

THE EULERIAN DISTRIBUTION ON THE INVOLUTIONS OF THE HYPEROCTAHEDRAL GROUP IS UNIMODAL

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ABSTRACT. The Eulerian distribution on the involutions of the symmetric group is unimodal, as shown by Guo and Zeng. In this paper we prove that the Eulerian distribution on the involutions of the hyperoctahedral group, when viewed as a colored permutation group, is unimodal in a similar way and we compute its generating function, using signed quasisymmetric functions.

1. INTRODUCTION AND RESULTS

An involution is a permutation $w \in \mathfrak{S}_n$ such that $w^{-1} = w$. When written in cycle notation, such a permutation consists only of one-cycles and two-cycles. Let I_n be the set of all involutions in the symmetric group \mathfrak{S}_n and let

$$I_n(x) = \sum_{w \in I_n} x^{\text{des}(w)},$$

where $\text{des}(w)$ is the number of descents (see Section 2 for missing definitions) of $w \in \mathfrak{S}_n$. For the first few values of n we have:

$$I_n(x) = \begin{cases} 1, & \text{if } n = 1 \\ 1 + x, & \text{if } n = 2 \\ 1 + 2x + x^2, & \text{if } n = 3 \\ 1 + 4x + 4x^2 + x^3, & \text{if } n = 4 \\ 1 + 6x + 12x^2 + 6x^3 + x^4, & \text{if } n = 5 \\ 1 + 9x + 28x^2 + 28x^3 + 9x^4 + x^5, & \text{if } n = 6. \end{cases}$$

Strehl [17] proved that $I_n(x)$ is symmetric with center of symmetry at $(n-1)/2$ (conjectured by D. Dumont). Dukes [6] proved partially that $I_n(x)$ is unimodal, which was later fully established by Guo and Zeng [9].

We will be concerned with a natural analogue of $I_n(x)$ for the hyperoctahedral group B_n of signed permutations. It is defined by the formula

$$I_n^B(x) = \sum_{w \in I_n^B} x^{\text{des}_B(w)}$$

where $\text{des}_B(w)$ is the number of descents of $w \in B_n$, when viewed as a colored permutation (see Section 2.3 for the definition) and I_n^B is the set of involutions in B_n . For the first few values of n we have:

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$$I_n^B(x) = \begin{cases} 1 + x, & \text{if } n = 1 \\ 1 + 4x + x^2, & \text{if } n = 2 \\ 1 + 9x + 9x^2 + x^3, & \text{if } n = 3 \\ 1 + 17x + 40x^2 + 17x^3 + x^4, & \text{if } n = 4 \\ 1 + 28x + 127x^2 + 127x^3 + 28x^4 + x^5, & \text{if } n = 5 \\ 1 + 43x + 331x^2 + 632x^3 + 331x^4 + 43x^5 + x^6, & \text{if } n = 6. \end{cases}$$

Désarménien and Foata [5] and later, using different methods, Gessel and Reutenauer [8] computed the generating function

$$(1) \quad \sum_{n \geq 0} \frac{I_n(x)}{(1-x)^{n+1}} t^n = \sum_{m \geq 0} \frac{x^m}{(1-t)^{m+1} (1-t^2)^{\frac{m(m+1)}{2}}}$$

of $I_n(x)$. Our first result computes the generating function of $I_n^B(x)$, using methods similar to those of [8].

Theorem 1.1.

$$(2) \quad \sum_{n \geq 0} \frac{I_n^B(x)}{(1-x)^{n+1}} t^n = \sum_{m \geq 0} \frac{x^m}{(1-t)^{2m+1} (1-t^2)^{m^2}}.$$

One key ingredient of the proof of Theorem 1.1 is a B_n -analogue of the well known expansion of the Schur symmetric functions in terms of fundamental quasisymmetric functions, obtained by Adin et al. in [1] (discussed in Subsection 2.3).

Using the generating function obtained in Theorem 1.1 we derive a linear recurrence formula for the sequence of coefficients of $I_n^B(x)$, which we use to prove the main result of this paper, namely the unimodality of $I_n^B(x)$. It is noted that a consequence of this formula is the palindromicity of $I_n^B(x)$, which can also be proved combinatorially using the natural B_n -analogue of the Robinson-Schensted correspondence (discussed in Subsection 2.3).

The structure of the paper is as follows. Section 2 fixes notation and reviews background material. Section 3 proves Theorem 1.1 and obtains a linear recurrence formula for the coefficients of $I_n^B(x)$. Section 4 proves the unimodality of $I_n(x)$. Section 5 discusses some remarks and open problems.

2. BACKGROUND AND NOTATION

This section fixes notation and briefly reviews background material regarding the combinatorics of (signed) permutations and Young (bi)tableaux, symmetric and unimodal polynomials and the theory of symmetric and quasisymmetric functions which will be needed in the sequel. More information on these topics can be found in [1], [14] and [15, Chapter 7].

For positive integer n we set $[n] := \{1, 2, \dots, n\}$ and $\Omega_n := \{1, -1, 2, -2, \dots, n, -n\}$. We denote by $|S|$ the cardinality of a finite set S .

2.1. Permutations, tableaux and unimodal polynomials. We will denote by \mathfrak{S}_n the symmetric group of all permutations of the set $[n]$, i.e. bijective maps $w : [n] \rightarrow [n]$. If $w = w_1 w_2 \cdots w_n \in \mathfrak{S}_n$ and $1 \leq i \leq n-1$, then i is a descent of w , if $w_i > w_{i+1}$ (otherwise, it is an ascent). Let $\text{Des}(w)$ be the set of descents of $w \in \mathfrak{S}_n$ and set $\text{des}(w) = |\text{Des}(w)|$. A statistic on \mathfrak{S}_n is called Eulerian, if it is equidistributed with the descent number statistic. The n th Eulerian polynomial [14, Section 1.4] is defined by the formula

$$A_n(x) = \sum_{w \in \mathfrak{S}_n} x^{\text{des}(w)}$$

for every positive integer n .

Let $p(x) = \sum_{k=0}^d a_k x^k$ be a polynomial with real coefficients. We recall that $p(x)$ is unimodal (and has unimodal coefficients) if there exists an index $0 \leq j \leq d$ such that $a_0 \leq a_1 \leq \cdots \leq a_j \geq a_{j+1} \geq \cdots \geq a_d$. The polynomial $p(x)$ is said to be log-concave (and has log-concave coefficients) if $a_i^2 \geq a_{i-1} a_{i+1}$ for $1 \leq i \leq d-1$. We will say that $p(x)$ is symmetric (and that it has symmetric coefficients) if there exists an integer $n \geq d$ such that $a_i = a_{n-i}$ for $0 \leq i \leq n$, where $a_k = 0$ for $n > d$. The center of symmetry of $p(x)$ is then defined to be $n/2$ (provided $p(x)$ is nonzero). We say that $p(x)$ is γ -positive if

$$p(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1+x)^{n-2i}$$

for some $n \in \mathbb{N}$ and nonnegative reals $\gamma_0, \gamma_1, \dots, \gamma_{\lfloor n/2 \rfloor}$. Every γ -positive polynomial is symmetric and unimodal, as a sum of symmetric and unimodal polynomials with a common center of symmetry. The n th Eulerian polynomial is an example of a γ -positive polynomial (hence unimodal) as proved by Foata and Schützenberger [7] (see [2, Theorem 2.1] for several combinatorial interpretations of the corresponding γ -coefficients). Unimodal polynomials arise often in combinatorics, geometry, and algebra, see for example [4] and [13]. The γ -positivity of $p(x)$ often provides a more elementary proof of the unimodality of $p(x)$. For a comprehensive survey of γ -positivity in combinatorics and geometry we refer the reader to [2].

Given a partition of n , written as $\lambda \vdash n$, we will denote by $\text{SYT}(\lambda)$ the set of all standard Young tableaux of shape λ . The descent set $\text{Des}(Q)$ of a standard Young tableau $Q \in \text{SYT}(\lambda)$, where $\lambda \vdash n$, is the set of all $i \in [n-1]$ for which $i+1$ appears in a lower row in Q than i does. We recall that the Robinson-Schensted correspondence [15, Theorem 7.13.5] is a bijection from \mathfrak{S}_n to the set of pairs (P, Q) of standard Young tableaux of the same shape and size n with the property [15, Lemma 7.23.1] that $\text{Des}(w) = \text{Des}(Q(w))$ and $P(w^{-1}) = Q(w)$, for all $w \in \mathfrak{S}_n$. So, restricting ourselves to I_n the Robinson-Schensted correspondence is a bijection from I_n to the set SYT_n of all standard Young tableaux of size n which preserves the descent set. Strehl [17] noticed that the map $Q \mapsto Q^t$, where Q^t is the transpose tableau of Q , i.e. the tableau whose rows coincide with the columns of Q , is a bijection from the set of $Q \in \text{SYT}_n$ with $\text{des}(Q) = k$ to the set of $Q \in \text{SYT}_n$ with $\text{des}(Q) = n-1-k$, thus proving that the polynomial $I_n(x)$ is symmetric with center of symmetry at $(n-1)/2$.

2.2. Symmetric and quasisymmetric functions. Our notation concerning these topics follows that of [15, Chapter 7]. Any unexplained terminology can be found there. Let $\mathbf{x} = (x_1, x_2, \dots)$ and $\mathbf{y} = (y_1, y_2, \dots)$ be sequences of pairwise commuting indeterminates. We will denote by Λ^n (respectively, QSym^n) the \mathbb{C} -vector space of homogeneous symmetric (respectively, quasisymmetric) functions of degree n in \mathbf{x} . The fundamental quasisymmetric function associated to $S \subseteq [n-1]$ is defined as

$$F_{n,S}(\mathbf{x}) = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ j \in S \Rightarrow i_j < i_{j+1}}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

The set $\{F_{n,S}(\mathbf{x}) : S \subseteq [n-1]\}$ is known to be a basis of QSym^n . The following well-known proposition expresses the Schur function $s_\lambda(\mathbf{x})$ associated to $\lambda \vdash n$ in terms of fundamental quasisymmetric functions.

Proposition 2.1. ([15, Theorem 7.19.7]) *For every $\lambda \vdash n$,*

$$(3) \quad s_\lambda(\mathbf{x}) = \sum_{Q \in \text{SYT}(\lambda)} F_{n, \text{Des}(Q)}(\mathbf{x}).$$

A consequence of the symmetry of the Robinson-Schensted correspondence is the following Cauchy-type identity for Schur functions, which will be useful in Section 3.

Proposition 2.2. ([15, Corollary 7.13.9]) *Let $\text{Par} := \cup_{n \geq 0} \text{Par}(n)$, where $\text{Par}(n)$ consists of all partitions of n . Then*

$$(4) \quad \sum_{\lambda \in \text{Par}} s_\lambda(\mathbf{x}) = \frac{1}{\prod_{i \geq 1} (1 - x_i) \prod_{1 \leq i < j} (1 - x_i x_j)}.$$

2.3. Signed permutations, bitableaux and signed quasisymmetric functions. Our notation concerning these topics mostly follows that of [1, Section 2].

The hyperoctahedral group B_n consists of all signed permutations of length n , i.e. bijective maps $w : \Omega_n \rightarrow \Omega_n$ such that $w(a) = b$ implies $w(-a) = -b$ for every $a \in \Omega_n$. In this paper we view the hyperoctahedral group as a colored permutation group and it is convenient to use the total order

$$-1 <_r -2 <_r \dots <_r 0 <_r 1 <_r 2 <_r \dots$$

on \mathbb{Z} . For $w \in B_n$, we define

$$\text{des}_B(w) = |\{i \in \{0, 1, \dots, n-1\} : w(i) >_r w(i+1)\}|$$

and

$$\text{des}^B(w) = |\{i \in \{0, 1, \dots, n-1\} : w(i) > w(i+1)\}|,$$

where $w(0) := 0$. Such an index is called a descent of w . In the first case, des_B coincides with the notion of descent for colored permutations [16, Section 2] and in the second case, des^B coincides with the notion of descent in Coxeter groups [10, Section 13.1]. The B_n -Eulerian polynomial is defined by the formula

$$B_n(x) = \sum_{w \in B_n} x^{\text{des}_B(w)} = \sum_{w \in B_n} x^{\text{des}^B(w)}$$

for every positive integer n . B_n -Eulerian polynomials share similar properties with the classic Eulerian polynomials. In fact, they are γ -positive (thus unimodal) (see [2, Theorem 2.10] for combinatorial interpretations of the corresponding γ -coefficients).

The signed descent set of $w \in B_n$, denoted $\text{sDes}(w)$, is defined as the pair $(\text{Des}(w), \epsilon)$, where $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \{-, +\}^n$ is the sign vector with i th coordinate equal to the sign of $w(i)$ and $\text{Des}(w)$ consists of the indices $i \in [n-1]$ for which either $\epsilon_i = +$ and $\epsilon_{i+1} = -$, or $\epsilon_i = \epsilon_{i+1}$ and $|w(i)| > |w(i+1)|$. For $w \in B_n$ we have $\text{des}_B(w) = |\text{Des}(w)|$, if $\epsilon_1 = +$ and $\text{des}_B(w) = |\text{Des}(w)| + 1$, if $\epsilon_1 = -$.

A bipartition of a positive integer n , written $(\lambda, \mu) \vdash n$, is any pair (λ, μ) of integer partitions of total sum n . A standard Young bitableau of shape $(\lambda, \mu) \vdash n$ and size n is any pair $Q = (Q^+, Q^-)$ of tableaux which are strictly increasing along rows and columns, such that Q^+ has shape λ , Q^- has shape μ and every element of $[n]$ appears exactly once as an entry of Q^+ or Q^- . The tableaux Q^+ and Q^- are called

the parts of Q . We will denote by $\text{SYT}(\lambda, \mu)$ the set of all standard Young bitableaux of shape (λ, μ) . The signed descent set of $Q \in \text{SYT}(\lambda, \mu)$, denoted $\text{sDes}(Q)$, is defined as the pair $(\text{Des}(Q), \epsilon)$, where $\epsilon = (\epsilon_1, \epsilon_2, \dots, \epsilon_n) \in \{-, +\}^n$ is the sign vector with i th coordinate equal to the sign of the part of Q in which i appears and $\text{Des}(Q)$ is the set of indices $i \in [n-1]$ for which either $\epsilon_i = +$ and $\epsilon_{i+1} = -$, or $\epsilon_i = \epsilon_{i+1}$ and $i+1$ appears in Q in a lower row than i . For $Q \in \text{SYT}(\lambda, \mu)$ we let $\text{des}_B(Q) = |\text{Des}(Q)|$, if $\epsilon_1 = +$ and $\text{des}_B(Q) = |\text{Des}(Q)| + 1$, if $\epsilon_1 = -$.

Remark 2.3. Our definitions of the signed descent set of a signed permutation and a standard Young bitableau are slightly different from, but equivalent to, the ones given in [1, Definitions 2.2 and 2.3].

The Robinson-Schensted correspondence has a natural B_n -analogue. The Robinson-Schensted correspondence of type B , as described in [12, Section 6] and [1, Section 5], is a bijection from B_n to the set of pairs (P^B, Q^B) of standard Young bitableaux of the same shape and size n such that [1, Proposition 5.1] $P^B(w^{-1}) = Q^B(w)$ and $\text{sDes}(w) = \text{sDes}(Q^B(w))$, for all $w \in B_n$. So, restricting ourselves to signed involutions, the Robinson-Schensted correspondence of type B is a bijection from I_n^B to the set SYB_n of all standard Young bitableaux of size n which preserves the signed descent set.

Proposition 2.4. *The polynomial $I_n^B(x)$ is symmetric with center of symmetry at $n/2$.*

Proof. It suffices to describe a bijection from the set of all standard Young bitableaux Q with $\text{des}_B(Q) = k$ to the set of all standard Young bitableaux Q with $\text{des}_B(Q) = n - k$. Given a standard Young bitableau $Q = (Q^+, Q^-)$, we define its transpose $Q^t = ((Q^t)^+, (Q^t)^-)$ by setting $(Q^t)^+ := (Q^-)^t$ and $(Q^t)^- := (Q^+)^t$. We leave it to the reader to verify that the map $Q \mapsto Q^t$ has the required properties. \square

Different B_n -analogues of quasisymmetric functions have been suggested. The B_n -analogue of the fundamental quasisymmetric functions that we use is the one introduced by Poirier [11, Section 3]. For $w \in B_n$ let

$$(5) \quad F_w(\mathbf{x}, \mathbf{y}) = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ j \in \text{Des}(w) \Rightarrow i_j < i_{j+1}}} z_{i_1} z_{i_2} \cdots z_{i_n}$$

where $z_{i_j} = x_{i_j}$ if $\epsilon_j = +$ and $z_{i_j} = y_{i_j}$ if $\epsilon_j = -$. For a standard Young bitableau Q , we can similarly define $F_Q(\mathbf{x}, \mathbf{y})$ as in (5) with w replaced by Q . The following B_n -analogue of Proposition 2.1, proved by Adin et al. [1, Proposition 4.2], plays a key role to the proof of Theorem 1.1.

Proposition 2.5. ([1, Proposition 4.2]) *For all partitions λ, μ*

$$(6) \quad s_\lambda(\mathbf{x}) s_\mu(\mathbf{y}) = \sum_{Q \in \text{SYB}(\lambda, \mu)} F_Q(\mathbf{x}, \mathbf{y}).$$

3. GENERATING FUNCTION FOR $I_n^B(x)$

This section provides a proof of (1) and proves Theorem 1.1. Then, using Theorem 1.1, it derives a linear recurrence formula for the coefficients of $I_n^B(x)$.

Proof of (1). For a power series $f(\mathbf{x})$, $m \in \mathbb{Z}_{>0}$ and indeterminate t we write

$$f(t^m) = f(x_1 = x_2 = \dots = x_m = t, x_{m+1} = x_{m+2} = \dots = 0).$$

For $S \subseteq [n-1]$ we have (see the discussion before Proposition 7.19.12 in [15])

$$\sum_{m \geq 1} F_{n,S}(1^m) x^{m-1} = \frac{x^{|S|}}{(1-x)^{n+1}}.$$

Thus, letting $S = \text{Des}(w)$, we get

$$(7) \quad \sum_{m \geq 1} F_{n,\text{Des}(w)}(1^m) x^{m-1} = \frac{x^{\text{des}(w)}}{(1-x)^{n+1}}$$

for every $w \in \mathfrak{S}_n$. Taking the sum over all $w \in I_n$, (7) becomes

$$\frac{I_n(x)}{(1-x)^{n+1}} = \sum_{w \in I_n} \sum_{m \geq 1} F_{n,\text{Des}(w)}(1^m) x^{m-1} = \sum_{m \geq 1} \sum_{Q \in \text{SYT}_n} F_{n,\text{Des}(Q)}(1^m) x^{m-1}$$

which by (3) becomes

$$\frac{I_n(x)}{(1-x)^{n+1}} = \sum_{m \geq 1} \sum_{\lambda \vdash n} s_\lambda(1^m) x^{m-1}.$$

Then, using (4), we get

$$\begin{aligned} \sum_{n \geq 0} \frac{I_n(x)}{(1-x)^{n+1}} t^n &= \sum_{m \geq 1} \sum_{\lambda \in \text{Par}} s_\lambda(1^m) x^{m-1} t^{|\lambda|} \\ &= \sum_{m \geq 1} \sum_{\lambda \in \text{Par}} s_\lambda(t^m) x^{m-1} \\ &= \sum_{m \geq 1} \frac{x^{m-1}}{(1-t)^m (1-t^2)^{\binom{m}{2}}}. \end{aligned}$$

□

The following lemma, stated without proof as Equation (43) in [2], will be used in the proof of Theorem 1.1. For a power series $f(\mathbf{x}, \mathbf{y})$, $m \in \mathbb{Z}_{>0}$ and indeterminates p, q we write $f(p^m, q^m)$, where $x_1 = x_2 = \dots = x_m = p$, $x_{m+1} = x_{m+2} = \dots = 0$ and $y_1 = y_2 = \dots = y_m = q$, $y_{m+1} = y_{m+2} = \dots = 0$.

Lemma 3.1. *For $w \in B_n$ we have*

$$(8) \quad \sum_{m \geq 1} F_w(1^m, 01^{m-1}) x^{m-1} = \frac{x^{\text{des}_B(w)}}{(1-x)^{n+1}}.$$

Proof. Let $w \in B_n$. First suppose that $w(1) > 0$. Then

$$F_w(1^m, 01^{m-1}) = \sum_{\substack{1 \leq i_1 \leq i_2 \leq \dots \leq i_n \leq m \\ j \in \text{Des}(w) \Rightarrow i_j < i_{j+1}}} 1 = \sum_{1 \leq i'_1 \leq i'_2 \leq \dots \leq i'_n \leq m - |\text{Des}(w)|} 1.$$

But, in this case $\text{des}_B(w) = |\text{Des}(w)|$. So

$$F_w(1^m, 01^{m-1}) = \sum_{1 \leq i'_1 \leq i'_2 \leq \dots \leq i'_n \leq m - \text{des}_B(w)} 1 = [x^{m-1}] \frac{x^{\text{des}_B(w)}}{(1-x)^{n+1}},$$

where $[x^n]f(x)$ is the coefficient of x^n in the formal power series $f(x)$, and (8) follows. Now suppose $w(1) < 0$. Then

$$F_w(1^m, 01^{m-1}) = \sum_{\substack{1 < i_1 \leq i_2 \leq \dots \leq i_n \leq m \\ j \in \text{Des}(w) \Rightarrow i_j < i_{j+1}}} 1 = \sum_{1 \leq i'_1 \leq i'_2 \leq \dots \leq i'_n \leq m - |\text{Des}(w)| - 1} 1.$$

But, in this case, $\text{des}_B(w) = |\text{Des}(w)| + 1$ and so

$$F_w(1^m, 01^{m-1}) = \sum_{1 \leq i'_1 \leq i'_2 \leq \dots \leq i'_n \leq m - \text{des}_B(w)} 1 = [x^{m-1}] \frac{x^{\text{des}_B(w)}}{(1-x)^{n+1}}$$

and (8) follows. \square

Proof of Theorem 1.1. Taking the sum over all $w \in I_n^B$, (8) becomes

$$\frac{I_n^B(x)}{(1-x)^{n+1}} = \sum_{w \in I_n^B} \sum_{m \geq 1} F_w(1^m, 01^{m-1}) x^{m-1} = \sum_{m \geq 1} \sum_{Q \in \text{SYB}_n} F_Q(1^m, 01^{m-1}) x^{m-1}.$$

By (6), we have

$$\frac{I_n^B(x)}{(1-x)^{n+1}} = \sum_{m \geq 1} \sum_{(\lambda, \mu) \vdash n} s_\lambda(1^m) s_\mu(1^{m-1}) x^{m-1}$$

and using (4) twice we conclude that

$$\begin{aligned} \sum_{n \geq 0} \frac{I_n^B(x)}{(1-x)^{n+1}} t^n &= \sum_{m \geq 1} \sum_{\lambda, \mu \in \text{Par}} s_\lambda(1^m) s_\mu(1^{m-1}) x^{m-1} t^{|\lambda|+|\mu|} \\ &= \sum_{m \geq 1} \left(\sum_{\lambda \in \text{Par}} s_\lambda(t^m) \right) \left(\sum_{\mu \in \text{Par}} s_\mu(t^{m-1}) \right) \\ &= \sum_{m \geq 1} \frac{x^{m-1}}{(1-t)^m (1-t^2)^{\binom{m}{2}} (1-t)^{m-1} (1-t^2)^{\binom{m-1}{2}}} \\ &= \sum_{m \geq 1} \frac{x^{m-1}}{(1-t)^{2m-1} (1-t^2)^{(m-1)^2}}. \end{aligned}$$

\square

Let $I_n^B(x) = \sum_{k=0}^n I_{n,k}^B x^k$. The following proposition provides a linear recurrence formula for the sequence $I_{n,0}^B, I_{n,1}^B, \dots, I_{n,n}^B$ for every $n \in \mathbb{Z}_{>0}$. For a similar formula regarding the coefficients of $I_n(x)$, see [9, Theorem 2.2].

Proposition 3.2. *For $n \geq 3$ and $k \geq 0$, the numbers $I_{n,k}^B$ satisfy the following recurrence formula:*

$$\begin{aligned} nI_{n,k}^B &= (2k+1)I_{n-1,k}^B + (2n-2k+1)I_{n-1,k-1}^B + (n-1+2k(k+1))I_{n-2,k}^B \\ &\quad + (2(n-1)+4(n-k-1)(k-1))I_{n-2,k-1}^B \\ (9) \quad &\quad + ((2n-3)(n-1)+2(k-2)(k-2n+1))I_{n-2,k-2}^B \end{aligned}$$

where $I_{n,k}^B = 0$ for every $k < 0$.

Proof. Extracting the coefficients of t^n of both sides in (2) we get

$$(10) \quad \frac{I_n^B(x)}{(1-x)^{n+1}} = \sum_{m \geq 0} \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m^2 + j - 1}{j} \binom{2m + n - 2j}{n - 2j} x^m.$$

Let

$$r(n, m) = \sum_{j=0}^{\lfloor n/2 \rfloor} \binom{m^2 + j - 1}{j} \binom{2m + n - 2j}{n - 2j}$$

for all $n, m \geq 0$. Clearly, $\sum_{n \geq 0} r(n, m) t^n = (1-t)^{-(2m+1)} (1-t^2)^{-m^2}$ and taking derivatives with respect to t we have

$$\sum_{n \geq 1} n r(n, m) t^{n-1} = \frac{(2m^2 + 2m + 1)t + 2m + 1}{1 - t^2} \sum_{n \geq 0} r(n, m) t^n.$$

Extracting the coefficients of t^n in the expression above we obtain the following recurrence formula for the numbers $r(n, m)$:

$$(11) \quad n r(n, m) = (2m + 1) r(n - 1, m) + (2m^2 + 2m + n - 1) r(n - 2, m),$$

for every $n \geq 3$. Substituting (11) to (10) we have

$$\begin{aligned} \frac{n I_n^B(x)}{(1-x)^{n+1}} &= 2x \left(\frac{I_n^B(x)}{(1-x)^{n+1}} \right)' + \frac{I_{n-1}^B(x)}{(1-x)^n} + 2x^2 \left(\frac{I_{n-2}^B(x)}{(1-x)^{n-1}} \right)'' \\ &\quad + 4x \left(\frac{I_{n-2}^B(x)}{(1-x)^{n-1}} \right)' + (n-2) \frac{I_{n-2}^B(x)}{(1-x)^{n-1}} \end{aligned}$$

or

$$\begin{aligned} n I_n^B(x) &= (2n-1) x I_{n-1}^B(x) + I_{n-1}^B(x) - 2x^2 I_{n-1}^{B'}(x) + 2x I_{n-1}^{B'}(x) \\ &\quad + (2n-3)(n-1) x^2 I_{n-2}^B(x) + 2(n-1) x I_{n-2}^B(x) + (n-1) I_{n-2}^B(x) \\ &\quad - 4(n-2) x^3 I_{n-2}^{B'}(x) + 4(n-3) x^2 I_{n-2}^{B'}(x) + 4x I_{n-2}^{B'}(x) \\ &\quad + 2x^4 I_{n-2}^{B''}(x) - 4x^3 I_{n-2}^{B''}(x) + 2x^2 I_{n-2}^{B''}(x). \end{aligned}$$

Comparing the coefficients of x^k in both sides of the above identity, we obtain the desired formula (9). \square

Remark 3.3. Another proof of Proposition 2.4 can be obtained from Proposition 3.2 as follows. The right-hand side of (9) is invariant under the substitution $k \rightarrow n - k$ provided the sequences $I_{n-1,k}^B$ and $I_{n-2,k}^B$ are symmetric. Thus, by induction the proof follows.

4. UNIMODALITY OF $I_n^B(x)$

This section proves the main result of this paper, namely the unimodality of $I_n^B(x)$ for every positive integer n . The method is similar to that of Guo and Zeng [9] who proved the unimodality of $I_n(x)$. First, we recall the following observation of Guo and Zeng [9, Lemma 3.1].

Lemma 4.1. *Let x_0, x_1, \dots, x_n and a_0, a_1, \dots, a_n be real numbers such that $x_0 \geq x_1 \geq \dots \geq x_n \geq 0$ and $a_0 + a_1 + \dots + a_n \geq 0$ for all $0 \leq k \leq n$. Then*

$$a_0x_1 + a_1x_1 + \dots + a_nx_n \geq 0.$$

By Proposition 2.4 the sequence $I_{n,0}^B, I_{n,1}^B, \dots, I_{n,n}^B$ is symmetric. So, to prove its unimodality, it suffices to show that $I_{n,k}^B \geq I_{n,k-1}^B$ for every $0 \leq k \leq \lfloor n/2 \rfloor$.

Theorem 4.2. *The sequence $I_{n,0}^B, I_{n,1}^B, \dots, I_{n,n}^B$ is unimodal for every $n \in \mathbb{Z}_{>0}$.*

Proof. We proceed by induction on n . For $n \leq 2$ we saw in Section 1 that the statement is true. Suppose that $n > 2$ and that the sequences $I_{n-1,k}^B$ and $I_{n-2,k}^B$ are unimodal. Substituting $k \rightarrow k-1$ in (9) we obtain

$$\begin{aligned} nI_{n,k-1}^B &= (2k-1)I_{n-1,k-1}^B + (2n-2k+3)I_{n-1,k-2}^B + (n-1+2k(k-1))I_{n-2,k-1}^B \\ &\quad + (2(n-1)+4(n-k)(k-2))I_{n-2,k-2}^B \\ (12) \quad &\quad + ((2n-3)(n-1)+2(k-3)(k-2n))I_{n-2,k-3}^B. \end{aligned}$$

Subtracting (12) from (9) we get

$$\begin{aligned} n(I_{n,k}^B - I_{n,k-1}^B) &= A_0I_{n-1,k}^B + A_1I_{n-1,k-1}^B + A_2I_{n-1,k-2}^B \\ (13) \quad &\quad + D_0I_{n-2,k}^B + D_1I_{n-2,k-1}^B + D_2I_{n-2,k-2}^B + D_3I_{n-2,k-3}^B, \end{aligned}$$

where

$$\begin{aligned} A_0 &= 2k+1 & D_1 &= 4nk-3n-6k^2+2k+3 \\ A_1 &= 2n-4k+2 & D_2 &= 2n^2-8nk+9n+6k^2-10k+1 \\ A_2 &= -2n+2k-3 & D_3 &= -2n^2+4nk-7n-2k^2+6k-3. \\ D_0 &= n+2k^2+2k-1 \end{aligned}$$

By the induction hypothesis we know that $I_{n-1,k}^B \geq I_{n-1,k-1}^B \geq I_{n-1,k-2}^B$ for every $0 \leq k \leq \lfloor n/2 \rfloor$. Also, $A_0 = 2k+1 \geq 0$, $A_0 + A_1 = 2(n-k)+3 \geq 0$ and $A_0 + A_1 + A_2 = 0$. Thus, by Lemma 4.1, it follows that

$$(14) \quad A_0I_{n-1,k}^B + A_1I_{n-1,k-1}^B + A_2I_{n-1,k-2}^B \geq 0$$

for every $0 \leq k \leq \lfloor n/2 \rfloor$. It remains to show that

$$(15) \quad D_0I_{n-2,k}^B + D_1I_{n-2,k-1}^B + D_2I_{n-2,k-2}^B + D_3I_{n-2,k-3}^B \geq 0.$$

Indeed, combining (14) and (15) yields $I_{n,k}^B - I_{n,k-1}^B \geq 0$ for every $0 \leq k \leq \lfloor n/2 \rfloor$ and the proof is completed.

We distinguish two cases. In the first case, we assume that $1 \leq k \leq \lfloor n/2 \rfloor - 1$. By the induction hypothesis we know that $I_{n-2,k}^B \geq I_{n-2,k-1}^B \geq I_{n-2,k-2}^B \geq I_{n-2,k-3}^B$. Also,

$$\begin{aligned} D_0 &= n + 2k^2 + 2k - 1 \geq 0 \\ D_0 + D_1 &= 2(k-1)(n-2k) + 2 \geq 0 \\ D_0 + D_1 + D_2 &= 2n(n-2k) + n + 6(n-k) + 2k^2 + 3 \geq 0 \\ D_0 + D_1 + D_2 + D_3 &= 0. \end{aligned}$$

Thus, by Lemma 4.1, (15) follows. For the second case, suppose that $k = \lfloor n/2 \rfloor$. If n is odd, then by the induction hypothesis and symmetry we have

$$I_{n-2,k}^B = I_{n-2,k-1}^B \geq I_{n-2,k-2}^B \geq I_{n-2,k-3}^B$$

and from the previous calculations (15) follows. If n is even, then by the induction hypothesis and symmetry we have

$$I_{n-2,k}^B \geq I_{n-2,k-1}^B = I_{n-2,k-2}^B \geq I_{n-2,k-3}^B.$$

In this case, we notice that $D_1 = \frac{1}{2}(n^2 - 4n + 6) = \frac{1}{2}((n-2)^2 + 2) > 0$. Therefore, by Lemma 4.1 and the previous calculations, (15) follows. \square

5. REMARKS AND OPEN PROBLEMS

This section discusses further properties of $I_n^B(x)$. Barnabei et al. [3, Theorem 7] proved that $I_n(x)$ is not log-concave, answering negatively a conjecture due to Brenti. The same holds for $I_n^B(x)$.

Proposition 5.1. *The sequence $I_{n,0}^B, I_{n,1}^B, \dots, I_{n,n}^B$ is not log-concave in general.*

Proof. It is known that the product of a unimodal and log-concave polynomial with a log-concave polynomial is log-concave as well (see [3, Proposition 6]). Formula 10 shows that the polynomial $r(x) = \sum_{k=0}^n r(n,k)x^k$ is equal to the product of $I_n^B(x)$ and $q(x) = \sum_{k=0}^n \binom{n+k}{k} x^k$. Clearly, $q(x)$ is log-concave. So, if $I_n^B(x)$ was log-concave then $r(x)$ has to be log-concave since $I_n^B(x)$ is unimodal by Theorem 4.2. But, $r(x)$ is not log-concave in general. For instance,

$$r(89, 2)^2 = 113789153706560010000 < 114890217312335629500 = r(89, 1)r(89, 3).$$

\square

Guo and Zeng [9, Conjecture 4.1] conjectured that $I_n(x)$ is γ -positive. The conjecture remains unsolved as far as we know. We notice that

$$I_n^B(x) = \begin{cases} 1+x, & \text{if } n=1 \\ (1+x)^2+2x, & \text{if } n=2 \\ (1+x)^3+6x(1+x), & \text{if } n=3 \\ (1+x)^4+13x(1+x)^2+8x^2, & \text{if } n=4 \\ (1+x)^5+23x(1+x)^3+48x^2(1+x), & \text{if } n=5 \\ (1+x)^6+37x(1+x)^4+168x^2(1+x)^2+56x^3, & \text{if } n=6. \end{cases}$$

The above calculations and the similarity between $I_n(x)$ and $I_n^B(x)$ suggest that $I_n^B(x)$ may be γ -positive for every n (see also [2, Section 2.1.5]).

Remark 5.2. As mentioned in [2, Section 2.1.5], the definition of $I_n^B(x)$ is presumably but not obviously unaffected when des_B is replaced by des^B . We have confirmed it for $n \leq 5$.

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