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Fractional cross intersecting families

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Abstract

Let $\mathcal{A} = \{A_1, ..., A_p\}$ and $\mathcal{B} = \{B_1, ..., B_q\}$ be two families of subsets of [n] such that for every $i \in [p]$ and $j \in [q]$, $|A_i \cap B_j| = \frac{c}{d}|B_j|$, where $\frac{c}{d} \in [0, 1]$ is an irreducible fraction. We call such families $\frac{c}{d}$ -cross intersecting families. In this paper, we find a tight upper bound for the product $|\mathcal{A}||\mathcal{B}|$ and characterize the cases when this bound is achieved for $\frac{c}{d} = \frac{1}{2}$. Also, we find a tight upper bound on $|\mathcal{A}||\mathcal{B}|$ when \mathcal{B} is k-uniform and characterize, for all $\frac{c}{d}$, the cases when this bound is achieved.

1 Introduction

Let [n] denote $\{1, ..., n\}$ and let $2^{[n]}$ denote the power set of [n]. We shall use $\binom{[n]}{k}$ to denote the set of all k-sized subsets of [n]. Let $\mathcal{F} \subseteq 2^{[n]}$. The family \mathcal{F} is an *intersecting family* if every two sets in \mathcal{F} intersect with each other. The famous Erdős-Ko-Rado Theorem [1] states that $|\mathcal{F}| \leq \binom{n-1}{k-1}$ if \mathcal{F} is a k-uniform intersecting family, where $2k \leq n$. Several variants of the notion of intersecting families have been extensively studied in the literature. Given a set $L = \{l_1, \ldots, l_s\}$ of nonnegative integers, a family $\mathcal{F} \subseteq 2^{[n]}$ is *L-intersecting* if for all $F_i, F_j \in \mathcal{F}, F_i \neq F_j, |F_i \cap F_j| \in L$. Ray-Chaudhuri and Wilson in [2] showed that if \mathcal{F} is k-uniform and *L*-intersecting, then $|\mathcal{F}| \leq \binom{n}{s}$ and the bound is tight. Frankl and Wilson in [3] showed a tight upper bound of $\binom{n}{s} + \binom{n}{s-1} + \cdots + \binom{n}{0}$ if the restriction on the cardinalities of the sets of an *L*-intersecting family is relaxed. Further, if *L* is a singleton set, then Fisher inequality [4] gives an upper bound of $|\mathcal{F}| \leq n$ for the cardinality of an *L*-intersecting family \mathcal{F} . Recently, in [5], Balachandran et al. introduced a fractional variant of the classical *L*-intersecting families. For a survey on intersecting families, see [6].

Two families $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are cross-intersecting if $|A \cap B| > 0, \forall A \in \mathcal{A}, B \in \mathcal{B}$. Pyber in [7] showed that if $n \geq 2k$, and $\mathcal{A}, \mathcal{B} \subseteq {\binom{[n]}{k}}$ is a cross-intersecting pair of families, then $|\mathcal{A}||\mathcal{B}| \leq {\binom{n-1}{k-1}}^2$. Frankl et al. in [8] showed that if $\mathcal{A}, \mathcal{B} \subset {\binom{[n]}{k}}$ such that $|A \cap B| \geq t$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$, then for all $n \geq (t+1)(k-t+1)$, $|\mathcal{A}||\mathcal{B}| \leq {\binom{n-t}{k-t}}^2$, the cross-intersecting version of the Erdős-Ko-Rado Theorem. A cross-intersecting pair of families $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ is said to be *l*-cross-intersecting if $\forall A \in \mathcal{A}, B \in \mathcal{B}, |A \cap B| = l$, for some positive integer *l*. Ahlswede, Cai and Zhang showed in [9], for all $n \geq 2l$, a simple construction of an *l*-cross-intersecting pair $(\mathcal{A}, \mathcal{B})$ of families of subsets of [n] with $|\mathcal{A}||\mathcal{B}| = {\binom{2l}{l}}2^{n-2l} = \Theta(\frac{2^n}{\sqrt{l}})$. Later Alon and Lubetzky in [10] showed that the $\Theta(\frac{2^n}{\sqrt{l}})$ bound is tight and characterized the cases when the bound is achieved.

In this paper, we introduce a fractional variant of the *l*-cross-intersecting families. Let $\mathcal{A} = \{A_1, ..., A_p\}$ and $\mathcal{B} = \{B_1, ..., B_q\}$ be two families of subsets of [n] such that for every $i \in [p]$ and $j \in [q]$, $|A_i \cap B_j| = \frac{c}{d}|B_j|$, where $\frac{c}{d} \in [0, 1]$ is an irreducible fraction. We call such an $(\mathcal{A}, \mathcal{B})$ pair a $\frac{c}{d}$ -cross-intersecting pair of families. Given c, d, and n, let $\mathcal{M}_{\frac{c}{d}}(n)$ denote the maximum value of $|\mathcal{A}||\mathcal{B}|$ where $(\mathcal{A}, \mathcal{B})$ is a $\frac{c}{d}$ -cross intersecting pair of families of subsets of [n]. We have the following results:

Theorem 1.1. $\mathcal{M}_{\frac{c}{d}}(n) = 2^n$

When $\frac{c}{d} = 0$, $\mathcal{A} = 2^{[n]}$, $\mathcal{B} = \{\emptyset\}$ is a maximal pair. In fact, $\mathcal{A} = 2^{[k]}$, $\mathcal{B} = \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the power set of $S = \{k + 1, \ldots, n\}$, are the only maximal pairs up to a relabelling of the elements, $0 \leq k \leq n$. When $\frac{c}{d} = 1$, $\mathcal{A} = \{[n]\}$ and $\mathcal{B} = 2^{[n]}$ is a maximal pair. In fact, $\mathcal{B} = 2^{[k]}$, $\mathcal{A} = \{A : A = [k] \cup T$, where $T \in \mathcal{P}(S)\}$, where $\mathcal{P}(S)$ is the power set of $S = \{k + 1, \ldots, n\}$, are the only maximal pairs up to a relabelling of the elements, $0 \leq k \leq n$. In Theorem 1.2, we characterize all maximal pairs when $\frac{c}{d} = \frac{1}{2}$.

Theorem 1.2. Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{1}{2}$ -cross intersecting pair of families of subsets of [n] with $|\mathcal{A}||\mathcal{B}| = 2^n$. Then $(\mathcal{A}, \mathcal{B})$ is one of the following $\lfloor \frac{n}{2} \rfloor + 1$ pairs of families $(\mathcal{A}_k, \mathcal{B}_k), 0 \le k \le \lfloor \frac{n}{2} \rfloor$, up to isomorphism.

$$\mathcal{A}_0 = 2^{[n]} and \mathcal{B}_0 = \{\emptyset\}$$

$$\mathcal{A}_k = \{A \in 2^{[n]} : |A \cap \{2i - 1, 2i\}| = 1 \quad \forall i, 1 \le i \le k\}$$

 $\mathcal{B}_{k} = \{ B \in 2^{[n]} : |B \cap \{2i - 1, 2i\}| \in \{0, 2\} \quad \forall i, 1 \le i \le k \text{ and } \forall j > 2k, \ j \notin B \},$ where $1 \le k \le \lfloor \frac{n}{2} \rfloor.$ It would be interesting to show a characterization theorem for any $\frac{c}{d} \in [0, 1]$. We do have such a general characterization theorem (along with a new tight upper bound) in Theorem 1.3 for the case when \mathcal{B} is k-uniform. The proof is a direct application of Theorem 1.1 in [10].

Theorem 1.3. Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$ -cross intersecting pair of families of subsets of [n]. Let \mathcal{B} be k-uniform. Then, there exists some $k_0 > 0$, such that for $k > k_0$ we have

$$|\mathcal{A}||\mathcal{B}| \le {\binom{2ck}{d}}{\frac{ck}{d}} 2^{n-\frac{2ck}{d}}$$

and the bound is tight if and only if, either (a) or (b) hold:

- (a) When $\frac{c}{d} = 1$, $\mathcal{A} = \{\{1, \dots, \kappa\}\} \times 2^{Y}$, $\mathcal{B} = {\binom{[\kappa]}{k}}$ where $Y = \{\kappa + 1, \dots, n\}$ and $\kappa \in \{2k 1, 2k\}$ up to a relabelling of the elements of [n].
- (b) When $\frac{c}{d} \neq 1$:
 - (i) If k is even, c = 1, d = 2, $\frac{ck}{d} = \lceil \frac{k}{2} \rceil$,
 - (ii) If k is odd, $c = \frac{k+1}{2}$, d = k, $\frac{ck}{d} = \lfloor \frac{k}{2} \rfloor$,

and for both the cases((i) and (ii)), there exists some τ such that, $k + \tau \leq n$ and up to a relabelling of the elements of [n],

$$\mathcal{A} = \{ \bigcup_{T \in J} T : J \subset \{\{1, k+1\}, \dots, \{\tau, k+\tau\}, \{\tau+1\}, \dots, \{k\}\}, |J| = \lceil \frac{k}{2} \rceil \} \times 2^X$$

where $X = \{k + \tau + 1, ..., n\}$ and

$$\mathcal{B} = \{L \cup \{\tau + 1, \dots, k\} : L \subset \{1, \dots, \tau, k + 1, \dots, k + \tau\}, |L \cap \{i, k + i\}| = 1 \text{ for all } i \in [\tau]\}.$$

2 Notations and definitions

Given any $S \subseteq [n]$, we shall use $\chi(S)$ to denote the *characteristic vector* of S which is a 0-1 vector of size n having its i^{th} entry equal to 1 if and only if $i \in S$. The *weight* of a vector is the number of non-zero entries it has, and hence weight of $\chi(S)$ is the same as |S|.

For any family $\mathcal{A} \subseteq 2^{[n]}$, we shall (ab)use \mathcal{A} to denote the collection of characteristic vectors of the members of \mathcal{A} as well. The meaning will be clearly stated if not clear from the context.

Let V be a collection of vectors in \mathbb{F}_2^n . Then, we define the following:

- 1. span(V): The collection of all the vectors that can be expressed as a linear combination in \mathbb{F}_2 of the vectors of V. We know that span(V) is a vector space over \mathbb{F}_2 .
- 2. basis(V): We use basis(V) to denote the basis of span(V).
- 3. dim(V): dim(V) = |basis(V)|

Definition 1. $V \subseteq \mathbb{F}_2^n$ is a linear code if V = span(V).

Definition 2. Given a linear code $C \subseteq \mathbb{F}_2^n$, the dual code C^{\perp} is defined as,

$$C^{\perp} = \{ x \in \mathbb{F}_2^n | \langle x, c \rangle = 0, \forall c \in C \}$$

where $\langle x, y \rangle$ is the standard inner product over \mathbb{F}_2 .

The following is a well-known fact that is easy to verify.

Lemma 2.1. If $C \subseteq \mathbb{F}_2^n$ is a linear code, then C^{\perp} is also a linear code.

Definition 3. Self orthogonal and self dual codes: A code C is self orthogonal if $C \subseteq C^{\perp}$ and it is self dual if $C = C^{\perp}$.

3 Bounding $\mathcal{M}_{\frac{c}{d}}(n)$

Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$ -cross-intersecting pair of families of subsets of [n], where $\frac{c}{d} \in [0, 1]$ is an irreducible fraction. We shall (ab)use \mathcal{A}, \mathcal{B} to denote the set of characteristic vectors of the sets in \mathcal{A}, \mathcal{B} respectively. For any $a \in \mathcal{A}, b \in \mathcal{B}$, we observe that $\langle a, b \rangle \equiv |A \cap B| \pmod{2}$, where $a = \chi(A), b = \chi(B)$.

Partition the family \mathcal{B} into two parts as,

$$\mathcal{B}_1 = \{ B \in \mathcal{B} : |B| \equiv 0 \pmod{2d} \}$$
(1)

$$\mathcal{B}_2 = \{ B \in \mathcal{B} : |B| \equiv d \pmod{2d} \}$$
(2)

As all the sets $B \in \mathcal{B}$ have their cardinality |B| divisible by d, $\{\mathcal{B}_1, \mathcal{B}_2\}$ is a valid partition of \mathcal{B} . Therefore $\forall a \in \mathcal{A}$, $b \in \mathcal{B}$, using the $\frac{c}{d}$ intersection property, we have:

$$\langle a, b \rangle = \begin{cases} 1, if \ b \in \mathcal{B}_2 \ and \ c \ is \ odd \\ 0, \ otherwise \end{cases}$$

Construction 1. Construct a set \mathcal{B}'_1 , by appending a 0 to the left of every vector in \mathcal{B}_1 , and a set \mathcal{B}'_2 by appending a 1 to the left of every vector in \mathcal{B}_2 . Let $\mathcal{B}' = \mathcal{B}'_1 \cup \mathcal{B}'_2$. Construct a set \mathcal{A}' by appending a 1 to the left of every vector in \mathcal{A} .

We now have, the value of

$$\langle a,b
angle=0 \;\; orall a\in \mathcal{A}^{'}, \, b\in \mathcal{B}^{'}$$

So, $(span(\mathcal{A}'), span(\mathcal{B}'))$ is a pair of mutually orthogonal subspaces of \mathbb{F}_2^{n+1} over \mathbb{F}_2 . We thus have,

$$\dim(span(\mathcal{A}')) + \dim(span(\mathcal{B}')) \le n+1$$

So, it follows that

$$|\operatorname{span}(\mathcal{A}')| \cdot |\operatorname{span}(\mathcal{B}')| = 2^{\dim(\operatorname{span}(\mathcal{A}'))} \cdot 2^{\dim((\operatorname{span}(\mathcal{B}')))}$$
$$= 2^{\dim(\operatorname{span}(\mathcal{A}')) + \dim(\operatorname{span}(\mathcal{B}'))}$$
$$< 2^{n+1}$$
(3)

Lemma 3.1. If the elements of a linear code $C \subseteq \mathbb{F}_2^n$ are arranged as rows of a matrix M_C with n columns, then for each column, one of the following holds,

- (i) All the entries in that column are 0
- (ii) Exactly half the entries in that column are 0, and the rest are 1.

Proof. As C is a linear code, if we pick any $a \in C$, and consider the set $S = \{a + x | x \in C\}$ where a + x is the vector addition in \mathbb{F}_2^n , then by the definition of a linear code S = C. Let M_S be a matrix whose rows are the vectors of S, taken in any order. M_S and M_C have the same set of rows (only their order may differ).

Let $j \in [n]$. Column j in M_C and M_S have the same number of 1's(and 0's). Suppose (i) does not hold for column j in M_C . Then, some row, say a, in M_C has its j^{th} entry as 1. Let S, and thereby M_S , be defined according to this vector a. From the definition of S, it is clear that the number of 1's in the j^{th} column of M_S is equal to the number of 1's in the j^{th} column of M_C . Since adding a to any $\{0,1\}$ vector flips the j^{th} coordinate of v, we conclude that (ii) holds for M_c . \Box

Corollary 3.2. $|span(\mathcal{A}')| \geq 2|\mathcal{A}'|$

Proof. The leftmost column of $\mathcal{M}_{\mathcal{A}'}$ does not contain any 0. As $span(\mathcal{A}')$ is a linear code and $\mathcal{A}' \subseteq span(\mathcal{A}')$, by condition (ii) of Lemma 3.1 above, $span(\mathcal{A}')$ must have at least $|\mathcal{A}'|$ more elements having their leftmost entry as 0.

Now we prove the main result of this section which is Theorem 1.1.

Statement of Theorem 1.1: $\mathcal{M}_{\frac{c}{d}}(n) = 2^n$

Proof. $\mathcal{A} = 2^{[n]}$, $\mathcal{B} = \{\emptyset\}$ is a trivial example of a $\frac{c}{d}$ cross-intersecting pair of families having $|\mathcal{A}||\mathcal{B}| = 2^n$. Thus, $\mathcal{M}_{\frac{c}{d}}(n) \geq 2^n$. The proof of the upper bound for $\mathcal{M}_{\frac{c}{d}}(n)$ follows from Inequality (3) and Corollary 3.2. Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$ cross-intersecting pair of families of subsets of [n]. Let $\mathcal{A}', \mathcal{B}'$ be constructed from \mathcal{A} , \mathcal{B} , respectively, as explained in the beginning of this section. Note that $|\mathcal{A}'| = |\mathcal{A}|$ and $|\mathcal{B}'| = |\mathcal{B}|$ by construction.

 $2^{n+1} \ge |\operatorname{span}(\mathcal{A}')| \cdot |\operatorname{span}(\mathcal{B}')| \qquad [\text{from (3)}]$ $\ge 2 \cdot |\mathcal{A}'| \cdot |\operatorname{span}(\mathcal{B}')| \qquad [\text{from Corollary 3.2}]$ $\ge 2 \cdot |\mathcal{A}'| \cdot |\mathcal{B}'|$ $= 2 \cdot |\mathcal{A}| \cdot |\mathcal{B}| \qquad [\text{by construction}]$

4 Characterization of maximal pairs when $\frac{c}{d} = \frac{1}{2}$

Definition 4. Cross bisecting pair of families: A pair of families of subsets of [n] is called a cross-bisecting pair if it is a $\frac{1}{2}$ cross-intersecting pair. $(\mathcal{A}, \mathcal{B})$ is called a maximal cross bisecting or simply a maximal pair, if it is a cross bisecting pair and $|\mathcal{A}||\mathcal{B}| = 2^n$.

For example, $\mathcal{A} = 2^{[n]}$ and $\mathcal{B} = \{\emptyset\}$ is a trivial maximal pair. In this section, we characterize all maximal pairs. Let $(\mathcal{A}, \mathcal{B})$ be a cross bisecting pair and let $(\mathcal{A}', \mathcal{B}')$ be the associated pair constructed by appending bits as defined in the previous section.

Definition 5. Let $f_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}'$ be a bijective mapping that maps every vector in \mathcal{A} to its corresponding vector in \mathcal{A}' , and let $g_{\mathcal{A}} : \mathcal{A}' \to \mathcal{A}$ be its inverse. Likewise, define functions $f_{\mathcal{B}}$ and $g_{\mathcal{B}}$ between \mathcal{B} and \mathcal{B}' . For any set $V \subseteq \mathcal{A}$, we shall use, $f_{\mathcal{A}}(V)$ to denote $\{f_{\mathcal{A}}(A) | A \in V\}$ and for any $V \subseteq \mathcal{A}'$, we use $g_{\mathcal{A}}(V)$ to denote $\{g_{\mathcal{A}}(A) | A \in V\}$. Similarly, for any $V \subseteq \mathcal{B}$, we use, $f_{\mathcal{B}}(V)$ to denote $\{f_{\mathcal{B}}(B) | B \in V\}$ and for any $V \subseteq \mathcal{B}'$, $g_{\mathcal{B}}(V)$ to denote $\{g_{\mathcal{B}}(B) | B \in V\}$

Observation 1. $f_{\mathcal{B}}(\mathcal{B}_1) = \mathcal{B}'_1$ and $f_{\mathcal{B}}(\mathcal{B}_2) = \mathcal{B}'_2$. Similarly, $g_{\mathcal{B}}(\mathcal{B}'_1) = \mathcal{B}_1$ and $g_{\mathcal{B}}(\mathcal{B}'_2) = \mathcal{B}_2$

Suppose $(\mathcal{A}, \mathcal{B})$ is a maximal pair. Then from the proof of Theorem 1.1, we must have :

$$|\operatorname{span}(\mathcal{A}')| = 2|\mathcal{A}'| \tag{4}$$

$$|\operatorname{span}(\mathcal{B}')| = |\mathcal{B}'| \tag{5}$$

$$\dim(span(\mathcal{A}')) + \dim(span(\mathcal{B}')) = n+1$$
(6)

Proposition 4.1. $\mathcal{B} = span(\mathcal{B})$. Further, $f_{\mathcal{B}}$ is a linear map.

Proof. This follows from equation (5). Let $x_1, x_2 \in \mathcal{B}$. We show that $x_3 = x_1 + x_2 \in \mathcal{B}$. This would imply \mathcal{B} is closed under addition in \mathbb{F}_2^n over \mathbb{F}_2 , and hence $\mathcal{B} = \operatorname{span}(\mathcal{B})$.

Let $x'_1 = f_{\mathcal{B}}(x_1)$ and $x'_2 = f_{\mathcal{B}}(x_2)$. From Equation (5), we have, $w = x'_1 + x'_2 \in \mathcal{B}'$. Since w and x_3 agree on each of the rightmost n bits of x_3 , we have $g_{\mathcal{B}}(w) = x_3$. Since $w \in \mathcal{B}'$, from the definition of the function $g_{\mathcal{B}}$ we have $x_3 = g_{\mathcal{B}}(w) \in \mathcal{B}$. Further, observe that $f_{\mathcal{B}}(x_1) + f_{\mathcal{B}}(x_2) = w = f_{\mathcal{B}}(x_3) = f_{\mathcal{B}}(x_1 + x_2)$ and hence $f_{\mathcal{B}}$ is a linear map.

That \mathcal{B} is a linear code from Proposition 4.1 implies closure of the family of subsets \mathcal{B} under symmetric difference. In fact, we have the following stronger result.

Proposition 4.2. Let vectors $b_1, b_2 \in \mathcal{B}$. Then, $b_1 + b_2 \in \mathcal{B}_1$ if and only if either $b_1, b_2 \in \mathcal{B}_1$, or $b_1, b_2 \in \mathcal{B}_2$. Otherwise, $b_1 + b_2 \in \mathcal{B}_2$.

Proof. We prove the 2-way implication, and rest of the proposition follows from Proposition 4.1. Let $b'_1 = f_{\mathcal{B}}(b_1), b'_2 = f_{\mathcal{B}}(b_2)$.

- $b_1 + b_2 \in \mathcal{B}_1 \Rightarrow b_1$ and b_2 are both from \mathcal{B}_1 , or both from \mathcal{B}_2 Since $f_{\mathcal{B}}$ is a linear map, we have $(b_1 + b_2 \in \mathcal{B}_1) \Rightarrow (f_{\mathcal{B}}(b_1 + b_2) = f_{\mathcal{B}}(b_1) + f_{\mathcal{B}}(b_2) = b'_1 + b'_2 \in \mathcal{B}'_1)$. So, the leftmost bit of $b'_1 + b'_2$ is 0. This means that the leftmost bit must be the same in b'_1 and b'_2 , which directly implies that either $b'_1, b'_2 \in \mathcal{B}'_1$, or $b'_1, b'_2 \in \mathcal{B}'_2$.
- Either $b_1, b_2 \in \mathcal{B}_1$, or $b_1, b_2 \in \mathcal{B}_2 \Rightarrow b_1 + b_2 \in \mathcal{B}_1$ Since b'_1 and b'_2 agree upon the leftmost bit, $b'_1 + b'_2$ has a 0 in its leftmost bit. So, $b'_1 + b'_2 \in \mathcal{B}'_1$. From the Observation 1 above, we have $b_1 + b_2 \in \mathcal{B}_1$.

Proposition 4.3. \mathcal{B} is a self-orthogonal code.

Proof. We prove the proposition by showing that $\forall b_1, b_2 \in \mathcal{B}, \langle b_1, b_2 \rangle = 0$. Let B_1, B_2 be the sets corresponding to the vectors b_1, b_2 , respectively. Since we are operating in the field \mathbb{F}_2 , it is enough to show that $|B_1 \cap B_2|$ is even.

Let $b_3 = b_1 + b_2$. We observe that b_3 is the characteristic vector of $B_3 = B_1 \Delta B_2$, the symmetric difference of B_1 and B_2 . We have,

$$|B_3| = |B_1 \Delta B_2| = |B_1| + |B_2| - 2|B_1 \cap B_2| \tag{7}$$

As $\frac{c}{d} = \frac{1}{2}$, $\forall B \in \mathcal{B}_1$, we have $|B| \equiv 0 \pmod{4}$. By Proposition 4.1, $B_1 \Delta B_2 = B_3 \in \mathcal{B}$ as \mathcal{B} is a linear code. Taking equation (7) modulo 4, if $B_3 \in \mathcal{B}_1$, then

$$|B_1| + |B_2| - 2|B_1 \cap B_2| \equiv 0 \pmod{4}$$

By Proposition 4.2, both B_1 and B_2 are either from \mathcal{B}_1 or from \mathcal{B}_2 . In both cases, $|B_1|+|B_2| \equiv 0 \pmod{4}$ Therefore, $2|B_1 \cap B_2| \equiv 0 \pmod{4}$ or $|B_1 \cap B_2| \equiv 0 \pmod{2}$. If $B_3 \in \mathcal{B}_2$, then

$$|B_1| + |B_2| - 2|B_1 \cap B_2| \equiv |B_3| \equiv 2 \pmod{4}$$

Again by Proposition 4.2, $|B_1| + |B_2| \equiv 2 \pmod{4}$.

So, we have $2|B_1 \cap B_2| \equiv 0 \pmod{4}$ or $|B_1 \cap B_2| \equiv 0 \pmod{2}$. Thus in both cases, $|B_1 \cap B_2|$ is even, so \mathcal{B} is a self-othogonal code.

Lemma 4.4. Let $(\mathcal{A}, \mathcal{B})$ be a maximal pair, then $|\mathcal{B}| \leq 2^{\lfloor \frac{n}{2} \rfloor}$

Proof. It is a known result (see [11]) that for a linear code $C \subseteq \mathbb{F}_2^n$ and its dual code C^{\perp} ,

$$\dim(C) + \dim(C^{\perp}) = n \tag{8}$$

For any self-orthogonal code $C, C \subseteq C^{\perp}$. So,

$$\dim(C) \le \dim(C^{\perp})$$

Applying equation (8) in this inequality, we get

$$n = \dim(C) + \dim(C^{\perp}) \ge 2\dim(C)$$

Therefore, $\dim(C) \le \frac{n}{2}$

Since \mathcal{B} is a self-orthogonal code (Proposition 4.3), we get dim $(\mathcal{B}) \leq \frac{n}{2}$. Hence,

$$|\mathcal{B}| \le 2^{\lfloor \frac{n}{2} \rfloor}$$

Proposition 4.5. If a set A bisects B_1 , B_2 and $B_1 \Delta B_2$, then A also bisects $B_1 \cap B_2$. *Proof.*

$$\begin{split} |A \cap (B_1 \wedge B_2)| &= \frac{|B_1 \wedge B_2|}{2} \text{ [A bisects } B_1 \wedge B_2] \\ \Rightarrow |A \cap ((B_1 \setminus B_2) \cup (B_2 \setminus B_1))| &= \frac{|B_1| + |B_2| - 2|B_1 \cap B_2|}{2} \\ \Rightarrow |A \cap (B_1 \setminus B_2)| + |A \cap (B_2 \setminus B_1)| &= \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2| \\ \Rightarrow |A \cap B_1| - |A \cap (B_1 \cap B_2)| + |A \cap (B_2)| - |A \cap (B_1 \cap B_2)| = \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2| \\ &= \frac{|B_1|}{2} + \frac{|B_2|}{2} - 2|A \cap (B_1 \cap B_2)| = \frac{|B_1|}{2} + \frac{|B_2|}{2} - |B_1 \cap B_2| \\ &= \text{[since } A \text{ bisects both } B_1 \text{ and } B_2] \\ &\Rightarrow 2|A \cap (B_1 \cap B_2)| = |B_1 \cap B_2| \\ &\Rightarrow |A \cap (B_1 \cap B_2)| = \frac{|B_1 \cap B_2|}{2} \end{split}$$

Proposition 4.6. \mathcal{B} is closed under intersection.

Proof. Let $B_1, B_2 \in \mathcal{B}$. We show that $B_1 \cap B_2 \in \mathcal{B}$. By Proposition 4.1, $b_1 + b_2 \in \mathcal{B}$ i.e., $B_1 \Delta B_2 \in \mathcal{B}$. Let A be any arbitrary member of \mathcal{A} . Now, A bisects B_1, B_2 and $B_1 \Delta B_2$ as $(\mathcal{A}, \mathcal{B})$ is a cross bisecting pair. By Proposition 4.5, A bisects $B_1 \cap B_2$. Since $(\mathcal{A}, \mathcal{B})$ is a maximal pair, we conclude that $B_1 \cap B_2 \in \mathcal{B}$.

Now, we prove the main result of this section, Theorem 1.2, the characterization of maximal pairs.

Statement of Theorem 1.2: Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{1}{2}$ -cross intersecting pair of families of subsets of [n] with $|\mathcal{A}||\mathcal{B}| = 2^n$. Then $(\mathcal{A}, \mathcal{B})$ is one of the following $\lfloor \frac{n}{2} \rfloor + 1$ pairs of families $(\mathcal{A}_k, \mathcal{B}_k), 0 \leq k \leq \lfloor \frac{n}{2} \rfloor$, up to isomorphism.

$$\mathcal{A}_0 = 2^{[n]} and \mathcal{B}_0 = \{\emptyset\}$$

$$\mathcal{A}_{k} = \{A \in 2^{[n]} : |A \cap \{2i - 1, 2i\}| = 1 \quad \forall i, 1 \le i \le k\}$$

 $\mathcal{B}_{k} = \{ B \in 2^{[n]} : |B \cap \{2i - 1, 2i\}| \in \{0, 2\} \ \forall i, 1 \le i \le k \text{ and } \forall j > 2k, j \notin B \},\$

where $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

By isomorphism, it is meant that for any maximal pair $(\mathcal{A}, \mathcal{B})$, \exists a bijective mapping $f : [n] \to [n]$ such that if every $A \in \mathcal{A}$ is replaced by $A_f = \{f(i) | i \in A\}$ and every $B \in \mathcal{B}$ is replaced by $B_f = \{f(i) | i \in B\}$ then the families $(\mathcal{A}_f, \mathcal{B}_f)$, where $\mathcal{A}_f = \{A_f | A \in \mathcal{A}\}$ and $\mathcal{B}_f = \{B_f | B \in \mathcal{B}\}$, is a maximal pair which is one of $(\mathcal{A}_k, \mathcal{B}_k)$, $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

Proof. Consider a maximal pair $(\mathcal{A}, \mathcal{B})$ where $\mathcal{B} \neq \{\emptyset\}$. We write the elements of \mathcal{B} as rows of a 0 - 1 matrix M_0 . Suppose n_0 columns have only 0 entries in all the rows $(n_0 \text{ may be } 0)$. As the characterization is up to isomorphism, we may assume that these are the rightmost n_0 columns of the matrix M_0 . In each of the remaining $n - n_0$ columns, from Lemma 3.1, there are exactly $\frac{|\mathcal{B}|}{2}$ 1's and $\frac{|\mathcal{B}|}{2}$ 0's as \mathcal{B} is a linear code. (by Proposition 4.1) Define

$$B_1 = \bigcap_{\substack{1 \in B, \\ B \in \mathcal{B}}} B$$

We write the $\frac{|\mathcal{B}|}{2}$ rows containing 1 in the leftmost column of M_0 as the top $\frac{|\mathcal{B}|}{2}$ rows to obtain a new matrix M_1 from M_0 . And B_1 is one of these rows according to Proposition 4.6. Moreover, as all intersections are of even cardinality (Proposition 4.3), $|B_1|$ is even.

Let $|B_1| = 2i_1, i_1 \ge 1$. So, there are $2i_1 - 1$ elements in B_1 other than the element 1. Due to isomorphism, we may assume them to be $2, 3, \ldots, 2i_1$. If $2i_1 + 1 \le n - n_0$, then define the set B_2 as:

$$B_2 = \bigcap_{\substack{2i_1 + 1 \in B, \\ B \in \mathcal{B}}} B$$

Claim 4.7. $1 \notin B_2$

Proof. Assume for the sake of contradiction, $1 \in B_2$. This implies that for all the $\frac{|\mathcal{B}|}{2}$ sets which contain the element $2i_1 + 1$ also contain the element 1. From Lemma 3.1, (number of sets in \mathcal{B} that contain the element 1) = (number of sets in \mathcal{B} that contain the element $2i_1 + 1$) = $\frac{|\mathcal{B}|}{2}$. Hence, for any $B \in \mathcal{B}$, $1 \in B \iff 2i_1 + 1 \in B$. This implies that $2i_1 + 1 \in B_1$, which is a contradiction. Hence, $1 \notin B_2$ and therefore B_2 does not belong to the top $\frac{|\mathcal{B}|}{2}$ rows of M_1 .

Claim 4.8. $B_1 \cap B_2 = \emptyset$

Proof. Assume for the sake of contradiction, $x \in B_1 \cap B_2$. Then x is present in the $\frac{|\mathcal{B}|}{2}$ rows of the matrix M_1 whose intersection yields B_1 . Since $x \in B_2$ and B_2 does not belong to these $\frac{|\mathcal{B}|}{2}$ rows of M_1 (by Claim 4.7). Thus, we have the element x present in at least $\frac{|\mathcal{B}|}{2} + 1$ rows of M_1 , contradicting Lemma 3.1.

We take the rows corresponding to the sets containing the $(2i_1 + 1)^{th}$ element that are not among the first $\frac{|\mathcal{B}|}{2}$ rows in M_1 and arrange them below the top $\frac{|\mathcal{B}|}{2}$ rows to create a matrix called M_2 from M_1 . Again from Proposition 4.3, $|B_2|$ is even, say $2i_2$. Due to isomorphism and Claim 4.8, we may assume that $2i_1 + 1, \ldots, 2i_1 + 2i_2$ are these $2i_2$ elements.

If $2i_1 + 2i_2 + 1 \le n - n_0$, then define,

$$B_3 = \bigcap_{\substack{2i_1 + 2i_2 + 1 \in B, \\ B \in \mathcal{B}}} B$$

Claim 4.9. $1 \notin B_3$ and $2i_1 + 1 \notin B_3$.

The proof is similar to that of Claim 4.7

Claim 4.10. $B_1 \cap B_3 = \emptyset$ and $B_2 \cap B_3 = \emptyset$.

The proof is again similar to that of Claim 4.8.

We take the rows corresponding to the sets containing the $(2i_1 + 2i_2 + 1)^{th}$ element that are not among the first r rows $(r > \frac{|\mathcal{B}|}{2})$ in M_2 which contain the elements 1 or $2i_1 + 1$ and arrange them below the top r rows of M_2 to create a matrix called M_3 from M_2 . From Proposition 4.3 and the definition of B_3 , we have $|B_3| = 2i_3, i_3 \ge 1$. Due to isomorphism and Claim 4.10, we may assume that $2i_1 + 2i_2 + 1, \ldots, 2i_1 + 2i_2 + 2i_3$ are these $2i_3$ elements.

We continue in this manner for k steps by constructing sets B_1, \ldots, B_k and matrices M_1, \ldots, M_k , where $k \ge 1$, until we have $2i_1 + \cdots + 2i_k = n - n_0$. Observe that B_1, \ldots, B_k and $P = \{n - n_0 + 1, \ldots, n\}$ is a partition of [n].

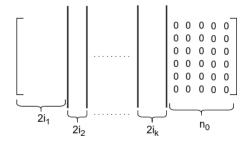


Figure 1: Partitioning the universe and thereby the columns of M_k

Claim 4.11. For any set $B \in \mathcal{B}$, $j \in [k]$, we have $B \cap B_j \in \{\emptyset, B_j\}$. Further, $B \cap P = \emptyset$.

Proof. From the definition of P, we have $B \cap P = \emptyset$. Let $j \in [k]$. Since B_j is equal to the intersection of some $\frac{|\mathcal{B}|}{2}$ sets in \mathcal{B} , we have B_j present as a subset of all these $\frac{|\mathcal{B}|}{2}$ sets. Applying Lemma 3.1, we can say that no element of B_j is present in any set in \mathcal{B} other than these $\frac{|\mathcal{B}|}{2}$ sets. Hence, the claim.

From Claim 4.11, observe that $S = \{B_1, \ldots, B_k\}$ forms a basis of the row space of the matrix M_k . The advantage of such a "disjoint basis" is that the bisection in one part is independent of another.

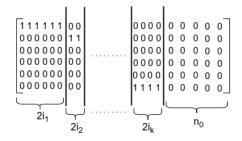


Figure 2: Basis for the code \mathcal{B}

Claim 4.12. A set $A \in \mathcal{A}$ bisects every set in \mathcal{B} if and only if it bisects every set in the basis S of \mathcal{B} .

Proof. The forward direction is straightforward as $S \subseteq \mathcal{B}$. For the opposite direction, let $A \in \mathcal{A}$ be a set that bisects every member of S. Since the sets corresponding to the members in S are disjoint, any $B \in \mathcal{B}$ can be written as a union of some of these sets.

Let $B = B_1 \cup \cdots \cup B_l$, where $\{B_1, \ldots, B_l\} \subseteq S$. Then,

$$|A \cap B| = |A \cap (\bigcup_{j=1}^{l} B_j)| = \sum_{j=1}^{l} |A \cap B_j| = \sum_{j=1}^{l} \frac{|B_j|}{2} = \frac{|\bigcup_{j=1}^{l} B_j|}{2} = \frac{|B|}{2}$$

Since each set $A \in \mathcal{A}$ bisects the sets B_1, \ldots, B_k and P, from Claim 4.12, the set A may contain any of the 2^{n_0} subsets of P, and $|A \cap B_1| = i_1, \ldots, |A \cap B_k| = i_k$. Since dim $(\mathcal{B}) = k$, by Proposition 4.1, we have $|\mathcal{B}| = 2^k$.

$$|\mathcal{A}||\mathcal{B}| = \left(2^{n_0} \cdot \prod_{j=1}^k \binom{2i_j}{i_j}\right) \cdot 2^k \tag{9}$$

Recall that $\sum_{j=1}^{k} 2i_j = n - n_0$. Right hand side of Equation (9), is equal to 2^n if and only if $i_j = 1$, $\forall j \in [k]$.

Thus, if $\mathcal{B} \neq \{\emptyset\}$, then $(\mathcal{A}_k, \mathcal{B}_k), k \geq 1$, defined in the statement of the theorem are the only maximal pairs. This completes the proof of Theorem 1.2.

5 Tight upper bound on $M_{\frac{c}{d}}(n)$ when \mathcal{B} is k-uniform and characterization of the cases when the bound is achieved

Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$ cross-intersecting pair of families of subsets of [n], where $\frac{c}{d} \in [0, 1]$ is an irreducible fraction. In this section, we deal with the scenario when \mathcal{B} is kuniform, where $0 < k \leq n$. Since \mathcal{B} is k-uniform, for any $A \in \mathcal{A}$ and any $B \in \mathcal{B}$, $|A \cap B| = \frac{ck}{d} = l$. Since c is relatively prime with d, and $|A \cap B|$ is an integer, we have k divisible by d. Therefore, we have a uniformly cross intersecting pair of families.

Alon and Lubetzky in [10] found a tight upper bound for the case of uniformly cross intersecting families and fully characterized the cases when the bound is achieved in the following theorem:

Theorem 5.1. [Theorem 1.1 in [10]] There exists some $l_0 > 0$ such that, for all $l \ge l_0$, every *l*-cross intersecting pair $\mathcal{A}, \mathcal{B} \subset 2^{[n]}$ satisfies:

$$|\mathcal{A}||\mathcal{B}| \le \binom{2l}{l} 2^{n-2l}$$

Furthermore, if $|\mathcal{A}||\mathcal{B}| = \binom{2l}{l}2^{n-2l}$, then there exists some choice of parameters κ, τ, n' :

$$\kappa \in \{2l-1, 2l\}, \tau \in \{0, \cdots, \kappa\}$$
$$\kappa + \tau \le n' \le n$$

such that upto a relabelling of the elements of [n] and swapping \mathcal{A}, \mathcal{B} , the following holds:

$$\mathcal{A} = \{\bigcup_{T \in J} T : J \subset \{\{1, \kappa + 1\}, \cdots, \{\tau, \kappa + \tau\}, \{\tau + 1\}, \cdots, \{\kappa\}\}, |J| = l\} \times 2^X, \\ \mathcal{B} = \{L \cup \{\tau + 1, \cdots, \kappa\} : L \subset \{1, \cdots, \tau, \kappa + 1, \cdots, \kappa + \tau\}, |L \cap \{i, \kappa + i\}| = 1 \text{ for } all \ i \in [\tau]\} \times 2^Y$$

where $X = \{\kappa + \tau + 1, \cdots, n'\}$ and $Y = \{n' + 1, \cdots, n\}.$

Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$ cross-intersecting family where \mathcal{B} is k-uniform. From Theorem 5.1, there exists a $k_0 > 0$ such that if $\frac{ck}{d} = l > k_0$, then $|\mathcal{A}||\mathcal{B}| \leq {\binom{2l}{l}}2^{n-2l}$. Consider the case when \mathcal{B} corresponds to \mathcal{B} of Theorem 5.1. If $|\mathcal{A}||\mathcal{B}| = {\binom{2l}{l}}2^{n-2l}$, then $n' = n, Y = \emptyset$, and $k = \kappa$ in the statement of Theorem 5.1. Since $l = \frac{ck}{d}$ and $k \in \{\frac{2ck}{d} - 1, \frac{2ck}{d}\}$, we have the following two cases:

Case 1: $k = \frac{2ck}{d} - 1$. Then, (k+1)d = 2ck. Since gcd(c, d) = 1 and gcd(k, k+1) = 1, we have k|d|2k. Thus, d = k or d = 2k. We claim that d = 2k is an invalid case.

This is because, when d = 2k, we have c = k + 1. Since gcd(c, d) = 1, k cannot be odd. And if k is even, then $l = \frac{ck}{d} = \frac{k+1}{2}$ is not an integer. So, the only valid case is d = k, $c = \frac{k+1}{2} = l$ and k is an odd integer. **Case 2:** $k = \frac{2ck}{d}$. Then, $\frac{c}{d} = \frac{1}{2}$, that is $(\mathcal{A}, \mathcal{B})$ is a cross bisecting pair. Since

Case 2: $k = \frac{2ck}{d}$. Then, $\frac{c}{d} = \frac{1}{2}$, that is $(\mathcal{A}, \mathcal{B})$ is a cross bisecting pair. Since $l = \frac{ck}{d} = \frac{k}{2}$ is an integer, k must be even in this case. If \mathcal{B} corresponds to \mathcal{A} of Theorem 5.1, we have $X = \emptyset$, $\tau = 0$, \mathcal{B} is k(=l)-uniform, $l = \frac{ck}{d}$. Thus, we have $\frac{c}{d} = 1$, $\mathcal{A} = \{\{1, \ldots, \kappa\}\} \times 2^{Y}$ where $Y = \{\kappa + 1, \ldots, n\}$

and $\mathcal{B} = {\binom{[\kappa]}{k}}, \kappa \in \{2k - 1, 2k\}$ up to a relabelling of the elements.

This leads us to the main result of this section.

Statement of Theorem 1.3: Let $(\mathcal{A}, \mathcal{B})$ be a $\frac{c}{d}$ -cross intersecting pair of families of subsets of [n]. Let \mathcal{B} be k-uniform. Then, there exists some $k_0 > 0$, such that for $k > k_0$ we have

$$|\mathcal{A}||\mathcal{B}| \le {\binom{2ck}{d}}{\frac{ck}{d}} 2^{n-\frac{2ck}{d}}$$

and the bound is tight if and only if, either (a) or (b) hold:

- (a) When $\frac{c}{d} = 1$, $\mathcal{A} = \{\{1, \dots, \kappa\}\} \times 2^{Y}$, $\mathcal{B} = \binom{[\kappa]}{k}$ where $Y = \{\kappa + 1, \dots, n\}$ and $\kappa \in \{2k 1, 2k\}$ up to a relabelling of the elements of [n].
- (b) When $\frac{c}{d} \neq 1$:

(i) If k is even,
$$c = 1$$
, $d = 2$, $\frac{ck}{d} = \lceil \frac{k}{2} \rceil$,
(ii) If k is odd, $c = \frac{k+1}{2}$, $d = k$, $\frac{ck}{d} = \lceil \frac{k}{2} \rceil$,

and for both the cases((i) and (ii)), there exists some τ such that, $k + \tau \leq n$ and up to a relabelling of the elements of [n],

$$\mathcal{A} = \{ \bigcup_{T \in J} T : J \subset \{\{1, k+1\}, \dots, \{\tau, k+\tau\}, \{\tau+1\}, \dots, \{k\}\}, |J| = \lceil \frac{k}{2} \rceil \} \times 2^X$$

where $X = \{k + \tau + 1, ..., n\}$ and

$$\mathcal{B} = \{L \cup \{\tau + 1, \dots, k\} : L \subset \{1, \dots, \tau, k + 1, \dots, k + \tau\}, |L \cap \{i, k + i\}| = 1 \text{ for} \\ all \ i \in [\tau]\}.$$

6 Discussion

What are those pairs of $\frac{c}{d}$ -cross intersecting families $(\mathcal{A}, \mathcal{B})$ which achieve $|\mathcal{A}||\mathcal{B}| = 2^n$ (equal to the upper bound for $\mathcal{M}_{\frac{c}{d}}(n)$ proved in Theorem 1.1)? In the introduction we characterize such families when $\frac{c}{d} = 0$ and $\frac{c}{d} = 1$. In Theorem 1.2, we characterize such families when $\frac{c}{d} = \frac{1}{2}$. From Theorem 1.3, we see that when \mathcal{B} is *k*-uniform, $|\mathcal{A}||\mathcal{B}|$ is maximized when $\frac{c}{d}$ is 1 or nearly $\frac{1}{2}(\frac{1}{2} \text{ or } \frac{1}{2} + \frac{1}{2k})$. For $\frac{c}{d} \in (0, 1)$, besides the case $\mathcal{A} = 2^{[n]}$, $\mathcal{B} = \{\emptyset\}$, is $|\mathcal{A}||\mathcal{B}| = 2^n$ achieved only when $\frac{c}{d}$ is close to $\frac{1}{2}$?

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