Ramsey and Gallai-Ramsey numbers for two classes of unicyclic graphs^{*}

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Abstract

Given a graph G and a positive integer k, define the *Gallai-Ramsey* number to be the minimum number of vertices n such that any k-edge coloring of K_n contains either a rainbow (all different colored) triangle or a monochromatic copy of G. In this paper, we consider two classes of unicyclic graphs, the star with an extra edge and the path with a triangle at one end. We provide the 2-color Ramsey numbers for these two classes of graphs and use these to obtain general upper and lower bounds on the Gallai-Ramsey numbers.

1 Introduction

In this work, we consider only edge-colorings of graphs. A coloring of a graph is called *rainbow* if no two edges have the same color.

Colorings of complete graphs that contain no rainbow triangle have very interesting and somewhat surprising structure. In 1967, Gallai [6] first examined this structure under the guise of transitive orientations. The result was reproven in [8] in the terminology of graphs and can also be traced to [1]. For the following statement, a trivial partition is a partition into only one part.

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Theorem 1 ([1, 6, 8]). In any coloring of a complete graph containing no rainbow triangle, there exists a nontrivial partition of the vertices (that is, with at least two parts) such that there are at most two colors on the edges between the parts and only one color on the edges between each pair of parts.

For ease of notation, we refer to a colored complete graph with no rainbow triangle as a *Gallai-coloring* and the partition provided by Theorem 1 as a *Gallai-partition*. The induced subgraph of a Gallai colored complete graph constructed by selecting a single vertex from each part of a Gallai partition is called the *reduced graph* of that partition. By Theorem 1, the reduced graph is a 2-colored complete graph.

Given two graphs G and H, let R(G, H) denote the 2-color Ramsey number for finding a monochromatic G or H, that is, the minimum number of vertices n needed so that every red-blue coloring of K_n contains either a red copy of G or a blue copy of H. Although the reduced graph of a Gallai partition uses only two colors, the original Gallai-colored complete graph could certainly use more colors. With this in mind, we consider the following generalization of the Ramsey numbers. Given two graphs G and H, the general k-colored Gallai-Ramsey number $gr_k(G:H)$ is defined to be the minimum integer m such that every k-coloring of the complete graph on m vertices contains either a rainbow copy of G or a monochromatic copy of H. With the additional restriction of forbidding the rainbow copy of G, it is clear that $gr_k(G:H) \leq R_k(H)$ for any graph G.

In this work, we consider the Gallai-Ramsey numbers for finding either a rainbow triangle or monochromatic graph coming from two classes of unicyclic graphs: a star with an extra edge that forms a triangle, and a path with an extra edge from an end vertex to an internal vertex formaing a triangle. In order to produce sharp results for the Gallai-Ramsey numbers of these graphs, we first prove the 2-color Ramsey numbers for these graphs.

These graphs are particularly interesting because although they are not bipartite, they are very close to being a tree (and therefore bipartite). The dichotomy between bipartite and non-bipartite graphs is critical in the study of Gallai-Ramsey numbers in light of the following result.

Theorem 2 ([7]). Let H be a fixed graph with no isolated vertices. If H is not bipartite, then $gr_k(K_3 : H)$ is exponential in k. If H is bipartite, then $gr_k(K_3 : H)$ is linear in k.

We refer the interested reader to [11] for a dynamic survey of small Ramsey numbers and [5] for a dynamic survey of rainbow generalizations of Ramsey theory, including topics like Gallai-Ramsey numbers.

Section 2 contains two known results that will be used as part of our proofs. Section 3 discusses the case where H is a star with the addition of an extra edge to form a triangle. Section 4 discusses the case where H is a path with the addition of an extra edge betwen the first vertex and the third vertex, again forming a triangle.

2 Preliminaries

In this section, we recall two results about cycles and a helpful lemma which will be used later in our proofs. First the known Ramsey number for cycles.

Theorem 3 ([3, 10, 12]). *Given integers* $m, n \ge 3$,

$$R(C_m, C_n) = \begin{cases} 2n - 1 \\ for \ 3 \le m \le n, \ m \ odd, \ (m, n) \ne (3, 3), \\ n - 1 + m/2 \\ for \ 4 \le m \le n, \ m \ and \ n \ even, \ (m, n) \ne (4, 4), \\ \max\{n - 1 + m/2, 2m - 1\} \\ for \ 4 \le m < n, \ m \ even \ and \ n \ odd. \end{cases}$$

Next the general k-color Gallai-Ramsey number for even cycles is not yet known but the following bound have been shown.

Theorem 4 ([4], [9]). Given integers $n \ge 2$ and $k \ge 1$,

$$(n-1)k + n + 1 \le gr_k(K_3:C_{2n}) \le (n-1)k + 3n.$$

Finally we present a lemma from [13].

Lemma 1 ([13]). Let $k \geq 3$, H be a graph with |H| = m, and let G be a Gallai coloring of the complete graph K_n containing no monochromatic copy of H. If $G = A \cup B_1 \cup B_2 \cup \cdots \cup B_{k-1}$ where A uses at most k colors (say from [k]), $|B_i| \leq m-1$ for all i, and all edges between A and B_i have color i, then $n \leq gr_k(K_3:H) - 1$.

Note that this lemma uses the assumed structure to provide a bound on |G| even if G itself uses more than k colors.

3 Star with an extra edge

Let S_t denote the star with t total vertices (and t-1 edges). Then for $t \ge 3$, let S_t^+ denote graph consisting of the star S_t with the addition of an edge between two of the pendant vertices, forming a triangle. Note that $S_3^+ = K_3$.

Before beginning the discussion of the Gallai-Ramsey number for S_t^+ , we must first find the 2-color Ramsey number.

Theorem 5. For $t \geq 3$,

$$R(S_t^+, S_t^+) = 2t - 1.$$

Proof. The lower bound follows from the graph constructed by taking two copies of K_{t-1} each colored entirely with red and adding all blue edges in between the two copies. Each red component is too small to contain a monochromatic copy of S_t^+ and the blue subgraph is bipartite so cannot contain a copy of S_t^+ . The

order of this constructed graph is 2t - 2, meaning that the Ramsey number is at least 2t - 1.

For the upper bound, consider an arbitrary vertex v in any 2-coloring of K_{2t-1} . If v has at least t-1 incident red edges, then the graph induced on the set of vertices at the opposite end of these red edges must contain no red edges to avoid a red S_t^+ . This induces a blue complete graph so there must be at most t-1 such vertices. The same holds for incident blue edges at v, meaning that v must have precisely t-1 incident red edges and t-1 incident blue edges. Let R and B be the corresponding sets of vertices, as above, inducing blue and red complete graphs respectively and let $r \in V(R)$ and $b \in V(B)$. The edge rb must be either red or blue, but either one creates the desired monochromatic copy of S_t^+ centered at one of r or b, completing the proof.

Next we prove a lemma which provides the lower bound on the Gallai-Ramsey number.

Lemma 2. For $k \ge 1$,

$$gr_k(K_3:S_t^+) \ge \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. We prove this result by inductively constructing a k-coloring of K_n where

$$n = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} & \text{if } k \text{ is odd,} \end{cases}$$

which contains no rainbow triangle and no monochromatic copy of S_t^+ .

If k is odd, let G_1 be a complete graph on t-1 vertices colored entirely with color 1. Note that with only t-1 vertices, this contains no monochromatic copy of S_t^+ . Suppose we have constructed a colored complete graph G_{2i-1} where i is a positive integer and 2i-1 < k, using the 2i-1 colors $1,2, \ldots,$ 2i-1 and having order $n_{2i-1} = (t-1) \cdot 5^{i-1}$. Construct G_{2i+1} by making five copies of G_{2i-1} and inserting edges of color 2i and 2i+1 between the copies to form a blow-up of the unique 2-colored K_5 which contains no monochromatic triangle. This coloring clearly contains no rainbow triangle and, since there is no monochromatic triangle in either of the two new colors, there can be no monochromatic copy of S_t^+ in G_{2i+1} . Ultimately, G_k is a k-colored complete graph containing no rainbow triangle and no monochromatic copy of S_t^+ with $|G_k| = (t-1) \cdot 5^{(k-1)/2}$.

If k is even, let G_2 be a 2-colored complete graph on 2t - 2 vertices containing no monochromatic copy of S_t^+ using colors 1 and 2. That is, G_2 is the sharpness example from Theorem 5. Suppose we have constructed a coloring of G_{2i} where i is a positive integer and 2i < k, using 2i colors 1, 2, ..., 2i and having order $n_{2i} = (2t-2) \cdot 5^{i-1}$ such that G_{2i} contains no rainbow triangle and no monochromatic copy of S_t^+ . Construct G_{2i+2} by making five copies of G_{2i} and inserting edges of colors 2i + 1 and 2i + 2 between the copies to form a blow-up of the unique 2-colored K_5 which contains no monochromatic triangle. This coloring clearly contains no rainbow triangle and, since there is no monochromatic triangle in either of the two new colors, there can be no monochromatic copy of S_t^+ in G_{2i+2} . Ultimately, G_k is a k-colored complete graph containing no rainbow triangle and no monochromatic copy of S_t^+ with $|G_k| = 2(t-1) \cdot 5^{(k-2)/2}$.

At last, the main result of this section, the precise Gallai-Ramsey number for S_t^+ .

Theorem 6. For $k \ge 1$ and $t \ge 3$,

$$gr_k(K_3:S_t^+) = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. The lower bound follows from Lemma 2. We prove the upper bound by induction on k. The case k = 1 is trivial and the case k = 2 is precisely Theorem 5, so suppose $k \ge 3$ and let G be a Gallai coloring of K_n where

$$n = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Since G is a Gallai coloring, by Theorem 1, there is a Gallai partition of G. Suppose red and blue are the two colors appearing in the Gallai partition. Let m be the number of parts in this partition and choose such a partition where m is minimized. By Theorem 5 applied on the reduced graph, we see that $m \leq 2t-2$. Let H_1, H_2, \ldots, H_m be the parts of this partition and suppose $|H_i| \geq |H_{i+1}|$ for all i with $1 \leq i \leq m-1$. Let r be the number of parts of the Gallai partition with order at least t-1, so $|H_r| \geq t-1$ and $|H_{r+1}| \leq t-2$.

First suppose $2 \le m \le 3$. If $m \le 3$, then by the minimality of m, we may assume m = 2, say with corresponding parts H_1 and H_2 . Without loss of generality, suppose all edges between H_1 and H_2 are blue. Since $k \ge 3$, we have $|G| = |H_1| + |H_2| \ge 5t - 4$, so there is at least one part of order at least t - 1, meaning that $|H_1| \ge t - 1$. If $|H_i| \ge t - 1$ for i = 1, 2, then to avoid a blue S_t^+ , there can be no blue edges within H_1 and H_2 . Since a color is missing within each H_i , apply induction on k within H_i . This means that

$$|G| = |H_1| + |H_2| \le 2[gr_{k-1}(K_3 : S_t^+) - 1] < n,$$

a contradiction. Otherwise if $|H_2| < t - 1$, then still there are no blue edges within H_1 so by induction on k within H_1 , we have

$$|G| = |H_1| + |H_2| \le [gr_{k-1}(K_3: S_t^+) - 1] + (t-2) < n,$$

a contradiction. We may therefore assume $m \ge 4$.

If $r \ge 4$ and $m \ge 6$, then any choice of 6 parts containing the 4 parts $\mathcal{H} = \{H_1, H_2, H_3, H_4\}$ will contain a monochromatic triangle in the reduced

graph. Such a triangle must contains at least one part from \mathcal{H} , meaning that the corresponding subgraph of G must contain a monochromatic copy of S_t^+ , a contradiction. Thus, we may assume either $4 \leq m \leq 5$ or $r \leq 3$. We consider cases based on the value of r. First a couple of claims that will be helpful in the later analysis.

Claim 1. If there are several parts of a Gallai partition, each of order at most t-2, such that all edges in between pairs of these parts are red, then there are at most a total of 2t - 4 vertices in these parts.

Proof. Let $H'_1, H'_2, \ldots, H'_{m'}$ be these parts. If $m' \leq 2$, then $|H'_1 \cup H'_2| \leq 2t - 4$ by assumption. If $m' \geq 3$, to avoid creating a red S_t^+ using a vertex of H'_1 as the center of the star, we have $|H'_2 \cup H'_3 \cup \cdots \cup H'_{m'}| \leq t-2$, so $|H'_1 \cup H'_2 \cup \cdots \cup H'_{m'}| \leq 2t - 4$, completing the proof.

Claim 2. If H_1 and H_2 are two parts of a Gallai partition each with order at least t-1, say with blue edges in between H_1 and H_2 , then there is at most one part with blue edges to H_1 and with red edges to H_2 , and similarly, there is at most one part with red edges to H_1 and with blue edges to H_2 . If c is the color of the edges between H_1 and H_2 , then there are no parts of the Gallai partition with edges of color c to both H_1 and H_2 .

Proof. For a contradiction, suppose there are two parts H_3 and H_4 with red edges to H_1 and blue edges to H_2 . If the edges from H_3 to H_4 are red, then $H_1 \cup H_3 \cup H_4$ contains a red copy of S_t^+ . Otherwise, if the edges from H_3 to H_4 are blue, then $H_2 \cup H_3 \cup H_4$ contains a blue copy of S_t^+ , either case providing a contradiction. The proof is symmetric for two parts with red edges to H_2 and blue edges to H_1 .

For the second conclusion, if there was a part H_3 with blue edges to both H_1 and H_2 , then $H_1 \cup H_2 \cup H_3$ contains a blue copy of S_t^+ , for a contradiction.

Case 1. r = 0.

Consider the colors of the edges from H_1 to $H_2 \cup H_3 \cup \cdots \cup H_m$. Let A be the union of the parts with blue edges to H_1 and let B be the union of the parts with red edges to H_1 . If |A| (or similarly |B|) is at least t - 1, then there can be no blue edges within A (respectively red edges within B). Then all edges between the parts in A (respectively B) must be red (respectively blue) so by Claim 1, we have $|A| \leq 2t - 4$ and $|B| \leq 2t - 4$. This means

$$|G| = |H_1| + |A| + |B| \le 5(t-2) < n,$$

a contradiction.

Case 2. r = 1.

Again let A be the union of the parts with blue edges to H_1 and let B be the union of the parts with red edges to H_1 . As in the previous case, we have $|A| \leq 2t - 4$ and $|B| \leq 2t - 4$. Since *m* is minimal, neither *A* nor *B* can be empty. There can therefore be no red or blue edges within H_1 , we have

$$|G| = |H_1| + |A| + |B|$$

$$\leq [gr_{k-2}(K_3:S_t^+) - 1] + 2(2t - 4)$$

$$< n,$$

a contradiction.

Case 3. r = 2.

Suppose blue is the color of the edges between H_1 and H_2 , so neither H_1 nor H_2 can contain blue edges and by Claim 2, there is no part with blue edges to both H_1 and H_2 . By Claim 2, there is at most one part H_3 with blue edges to H_1 and with red edges to H_2 , and there is at most one part H_4 with red edges to H_1 and with blue edges to H_2 . Let A be the union of the remaining parts, with all red edges to $H_1 \cup H_2$. By Claim 1, we have $|A| \leq 2t - 4$. By the minimality of m, all parts have incident edges from other parts in both red and blue so both H_1 and H_2 contain no red edges or blue edges. This means $|H_i| \leq gr_{k-2}(K_3:S_t^+) - 1$, so

$$|G| = |H_1| + |H_2| + |H_3| + |H_4| + |A|$$

$$\leq 2[gr_{k-2}(K_3:S_t^+) - 1] + 4(t-2)$$

$$< n,$$

a contradiction.

Case 4. r = 3.

To avoid a monochromatic copy of S_t^+ , the triangle in the reduced graph corresponding to the parts $\{H_1, H_2, H_3\}$ must not be monochromatic. Without loss of generality, suppose the edges from H_2 to H_3 are red and all edges from H_1 to $H_2 \cup H_3$ are blue. Then H_2 and H_3 contain neither red nor blue edges, and H_1 contains no blue edges.

First we claim that there is no part having blue edges to H_1 . Otherwise suppose there is such a part, say H' with blue edges to H_1 . To avoid a blue copy of S_t^+ , all edges from H' to $H_2 \cup H_3$ must be red, but then $H' \cup H_2 \cup H_3$ contains a red copy of S_t^+ , a contradiction. Thus, all edges from H_1 to $H_4 \cup \ldots \cup H_m$ must be red. Since $m \ge 4$, this means H_1 also contains no red edges so $|H_i| \le gr_{k-2}(K_3:S_t^+) - 1$ for $1 \le i \le 3$.

By Claim 2, there is at most one part with blue edges to H_2 and red edges to H_3 , say H_4 , and there is at most one part with red edges to H_2 and blue edges to H_3 , say H_5 . Also by Claim 2, there is at most one part with blue edges to $H_2 \cup H_3$, say H_6 . Note that $H_4 \cup H_5 \cup H_6 \neq \emptyset$ by the minimality of m. This covering all the possibilities, we get

$$|G| = \sum_{i=1}^{6} |H_i|$$

$$\leq 3[gr_{k-2}(K_3:S_t^+) - 1] + 3(t-2)$$

$$< n,$$

a contradiction.

Case 5. $r \ge 4$.

As observed previously, this implies that $4 \leq m \leq 5$. Within the subgraph of the reduced graph induced on the r parts of order at least t-1, there can be no monochromatic triangle. If r = 5, there is only one coloring of K_5 with no monochromatic triangle and if r = 4, there are two colorings of K_4 with no monochromatic triangle. In each of these colorings, every vertex has at least one incident edge in both colors, meaning that all of the r corresponding parts of order at least t-1 must have no red or blue edges. Then

$$\begin{array}{rcl}
G| &=& \sum_{i=1}^{m} |H_i| \\
&\leq & 5[gr_{k-2}(K_3:S_t^+) - 1] \\
&<& n.
\end{array}$$

a contradiction, completing the proof of this case and the proof of Theorem 6. $\hfill \Box$

4 Path with a triangle end

Let P_t denote the path of order t. Then for $t \ge 3$, let P_t^+ denote the graph consisting of the path P_t with the addition of an edge between one end and the vertex at distance 2 along the path from that end, forming a triangle. Note that $P_3^+ = K_3$ and $P_4^+ = S_4^+$.

Before beginning the study of the Gallai-Ramsey number, we first establish the 2-color Ramsey number for P_t^+ .

Theorem 7. For $t \ge 4$,

$$R(P_t^+, P_t^+) = 2t - 1.$$

Note that if t = 3, then $P_3^+ = K_3$ so $R(P_3^+, P_3^+) = 6$.

Proof. The lower bound follows from the graph constructed by taking two copies of K_{t-1} each colored entirely with red and adding all edges in blue in between. Each red component is too small to contain a monochromatic copy of P_t^+ and the blue subgraph is bipartite so cannot contain a copy of P_t^+ . The order of

this constructed graph is 2t - 2, meaning that the Ramsey number is at least 2t - 1.

If t = 4, then $P_4^+ = S_4^+$ so $R(P_4^+, P_4^+) = 7$ by Theorem 6, so suppose $t \ge 5$. For the upper bound in general, let G be a 2-coloring of K_{2t-1} , say using red and blue.

First suppose t is odd, so by Theorem 3, there is a monochromatic copy of C_t in G, say with C being a red copy of C_t . If we let $C = v_1 v_2 \cdots v_t v_1$, then to avoid creating a red copy of P_t^+ , all edges of the form $v_i v_{i+2}$ must be blue, where indices are taken modulo t. Since t is odd, these edges form a blue copy of C_t . Let v be an arbitrary vertex in $G \setminus C$. Without loss of generality, suppose vv_2 is red. Then to avoid creating a red copy of P_t^+ , the edges vv_1 and vv_3 must be blue. Then vv_1v_3 forms a blue triangle and, along with the rest of the blue cycle, this structure contains the desired blue copy of P_t^+ .

Next suppose t is even and additionally we first suppose $t \ge 10$. Then by Theorem 3, there is a monochromatic copy of C_{t+2} in G, say with C being a red copy of C_{t+2} . We again let $C = v_1 v_2 \cdots v_{t+2} v_1$ and note that, as in the previous case, every edge of the form $v_i v_{i+2}$ must be blue where indices are taken modulo t+2. Since t is even, these blue edges induce two copies of $C_{(t+2)/2}$.

If no vertex in $G \setminus V(C)$ has red edges to C, then it is easy to construct a blue copy of P_t^+ so let $v \in G \setminus V(C)$ with v having a red edge to C, say to v_2 . Then both edges vv_1 and vv_3 must be blue to avoid a red copy of P_t^+ . Let C_{even} (and C_{odd}) be the two blue cycles on the even (respectively odd) indexed vertices. Since these edges form a blue triangle with the edge v_1v_3 , this restricts the blue edges that can go between the two blue cycles. In fact, all edges from $\{v_{t+1}, v_5\}$ to C_{even} must be red. To avoid a red P_t^+ , the edges vv_5 and vv_{t+1} must also be blue so, as above, v_1 and v_3 must also have all red edges to C_{even} . In order to avoid a red copy of P_t^+ , these red edges imply that all edges between pairs of even indexed vertices must be blue, inducing a blue complete graph of order $\frac{t+2}{2}$. To avoid a blue copy of P_t^+ , all edges between C_{even} and C_{odd} must be red, in turn meaning that all edges between pairs of odd indexed vertices must be blue. Let A and B be the two blue cliques. Each vertex in $G \setminus V(C)$ can have red edges to only one of A or B, say A. This means that each such vertex must have all blue edges to the opposite clique B, which in turn means that it must have all red edges to A. Putting all of this together, we see that the red subgraph is a complete bipartite graph. With |G| = 2t - 1, one part must have order at least t and induce a blue complete graph, containing the desired copy of P_t^+ .

If t = 6, then |G| = 11. By Theorem 3, there is a monochromatic copy of C_6 , say C in red with vertices v_1, v_2, \ldots, v_6 in this order. As above, every edge of the form $v_i v_{i+2}$ must be blue where indices are taken modulo t. These blue edges induce two blue triangles. To avoid creating a blue copy of P_6^+ , all edges between these two triangles must be red. To avoid creating a red copy of P_6^+ , no vertex in $G \setminus C$ can have at least one red edge to both blue triangles, meaning that every vertex has all blue edges to at least one of the two triangles. Since there are 5 vertices in $G \setminus C$, at least three of them must have all blue edges to a single blue triangle, say vertices x, y, z have all blue edges to the blue triangle

 $v_2v_4v_6$. Then the graph induced on $\{x, y, z, v_2, v_4, v_6\}$ contains a blue copy of P_6^+ , for a contradiction.

If t = 8, then |G| = 15. By Theorem 3, there is a monochromatic copy of C_8 , say C in red with vertices v_1, v_2, \ldots, v_8 in this order. As above, every edge of the form $v_i v_{i+2}$ must be blue where indices are taken modulo t. These blue edges induce two blue copies of C_4 with vertices $v_1 v_3 v_5 v_7$ and $v_2 v_4 v_6 v_8$ respectively.

First, suppose there is a red edge within each of these copies of C_4 , say without loss of generality, that v_1v_5 and v_2v_6 are red. Then to avoid creating a red copy of P_8^+ , the edges v_1v_4, v_2v_7, v_3v_6 , and v_5v_8 must be blue. These edges together with the two copies of C_4 form a blue cube $P_2 \times P_2 \times P_2$. If any face of this cube contains a blue edge, say for example the edge v_1v_6 , then there would be a blue copy of P_8^+ with path vertices $v_1v_3v_6v_4v_2v_8v_7v_7$ and extra edge v_1v_6 . Thus, every face of this cube contains only red edges. Let v be an arbitrary vertex in $G \setminus C$. If v has a blue edge to any vertex of C, without loss of generality say v_1 , then to avoid creating a blue copy of P_8^+ , all edges from v to $\{v_3, v_4, v_7\}$ must be red. This makes a red copy of P_8^+ with path vertices $vv_3v_7v_8v_1v_2v_6v_5$ and extra edge v_3v_7 . Thus, every vertex in $G \setminus C$ must have only red edges to C but this again creates the same red P_8^+ , a contradiction.

Finally, we may assume there is no red edge within one of the copies of C_4 , say without loss of generality, that $A = \{v_1, v_3, v_5, v_7\}$ induces a blue copy of K_4 . Any blue edge from A to $B = \{v_2, v_4, v_6, v_8\}$ would create a blue copy of P_8^+ so all such edges must be red. This, in turn, means that B also induces a blue copy of K_4 . In order to avoid a red copy of P_8^+ , no vertex of $G \setminus C$ can have red edges to both A and B, so each vertex in $G \setminus C$ must have all blue edges to at least one of A or B. Since there are 7 vertices in $G \setminus C$, there must be at least 4 of them with all blue edges to the same set, say A. The blue graph induced on these vertices along with A easily contains a blue copy of P_8^+ , a contradiction completing the proof.

In fact, the same proof yields a slightly more general result.

Corollary 8. For $t \ge s \ge 4$,

$$R(P_s^+, P_t^+) = 2t - 1.$$

We now begin the discussion of the Gallai-Ramsey number for P_t^+ by stating the lower bound. Indeed, the same construction as used in the proof of Lemma 2 also contains no monochromatic copy of P_t^+ so this result is an immediate corollary.

Lemma 3. For $t \ge 4$ and $k \ge 1$,

$$gr_k(K_3: P_t^+) \ge \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Theorem 9. For $t \ge 4$ and $k \ge 1$,

$$gr_k(K_3:P_t^+) = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Proof. The case k = 1 is trivial. From Theorem 7, we have $R(P_t^+, P_t^+) = 2t - 1$, and hence the result is true for k = 2. We therefore suppose $k \ge 3$ and let G be a coloring of K_n where

$$n = n_k = \begin{cases} 2(t-1) \cdot 5^{\frac{k-2}{2}} + 1 & \text{if } k \text{ is even,} \\ (t-1) \cdot 5^{\frac{k-1}{2}} + 1 & \text{if } k \text{ is odd.} \end{cases}$$

Let T be a maximal set of vertices $T = T_1 \cup T_2 \cup \cdots \cup T_k$ where each subset T_i has all incident edges to $G \setminus T$ in color i and $|G \setminus T| \ge \lfloor \frac{t}{2} \rfloor$ constructed iteratively by adding at most $2 \lfloor \frac{t}{2} \rfloor$ vertices to T at a time, with at most $\lfloor \frac{t}{2} \rfloor$ vertices being added to each of two sets T_i at a time. We first claim that each $|T_i|$ is small.

Claim 3. For each *i* with $1 \le i \le k$, we have $|T_i| \le \lfloor \frac{t}{2} \rfloor - 1$ and furthermore, $T_i = \emptyset$ for some *i*.

Note that this implies that $|T| \leq (k-1) \left(\left\lfloor \frac{t}{2} \right\rfloor - 1 \right)$. The proof of Claim 3 is similar to the corresponding proof in [13].

Proof. By the iterative definition of T, we may assume that this is the first step in the iterative construction where the set T violates either of these assumptions. That is, assume that $|T_i| \leq 2 \lfloor \frac{t}{2} \rfloor - 1$ for all i and either

- at most two sets T_i and T_j have $|T_i|, |T_j| > \lfloor \frac{t}{2} \rfloor 1$, or
- no set is empty and at most one set T_i has $|T_i| > \lfloor \frac{t}{2} \rfloor 1$.

In either case, we certainly have $|T| \leq (k+1) \left| \frac{t}{2} \right|$.

We first show that $T_i=\emptyset$ for some i so suppose the latter item above. If $k\geq 4,$ then

$$|G \setminus T| \ge n - (k+1) \left\lfloor \frac{t}{2} \right\rfloor \ge \left(\frac{t-1}{2} - 1\right)k + 3\left(\frac{t-1}{2}\right).$$

By Theorem 4, there is a monochromatic even cycle of length at least t-1 contained within $G \setminus T$. Since $T_i \neq \emptyset$ for all *i*, this cycle can easily be extended to a monochromatic copy of P_t^+ , for a contradiction.

Next we show that $|T_i| \leq \lfloor \frac{t}{2} \rfloor - 1$ for all *i*. Thus, suppose there are at most two sets, T_i and T_j , with $\lfloor \frac{t}{2} \rfloor \leq |T_i| \leq t$ and possibly $\lfloor \frac{t}{2} \rfloor \leq |T_j| \leq t$ (noting that one of these sets, say T_j , may not be large). Any edge of color *i* (or possibly *j*) within $G \setminus T$ would produce a monochromatic copy of P_t^+ so

 $G \setminus T$ contains no edge of color *i* (or possibly *j*). Then by Lemma 1, we have $|G \setminus (T_i \cup T_j)| \le n_{k-1} - 1$ so

$$n = |T_i| + |G \setminus (T_i \cup T_j)|$$

$$\leq 2t + n_{k-1} - 1$$

$$< n_k,$$

a contradiction.

Let $G' = G \setminus T$. Since G' is a Gallai coloring, it follows from Theorem 1 that there is a Gallai partition of V(G'). Suppose that the two colors appearing in the Gallai partition are red and blue. Let m be the number of parts in this partition and choose such a partition where m is minimized. Let H_1, H_2, \ldots, H_m be the parts of this partition, say with $|H_1| \ge |H_2| \ge \cdots \ge |H_m|$. When the context is clear, we also abuse notation and let H_i denote the vertex of the reduced graph corresponding to the part H_i .

If $2 \le m \le 3$, then by the minimality of m, we may assume m = 2. Let H_1 and H_2 be the corresponding parts. Suppose all edges from H_1 to H_2 are red. If $|H_1| \ge \lceil t/2 \rceil$ and $|H_2| \ge \lceil t/2 \rceil$, then to avoid creating a red copy of P_t^+ , there is no red edge in each H_i with i = 1, 2 and the corresponding subset of T is also empty, say $T_1 = \emptyset$. This means that, by Lemma 1, we have

$$\begin{aligned} |G| &= |T| + |H_1| + |H_2| \\ &\leq |H_1 \cup T_2| + |H_2 \cup T_3 \cup T_4 \cup \dots \cup T_k| \\ &\leq 2(n_{k-1} - 1) \\ &< n, \end{aligned}$$

a contradiction. If $|H_1| \leq \lfloor t/2 \rfloor - 1$ and $|H_2| \leq \lfloor t/2 \rfloor - 1$, then

$$|G| = |T| + |H_1| + |H_2| \\ \leq \frac{k+1}{2}t \\ < n_k,$$

a contradiction. We may therefore assume that $|H_1| \ge \lceil t/2 \rceil$ and $|H_2| \le \lceil t/2 \rceil - 1$. Adding $|H_2|$ to T contradicts the maximality of T and completes the proof when $2 \le m \le 3$.

From now on, we assume $m \geq 4$. Let r be the number of parts of the Gallai partition with order at least $\lceil t/2 \rceil$ and call these parts "large" while other parts are called "small". Then $|H_r| \geq \lceil t/2 \rceil$ and $|H_{r+1}| \leq \lceil t/2 \rceil - 1$. To avoid a monochromatic copy of P_t^+ , there can be no monochromatic triangle among these r large parts, leading to the following immediate fact.

Fact 1. $r \leq 5$.

The remainder of the proof is broken into cases based on the value of r.

Case 1. r = 5.

In this case, t = 5 since otherwise any monochromatic triangle in the reduced graph restricted to H_1, H_2, \ldots, H_6 would yield a monochromatic copy of P_t^+ . To avoid the same construction, the reduced graph on the parts H_1, H_2, H_3, H_4, H_5 must be the unique 2-coloring of K_5 with no monochromatic triangle with two complementary monochromatic cycles with in red and blue respectively. To avoid creating a monochromatic copy of P_t^+ , for each *i* with $1 \le i \le 5$, the part H_i contains neither red edges nor blue edges. Then, by Lemma 1,

$$|G| = |T| + \sum_{i=1}^{5} |H_i| \le 5[n_{k-2} - 1] < n_k,$$

a contradiction.

Case 2. r = 4.

To avoid monochromatic triangle in K_4 , without loss of generality, the four largest parts must form one of two structures:

- Type 1: There is a red cycle $H_1H_3H_4H_2H_1$ and a blue 2-matching on the edges H_1H_4 , and H_2H_3 in the reduced graph, or
- Type 2: There is a red path $H_3H_2H_1H_4$ and a blue path $H_1H_3H_4H_2$ in the reduced graph.

If r = m = 4, then using Lemma 1,

$$|G| = |T| + \sum_{i=1}^{4} |H_i| \le 4[n_{k-2} - 1] < n_k,$$

a contradiction. So suppose m > r = 4. This proof focuses on the reduced graph. For Type 1, the subgraph of the reduced graph restricted to $\{H_1, H_2, H_3, H_4\}$ is not a subgraph of the unique 2-colored copy of K_5 containing no rainbow triangle. This means that the subgraph of the reduced graph restricted to $\{H_1, H_2, \ldots, H_5\}$ contains a monochromatic triangle, leading to a monochromatic copy of P_t^+ in G, a contradiction.

For Type 2, outside of $\{H_1, H_2, H_3, H_4\}$, there are small parts H_5, H_6, \ldots, H_m . We may therefore assume that for all H_i with $5 \le i \le m$, we have that the edges H_3H_i and H_4H_i are blue. To avoid a blue triangle, the edges H_1H_i and H_2H_i are red. By minimality of t, we have t = 5 since all parts H_i for $i \ge 5$ have the same color on edges to H_j for $j \le 4$. To avoid creating a monochromatic copy of P_t^+ , none of these large parts contains any red or blue edges. Then using Lemma 1,

$$|G| = |T| + \sum_{i=1}^{4} |H_i| \le 4[gr_{k-2}(K_3; P_t^+) - 1] + \lceil t/2 \rceil - 1 < n,$$

a contradiction.

Case 3. r = 3.

The triangle in the reduced graph cannot be monochromatic so without loss of generality, suppose H_1H_2 , H_1H_3 are red, and H_2H_3 is blue. To avoid a red or blue triangle, any remaining parts are partitioned into the following sets.

- Let A be the set of parts outside H_1, H_2, H_3 such that each has blue edges to H_1, H_3 and red edges to H_2 ,
- Let B be the set of parts outside H_1, H_2, H_3 such that each has red edges to H_2, H_3 and blue edges to H_1 , and
- Let C be the set of parts outside H_1, H_2, H_3 such that each has blue edges to H_1, H_2 and red edges to H_3 .

Note that $|G| = |T| + |A| + |B| + |C| + |H_1| + |H_2| + |H_3|$. If A contains a blue edge, then using a long blue path between H_2 and H_3 along with a triangle formed using the blue edge in A, there would be a blue copy of P_t^+ . This and similar easy arguments lead to the following fact.

Fact 2.

- A contains no red or blue edges,
- B contains no red edges,
- C contains no red or blue edges,
- H₁ contains no red edges,
- H_2 contains no red or blue edges, and
- H₃ contains no red or blue edges.

Furthermore, we have the following claim.

Claim 4. If $B \neq \emptyset$, then $A = \emptyset$ and $C = \emptyset$.

Proof. Assume, to the contrary, that $B \neq \emptyset$ and either $A \neq \emptyset$ or $C \neq \emptyset$, without loss of generality, say $A \neq \emptyset$. there is an red edge between A and B, then there is a red triangle among A, B, H_2 . This red triangle together with the red edges between H_1 and H_2 forms a red copy of P_t^+ , a contradiction. If there is a blue edge between A and B, then there is a blue triangle among A, B, H_1 . This blue triangle together a with blue path of the form H_1CH_2 and the edges between H_2 and H_3 forms a blue copy of P_t^+ , a contradiction.

From Claim 4, if $B \neq \emptyset$, then $A = \emptyset$ and $C = \emptyset$. We can then regard $H_1 \cup B$ and $H_2 \cup H_3$ as two parts of a Gallai partition of G and the edges between these parts are all red, which contradicts the minimality of t.

We may therefore assume that $B = \emptyset$. Then we have the following claim.

Claim 5. There is only one part in A, and there is only one part in C.

Proof. The edges between any pair of parts must be either red or blue but neither A nor C contain any red or blue edges by Fact 2. \Box

Then by Lemma 1,

$$|G| = |T| + |A| + |B| + |C| + |H_1| + |H_2| + |H_3|$$

$$\leq 3[n_{k-2} - 1] + 2(\lceil t/2 \rceil - 1)$$

$$< n_k,$$

a contradiction.

Case 4. r = 2.

Suppose all edges from H_1 to H_2 are red. To avoid creating a monochromatic copy of P_t^+ , there is no part outside H_1 and H_2 with red edges to all of $H_1 \cup H_2$. Therefore, any remaining parts are partitioned into the following sets.

- Let A be the set of parts outside H_1, H_2 with blue edges to H_2 and red edges to H_1 ,
- Let B be the set of parts outside H_1, H_2 with blue edges to $H_1 \cup H_2$,
- Let C be the set of parts outside H_1, H_2 with blue edges to H_1 and red edges to H_2 .

Note that $|G| = |A| + |B| + |C| + |H_1| + |H_2|$. Then we have the following fact. Furthermore, we have the following claims.

Claim 6. $|A| \leq \lfloor t/2 \rfloor - 1$ and $|C| \leq \lfloor t/2 \rfloor - 1$.

Proof. Assume, to the contrary, that $|A| \ge \lceil t/2 \rceil$. Then there are two parts in A, say A', A''. If the edges from A' to A'' are red, then there is a red triangle among A', A'', H_1 , together with the edges between H_1 and H_2 , there is a red P_t^+ , a contradiction. If the edges from A' to A'' are blue, then there is a blue triangle among A', A'', H_2 , there is a blue P_t^+ since $|A| \ge \lceil t/2 \rceil$, a contradiction. Similarly, there is only one part in C.

Claim 7. $|A| + |B| \le t - 1$ and $|B| + |C| \le t - 1$.

Proof. Assume, to the contrary, that $|A| + |B| \ge t$. Then there are at least three parts in $A \cup B$. Since both A and B have blue edges to H_2 , there can be no blue edges within $A \cup B$. This means there are only red edges appearing between the parts of the Gallai partition within $A \cup B$. With at least t verices, there is a red copy of P_t^+ within $A \cup B$ for a contradiction.

From Claims 6 and 7, we have $|A| \leq \lceil t/2 \rceil - 1$, $|C| \leq \lceil t/2 \rceil - 1$, $|A| + |B| \leq t - 1$, and $|B| + |C| \leq t - 1$. By Lemma 1, we have $|H_i| \leq gr_{k-1}(K_3; P_t^+) - 1$ with $i = 1, 2, \text{ and } |H_1| + |C| \leq gr_{k-1}(K_3; P_t^+) - 1$ and $|H_1| + |A| + |B| \leq gr_{k-1}(K_3; P_t^+) - 1$. Again using Lemma 1, we get

$$|G| = |T| + |A| + |B| + |C| + |H_1| + |H_2| \le 2(n_{k-1} - 1) < n_k,$$

a contradiction.

Case 5. r = 1.

Let A be the set of parts with blue edges to H_1 , and B be the set of parts with red edges to H_1 .

If $|A| \ge \lfloor \frac{t}{2} \rfloor$ and $|B| \ge \lfloor \frac{t}{2} \rfloor$, then it follows from Claim 7 that $|A| \le t + 2$ and $|B| \le t + 2$. Since A and B are each large, it follows that there are neither red edges nor blue edges in H_1 , and hence $|H_1| \le n_{k-2} - 1$. Then by Lemma 1,

$$|G| = |T| + |H_1| + |A| + |B| \le n_{k-2} - 1 + (2t+4) < n_k,$$

a contradiction.

If $|A| \ge \lfloor \frac{t}{2} \rfloor$ and $|B| \le \lfloor \frac{t}{2} \rfloor - 1$, then it follows from Claim 4 that $|A| \le t+2$. Since A is big, it follows that there are no red edges in H_1 , and hence $|H_1| \le n_{k-1} - 1$. By Lemma 1,

$$|G| = |T| + |H_1| + |A| + |B| \le n_{k-1} - 1 + (t+2) + \left(\left\lfloor \frac{t}{2} \right\rfloor - 1\right) < n_k,$$

a contradiction.

Finally if $|A|, |B| \leq \lfloor \frac{t}{2} \rfloor - 1$, both sets can be added to T, contradicting the maximality of T.

Case 6. r = 0.

In this case, we have $|H_i| \leq \lceil t/2 \rceil - 1$ for all i with $1 \leq i \leq m$. We need only consider the case k = 3 since the parts are too small to contain a monochromatic copy of P_t^+ in a color other than red or blue. Then n = 5t - 4 so $|G'| \geq 4t - 2$. Note that this means there are at least 9 parts in the Gallai partition of G', so in particular, there must be either a red or blue triangle in the reduced graph.

Suppose first that there is both a red triangle and blue triangle in G'. Select one such triangle in each color and remove their vertices. By deleting at most six vertices, we still have at least 4t-8 vertices in G'. Recall that by minimality of m, the edges of G restricted to either red or blue induce a connected subgraph. From Theorem 4, if t-2 is even, then $gr_k(K_3; C_{t-2}) \leq 3t-9$ and if t-3 is even, then $gr_k(K_3; C_{t-3}) \leq 3t-9$. This means G' contains an even cycle C_{t-2} or C_{t-3} . This cycle together with the deleted triangle form a red P_t^+ or blue P_t^+ (since each color induces a connected subgraph), a contradiction.

Thus, suppose that there is a red triangle but no blue triangles in G'. Choose an arbitrary set H_i and let G_R be the set of vertices with red edges to H_i and G_B be the set of vertices with blue edges to H_i . Note that G_B contains no blue edge so if $|G_B| \ge t$, then since all parts have order at most $\lfloor t/2 \rfloor - 1$, there are at least 3 parts in G_B and all red edges in between these parts, creating a red copy of P_t^+ . Thus, $|G_B| \le t - 1$.

Since H_i was chosen arbitrarily, this is true about every such part. Let D be the assumed red triangle. This means that every vertex in $G' \setminus D$, say $v \in H_i$, has red degree at least

$$n_k - |T| - 3 - |H_i| - (t - 1) \ge \frac{|G' \setminus D|}{2}$$

By Dirac's Theorem [2], there exists a red Hamiltonian cycle within $G' \setminus D$. This cycle along with the red triangle D (since the red subgraph is connected), produces a red copy of P_t^+ , a contradiction to complete the proof.

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