;$$
$$\beta = y + \sum_{i=1}^{2r'+2} w_i + a + b + c + d + g + h.$$

Then $Q(\alpha) = Q(\beta) = 0$ and $B(\alpha, \beta) = 1$. Also for $1 \leqslant i \leqslant 2(r'+1)$ we have $B(\alpha, w_i) = B(\beta, w_i) = 1$, and $B(\alpha, u_j) = B(\beta, u_j) = 1$ for $1 \leqslant j \leqslant 6$. Hence $\mathcal{C}_W \cup \{u_1, \ldots, u_6\} \cup \{\alpha, \beta\}$ forms a symmetric basis of V.

(ii) Suppose that Q is elliptic and $r \equiv 2$ or $3 \pmod 4$. When $r = 2$, let $\{e_1, f_1, x, y\}$ be an elliptic basis of V. Let

$$c_1 = e_1,\ c_2 = f_1,\ c_3 = x + c_1 + c_2,\ c_4 = y + c_1 + c_2.$$

Then $Q(c_i) = 0$ and $B(c_i, c_j) = 1$ for $1 \leqslant i < j \leqslant 4$. So $\{c_1, c_2, c_3, c_4\}$ is a symmetric basis of V.

When $r = 3$, let $\{e_1, e_2, f_1, f_2, x, y\}$ be an elliptic basis. Let c_1, c_2, c_3, c_4 be defined as above and let

$$c_5 = e_2 + c_1 + c_2 + c_3 + c_4,\ c_6 = f_2 + c_1 + c_2 + c_3 + c_4.$$

Then $\{c_1, c_2, c_3, c_4, c_5, c_6\}$ is a symmetric basis of V.

Now assume the lemma holds for all $r \leqslant 4\ell + 3$ with $r \equiv 2$ or $3 \pmod 4$ where ℓ is a nonnegative integer. Note that we have seen that the lemma holds when $\ell = 0$. Let $r' = 4\ell + 2$ and suppose that $r = r' + 4 = 4(\ell+1)+2$. Let $W \leqslant V$ be a subspace of dimension $2(r'+1)$ such that Q is elliptic on W and

$$V = W \perp \langle a, b\rangle \perp \langle c, d\rangle \perp \langle g, h\rangle,$$

where $\{a, b\}, \{c, d\}, \{g, h\}$ are hyperbolic. By our induction W has a symmetric basis

$$\mathcal{C}_W = \{w_1, \ldots, w_{2r'+1}, w_{2r'+2}\}.$$

Then by Lemma 3.2, when $r = 4(\ell+1)+2$, V contains a symmetric basis. When $r = 4(\ell+1)+3$, let x, y, α, β be vectors of V as defined in (i). Then $\mathcal{C}_W \cup \{u_1, \ldots, u_6\} \cup \{\alpha, \beta\}$ forms a symmetric basis of V. ∎

Let V be a d-dimensional vector space over field $\mathbb{F}_q$ equipped with a quadratic form Q and U be a d-dimensional vector spaces over field $\mathbb{F}_q$ equipped with a quadratic form Q'. An isometry from V to U is an invertible linear map $\sigma : V \to U$ such that

$$Q'(v^\sigma) = Q(v) \tag{1}$$

for all $v \in V$. Notice that (1) implies that

$$B_{Q'}(u^\sigma, v^\sigma) = B_Q(u, v)$$

for all $u, v \in V$. If such an isometry exists, then both U and V, and Q and Q' are said to be *isometric*. Let $W \subseteq V$ and $W' \subseteq U$. Then W and W' are isometric if the restrictions $Q \mid_W$ and $Q' \mid_{W'}$ are isometric. We say that σ is an *isometry of* Q if $U = V$. The *isometry group* of Q is the set of isometries of Q under composition. Notice that the isometry group of Q is a subgroup of the isometry group of the associated bilinear form B_Q.

Lemma 3.4. *Let V be a vector space of dimension $2r$ with nondegenerate quadratic form Q and a symmetric basis $\mathcal{C} = \{v_1, v_2, \ldots, v_{2r}\}$.*

(i). If $r \equiv 0$ or 1 (mod 4), then Q is hyperbolic;

(ii). If $r \equiv 2$ or 3 (mod 4), then Q is elliptic.

Proof. Let Q_1 and Q_2 be quadratic forms on V such that V has a symmetric basis $\mathcal{C} = \{c_1, \ldots, c_{2r}\}$ with respect to Q_1 and a symmetric basis $\mathcal{C}' = \{c'_1, \ldots, c'_{2r}\}$ with respect to Q_2. Let $\sigma : V \to V$ be the linear map defined by

$$\sigma : c_i \to c'_i, \text{ for } 1 \le i \le 2r.$$

Then we have

$$B_{Q_2}(c_i^\sigma, c_j^\sigma) = B_{Q_1}(c_i, c_j) \text{ and } Q_2(c_i^\sigma) = Q_1(c_i), \text{ for all } 1 \le i, j \le 2r.$$

Thus σ is an isometry of Q_1 and hence Q_1 and Q_2 have the same type. Therefore by Lemma 3.3 either Q_2 is hyperbolic with $r \equiv 0$ or 1 (mod 4), or Q_2 is elliptic with $r \equiv 2$ or 3 (mod 4). ∎

Combining the results of this section, we obtain the following theorem.

Theorem 3.1. *Let V be a vector space of dimension $2r$ over $\mathbb{F}_2$ with nondegenerate quadratic form Q. Then V has a symmetric basis if and only if either Q is hyperbolic and $r \equiv 0$ or 1 (mod 4), or Q is elliptic and $r \equiv 2$ or 3 (mod 4).*

4 A 2-Arc-Transitive Cover of Hypercubes

Let G be an extraspecial 2-group of order 2^{2r+1} with identity $\mathbf{1}$ ($r \geqslant 1$). Let $Z = \langle z \rangle$ be the center of G. Then $Z \cong \mathbb{Z}_2$ and $G/Z \cong \mathbb{Z}_2^{2r}$ is elementary abelian. The commutator of any two elements in G or the square of any element in G lies in Z. So G is a nilpotent group of class 2. Define two functions $B : G/Z \times G/Z \to Z$ and $Q : G/Z \to Z$ as below: for any Zx, Zy in G/Z,

$$B(Zx, Zy) = [x, y],$$
$$Q(Zx) = x^2.$$

Then Q is a quadratic form on $V = G/Z$ with associated bilinear form B. Note that if $B(Zx, Zy) = \mathbf{1}$ for some $x, y \in G$, then x, y commute. So if $B(Zx, Zy) = \mathbf{1}$ for all $y \in G$, then Zx must be the identity in G/Z. Therefore Q is nondegenerate on G/Z. Furthermore B is symmetric as $[y, x] = [x, y]^{-1} = [x, y]$ for all $x, y \in G$.

We say that G is an extraspecial 2-group of *plus type* if Q is hyperbolic, denoted by 2_+^{2r+1}, and G is an extraspecial 2-group of *minus type* if Q is elliptic, denoted by 2_-^{2r+1}. It is known [17] that if $G = 2_+^{2r+1}$, then it is the central product of r dihedral groups D_8, otherwise G is the central product of $r - 1$ dihedral groups D_8 with one quaternion group Q_8. Also Winter proved that $\mathrm{Aut}(2_\epsilon^{2r+1}) = 2^{2r}.O^\epsilon(2r, 2)$ where $O^\epsilon(2r, 2)$ with $\epsilon \in \{+, -\}$ is an orthogonal group (see [17, Theorem 1]).

By Theorem 3.1, G/Z has a symmetric basis if and only if either G is of plus type with $r \equiv 0$ or 1 (mod 4), or G is of minus type with $r \equiv 2$ or 3 (mod 4). Let $\mathcal{B} = \{Zg_1, Zg_2, \ldots, Zg_{2r}\}$ be a symmetric basis of G/Z, and so for all $1 \leqslant i \neq j \leqslant 2r$ we have that $Q(Zg_i) = (g_i)^2 = \mathbf{1}$ and $B(Zg_i, Zg_j) = [g_i, g_j] = z$. This implies that $g_1, g_2, \ldots, g_{2r}$ are involutions of G, and $g_i g_j = g_j g_i z$ for all distinct i and j with $1 \leqslant i, j \leqslant 2r$. Let $\Sigma = \mathrm{Cay}(G/Z, \mathcal{B})$ and $\Gamma = \mathrm{Cay}(G, S)$ with $S = \{g_1, g_2, \ldots, g_{2r}\}$. Thus Σ is a $2r$-dimensional hypercube.

To prove Theorem 1.1, we first show that Γ is a 2-arc-transitive cover of Σ. The *Frattini subgroup* $\Phi(M)$ of a group M is the intersection of all maximal subgroups of M. If M is a p-group, then the *Frattini quotient* $M/\Phi(M)$ *of* M is isomorphic to Z_p^k where k is the smallest number of generators for M. Since G is an extraspecial 2-group, we have that $\Phi(G) = Z$. Since $G/Z \cong \mathbb{Z}_2^{2r}$ has a symmetric basis $\mathcal{B}$, by the Burnside Basis Theorem (see [13, Theorem 11.12]) we have that $G = \langle g_1, g_2, \ldots, g_{2r}\rangle$.

Let $g \in G$ where $\mathbf{1} \neq g \neq z$. Then Zg can be uniquely written as $Zg_{s_1} Zg_{s_2} \cdots Zg_{s_t}$ where $1 \leqslant s_1 < s_2 < \cdots < s_t \leqslant 2r$. So $g = z^j g_{s_1} \cdots g_{s_t}$ for $j = 0$ or 1.

For each $\sigma \in S_{2r}$, define a map $\tilde{\sigma} : G \to G$ by $z^{\tilde{\sigma}} = z$, and for all $g \in G$ with $g \neq z$,

$$g^{\tilde{\sigma}} = z^j g_{s_1^\sigma} \cdots g_{s_t^\sigma}$$

where $g = z^j g_{s_1} \cdots g_{s_t}$ for $j = 0$ or 1. Note that $(Zg_1)(Zg_2) = Zg_1g_2 = Zg_2g_1$ for any g_1 and g_2 in G. In particular, for all s_i, s_k we have $g_{s_i} g_{s_k} = z^j g_{s_k} g_{s_i}$ for some j and so we can deduce that $\tilde{\sigma}$ is a homomorphism. Suppose that $h \in \mathrm{Ker}(\tilde{\sigma})$ and $h \neq \mathbf{1}$. Since $h \neq z$, the element h can be uniquely written as $z^j g_{s_1} \cdots g_{s_t}$ for some $s_i \in \{1, \ldots, 2r\}$. Since $h^{\tilde{\sigma}} = \mathbf{1}$, we have

$$Z = Z(h^{\tilde{\sigma}}) = Zg_{s_1^\sigma} \cdots Zg_{s_t^\sigma}.$$

This is a contradiction as $\{Zg_1, \ldots, Zg_{2r}\}$ is a basis of G/Z. Therefore, $\mathrm{Ker}(\tilde{\sigma}) = \mathbf{1}$. Thus, $\tilde{\sigma}$ is injective and as G is finite, it follows that $\tilde{\sigma}$ is surjective. Hence, $\tilde{\sigma} \in Aut(G)$.

Theorem 4.1. $S_{2r} \leqslant Aut(G)$.

Proof. Let $\phi : S_{2r} \to Aut(G)$ be the map defined as below:

$$\phi : \sigma \to \tilde{\sigma}, \text{ for each } \sigma \in S_{2r}.$$

It is not hard to prove that ϕ is a homomorphism from S_{2r} into $Aut(G)$. Let K be the kernel of ϕ, that is,

$$K = \{\sigma \in S_{2r} \mid \tilde{\sigma} = \mathbf{1}\}.$$

Since $\tilde{\sigma} = \mathbf{1}$, we have that $g_i^{\tilde{\sigma}} = g_{i^\sigma} = g_i$ for all $1 \leqslant i \leqslant 2r$. Thus $\sigma = \mathbf{1}$, and so $K = \{\mathbf{1}\}$. Therefore, $S_{2r} \leqslant Aut(G)$. ∎

Theorem 4.2. *The graph Γ is a 2-arc-transitive Cayley graph of G.*

Proof. Let $A = Aut(\Gamma)$. Then $S_{2r} \leqslant A_{\mathbf{1}}$ as S_{2r} fixes the identity element $\mathbf{1}$ of G. Let $N(\mathbf{1})$ be the set of neighbours of $\mathbf{1}$ in Γ. Thus S_{2r} is 2-transitive on $N(\mathbf{1})$. Since Γ is vertex-transitive, we have that Γ is 2-arc-transitive. ∎

Theorem 4.3. *Γ is a 2-arc-transitive cover for an even-dimensional hypercube.*

Proof. It follows from Theorem 4.2 that $G \rtimes S_{2r} \leqslant \mathrm{Aut}(\Gamma)$ is 2-arc-transitive on Γ. Since $|G : Z(G)| \geqslant 4$, by [11, Theorem 4.1] we have that Γ is a cover of $\Gamma_{Z(G)} \cong \Sigma$. ∎

Now we prove that Γ is a normal Cayley graph for G. By a computation in Magma [3], we found that Γ is a normal Cayley graph for G for all $r \in \{1, 2, 3\}$. Next, we show that this is true for the general case, that is, for all $r \geqslant 4$.

Let C be a cycle in Γ with $V(C) = \{c_1, \ldots, c_{t_c}\}$ and $E = \{\{c_{t_c}, c_1\}\} \cup \{\{c_i, c_{i+1}\} \mid 1 \leqslant i \leqslant t_c - 1\}$ where t_c is the length of C. There is a sequence $(s_1, \ldots, s_{t_c})$ induced by C where $s_i \in S$ for $1 \leqslant i \leqslant t_c$, such that $c_1 = s_{t_c} c_{t_c}$ and $c_{i+1} = s_i c_i$ for $1 \leqslant i \leqslant t_c - 1$. Since s_i is an involution for all $1 \leqslant i \leqslant 2r$, we have that

$$s_1 s_2 \cdots s_{t_c} = \mathbf{1}, \tag{2}$$

and

$$c_i = \begin{cases} s_{t_c} c_{t_c}, & i = 1, \\ (s_{i-1} s_{i-2} \cdots s_1) c_1, & 2 \leqslant i \leqslant t_c. \end{cases} \tag{3}$$

The sequence is uniquely determined by C. Since Γ is simple, we have the following lemma.

Lemma 4.1. *$s_1 \neq s_{t_c}$, and $s_i \neq s_{i+1}$ for all $1 \leqslant i \leqslant t_c - 1$.*

We call the sequence induced by C the *cycle-sequence for C*.

Lemma 4.2. *Suppose that $(s_1, \ldots, s_n)$ is a sequence with $s_i \in S$ for all $1 \leqslant i \leqslant n$ such that $s_1 \cdots s_n \in Z$. Then for each $1 \leqslant i \leqslant n$, the element s_i appears an even number of times in the sequence.*

Proof. Note that

$$Z = Z s_1 \cdots s_n = (Z s_1) \cdots (Z s_n) = (Z s_{m_1})^{k_1} \cdots (Z s_{m_t})^{k_t},$$

where $\sum_{1 \leqslant i \leqslant t} k_i = n$ and k_i is the number of s_{m_i} in $(s_1, \ldots, s_n)$. Since $\{Z s_1, \ldots, Z s_{2r}\}$ is a basis and s_i is an involution for all $1 \leqslant i \leqslant 2r$, we have that k_i is even for all $1 \leqslant i \leqslant t$. ∎

Let $A_{\mathbf{1}}^{[1]}$ be the automorphisms in $A_{\mathbf{1}}$ that fix each vertex in $N(\mathbf{1})$. Recall that $N(\mathbf{1}) = S$. For each distinct i and j with $1 \leqslant i, j \leqslant 2r$, note that the sequence $(s_1, \ldots, s_8)$ defined by

$$s_k = \begin{cases} g_i, & \textit{if } k \textit{ is odd}, \\ g_j, & \textit{if } k \textit{ is even}. \end{cases}$$

is an 8-cycle in Γ which we will denote by C_{ij} (see Figure 1(a)). Let $c_1 = \mathbf{1}$ and for each $2 \leqslant k \leqslant 8$, let $c_k = s_{k-1} s_{k-2} \cdots s_1$, in particular, we have that $c_5 = g_j g_i g_j g_i = z$ and $c_8 = g_j$.

Let $\rho \in A_{\mathbf{1}}^{[1]}$ and let C_{ij}^{ρ} be the image of C_{ij} under ρ with $V(C_{ij}^{\rho}) = \{u_k \mid u_k = c_k^{\rho}, 1 \leqslant k \leqslant 8\}$. Since $g_i, g_j \in N(1)$ and $\rho \in A_{\mathbf{1}}^{[1]}$, we have that $u_1 = \mathbf{1}$, $u_2 = g_i$ and $u_8 = g_j$. Let $\{a_1, \ldots, a_8\}$ be the cycle-sequence for C_{ij}^{ρ}, and so $a_1 = g_i$ and $a_8 = g_j$ (see Figure 1(b)).

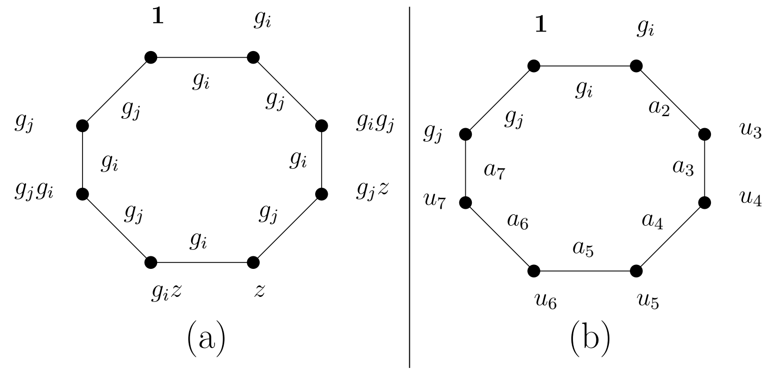

Figure 1: The 8-cycle C_{ij} and its image under $\rho \in A_{\mathbf{1}}^{[1]}$ $(1 \leqslant i, j \leqslant 2r,\ i \neq j)$

Lemma 4.3. *Suppose that $z^{\rho} \neq z$. Then for $1 \leqslant i \leqslant 8$, the element a_i appears exactly twice in the cycle sequence $(s_1, \ldots, s_8)$. Moreover, a_1, a_2, a_3, a_4 are pairwise distinct, and a_5, a_6, a_7, a_8 are pairwise distinct.*

Proof. By Lemma 4.1 we have that $a_1 \neq a_8$ and $a_i \neq a_{i+1}$ for all $1 \leqslant i \leqslant 7$. Suppose that $a_3 = a_1$. Then by (3), we have that

$$\begin{aligned} u_5 &= a_4 a_3 a_2 a_1 \\ &= a_4 a_1 a_2 a_1 \\ &= a_4 a_2 z \\ &= a_2 a_4. \end{aligned}$$

If $a_2 = a_4$, then $u_5 = \mathbf{1}$, which is a contradiction. Thus $a_2 \neq a_4$, which implies that there is a 2-path in Γ connecting $\mathbf{1}$ and u_5. Since $\rho \in A$ and $z^{\rho} = u_5$, we have that there is a 2-path in Γ connecting $\mathbf{1}$ and z, which leads to a contradiction. Hence $a_3 \neq a_1$. If $a_4 = a_1$, then

$$\begin{aligned} u_5 &= a_4 a_3 a_2 a_1 \\ &= a_1 a_3 a_2 a_1 \\ &= a_3 a_2, \end{aligned}$$

and by the same arguments, we deduce that $a_4 \neq a_1$. Suppose that $a_2 = a_4$. Then $u_5 = a_4 a_3 a_2 a_1 = a_1 a_3$, which leads to a contradiction by the same arguments. Thus a_1, a_2, a_3, a_4 are pairwise distinct. By (3) we have that

$$u_8 = a_7 a_6 a_5 u_5.$$

Since $\mathbf{1} = u_1 = a_8 u_8$, we have that $\mathbf{1} = a_8 a_7 a_6 a_5 u_5$. Since S consists of involutions, we have that $u_5 = a_5 a_6 a_7 a_8$. Then by the same arguments, we may conclude that a_5, a_6, a_7, a_8 are pairwise distinct. Therefore, by Lemma 4.2, we conclude that each term in $\{a_1, \ldots, a_8\}$ appears exactly twice. ∎

Lemma 4.4. *The cycle C_{ij} is fixed pointwise by ρ.*

Proof. We first show that $z^\rho = z$. Suppose to the contrary that $z^\rho \neq z$. Since $u_5 = z^\rho$, we have that $(N(z) \cap C_{ij})^\rho = \{g_i z, g_j z\}^\rho = \{u_4, u_6\}$. Recall that $u_4 = a_3a_2a_1$ and $u_6 = a_5a_4a_3a_2a_1$.

Suppose that $(g_i z)^\rho = u_4$. Let $g_k \in S$ be such that $g_k \neq a_i$ for all $1 \leqslant i \leqslant 3$, and let C_{ik} be an 8-cycle. Note that C_{ik} has the same shape as in Figure 1(a). Thus $g_i z$ and g_k are connected by a 2-path in C_{ik}, and so we have that $(g_i z)^\rho$ and g_k^ρ are connected by a 2-path, that is, u_4 and g_k are connected by a 2-path in Γ. We may assume that $x_2x_1u_4 = g_k$ for some $x_1, x_2 \in S$ where $x_1 \neq x_2$, and so $x_2x_1a_3a_2a_1g_k = \mathbf{1}$. By Lemma 4.2 and Lemma 4.3, we must have that $g_k = a_i$ for some $1 \leqslant i \leqslant 3$, which leads to a contradiction. Thus $(g_i z)^\rho \neq u_4$, and so $(g_i z)^\rho = u_6$.

By Lemma 4.3, $a_5 = a_j$ for some $1 \leqslant j \leqslant 3$. Suppose that $a_5 = a_1$. Then $(g_i z)^\rho = u_6 = a_1a_4a_3a_2a_1 = a_4a_3a_2z$. Let $g_{k'} \in S$ such that $g_{k'} \neq a_i$ for $2 \leqslant i \leqslant 4$, and let $C_{ik'}$ be the corresponding 8-cycle. Since $g_i z$ and $g_{k'}$ are joined by a 2-path, we have that u_6 and $g_{k'}$ are joined by a 2-path, and so we have $y_2y_1a_4a_3a_2g_{k'} = z$ for some $y_1, y_2 \in S$ where $y_1 \neq y_2$. Then by similar arguments, we conclude that $g_{k'} = a_j$ for some $2 \leqslant j \leqslant 4$, which is a contradiction. For the remaining two cases where $a_5 = a_2$ or $a_5 = a_3$, we can obtain contradictions following similar arguments. Hence $(g_i z)^\rho \neq u_6$, which leads to a contradiction to the fact that $(g_i z)^\rho \in \{u_4, u_6\}$. Therefore, $z^\rho = z$, that is, $u_5 = z$.

Recall that $u_5 = a_4a_3a_2a_1 = a_5a_6a_7a_8$ where $a_1 = g_i$ and $a_8 = g_j$. By Lemma 4.2 we have that $a_1 = a_3 = g_i$, $a_2 = a_4$, $a_5 = a_7$ and $a_6 = a_8 = g_j$ as $u_5 = z$ and $a_i \neq a_{i+1}$ for all $1 \leqslant i \leqslant 7$. If $a_2 = a_4 = g_j$ and $a_5 = a_7 = g_i$, then C_{ij} and C_{ij}^ρ have the same cycle sequence, that is, $C_{ij}^\rho = C_{ij}$.

Suppose to the contrary that $a_2 = a_4 \neq g_j$. Since $u_5 = z$, we have that $u_4 = a_4 z = a_2 z$ and $u_6 = a_5 z$. Recall that $\{g_i z, g_j z\}^\rho = \{u_4, u_6\}$. Since $a_2 \in S$, we may assume that there exists $1 \leqslant t \leqslant 2r$ such that $a_2 = g_t$ with $t \neq i$ and $t \neq j$. Suppose that $(g_i z)^\rho = u_4 = a_2 z$, that is, $(g_i z)^\rho = g_t z$. Let C_{it} be the corresponding 8-cycle. Thus g_t and $g_i z$ are connected by a 2-path in C_{it}, and so we have that g_t and $(g_i z)^\rho$ are connected by a 2-path. Thus there exists $w_1, w_2 \in S$ such that $w_1 \neq w_2$ and $w_2w_1g_t = g_t z$, that is, $w_1w_2 = z$, which is a contradiction as Zw_1 and Zw_2 are base elements. Hence $(g_i z)^\rho = u_6 = a_5 z$, that is, $(g_j z)^\rho = u_4 = a_2 z$.

Recall that $a_2 = g_t$ where $t \neq i$ and $t \neq j$. Let C_{jt} be the corresponding 8-cycle. Note that C_{jt} has the same shape as in Figure 1(a). Thus $g_j z$ and g_t are connected by a 2-path. Hence $(g_j z)^\rho = a_2 z = g_t z$ and g_t are connected by a 2-path. Suppose that there exist $u_1, u_2 \in S$ such that $u_2u_1g_t = g_t z$. Thus we have that $u_1u_2 = z$, which is a contradiction. Hence $a_2 = a_4 = g_j$.

By a similar argument, we may conclude that $a_5 = a_7 = g_i$. Therefore $C_{ij}^\rho = C_{ij}$. ∎

Corollary 4.1. *Let $v \in N(\mathbf{1})$. Then $\rho \in A_v^{[1]}$.*

Proof. Let $w \in N(v)$. We may assume that $v = g_1$, and so $w = g_1 g_i$ for some $1 \leqslant i \leqslant 2r$. If $i = 1$, then $w = \mathbf{1}$, and so ρ fixes $\mathbf{1}$. Suppose that $i \neq 1$, and consider the 8-cycle C_{1i}. By Lemma 4.4, $C_{1i}^\rho = C_{1i}$, that is, ρ fixes each vertex on C_{1i}, and so $w^\rho = w$. Therefore $\rho \in A_v^{[1]}$. ∎

Lemma 4.5. $A_{\mathbf{1}}^{[1]} = \{\mathbf{1}\}$.

Proof. By Corollary 4.1 we have that $A_{\mathbf{1}}^{[1]} \leqslant A_v^{[1]}$. Since Γ is vertex-transitive and finite, we have that $|A_{\mathbf{1}}^{[1]}| = |A_v^{[1]}|$, which implies that $A_{\mathbf{1}}^{[1]} = A_v^{[1]}$. Since Γ is vertex-transitive, it follows that $A_u^{[1]} = A_w^{[1]}$ for each $u \in V(\Gamma)$ and $w \in N(u)$. Thus by connectivity $A_{\mathbf{1}}^{[1]} = A_u^{[1]}$ for all $u \in V(\Gamma)$, and so $A_{\mathbf{1}}^{[1]} = \{\mathbf{1}\}$. ∎

Proof of Theorem 1.1. By Lemma 4.5 we have $A_{\mathbf{1}} \cong A_{\mathbf{1}}^{N(\mathbf{1})} = S_{2r}$. Further by Theorem 4.1 we obtain $A_{\mathbf{1}} = Aut(G, S)$. Therefore $A = G \rtimes S_{2r}$, namely, Γ is a normal Cayley graph for G. Also in

Theorem 4.3 we have proved that Γ is a 2-arc-transitive cover of a hypercube of dimension $2r$. This completes the proof of Theorem 1.1. ∎

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