



# On the Connectivity and Independence Number of Power Graphs of Groups

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## Abstract

Let  $G$  be a group. The power graph of  $G$  is a graph with vertex set  $G$  in which two distinct elements  $x, y$  are adjacent if one of them is a power of the other. We characterize all groups whose power graphs have finite independence number, show that they have clique cover number equal to their independence number, and calculate this number. The proper power graph is the induced subgraph of the power graph on the set  $G - \{1\}$ . A group whose proper power graph is connected must be either a torsion group or a torsion-free group; we give characterizations of some groups whose proper power graphs are connected.

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## 1 Introduction

There are a number of graphs associated with algebraic objects, including the commuting graph and the generating graph of a group. Such graphs can give information about the underlying algebraic structure; in addition, the use of algebraic techniques can give new constructions and analysis of graphs with

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important properties. (For one example, the first explicit constructions of expander graphs was as Cayley graphs of groups, see [13, 14].)

Our topic here is the *power graph*, introduced in the context of semigroup theory by Kelarev and Quinn [10]. This can be defined for any magma  $A$  (set with binary operation, if we define  $a^n$  to be the set of all  $n$ -th powers of  $a$  (all products of  $n$  factors  $a$ , with arbitrary bracketing)); the vertex set is  $A$ , and there is an edge from  $a$  to  $b$  for all  $b \in a^n$  and all  $n \in \mathbb{N}$ . It is perhaps most natural to consider it for *power-associative* magmas, where all  $n$ th powers of an element  $a$  are equal, and in particular in semigroups and groups (where the associative law holds in general). We consider groups in this paper.

The power graph of a group  $G$  arises naturally as a directed graph (as defined above); the *undirected power graph* is obtained by ignoring directions. It forms part of a hierarchy of graphs, each one contained in the next: the power graph, the *enhanced power graph* (two vertices  $a$  and  $b$  joined if both are powers of a common element  $c$ ), the *commuting graph* (two vertices  $a$  and  $b$  joined if  $ab = ba$ ), and, in the case where  $G$  is a non-abelian group generated by two elements, the complement of the *generating graph*. (In the generating graph, two elements  $a$  and  $b$  are joined if  $\langle a, b \rangle = G$ .) The commuting and generating graphs have been much studied; to mention just one striking property, the uniform random walk on the commuting graph (including loops) has a limiting distribution which is uniform on conjugacy classes (this is a special case of Mark Jerrum's "Burnside process" [9]). There is a growing body of literature on the power graph also; we refer to [2] for a survey, and also [1, 4, 8, 10, 11].

In this paper we study the connectedness, independence number, and clique cover number of power graphs of groups.

We begin with some standard definitions from graph theory and group theory.

Let  $G$  be a graph with vertex set  $V(G)$ . An *independent set* is a set of vertices in a graph, no two of which are adjacent; that is, a set whose induced subgraph is null. The *independence number* of a graph  $G$  is the cardinality of the largest independent set and is denoted by  $\alpha(G)$ . The *chromatic number* of  $G$  is the minimum number of parts in a partition of  $V(G)$  into independent sets. Dually, a *clique* is a set of vertices with all pairs adjacent; the *clique cover number* is the minimum number of parts in a partition of  $V(G)$  into cliques. Clearly we have

- (a) the clique number and chromatic number of a graph are equal to the independence number and clique cover number of the complementary graph;
- (b) for any graph, the clique number is at most the chromatic number, and the independence number is at most the clique cover number.

The cyclic group of order  $n$  is denoted by  $C_n$ . A group  $G$  is called *periodic* if every element of  $G$  has finite order. For every element  $g \in G$ , the order of  $g$  is denoted by  $o(g)$ . If there exists an integer  $n$  such that for all  $g \in G$ ,  $g^n = 1$ , where  $1$  is the identity element of  $G$ , then  $G$  is said to be of *bounded exponent* and the *exponent* of  $G$  is  $\exp(G) = \min\{n \mid g^n = 1 \text{ for } g \in G\}$ .

A group  $G$  is said to be *torsion-free* if apart from the identity every element of  $G$  has infinite order.

Let  $p$  be a prime number. The  $p$ -quasicyclic group (known also as the *Prüfer group*) is the  $p$ -primary component of  $\mathbb{Q}/\mathbb{Z}$ . It is denoted by  $C_{p^\infty}$ .

The center of a group  $G$ , denoted by  $Z(G)$ , is the set of elements that commute with every element of  $G$ . A group  $G$  is called *locally finite* if every finitely generated subgroup of  $G$  is finite. Obviously any locally finite group is periodic. We say a group  $G$  is *locally center-by-finite* if for any finite subset  $X$  of  $G$ ,  $[X : Z(\langle X \rangle)] < \infty$ .

A module  $M$  is an *essential extension* of a submodule  $N$ , if any nonzero submodule of  $M$  has nonzero intersection with  $N$ . A module  $M$  is a maximal essential extension of  $N$  if, whenever  $K$  is an arbitrary essential extension of  $N$  and  $M \subseteq K$ , then  $K = M$ .

Now we give definitions of the graphs to be considered in this paper.

The *directed power graph* of  $G$  is the directed graph  $\mathbf{P}(G)$  which takes  $G$  as its vertex set with an edge from  $x$  to  $y$  if  $y \neq x$  and  $y$  is a positive power of  $x$ . The (undirected) *power graph* of  $G$ , denoted  $P(G)$ , takes  $G$  as its vertex set with an edge between distinct elements if one is a positive power of the other. For example, if  $G = C_{p^\infty}$ , then  $P(G)$  is a countably infinite complete graph. The *proper power graph* of  $G$  is  $P^*(G) = P(G) - 1$ .

Directed power graphs were first define by Kelarev and Quinn [10] to study semigroups, and undirected power graphs of groups were introduced by Chakrabarty et al. [5]. The reference [2] surveys a number of results on power graphs. In [1], Aalipour et al. provide some results on the finiteness of the independence number of power graphs. They proved, if  $G$  is an infinite nilpotent group, then  $\alpha(P(G)) = \alpha(P^*(G)) < \infty$  if and only if  $G \cong C_{p^\infty} \times H$ , where  $H$  is a finite group and  $p \nmid |H|$ . They posed the question, does this hold without assuming nilpotence?

In Sect. 2 of this paper we give an affirmative answer to this question, and compute the independence number and clique cover number of the power graphs of infinite groups of this form (they must be equal).

In Sect. 3, we consider the question of connectivity of proper power graphs of infinite groups.

## 2 Independence Number

In this section we give a description of groups whose power graph has finite independence number, and show that, for such groups, the independence number and clique cover number of the power graph are equal.

We need the following theorems [1, Theorems 1 and 2].

**Theorem 1** *Let  $G$  be a group satisfying  $\alpha(P(G)) < \infty$ . Then*

- (a)  $[G : Z(G)] < \infty$ .
- (b)  $G$  is locally finite.

**Theorem 2** *Let  $G$  be an abelian group satisfying  $\alpha(P(G)) < \infty$ . Then either  $G$  is finite or  $G \cong C_{p^\infty} \times H$ , where  $H$  is a finite group and  $p \nmid |H|$ .*

Now we settle an open problem of Aalipour et al. [1, Question 38].

**Theorem 3** *Let  $G$  be a group satisfying  $\alpha(P(G)) < \infty$ . Then either  $G$  is finite, or  $G \cong C_{p^\infty} \times H$ , where  $H$  is a finite group and  $p \nmid |H|$ .*

*Conversely, these groups  $G$  do satisfy  $\alpha(P(G)) < \infty$ .*

**Proof** First suppose that  $G \cong C_{p^\infty} \times H$ , where  $H$  is a finite group and  $p \nmid |H|$ . Assume that  $S$  is a infinite independent set of  $P(G)$ . Since  $H$  is finite,  $S$  has an infinite subset  $S_1 = \{(a, h) \mid a \in K\}$  for some infinite  $K \subseteq C_{p^\infty}$  and some  $h \in H$ . Let  $(a, h), (b, h) \in S_1$  and  $\langle a \rangle \subseteq \langle b \rangle$ . Since  $\langle (a, h) \rangle = \langle a \rangle \times \langle h \rangle$  then  $\langle (a, h) \rangle \subseteq \langle (b, h) \rangle$ . Thus all vertices in  $S_1$  are adjacent, a contradiction.

Conversely, suppose that  $\alpha(P(G)) < \infty$  and  $G$  is infinite. Then by Theorem 1,  $[G : Z(G)] < \infty$  and so  $G = Z(G)H_1$ , where  $H_1$  is a finitely generated subgroup of  $G$ . Theorem 1 implies that  $H_1$  is finite. By Theorem 2,  $Z(G) = AB$ , where  $A \cong C_{p^\infty}$  and  $B$  is a finite group such that  $p \nmid |B|$ . Consequently,  $G = ABH_1 = AH_2$ , where  $H_2 = BH_1$ . Since  $B$  is a central subgroup of  $G$ , then  $H_2$  is a subgroup of  $G$ .

Note that  $A$  is a normal subgroup of finite index in  $G$ . If  $p \mid |G/A|$ , then  $G$  has an element  $a$  of order  $p$  outside  $A$ . (For let  $A$  be generated by elements  $b_i$  for  $i \in \mathbb{N}$ , where  $b_1^p = 1$  and  $b_{i+1}^p = b_i$  for  $i \geq 1$ . Suppose that  $a^p = b_i^d = b_{i+1}^{pd}$ , where  $p \nmid d$ . Since  $A$  is central,  $(ab_{i+1}^{-d})^p = 1$ ; we can use  $ab_{i+1}^{-d}$  in place of  $a$ .) Then  $A\langle a \rangle$  is an abelian  $p$ -group; so  $A\langle a \rangle \cong C_{p^\infty} \times C_p$ , which contradicts Theorem 2.

Finally, the Sylow  $p$ -subgroup of  $G$  is central, so by Burnside's transfer theorem,  $G$  has a normal  $p$ -complement, which is the required  $H$ .  $\square$

As a corollary of this result, we show the following:

**Corollary 1** *Let  $G$  be a group whose power graph  $P(G)$  has finite independence number. Then the independence number and clique cover number of  $P(G)$  are equal.*

**Remark 1** This theorem can be deduced using [1, Theorem 12], asserting that the power graph of a group of finite exponent is perfect, together with the Weak Perfect Graph Theorem of Lovász [12], asserting that the complement of a finite perfect graph is perfect; this argument would also require a compactness argument to show that the clique cover number of  $P(G)$  is equal to the maximum clique cover number of its finite subgroups. However, given our Theorem 3, the argument below is substantially more elementary.

**Proof** We know that  $G = C_{p^\infty} \times H$ , where  $H$  is a finite group and  $p$  a prime not dividing  $|H|$ . We examine the structure of the power graph of such a group. Let  $\mathbf{P}(H)$  be the directed power graph of  $H$ ; write  $h \leftrightarrow h'$  if  $h \rightarrow h'$  and  $h' \rightarrow h$  in  $\mathbf{P}(H)$ . Let  $Z_i$  be the set of elements of order  $p^i$  in  $C_{p^\infty}$ .

Note that  $(z', h')$  is a power of  $(z, h)$  if and only if  $z'$  is a power of  $z$  and  $h'$  a power of  $h$ . One way round is trivial. In the other direction, suppose that  $z' = z^a$  and  $h' = h^b$ . By the Chinese remainder theorem, choose  $c$  such that

$$c \equiv a \pmod{o(z)}, \quad c \equiv b \pmod{\exp(H)}.$$

Then  $(z, h)^c = (z^a, h^b) = (z', h')$ .

It follows that two elements  $(z, h)$  and  $(z', h')$  are adjacent in  $P(G)$  if and only if one of the following happens:

- (a)  $h \leftrightarrow h'$  in  $\mathbf{P}(G)$ ;
- (b)  $h \rightarrow h'$  and  $h' \not\rightarrow h$  in  $\mathbf{P}(G)$ , and  $z \in Z_i, z' \in Z_j$  with  $i \geq j$ ;
- (c)  $h \not\rightarrow h'$  and  $h' \rightarrow h$  in  $\mathbf{P}(G)$ , and  $z \in Z_i, z' \in Z_j$  with  $i \leq j$ .

Now the relation  $\leftrightarrow$  is an equivalence relation on  $H$ . If  $E$  is an equivalence class, then  $C_{p^\infty} \times E$  is a clique in  $P(G)$ , so we have a clique cover of size equal to the number of  $\leftrightarrow$  classes. We have to find an independent set of the same size.

To do this, we note that the  $\leftrightarrow$  classes are partially ordered by  $\rightarrow$ . Extend this partial order to a total order  $<$ , and number the classes  $E_1, \dots, E_r$  with  $E_1 < E_2 < \dots < E_r$ . Now take  $h_i \in E_i$  and  $z_j \in Z_j$ ; the set

$$\{(h_i, z_{r-i}) : 1 \leq i \leq r\}$$

is an independent set, since if  $i < j$  then  $h_j \not\rightarrow h_i$  and  $r - i > r - j$ . □

**Remark 2** This argument also gives us a formula for the independence number of  $P(C_{p^\infty} \times H)$ , where  $p \nmid |H|$ . Since  $x \leftrightarrow y$  if and only if  $\langle x \rangle = \langle y \rangle$ , we see that  $\alpha(P(C_{p^\infty} \times H))$  is equal to the number of cyclic subgroups of  $H$ .

**Corollary 2** *Let  $G$  be a group for which  $P(G)$  has finite independence number. Then  $P(G)$  is a perfect graph.*

**Proof** This is true if  $G$  is finite, by [1, Theorem 12], so suppose that  $G = C_{p^\infty} \times H$ , where  $p \nmid |H|$ . Any finite induced subgraph of  $G$  is contained in  $C_{p^n} \times H$ , for some  $n$ , and so is perfect. This gives the result, since an infinite graph is defined to be perfect if all its finite induced subgraphs are perfect. □

### 3 Connectivity

In a torsion-free group  $G$ , the identity is an isolated vertex of  $P(G)$ , while in a torsion group, it is joined to every vertex. So for questions of connectivity, we use the proper power graph  $P^*(G)$  instead.

However, a group may have elements other than the identity which are joined to all vertices in the power graph. Our first result explains when this can happen in a finite group.

Then we provide some results on the connectivity of proper power graphs, and extend the just-mentioned result to infinite groups.

#### 3.1 Vertices Joined to Every Vertex

In this subsection we consider finite groups only. We classify those groups in which some non-identity vertex is joined to all others, and decide whether the graphs remain connected when all such vertices are deleted.

Note that this is similar to the usual convention in studying the *commuting graph* of a group, the graph in which group elements  $x$  and  $y$  are joined if and only if  $xy = yx$ . In this case, the set of vertices joined to all others is precisely the *center* of the group, and in studying the connectedness of the commuting graph it is customary to delete the center: see for example [6, 15]. (We remark that the power graph of  $G$  is a spanning subgraph of its commuting graph.)

The *generalized quaternion group*  $Q_{2^n}$  is defined by

$$Q_{2^n} = \langle a, b \mid a^{2^{n-1}} = b^2, bab^{-1} = a^{-1}, b^4 = 1 \rangle.$$

**Theorem 4** *Let  $G$  be a finite group. Suppose that  $x \in G$  has the property that for all  $y \in G$ , either  $x$  is a power of  $y$  or vice versa. Then one of the following holds:*

- (a)  $x = 1$ ;
- (b)  $G$  is cyclic and  $x$  is a generator;
- (c)  $G$  is a cyclic  $p$ -group for some prime  $p$  and  $x$  is arbitrary;
- (d)  $G$  is a generalized quaternion group and  $x$  has order 2.

**Proof** We observe first that the converse is true; each of the four cases listed implies that  $x$  satisfies the hypothesis.

Note that the condition is inductive; that is, if  $H$  is a subgroup of  $G$  with  $x \in H$ , then  $x$  satisfies the same hypothesis in  $H$ . If no such subgroup apart from  $G$  exists, then  $G$  is cyclic and  $x$  is a generator. So we may inductively suppose that the theorem is true for any group smaller than  $G$ . We may clearly assume that  $x \neq 1$ .

We observe that, since an element and any power commute,  $x$  belongs to the center  $Z(G)$  of  $G$ . Moreover, if  $G$  is abelian, then it is cyclic. For if not, then for some prime  $p$ ,  $G$  contains elements of order  $p$  neither of which is a power of the other; then they cannot be both powers of  $x$ .

Suppose first that the order of  $x$  is a power of a prime  $p$ . Let  $z$  be a power of  $x$  which has order  $p$ . Then clearly  $\langle z \rangle$  is the only subgroup of order  $p$  in  $G$ . If  $G$  is not a  $p$ -group, it contains an element  $u$  of prime order  $q \neq p$ . Clearly neither  $x$  nor  $z$  is a power of the other.

If  $G$  is a  $p$ -group, then a theorem of Burnside [7, Theorem 12.5.2] shows that  $G$  is either cyclic or generalized quaternion; in the latter case,  $x$  has order 2.

So we may suppose that the order of  $x$  is not a prime power. If  $x \in H$  and  $H < G$ , then  $H$  must be cyclic generated by  $x$ . So  $\langle x \rangle$  is a maximal subgroup of  $G$ . In particular, the center of  $G$  is generated by  $x$ , but  $G$  itself is not cyclic. Now elements outside  $\langle x \rangle$  are not powers of  $x$ , and  $x$  cannot be a power of such an element (else  $G$  would be cyclic).  $\square$

Now we consider the result of deleting all such vertices. The power graph of a cyclic group of prime power order is complete, so nothing remains; but these groups present no difficulty.

Suppose that  $G$  is cyclic of non-prime-power order. If the order of  $G$  is the product of two primes  $p$  and  $q$ , then removing the identity and the generators leaves a disconnected graph consisting of complete graphs of sizes  $p - 1$  and  $q - 1$  with no edges between them. But in any other case, the graph is connected. For any element

of  $G$  has a power which has prime order, and if  $x$  and  $y$  are elements of distinct prime orders  $p$  and  $q$  then  $x$  and  $y$  are both joined to  $xy$ .

Finally, if we remove the identity and the involution from the generalized quaternion group of order  $2^n$ , we obtain a complete graph of cardinality  $2^{n-1} - 2$  together with  $2^{n-2}$  disjoint edges.

In the remaining case, we delete the identity and obtain the proper power graph  $P^*(G)$ . So from now on we consider only this case.

We extend Theorem 4 to infinite groups at the end of the next subsection.

### 3.2 Connectivity of the Proper Power Graph

Since an element of finite order is not adjacent to any element of infinite order, we have the following elementary result.

**Lemma 1** *If  $P^*(G)$  is connected then  $G$  is torsion-free or periodic.* □

For characterizing some torsion-free groups with connected proper power graphs we need the following famous theorems.

**Theorem 5** (Schur's theorem) *Let  $G$  be a group. If  $[G : Z(G)]$  is finite then  $G'$  is finite.*

**Theorem 6** *The additive group  $\mathbb{Q}$  of rational numbers is a (unique) maximal essential extension of the group  $\mathbb{Z}$  of integers, as a  $\mathbb{Z}$ -module.*

Now we can show the following:

**Theorem 7** *Let  $G$  be a locally center-by-finite group which is torsion-free. Then  $P^*(G)$  is connected if and only if  $G$  is isomorphic to a subgroup of  $\mathbb{Q}$ .*

**Proof** Let  $P^*(G)$  be connected, and  $x, y \in G$  nontrivial. There is a path  $x = a_1 - a_2 - \cdots - a_t = y$  in  $P^*(G)$ . Then  $\langle a_i \rangle \cap \langle a_{i+1} \rangle \neq \{1\}$  for  $1 \leq i \leq t - 1$ . This implies that  $\langle a_i \rangle$  and  $\langle a_{i+1} \rangle$  are *commensurable*, that is, their index has finite intersection in each. Since commensurability is an equivalence relation,  $\langle x \rangle$  and  $\langle y \rangle$  are commensurable, so their intersection is not trivial.

Firstly we prove that  $G$  is abelian. Suppose to the contrary, we assume that  $x, y$  are two elements with  $xy \neq yx$ . Set  $H = \langle x, y \rangle$ . By the hypothesis,  $[H : Z(H)]$  is finite. Now by Theorem 5,  $H'$  is a nontrivial finite group, which contradicts the assumption that  $G$  is torsion-free. We deduce that  $G$  is abelian.

Let  $a$  be a nontrivial element of  $G$  and  $H = \langle a \rangle$ . By assumption  $G$  is an essential extension of  $H$  as  $\mathbb{Z}$ -module. Let  $L$  be a maximal essential extension of  $G$ . By Theorem 6,  $L \cong \mathbb{Q}$ , which completes the proof.

For the converse, it is obvious that for any elements  $x = m/n, y = m_1/n_1 \in \mathbb{Q}$ ,  $nm_1x = n_1my$ , as desired. □

There are examples of non-abelian torsion-free groups in [3, 16], in which the intersection of any two non-trivial subgroups is non-trivial. Thus their proper power graphs are connected.

For further investigation of the power graphs of torsion-free groups, we refer to [4].

For periodic groups, it seems the following result is the best for connectivity.

**Theorem 8** [8, Lemma 2.1] *Let  $G$  be a periodic group. Then  $P^*(G)$  is connected if and only if for any two elements  $x, y$  of prime orders where  $\langle x \rangle \neq \langle y \rangle$ , there exist elements  $x = x_0, x_1, \dots, x_t = y$  such that  $o(x_{2i})$  is prime,  $o(x_{2i+1}) = o(x_{2i})o(x_{2i+2})$  for  $i \in \{0, \dots, t/2\}$  and,  $x_i$  is adjacent to  $x_{i+1}$  for  $i \in \{0, 1, \dots, t-1\}$ .*

By the above theorem, it can be seen that if  $|\pi(Z(G))| \geq 2$ , then  $P^*(G)$  is connected, where  $\pi(G)$  is the set of all prime numbers  $p$  such that  $G$  has an element of order  $p$ . (For suppose that  $p, q \in \pi(Z(G))$ . For any prime  $r \in \pi(G)$ , if  $r \neq p$ , there is a path between any element of order  $r$  and any central element of order  $p$ ; then it follows that all elements of prime order are in a single component, so  $P^*(G)$  is connected.)

Also the proper power graph of a  $p$ -group  $G$  is connected if and only if  $G$  has exactly one subgroup of order  $p$ . Moreover, if  $G$  is a finite  $p$ -group, then  $P^*(G)$  is connected if and only if  $G$  is cyclic or a generalized quaternion group  $Q_{2^n}$ . (This follows from the theorem of Burnside, that a  $p$ -group with a unique subgroup of order  $p$  is cyclic or generalized quaternion: see [7, 12.5.2].)

We have the following result for the infinite case. The group  $Q_{2^\infty}$  is defined to be

$$Q_{2^\infty} = \bigcup_{i \geq 3} Q_{2^i}$$

where  $Q_{2^i}$  is a subgroup of index 2 in  $Q_{2^{i+1}}$  (containing the central involution).

**Theorem 9** *Let  $G$  be an infinite locally finite  $p$ -group. Then  $P^*(G)$  is connected if and only if  $G \cong C_{p^\infty}$  for some prime number  $p$ , or  $G \cong Q_{2^\infty}$ .*

**Proof** By Theorem 8, if  $P^*(G)$  is connected, then  $G$  has a unique subgroup of order  $p$ , so by Burnside's Theorem it is a union of cyclic or generalized quaternion groups. If  $G$  is abelian then  $G \cong C_{p^\infty}$ . If  $G$  is non-abelian, it contains arbitrarily large finite generalized quaternion groups, and so  $G \cong Q_{2^\infty}$ .

Conversely, for the groups  $G$  of these two types,  $P^*(G)$  is connected, since the unique subgroup of order  $p$  contains a power of each non-identity element.  $\square$

Now we give a general class of examples.

**Example 1** Let  $G = \langle A, t \rangle$ , where  $A$  is an abelian torsion group of exponent greater than 2 and  $t$  is an element of order 2 inverting  $A$ . (This includes dihedral groups of order greater than 4.) Then  $G$  contains non-central involutions, for example  $t$ ; these are isolated vertices in  $P^*(G)$ . We note that the center of  $G$  is a 2-group.

Finally we return to the question of vertices joined to everything in infinite groups. Our result for locally finite groups is similar to the finite case.

**Theorem 10** *Let  $G$  be an infinite group, and suppose that  $x \in G$  has the property that for any  $y \in G$ , either  $y$  is a power of  $x$  or vice versa. Assume that  $x \neq 1$ . Then the following hold:*

- (a) *If  $G$  is not a torsion group, then  $G$  is infinite cyclic, and  $x$  is a generator.*



- (b) If  $G$  is locally finite, then either  $G = C_{p^\infty}$  for some prime  $p$  and  $x$  is arbitrary, or  $G = Q_{2^\infty}$  and  $x$  has order 2.

**Proof** The hypothesis implies that  $P^*(G)$  (obtained by deleting the identity from  $P(G)$ ) is connected; so by Lemma 1,  $G$  is either torsion-free or a torsion group.

Suppose that  $G$  is torsion-free; we claim that  $G = \langle x \rangle$ . If not, take  $y \notin \langle x \rangle$ . Then  $y$  is not a power of  $x$ , so  $x$  is a power of  $y$ . But the only elements in a cyclic group which are joined to all elements are the identity and the generators; so  $\langle x \rangle = \langle y \rangle$ , a contradiction.

Now suppose that  $G$  is a locally finite group. By Theorem 9,  $G$  is either  $C_{p^\infty}$ , or  $G = Q_{2^\infty}$ . In the first case,  $P^*(G)$  is complete, so any element satisfies the requirement for  $x$ ; in the second case, it is clear that  $x$  is the central involution.  $\square$

**Problem** Do the conclusions of part (b) of Theorem 10 hold under the weaker assumption that  $G$  is a torsion group?

Now we can, as before, decide whether deleting all vertices which are joined to everything leaves a connected graph, at least under the assumptions of Theorem 10:

- (a) If  $G = \langle x \rangle$  is infinite cyclic, then  $P(G) \setminus \{1, x^{\pm 1}\}$  is connected with diameter 2, since  $x^m$  and  $x^n$  are both joined to  $x^{mn}$ .
- (b) If  $G = C_{p^\infty}$ , then  $P(G)$  is complete.
- (c) If  $G = Q_{2^\infty}$  and  $x$  has order 2, then  $P(G) \setminus \{1, x\}$  consists of an infinite complete graph and infinitely many disjoint edges.

## 4 A Problem

We mentioned earlier that the limiting distribution of the uniform random walk on the commuting graph of a finite group (including loops) is uniform on conjugacy classes. What is the limiting distribution of the random walk on the power graph, or the extended power graph, of a finite group?

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