# The largest graphs with given order and diameter: A simple proof* 

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#### Abstract

A consequence of Ore's classic theorem characterizing the maximal graphs with given order and diameter is a determination of the largest such graphs. We give a very short and simple proof of this smaller result, based on a well-known elementary observation.


Key words. Diameter; size; extremal graphs

We consider finite simple graphs. For terminology and notations we follow the book [2]. The order of a graph is its number of vertices, and the size its number of edges. Denote by $V(G)$ and $E(G)$ the vertex set and edge set of a graph $G$ respectively. We give a very short and simple proof of the following theorem of Ore [1].

Theorem (Ore). For $d \geq 2$, the maximum size of a simple graph of order $n$ and diameter $d$ is $d+(n-d-1)(n-d+4) / 2$. This size is attained by a graph $G$ if and only if $G$ consists of a path $P$ of length $d$ such that the vertices outside $P$ form a clique and are each adjacent to the first three or last three among some three or four consecutive vertices on $P$.

Proof. Let $G$ be a simple graph of order $n$ and diameter $d$. Since $G$ is of diameter $d$, there are vertices $x$ and $y$ which are at distance $d$. Let $P$ be an $(x, y)$-path of length $d$ and denote $S=V(G) \backslash V(P)$. To avoid bringing $x$ and $y$ closer, every vertex of $S$ has at

[^0]most three neighbors on $P$, and if there are three then they are consecutive on $P$. Also, the $n-d-1$ vertices of $S$ induce at most $\binom{n-d-1}{2}$ edges. Counting also the edges on $P$, we thus have $|E(G)| \leq d+3(n-d-1)+\binom{n-d-1}{2}=d+(n-d-1)(n-d+4) / 2$.

To achieve equality and thus prove sharpness of the bound, $S$ must be a clique, and each vertex of $S$ must have three consecutive neighbors along $P$. Since the vertices of $S$ are pairwise adjacent, their neighborhoods on $P$ together can include only at most four consecutive vertices without providing a shorter $(x, y)$-path. Hence the extremal graphs are formed by choosing three or four consecutive vertices along $P$ and making each vertex of $S$ adjacent to the first three or the last three of them.

This proof makes it clear why there is the term $d$ and where the factor $n-d-1$ comes from in the expression of the maximum size. Ore [1] proved the result by first characterizing the maximal $n$-vertex graphs with diameter $d$. Zhou, Xu and Liu [3] gave a different proof of the maximum size by considering the complement graph, but they did not treat the extremal graphs.

Ore [1] also considered $k$-connected graphs. One would like to generalize the above argument to a simple proof of Ore's extremal result for $k$-connected graphs with diameter $d$, but this does not work. The problem is that in applying Menger's Theorem [2, p.167] to obtain $k$ internally disjoint paths joining two vertices at distance $d$, some of the paths may have length greater than $d$. The simplest example is an odd cycle. Ore was able to solve the more general problem by characterizing all the diameter-critical graphs.

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