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ORIGINAL PAPER

Strong Subgraph Connectivity of Digraphs

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Abstract

Let $D = (V, A)$ be a digraph of order n, S a subset of V of size k and $2 \leq k \leq n$. A strong subgraph H of D is called an S-*strong* subgraph if $S \subseteq V(H)$. A pair of S-strong subgraphs D_1 and D_2 are said to be *arc-disjoint* if $A(D_1) \cap A(D_2) = \emptyset$. A pair of arc-disjoint S-strong subgraphs D_1 and D_2 are said to be *internally disjoint* if $V(D_1) \cap V(D_2) = S$. Let $\kappa_S(D)$ (resp. $\lambda_S(D)$) be the maximum number of internally disjoint (resp. arc-disjoint) S-strong subgraphs in D. The *strong subgraph k -connectivity* is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V, |S| = k\}.$$

As a natural counterpart of the strong subgraph k-connectivity, we introduce the concept of *strong subgraph k -arc-connectivity* which is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$

A digraph $D = (V, A)$ is called *minimally strong subgraph* (k, ℓ)-*(arc-)connected* if $\kappa_k(D) \geq \ell$ (resp. $\lambda_k(D) \geq \ell$) but for any arc $e \in A$, $\kappa_k(D - e) \leq \ell - 1$ (resp. $\lambda_k(D - e) \leq \ell - 1$). In this paper, we first give complexity results for $\lambda_k(D)$, then obtain some sharp bounds for the parameters $\kappa_k(D)$ and $\lambda_k(D)$. Finally, minimally strong subgraph (k, ℓ)-connected digraphs and minimally strong subgraph (k, ℓ)-arc-connected digraphs are studied.

Keywords Directed graph connectivity · Strong subgraph connectivity · Strong subgraph arc connectivity · Generalized connectivity · Arc-disjoint subgraph decomposition

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1 Introduction

The generalized k-connectivity $\kappa_k(G)$ of a graph $G = (V, E)$ was introduced by Hager [8] in 1985 ($2 \leq k \leq |V|$). For a graph $G = (V, E)$ and a set $S \subseteq V$ of at least two vertices, an *S-Steiner tree* or, simply, an *S-tree* is a subgraph T of G which is a tree with $S \subseteq V(T)$. Two S-trees T_1 and T_2 are said to be *internally disjoint* if $E(T_1) \cap E(T_2) = \emptyset$ and $V(T_1) \cap V(T_2) = S$. The *generalized local connectivity* $\kappa_S(G)$ is the maximum number of internally disjoint S-trees in G. For an integer k with $2 \leq k \leq n$, the *generalized k-connectivity* is defined as

$$\kappa_k(G) = \min\{\kappa_S(G) \mid S \subseteq V(G), |S| = k\}.$$

Observe that $\kappa_2(G) = \kappa(G)$. If G is disconnected and vertices of S are placed in different connectivity components, we have $\kappa_S(G) = 0$. Thus, $\kappa_k(G) = 0$ for a disconnected graph G. Generalized connectivity of graphs has become an established area in graph theory, see a recent monograph [9] by Li and Mao on generalized connectivity of undirected graphs.

To extend generalized k-connectivity to directed graphs, Sun et al. [13] observed that in the definition of $\kappa_S(G)$, one can replace “an S-tree” by “a connected subgraph of G containing S”. Therefore, Sun et al. [13] defined *strong subgraph k-connectivity* by replacing “connected” with “strongly connected” (or, simply, “strong”) as follows. Let $D = (V, A)$ be a digraph of order n, S a subset of V of size k and $2 \leq k \leq n$. A subgraph H of D is called an *S-strong subgraph* if $S \subseteq V(H)$. A pair of S-strong subgraphs D_1 and D_2 are said to be *arc-disjoint* if $A(D_1) \cap A(D_2) = \emptyset$. A pair of arc-disjoint S-strong subgraphs D_1 and D_2 are said to be *internally disjoint* if $V(D_1) \cap V(D_2) = S$. Let $\kappa_S(D)$ be the maximum number of internally disjoint S-strong subgraphs in D. The *strong subgraph k -connectivity* [13] is defined as

$$\kappa_k(D) = \min\{\kappa_S(D) \mid S \subseteq V(D), |S| = k\}.$$

By definition, $\kappa_2(D) = 0$ if D is not strong.

As a natural counterpart of the strong subgraph k-connectivity, we now introduce the concept of strong subgraph k-arc-connectivity. Let $\lambda_S(D)$ be the maximum number of arc-disjoint S-strong digraphs in D. The *strong subgraph k-arc-connectivity* is defined as

$$\lambda_k(D) = \min\{\lambda_S(D) \mid S \subseteq V(D), |S| = k\}.$$

By definition, $\lambda_2(D) = 0$ if D is not strong.

For a digraph D, its *reverse* D^{rev} is a digraph with same vertex set and such that $xy \in A(D^{\text{rev}})$ if and only if $yx \in A(D)$. A digraph D is *symmetric* if $D^{\text{rev}} = D$. In other words, a symmetric digraph D can be obtained from its underlying undirected graph G by replacing each edge of G with the corresponding arcs of both directions, that is, $D = \overleftrightarrow{G}$.

The strong subgraph k-(arc-)connectivity is not only a natural extension of the concept of generalized k-(edge-)connectivity, but also relates to important problems in graph theory. For $k=2$, $\kappa_2(\overleftrightarrow{G})=\kappa(G)$ [13] and $\lambda_2(\overleftrightarrow{G})=\lambda(G)$ (Theorem 3.6). Hence, $\kappa_k(D)$ and $\lambda_k(D)$ could be seen as generalizations of connectivity and edge-connectivity of undirected graphs, respectively. For $k=n$, $\kappa_n(D)=\lambda_n(D)$ is the maximum number of arc-disjoint spanning strong subgraphs of D. Moreover, since $\kappa_S(G)$ and $\lambda_S(G)$ are the number of internally disjoint and arc-disjoint strong subgraphs containing a given set S, respectively, these parameters are relevant to the problem of finding the maximum number of strong spanning arc-disjoint subgraphs in a digraph studied, e.g., in [3–5, 12].

In what follows, n will denote the number of vertices of the digraph under consideration.

A digraph $D=(V(D),A(D))$ is called *minimally strong subgraph* (k,ℓ)*-connected* if $\kappa_k(D)\geq \ell$ but for any arc $e\in A(D)$, $\kappa_k(D-e)\leq \ell-1$. Similarly, a digraph $D=(V(D),A(D))$ is called *minimally strong subgraph* (k,ℓ)*-arc-connected* if $\lambda_k(D)\geq \ell$ but for any arc $e\in A(D)$, $\lambda_k(D-e)\leq \ell-1$.

A 2-cycle xyx of a strong digraph D is called a *bridge* if $D-\{xy,yx\}$ is disconnected. Thus, a bridge corresponds to a bridge in the underlying undirected graph of D. An *orientation* of a digraph D is a digraph obtained from D by deleting an arc in each 2-cycle of D. A digraph D is *semicomplete* if for every distinct $x,y\in V(D)$ at least one of the arcs xy, yx is in D. A digraph D is *k-regular* if the in- and out-degree of every vertex of D is equal to k. We refer the readers to [2] for graph theoretical notation and terminology not given here.

Let $k\geq 2$ and $\ell\geq 2$ be fixed integers. By reduction from the Directed 2-Linkage problem, Sun et al. [13] proved that deciding whether $\kappa_S(D)\geq \ell$ is NP-complete for a k-subset S of $V(D)$. Thomassen [14] showed that for every positive integer p there are digraphs which are strongly p-connected, but which contain a pair of vertices not belonging to the same cycle. This implies that for every positive integer p there are strongly p-connected digraphs D such that $\kappa_2(D)=1$ [13].

The above negative results motivate studying strong subgraph k-connectivity for special classes of digraphs. In Sun et al. [13], showed that the problem of deciding whether $\kappa_k(D)\geq \ell$ for every semicomplete digraphs is polynomial-time solvable for fixed k and ℓ. The main tool used in their proof is a recent Directed k-Linkage theorem of Chudnovsky, Scott and Seymour [7]. Sun et al. [13] showed that for any connected graph G, the parameter $\kappa_2(\overleftrightarrow{G})$ can be computed in polynomial time. This result is best possible in the following sense. Let D be a symmetric digraph and $k\geq 3$ a fixed integer. Then it is NP-complete to decide whether $\kappa_S(D)\geq \ell$ for $S\subseteq V(D)$ with $|S|=k$ [13]. Let D be a strong digraph with n vertices. Sun et al. [13] proved that $1\leq \kappa_k(D)\leq n-1$ for $2\leq k\leq n$. The bounds are sharp; Sun et al. [13] also characterized those digraphs D for which $\kappa_k(D)$ attains the upper bound. The main tool used in their proof is a Hamiltonian cycle decomposition theorem of Tillson [15].

In this paper, we prove that for fixed integers $k,\ell\geq 2$, the problem of deciding whether $\lambda_S(D)\geq \ell$ is NP-complete for a digraph D and a set $S\subseteq V(D)$ of size k. This result is proved in Sect. 3 using the corresponding result for $\kappa_S(D)$ proved in [13]. In the same section, we also consider classes of digraphs. We characterize when $\lambda_k(D)\geq 2$, $2\leq k\leq n$, for both semicomplete and symmetric digraphs D of order n. The characterizations imply that the problem of deciding whether $\lambda_k(D) \geq 2$ is polynomial-time solvable for both semicomplete and symmetric digraphs. For fixed $\ell \geq 3$ and $k \geq 2$, the complexity of deciding whether $\lambda_k(D) \geq \ell$ remains an open problem for both semicomplete and symmetric digraphs. It was proved in [13] that for fixed $k, \ell \geq 2$ the problem of deciding whether $\kappa_k(D) \geq \ell$ is polynomial-time solvable for both semicomplete and symmetric digraphs, but it appears that the approaches to prove the two results cannot be used for $\lambda_k(D)$. In fact, we would not be surprised if the $\lambda_k(D) \geq \ell$ problem turns out to be NP-complete at least for one of the two classes of digraphs.

In Sect. 4, we first give sharp upper bounds for the parameters $\kappa_k(D)$ and $\lambda_k(D)$ in terms of classical connectivity. Then we get some lower and upper bounds for the parameter $\lambda_k(D)$ including a lower bound whose analog for $\kappa_k(D)$ does not hold as well as Nordhaus-Gaddum type bounds.

In Sect. 5, we characterize minimally strong subgraph $(2, n-2)$-connected digraphs and minimally strong subgraph $(2, n-2)$-arc-connected digraphs. Also, we bound the sizes of minimally strong subgraph $(2, n-2)$-connected digraphs.

We conclude the paper in Sect. 6 by discussing open problems.

2 Preliminaries

Let us start this section from observations that can be easily verified using definitions of $\lambda_k(D)$ and $\kappa_k(D)$. Note that the first inequality of the following inequalities (2) can be found in [13].

Proposition 2.1 *Let D be a digraph of order n, and let $k \geq 2$ be an integer. Then*

$$\lambda_{k+1}(D) \leq \lambda_k(D) \text{ for every } k \leq n-1 \tag{1}$$

For a spanning subgraph D' of D, we have

$$\kappa_k(D') \leq \kappa_k(D), \lambda_k(D') \leq \lambda_k(D) \tag{2}$$

$$\kappa_k(D) \leq \lambda_k(D) \leq \min\{\delta^+(D), \delta^-(D)\} \tag{3}$$

The inequality (1) means that the parameter λ_k has a monotonically non-increasing with respect to k. However, this property may not hold for κ_k, that is, $\kappa_n(D) \leq \kappa_{n-1}(D) \leq \cdots \leq \kappa_3(D) \leq \kappa_2(D) = \kappa(D)$ may not be true. Consider the following example: Let D be a digraph obtained from two copies D_1 and D_2 of the complete digraph $\overleftrightarrow{K}_t (t \geq 4)$ by identifying one vertex in each of them. Clearly, D is a strong digraph with a cut vertex, say u. For $2 \leq k \leq 2t-2$, let S be a subset of $V(D) \setminus \{u\}$ with $|S| = k$ such that $S \cap V(D_i) \neq \emptyset$ for every $i \in \{1, 2\}$. Since each S-strong subgraph must contain u, we have $\kappa_k(D) \leq 1$, furthermore, we deduce that $\kappa_k(D) = 1$ for $2 \leq k \leq 2t-2$. Let G_i be the underlying undirected graph of D_i for $i \in \{1, 2\}$. Each G_i contains $\lfloor \frac{t}{2} \rfloor$ edge-disjoint spanning trees, say $T_{i,j} (1 \leq j \leq \lfloor \frac{t}{2} \rfloor)$, since G_i is a complete graph of order t (see, e.g., (3.1) in [10]). Now in D, let H_j be a subgraph of D obtained from the tree T_j which is the union of $T_{1,j}$ and $T_{2,j}$ by replacing each edge with two arcs of the opposite directions. Clearly, these subgraphs are strong, spanning and arc-disjoint. Hence, $\kappa_{2t-1}(D) \geq \lfloor \frac{t}{2} \rfloor > 1 = \kappa_k(D)$ for $2 \leq k \leq 2t-2$.

We will use the following decomposition theorem by Tillson.

Theorem 2.2 [15] *The arcs of $\overleftrightarrow{K}_n$ can be decomposed into Hamiltonian cycles if and only if $n \neq 4, 6$.*

3 Complexity

Yeo proved that it is an NP-complete problem to decide whether a 2-regular digraph has two arc-disjoint hamiltonian cycles (see, e.g., Theorem 6.6 in [5]). Thus, the problem of deciding whether $\lambda_n(D) \geq 2$ is NP-complete, where n is the order of D. We will extend this result in Theorem 3.1.

Let D be a digraph and let $s_1, s_2, \ldots, s_k, t_1, t_2, \ldots, t_k$ be a collection of not necessarily distinct vertices of D. A *weak k-linkage* from $(s_1, s_2, \ldots, s_k)$ to $(t_1, t_2, \ldots, t_k)$ is a collection of k arc-disjoint paths $P_1, \ldots, P_k$ such that P_i is an (s_i, t_i)-path for each $i \in [k]$. A digraph $D = (V, A)$ is *weakly k-linked* if it contains a weak k-linkage from $(s_1, s_2, \ldots, s_k)$ to $(t_1, t_2, \ldots, t_k)$ for every choice of (not necessarily distinct) vertices $s_1, \ldots, s_k, t_1, \ldots, t_k$. The WEAK k -LINKAGE PROBLEM is the following. Given a digraph $D = (V, A)$ and distinct vertices $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$; decide whether D contains k arc-disjoint paths $P_1, \ldots, P_k$ such that P_i is an (x_i, y_i)-path. The problem is well-known to be NP-complete already for $k = 2$ [2].

Theorem 3.1 *Let $k \geq 2$ and $\ell \geq 2$ be fixed integers. Let D be a digraph and $S \subseteq V(D)$ with $|S| = k$. The problem of deciding whether $\lambda_S(D) \geq \ell$ is NP-complete.*

Proof Clearly, the problem is in NP. We will show that it is NP-hard using a reduction similar to that in Theorem 2.1 of [13]. Let us first deal with the case of $\ell = 2$ and $k = 2$. Consider the digraph D' used in the proof of Theorem 2.1 of [13] (see Fig. 1), where D is an arbitrary digraph, x, y are vertices not in D, and $t_1x, xs_1, t_2y, ys_2, xs_2, s_2x, yt_1, t_1y$ are additional arcs. To construct a new digraph D'' from D', replace every vertex u of D by two vertices u^- and u^+ such that u^-u^+ is an arc in D'' and for every $uv \in A(D)$ add an arc u^+v^- to D''. Also, for $z \in \{x, y\}$, for every arc zu in D' add an arc zu^- to D'' and for every arc uz add an arc u^+z to D''.

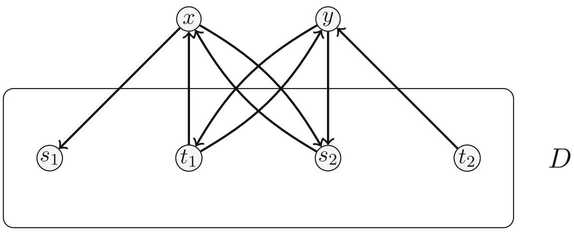

Fig. 1 The digraph D'

Let $S=\{x,y\}$. It was proved in Theorem 2.1 of [13] that $\kappa_S(D')\geq 2$ if and only if there are vertex-disjoint paths from s_1 to t_1 and from s_2 to t_2. It follows from this result and definition of D'' that $\lambda_S(D'')\geq 2$ if and only if there are arc-disjoint paths from s_1^- to t_1^+ and from s_2^- to t_2^+. Since the WEAK 2-LINKAGE PROBLEM is NP-complete, we conclude that the problem of deciding whether $\lambda_S(D'')\geq 2$ is NP-hard.

Now let us consider the case of $\ell\geq 3$ and $k=2$. Add to D'' $\ell-2$ copies of the 2-cycle xyx and subdivide the arcs of every copy to avoid parallel arcs. Let us denote the new digraph by D'''. Similarly to that in Theorem 2.1 of [13], we can show that $\lambda_S(D''')\geq \ell$ if and only if $\lambda_S(D'')\geq 2$.

It remains to consider the case of $\ell\geq 2$ and $k\geq 3$. Add to D''' (where $D'''=D''$ for $\ell=2$) $k-2$ new vertices $x_1,\ldots,x_{k-2}$ and arcs of ℓ 2-cycles xx_ix for each $i\in[k-2]$. Subdivide the new arcs to avoid parallel arcs. Denote the obtained digraph by D''''. Let $S=\{x,y,x_1,\ldots,x_{k-2}\}$. Similarly to that in Theorem 2.1 of [13], we can show that $\lambda_S(D'''')\geq \ell$ if and only if $\lambda_S(D'')\geq 2$.

Bang-Jensen and Yeo [5] conjectured the following:

Conjecture 1 *For every $\lambda\geq 2$ there is a finite set $\mathcal{S}_\lambda$ of digraphs such that a λ-arc-strong semicomplete digraph D contains λ arc-disjoint spanning strong subgraphs unless $D\in\mathcal{S}_\lambda$.*

Bang-Jensen and Yeo [5] proved the conjecture for $\lambda=2$ by showing that $|\mathcal{S}_2|=1$ and describing the unique digraph S_4 of $\mathcal{S}_2$ of order 4. Now we have the following characterization:

Theorem 3.2 *For a semicomplete digraph D, of order n and an integer k such that $2\leq k\leq n$, $\lambda_k(D)\geq 2$ if and only if D is 2-arc-strong and the following does not hold: $D\cong S_4$ and $k=4$.*

Proof We first consider the direction "only if". Suppose that D is not a 2-arc-strong and $xy\in A(D)$ such that $D-xy$ is not strong. Thus, for $S=\{x,y\}$ we have $\lambda_S(D)=1$. Hence $\lambda_2(D)=1$ and by (1) $\lambda_k(D)=1$ for each k, $2\leq k\leq n$. Furthermore, by the result of Bang-Jensen and Yeo, the following does not hold: $D\cong S_4$ and $k=4$. □

We next prove the direction "if". If D is 2-arc-strong and $D\ncong S_4$, then D contains two arc-disjoint spanning strong subgraphs by the result of Bang-Jensen and Yeo, that is, $\lambda_n(D)\geq 2$. Furthermore, we have $\lambda_k(D)\geq 2$ for all $2\leq k\leq n$ by (1). Now we consider the case that $D\cong S_4$. Let S be any subset of $V(D)$ with $|S|=3$; by symmetry of S_4 it suffices to assume that $S=\{v_1,v_2,v_3\}$ (see Fig. 2). Let D_1 be the cycle v_1,v_2,v_3,v_1 and D_2 be subgraph of D with $A(D_2)=A(D)\setminus A(D_1)$. It can be easily checked that both D_1 and D_2 are S-strong subgraphs, so $\lambda_3(D)\geq 2$. Furthermore by (1), we have $\lambda_2(D)\geq 2$.

Now we turn our attention to symmetric digraphs. We start from characterizing symmetric digraphs D with $\lambda_k(D)\geq 2$, an analog of Theorem 3.2. To prove it we will use the following result of Boesch and Tindell [6] translated from the language of mixed graphs to that of digraphs.

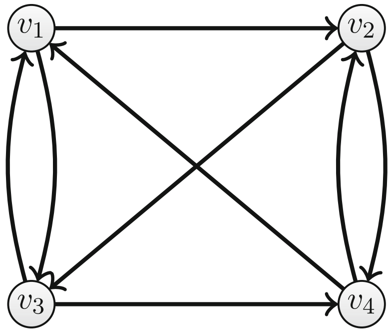

Fig. 2 Digraph S_4

Theorem 3.3 *A strong digraph D has a strong orientation if and only if D has no bridge.*

Here is our characterization.

Theorem 3.4 *For a strong symmetric digraph D of order n and an integer k such that* $2 \le k \le n$, $\lambda_k(D) \ge 2$ *if and only if D has no bridge.*

Proof Let D have no bridge. Then, by Theorem 3.3, D has a strong orientation H. Since D is symmetric, H^{rev} is another orientation of D. Clearly, H^{rev} is strong and hence $\lambda_k(D) \ge 2$. $\square$

Suppose that D has a bridge xyx. Choose a set S of size k such that $\{x, y\} \subseteq S$ and observe that any strong subgraph of D containing vertices x and y must include both xy and yx. Thus, $\lambda_S(D) = 1$ and $\lambda_k(D) = 1$.

Theorems 3.2 and 3.4 imply the following complexity result, which we believe to be extendable from $\ell = 2$ to any natural ℓ.

Corollary 3.5 *The problem of deciding whether* $\lambda_k(D) \ge 2$ *is polynomial-time solvable if D is either semicomplete or symmetric digraph of order n and* $2 \le k \le n$.

Now we give a lower bound on $\lambda_k(D)$ for symmetric digraphs D.

Theorem 3.6 *For every graph G, we have*

$$\lambda_k(\overleftrightarrow{G}) \ge \lambda_k(G).$$

Moreover, this bound is sharp. In particular, we have $\lambda_2(\overleftrightarrow{G}) = \lambda_2(G)$.

Proof We may assume that G is a connected graph. Let $S = \{x, y\}$, where x, y are distinct vertices of $\overleftrightarrow{G}$. Observe that $\lambda_S(G) \ge \lambda_S(\overleftrightarrow{G})$. Indeed, let $p = \lambda_S(\overleftrightarrow{G})$ and let $D_1, \ldots, D_p$ be arc-disjoint S-strong subgraphs of $\overleftrightarrow{G}$. Thus, by choosing a path from x to y in each D_i, we obtain p arc-disjoint paths from x to y, which correspond to p arc-disjoint paths between x and y in G. Thus, $\lambda(G) = \lambda_2(G) \ge \lambda_2(\overleftrightarrow{G})$.

We now consider the general k. Let $\lambda_S(\overleftrightarrow{G}) = \lambda_k(\overleftrightarrow{G})$ for some $S \subseteq V(\overleftrightarrow{G})$ with $|S| = k$. We know that there are at least $\lambda_k(G)$ edge-disjoint trees containing S in G, say $T_i (i \in [\lambda_k(G)])$. For each $i \in [\lambda_k(G)]$, we can obtain a strong subgraph containing S, say D_i, in $\overleftrightarrow{G}$ by replacing each edge of T_i with the corresponding arcs of both directions. Clearly, any two such subgraphs are arc-disjoint, so we have $\lambda_k(\overleftrightarrow{G}) = \lambda_S(\overleftrightarrow{G}) \geq \lambda_k(G)$, and we also have $\lambda_2(\overleftrightarrow{G}) = \lambda_2(G) = \lambda(G)$.

For the sharpness of the bound, consider the tree T with order n. Clearly, we have $\lambda_k(T) = 1$. Furthermore, $1 \leq \lambda_k(\overleftrightarrow{T}) \leq \min\{\delta^+(D), \delta^-(D)\} = 1$ by Inequality (3).□

Note that for the case that $3 \leq k \leq n$, the equality $\lambda_k(\overleftrightarrow{G}) = \lambda_k(G)$ does not always hold. For example, consider the cycle C_n of order n; it is not hard to check that $\lambda_k(\overleftrightarrow{C}_n) = 2$, but $\lambda_k(C_n) = 1$.

Theorem 3.6 immediately implies the next result, which follows from the fact that $\lambda(G)$ can be computed in polynomial time.

Corollary 3.7 *For a symmetric digraph D, $\lambda_2(D)$ can be computed in polynomial time.*

4 Sharp bounds of $\kappa_k(D)$ and $\lambda_k(D)$

To prove a new bound on $\kappa_k(D)$ in Theorem 4.2, we will use the following result of Sun et al. [13].

Theorem 4.1 *Let $2 \leq k \leq n$. For a strong digraph D of order n, we have*

$$1 \leq \kappa_k(D) \leq n - 1.$$

Moreover, both bounds are sharp, and the upper bound holds if and only if $D \cong \overleftrightarrow{K}_n$, $2 \leq k \leq n$ and $k \notin \{4, 6\}$.

The following result concerns the relation between $\kappa_k(D)$ (resp. $\lambda_k(D)$) and $\kappa(D)$ (resp. $\lambda(D)$).

Theorem 4.2 *Let $k \in \{2, \ldots, n\}$. The following assertions hold*:

(i) For $n \geq \kappa(D) + k$, we have $\kappa_k(D) \leq \kappa(D)$;
(ii) $\lambda_k(D) \leq \lambda(D)$. Moreover, both bounds are sharp.

Proof **Part (*i*).** For $k = 2$, assume that $\kappa(D) = \kappa(x, y)$ for some $\{x, y\} \subseteq V(D)$. It follows from the strong subgraph connectivity definition that $\kappa_{\{x,y\}}(D) \leq \kappa(x, y)$, so $\kappa_2(D) \leq \kappa_{\{x,y\}}(D) \leq \kappa(x, y) = \kappa(D)$.

We now consider the case of $k \geq 3$. If $\kappa(D) = n - 1$, then we have $\kappa_k(D) \leq n - 1 = \kappa(D)$ by Theorem 4.1. If $\kappa(D) = n - 2$, then there are two vertices, say u and v, such that $uv \notin A(D)$. So we have $\kappa_k(D) \leq n - 2 = \kappa(D)$ by Theorem 4.1. If $1 \leq \kappa(D) \leq n - 3$, then there exists a $\kappa(D)$-vertex cut, say Q, for two vertices u, v in D such that there is no $u - v$ path in $D - Q$. Let $S = \{u, v\} \cup S'$ where $S' \subseteq V(D) \setminus (Q \cup \{u, v\})$ and $|S'| = k - 2$. Since u and v are in different strong components of $D - Q$, any S-strong subgraph in D must contain a vertex in Q. By the definition of $\kappa_S(D)$ and $\kappa_k(D)$, we have $\kappa_k(D) \leq \kappa_S(D) \leq |Q| = \kappa(D)$.

For the sharpness of the bound, consider the following digraph D. Let D be a symmetric digraph whose underlying undirected graph is $K_k \bigvee \overline{K}_{n-k}$ $(n \geq 3k)$, i.e. the graph obtained from disjoint graphs K_k and $\overline{K}_{n-k}$ by adding all edges between the vertices in K_k and $\overline{K}_{n-k}$.

Let $V(D) = W \cup U$, where $W = V(K_k) = \{w_i \mid 1 \leq i \leq k\}$ and $U = V(\overline{K}_{n-k}) = \{u_j \mid 1 \leq j \leq n-k\}$. Let S be any k-subset of vertices of $V(D)$ such that $|S \cap U| = s$ $(s \leq k)$ and $|S \cap W| = k - s$. Without loss of generality, let $w_i \in S$ for $1 \leq i \leq k-s$ and $u_j \in S$ for $1 \leq j \leq s$. For $1 \leq i \leq k-s$, let D_i be the symmetric subgraph of D whose underlying undirected graph is the tree T_i with edge set

$$\{w_iu_1, w_iu_2, \ldots, w_iu_s, u_{k+i}w_1, u_{k+i}w_2, \ldots, u_{k+i}w_{k-s}\}.$$

For $k-s+1 \leq j \leq k$, let D_j be the symmetric subgraph of D whose underlying undirected graph is the tree T_j with edge set

$$\{w_ju_1, w_ju_2, \ldots, w_ju_s, w_jw_1, w_jw_2, \ldots, w_jw_{k-s}\}.$$

Observe that $\{D_i \mid 1 \leq i \leq k-s\} \cup \{D_j \mid k-s+1 \leq j \leq k\}$ is a set of k internally disjoint S-strong subgraph, so $\kappa_S(D) \geq k$, and then $\kappa_k(D) \geq k$. Combining this with the bound that $\kappa_k(D) \leq \kappa(D)$ and the fact that $\kappa(D) \leq \min\{\delta^+(D), \delta^-(D)\} = k$, we can get $\kappa_k(D) = \kappa(D) = k$.

Part (ii) Let A be a $\lambda(D)$-arc-cut of D, where $1 \leq \lambda(D) \leq n-1$. We choose $S \subseteq V(D)$ such that at least two of these k vertices are in different strong components of $D - A$. Thus, any S-strong subgraph in D must contain an arc in A. By the definition of $\lambda_S(D)$ and $\lambda_k(D)$, we have $\lambda_k(D) \leq \lambda_S(D) \leq |A| = \lambda(D)$.

For the sharpness of the bound, consider the the digraph D in part (i). Recall that $\{D_i \mid 1 \leq i \leq k\}$ is a set of k internally disjoint S-strong subgraph, so $\lambda_S(D) \geq \kappa_S(D) \geq k$, and then $\lambda_k(D) \geq k$. Combining this with the bound that $\lambda_k(D) \leq \lambda(D)$ and the fact that $\lambda(D) \leq \min\{\delta^+(D), \delta^-(D)\} = k$, we can get $\lambda_k(D) = \lambda(D) = k$. □

Note that the condition "$n \geq \kappa(D) + k$" in Theorem 4.2 cannot be removed. Consider the example after Proposition 2.1. We have $n = 2t - 1 < 2t = \kappa(D) + k$ when $k = n$, but now $\kappa_n(D) > \kappa(D)$.

In the proof of Theorem 4.1, they used the following result on $\kappa_k(\overleftrightarrow{K}_n)$.

Lemma 4.3 [13] *For $2 \leq k \leq n$, we have*

$$\kappa_k(\overleftrightarrow{K}_n) = \begin{cases} n-1, & \text{if } k \notin \{4, 6\}; \\ n-2, & \text{otherwise.} \end{cases}$$

We can now compute the exact values of $\lambda_k(\overleftrightarrow{K}_n)$.

Lemma 4.4 *For $2 \leq k \leq n$, we have*

$$\lambda_k(\overleftrightarrow{K}_n)=\begin{cases} n-1, & \text{if } k\notin\{4,6\}, \text{ or, } k\in\{4,6\} \text{ and } k<n;\\ n-2, & \text{if } k=n\in\{4,6\}.\end{cases}$$

Proof For the case that $2\leq k\leq n$ and $k\notin\{4,6\}$, by (3) and Lemma 4.3, we have $n-1\leq\kappa_k(\overleftrightarrow{K}_n)\leq\lambda_k(\overleftrightarrow{K}_n)\leq n-1$. Hence, $\lambda_k(\overleftrightarrow{K}_n)=n-1$ and in the following argument we assume that $2\leq k\leq n$ and $k\in\{4,6\}$.

We first consider the case of $2\leq k=n$. For $n=4$, since K_n contains a Hamiltonian cycle, the two orientations of the cycle imply that $\lambda_n(\overleftrightarrow{K}_n)\geq 2=n-2$. To see that there are at most two arc-disjoint strong spanning subgraphs of $\overleftrightarrow{K}_n$, suppose that there are three arc-disjoint such subgraphs. Then each such subgraph must have exactly four arcs (as $|A(\overleftrightarrow{K}_n)|=12$), and so all of these three subgraphs are Hamiltonian cycles, which means that the arcs of $\overleftrightarrow{K}_n$ can be decomposed into Hamiltonian cycles, a contradiction to Theorem 2.2). Hence, $\lambda_n(\overleftrightarrow{K}_n)=n-2$ for $n=4$. Similarly, we can prove that $\lambda_n(\overleftrightarrow{K}_n)=n-2$ for $n=6$, as K_n contains two edge-disjoint Hamiltonian cycles, and therefore $\overleftrightarrow{K}_n$ contains four arc-disjoint Hamiltonian cycles.

We next consider the case of $2\leq k\leq n-1$. We assume that $k=6$ as the case of $k=4$ can be considered in a similar and simpler way. Let $S\subseteq V(\overleftrightarrow{K}_n)$ be any vertex subset of size six. Let $S=\{u_i\mid 1\leq i\leq 6\}$ and $V(\overleftrightarrow{K}_n)\setminus S=\{v_j\mid 1\leq j\leq n-6\}$. Let D_1 be the cycle $u_1u_2u_3u_4u_5u_6u_1$; let $D_2=D_1^{\mathrm{rev}}$; let D_3 be the cycle $u_1u_3u_6u_4u_2u_5u_1$; let $D_4=D_3^{\mathrm{rev}}$; let D_5 be a subgraph of $\overleftrightarrow{K}_n$ with vertex set $S\cup\{v_1\}$ and arc set $\{u_1v_1, v_1u_2, u_2u_6, u_6v_1, v_1u_5, u_5u_3, u_3v_1, v_1u_4, u_4u_1\}$; let $D_6=D_5^{\mathrm{rev}}$; for each $x\in\{v_j\mid 2\leq j\leq n-6\}$, let D_x be a subgraph of $\overleftrightarrow{K}_n$ with vertex set $S\cup\{x\}$ and arc set $\{xu_i, u_ix\mid 1\leq i\leq 6\}$. Hence, we have $\lambda_S(D)\geq n-1$ for any $S\subseteq V(\overleftrightarrow{K}_n)$ with $|S|=6$ and so $\lambda_k(D)\geq n-1$. We clearly have $\lambda_k(D)\leq n-1$ by (3), then our result holds.$\square$

Now we obtain sharp lower and upper bounds for $\lambda_k(D)$ for $2\leq k\leq n$.

Theorem 4.5 *Let $2\leq k\leq n$. For a strong digraph D of order n, we have*

$$1\leq\lambda_k(D)\leq n-1.$$

Moreover, both bounds are sharp, and the upper bound holds if and only if $D\cong\overleftrightarrow{K}_n$, where $k\notin\{4,6\}$, or, $k\in\{4,6\}$ and $k<n$.

Proof The lower bound is clearly correct by the definition of $\lambda_k(D)$, and for the sharpness, a cycle is our desired digraph. The upper bound and its sharpness hold by (2) and Lemma 4.4.

If D is not equal to $\overleftrightarrow{K}_n$ then $\delta^+(D)\leq n-2$ and by (3) we observe that $\lambda_k(D)\leq\delta^+(D)\leq n-2$. Therefore, by Lemma 4.4, the upper bound holds if and only if $D\cong\overleftrightarrow{K}_n$, where $k\notin\{4,6\}$, or, $k\in\{4,6\}$ and $k<n$.

Shiloach [11] proved the following:

Theorem 4.6 [11] *A digraph D is weakly k-linked if and only if D is k-arc-strong.*

Using Shiloach's Theorem, we will prove the following lower bound for $\lambda_k(D)$. Such a bound does not hold for $\kappa_k(D)$ since it was shown in [13] using Thomassen's result in [14] that for every ℓ there are digraphs D with $\kappa(D) = \ell$ and $\kappa_2(D) = 1$.

Proposition 4.7 *Let $k \leq \ell = \lambda(D)$. We have $\lambda_k(D) \geq \lfloor \ell/k \rfloor$.*

Proof Choose an arbitrary vertex set $S = \{s_1, \ldots, s_k\}$ of D and let $t = \lfloor \ell/k \rfloor$. By Theorem 4.6, there is a weak kt-linkage L from $x_1, x_2, \ldots, x_{kt}$ to $y_1, y_2, \ldots, y_{kt}$, where $x_i = s_{i \bmod k}$ and $y_i = s_{i \bmod k+1}$ and $s_{k+1} = s_1$. Note that the paths of L form t arc-disjoint strong subgraphs of D containing S. □

For a digraph $D = (V(D), A(D))$, the *complement digraph*, denoted by D^c, is a digraph with vertex set $V(D^c) = V(D)$ such that $xy \in A(D^c)$ if and only if $xy \notin A(D)$.

Given a graph parameter $f(G)$, the Nordhaus-Gaddum Problem is to determine sharp bounds for (a) $f(G) + f(G^c)$ and (b) $f(G)f(G^c)$, and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide attention; see a recent survey paper [1] by Aouchiche and Hansen. Theorem 4.9 concerns such type of a problem for the parameter λ_k. To prove the theorem, we will need the following:

Proposition 4.8 *A digraph D with order n is strong if and only if $\lambda_k(D) \geq 1$, where $2 \leq k \leq n$.*

Proof If D is strong, then for every vertex set S of size k, D has a strong subgraph containing S. If $\lambda_k(D) \geq 1$, for each vertex set S of size k construct D_S, a strong subgraph of D containing S. The union of all D_S is a strong subgraph of D as there are sets $S_1, S_2, \ldots, S_p$ such that the union of $S_1, S_2, \ldots, S_p$ is $V(D)$ and for each $i \in [p-1]$, D_{S_i} and $D_{S_{i+1}}$ share a common vertex. □

Theorem 4.9 *For a digraph D with order n, the following assertions hold:*

(i) $0 \leq \lambda_k(D) + \lambda_k(D^c) \leq n-1$. Moreover, both bounds are sharp. In particular, the lower bound holds if and only if $\lambda(D) = \lambda(D^c) = 0$.

(ii) $0 \leq \lambda_k(D)\lambda_k(D^c) \leq \left(\frac{n-1}{2}\right)^2$. Moreover, both bounds are sharp. In particular, the lower bound holds if and only if $\lambda(D) = 0$ or $\lambda(D^c) = 0$.

Proof We first prove (*i*). Since $D \cup D^c = \overleftrightarrow{K}_n$, by definition of λ_k, $\lambda_k(D) + \lambda_k(D^c) \leq \lambda_k(\overleftrightarrow{K}_n)$. Thus, by Lemma 4.4, the upper bound for the sum $\lambda_k(D) + \lambda_k(D^c)$ holds. Let $H \cong \overleftrightarrow{K}_n$. When $k \notin \{4, 6\}$, or, $k \in \{4, 6\}$ and $k < n$, by Lemma 4.4, we have $\lambda_k(H) = n-1$ and we clearly have $\lambda_k(H^c) = 0$, so the upper bound is sharp.

The lower bound is clear. Clearly, the lower bound holds, if and only if $\lambda_k(D) = \lambda_k(D^c) = 0$, if and only if $\lambda(D) = \lambda(D^c) = 0$ by Proposition 4.8.

We now prove (*ii*). The lower bound is clear, and it holds, if and only if $\lambda_k(D) =$

0 or $\lambda_k(D^c) = 0$, if and only if $\lambda(D) = 0$ or $\lambda(D^c) = 0$ by Proposition 4.8. For the upper bound, we have

$$\lambda_k(D)\lambda_k(D^c) \le \left(\frac{\lambda_k(D) + \lambda_k(D^c)}{2}\right)^2 \le \left(\frac{n-1}{2}\right)^2.$$

Let $H \cong \overleftrightarrow{K}_n$ with $n = 2h + 1 \ge 7$. By Theorem 2.2, H contains $2h$ arc-disjoint Hamiltonian cycles: $H_1, \ldots, H_{2h}$. Let D_1 be the union of the former h cycles, and D_2 be the union of the remaining h cycles. Clearly, $D_1^c = D_2$ and $\lambda_n(D_i) \ge h$ and so $\lambda_k(D_i) \ge h$ for $1 \le i \le 2, 2 \le k \le n$ by (1). Furthermore, D_i is h-regular, so $\lambda_k(D_i) \le h$ by (3). Hence, $\lambda_k(D_i) = h$ for $1 \le i \le 2, 2 \le k \le n$. Now $\lambda_k(D_1)\lambda_k(D_1^c) = \lambda_k(D_1)\lambda_k(D_2) = h^2 = \left(\frac{n-1}{2}\right)^2$, so the upper bound is sharp. □

5 Minimally Strong Subgraph (k,ℓ)-(arc-)connected Digraphs

In this section, we will first study the minimally strong subgraph (k,ℓ)-connected digraphs. By the definition of a minimally strong subgraph (k,ℓ)-connected digraph, we can get the following observation.

Proposition 5.1 *A digraph D is minimally strong subgraph (k,ℓ)-connected if and only if $\kappa_k(D) = \ell$ and $\kappa_k(D - e) = \ell - 1$ for any arc $e \in A(D)$.*

Proof The direction "if" is clear by definition, and we only need to prove the direction "only if". Let D be a minimally strong subgraph (k,ℓ)-connected digraph. By definition, we have $\kappa_k(D) \ge \ell$ and $\kappa_k(D - e) \le \ell - 1$ for any arc $e \in A(D)$. Then for any set $S \subseteq V(D)$ with $|S| = k$, there is a set $\mathcal{D}$ of ℓ internally disjoint S-strong subgraphs. As e must belong to one and only one element of $\mathcal{D}$, we are done. □

A digraph D is *minimally strong* if D is strong but $D - e$ is not for every arc e of D.

Proposition 5.2 *The following assertions hold:*

(i) A digraph D is minimally strong subgraph $(k, 1)$-connected if and only if D is a minimally strong digraph;

(ii) For $k \ne 4, 6$, a digraph D is minimally strong subgraph $(k, n-1)$-connected if and only if $D \cong \overleftrightarrow{K}_n$.

Proof To prove (i), it suffices to show that a digraph D is strong if and only if $\kappa_k(D) \ge 1$. If D is strong, then for every vertex set S of size k, D has an S-strong subgraph. If $\kappa_k(D) \ge 1$, for each vertex set S of size k construct D_S, an S-strong subgraph of D. The union of all D_k is a strong subgraph of D as there are sets $S_1, S_2, \ldots, S_p$ such that the union of $S_1, S_2, \ldots, S_p$ is $V(D)$ and for each $i \in [p-1]$, D_{S_i} and $D_{S_{i+1}}$ share a common vertex.

Part (ii) follows from Theorem 4.1. □

The following result characterizes minimally strong subgraph $(2, n-2)$-connected digraphs.

Theorem 5.3 *A digraph D is minimally strong subgraph $(2, n-2)$-connected if and only if D is a digraph obtained from the complete digraph $\overleftrightarrow{K}_n$ by deleting an arc set M such that $\overleftrightarrow{K}_n[M]$ is a 3-cycle or a union of $\lfloor n/2 \rfloor$ vertex-disjoint 2-cycles. In particular, we have $f(n, 2, n-2) = n(n-1) - 2\lfloor n/2 \rfloor$, $F(n, 2, n-2) = n(n-1) - 3$.*

Proof Let $D \cong \overleftrightarrow{K}_n - M$ be a digraph obtained from the complete digraph $\overleftrightarrow{K}_n$ by deleting an arc set M. Let $V(D) = \{u_i \mid 1 \le i \le n\}$.

Firstly, we will consider the case that $\overleftrightarrow{K}_n[M]$ is a 3-cycle $u_1u_2u_3u_1$. We now prove that $\kappa_2(D) = n-2$. By (3), we have $\kappa_2(D) \le \min\{\delta^+(D), \delta^-(D)\} = n-2$. Let $S = \{u, v\} \subseteq V(D)$; we just consider the case that $u = u_1, v = u_2$ since the other cases are similar. Let D_1 be a subgraph of D with $V(D_1) = \{u_1, u_2, u_3\}$ and $A(D_1) = \{u_1u_3, u_3u_2, u_2u_1\}$; for $2 \le i \le n-2$, let D_i be a subgraph of D with $V(D_i) = \{u_1, u_2, u_{i+2}\}$ and $A(D_i) = \{u_1u_{i+2}, u_2u_{i+2}, u_{i+2}u_1, u_{i+2}u_2\}$. Clearly, $\{D_i \mid 1 \le i \le n-2\}$ is a set of $n-2$ internally disjoint S-strong subgraphs, so $\kappa_S(D) \ge n-2$ and $\kappa_2(D) \ge n-2$. Hence, $\kappa_2(D) = n-2$.

For any $e \in A(D)$, without loss of generality, one of the two digraphs in Fig. 3 is a subgraph of $\overleftrightarrow{K}_n[M \cup \{e\}]$, so if the following claim holds, then we must have $\kappa_2(D-e) \le \kappa_2(D') \le n-3$ by Proposition 4.3, and so D is minimally strong subgraph $(2, n-2)$-connected. Now it suffices to prove the following claim. □

Claim 1 *If $\overleftrightarrow{K}_n[M']$ is isomorphic to one of two graphs in Fig. 3, then $\kappa_2(D') \le n-3$, where $D' = \overleftrightarrow{K}_n - M'$.*

Proof of Claim 1 We first show that $\kappa_2(D') \le n-3$ if M' is the digraph of Fig. 3a. Let $S = \{u_2, u_4\}$; we will prove that $\kappa_S(D') \le n-3$, and then we are done. Suppose that $\kappa_S(D') \ge n-2$, then there exists a set of $n-2$ internally disjoint S-strong subgraphs, say $\{D_i \mid 1 \le i \le n-2\}$. If both of the two arcs u_2u_4 and u_4u_2 belong to the same D_i, say D_1, then for $2 \le i \le n-2$, each D_i contains at least one vertex and at most two vertices of $\{u_i \mid 1 \le i \le n, i \ne 2, 4\}$. Furthermore, there is at most one D_i, say D_2, contains (exactly) two vertices of $\{u_i \mid 1 \le i \le n, i \ne 2, 4\}$. We just consider the case that $u_1, u_3 \in V(D_2)$ since the other cases are similar. In this case, we must have that each vertex of $\{u_i \mid 5 \le i \le n\}$ belongs to exactly one digraph from $\{D_i \mid 3 \le i \le n-2\}$ and vice versa. However, this is impossible since the vertex set $\{u_2, u_4, u_5\}$ cannot induce an S-strong subgraph of D', a contradiction.

So we now assume that each D_i contains at most one of u_2u_4 and u_4u_2. Without

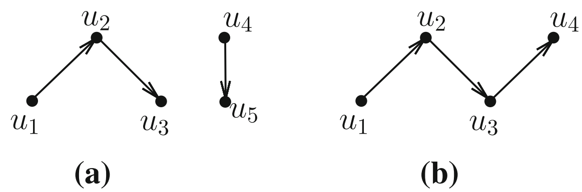

Fig. 3 Two graphs for Claim 1

loss of generality, we may assume that $u_2u_4 \in A(D_1)$ and $u_4u_2 \in A(D_2)$. In this case, we must have that each vertex of $\{u_i \mid 1 \le i \le n, i \ne 2, 4\}$ belongs to exactly one digraph from $\{D_i \mid 1 \le i \le n-2\}$ and vice versa. However, this is also impossible since the vertex set $\{u_2, u_4, u_5\}$ cannot induce an S-strong subgraph of D', a contradiction.

Hence, we have $\kappa_2(D') \le n-3$ in this case. For the case that M' is the digraph of Fig. 3b, we can choose $S = \{u_2, u_3\}$ and prove that $\kappa_S(D') \le n-3$ with a similar argument, and so $\kappa_2(D') \le n-3$ in this case. This completes the proof of the claim.

Secondly, we consider the case that $\overleftrightarrow{K}_n[M]$ is a union of $\lfloor n/2 \rfloor$ vertex-disjoint 2-cycles. Without loss of generality, we may assume that $M = \{u_{2i-1}u_{2i}, u_{2i}u_{2i-1} \mid 1 \le i \le \lfloor n/2 \rfloor\}$. We just consider the case that $S = \{u_1, u_3\}$ since the other cases are similar. In this case, let D_1 be the subgraph of D with $V(D_1) = \{u_1, u_3\}$ and $A(D_1) = \{u_1u_3, u_3u_1\}$; let D_2 be the subgraph of D with $V(D_2) = \{u_1, u_2, u_3, u_4\}$ and $A(D_2) = \{u_1u_4, u_4u_1, u_2u_4, u_4u_2, u_2u_3, u_3u_2\}$; for $3 \le i \le n-2$, let D_i be the subgraph of D with $V(D_i) = \{u_1, u_2, u_{i+2}\}$ and $A(D_i) = \{u_1u_{i+2}, u_3u_{i+2}, u_{i+2}u_1, u_{i+2}u_3\}$. Clearly, $\{D_i \mid 1 \le i \le n-2\}$ is a set of $n-2$ internally disjoint S-strong subgraphs, so $\kappa_S(D) \ge n-2$ and then $\kappa_2(D) \ge n-2$. By (3), we have $\kappa_2(D) \le \min\{\delta^+(D), \delta^-(D)\} = n-2$. Hence, $\kappa_2(D) = n-2$. Let $e \in A(D)$; clearly e must be incident with at least one vertex of $\{u_i \mid 1 \le i \le 2\lfloor n/2 \rfloor\}$. Then we have that $\kappa_2(D-e) \le \min\{\delta^+(D-e), \delta^-(D-e)\} = n-3$ by (3). Hence, D is minimally strong subgraph $(2, n-2)$-connected.

Now let D be minimally strong subgraph $(2, n-2)$-connected. By Theorem 4.1, we have that $D \ncong \overleftrightarrow{K}_n$, that is, D can be obtained from a complete digraph $\overleftrightarrow{K}_n$ by deleting a nonempty arc set M. To end our argument, we need the following three claims. Let us start from a simple yet useful observation.

Proposition 5.4 *No pair of arcs in M has a common head or tail.*

Proof of Proposition 5.4. By (3) no pair of arcs in M has a common head or tail, as otherwise we would have $\kappa_2(D) \le n-3$.

Claim 2 $|M| \ge 3$.

Proof of Claim 2 Let $|M| \le 2$. We may assume that $|M| = 2$ as the case of $|M| = 1$ can be considered in a similar and simpler way.

Let the arcs of M have no common vertices; without loss of generality, $M = \{u_1u_2, u_3u_4\}$. Then $\kappa_2(D - u_2u_1) = n-2$ as $D - u_2u_1$ is a supergraph of $\overleftrightarrow{K}_n$ without a union of $\lfloor n/2 \rfloor$ vertex-disjoint 2-cycles including the cycles $u_1u_2u_1$ and $u_3u_4u_3$. Thus, D is not minimally strong subgraph $(2, n-2)$-connected. Let the arcs of M have no common vertex. By Proposition 5.4, without loss of generality, $M = \{u_1u_2, u_2u_3\}$. Then $\kappa_2(D - u_3u_1) = n-2$ as we showed in the beginning of the proof of this theorem. Thus, D is not minimally strong subgraph $(2, n-2)$-connected. Now let the arcs of M have the same vertices, i.e., without loss of generality, $M = \{u_1u_2, u_2u_1\}$. As above, $\kappa_2(D - u_2u_1) = n-2$ and D is not minimally strong subgraph $(2, n-2)$-connected.

Claim 3 *If $|M| = 3$, then $\overleftrightarrow{K}_n[M]$ is a* 3*-cycle.*

Proof of Claim 3 Suppose that D is minimally strong subgraph $(2, n-2)$-connected, but $\overleftrightarrow{K}_n[M]$ is not a 3-cycle. By Proposition 5.4, no pair of arcs in M has a common head or tail. Thus, $\overleftrightarrow{K}_n[M]$ must be isomorphic to one of graphs in Figs. 3 and 4. If $\overleftrightarrow{K}_n[M]$ is isomorphic to one of graphs in Fig. 3, then $\kappa_2(D) \leq n-3$ by Claim 1 and so D is not minimally strong subgraph $(2, n-2)$-connected, a contradiction. For an arc set M_0 such that $\overleftrightarrow{K}_n[M_0]$ is a union of $\lfloor n/2 \rfloor$ vertex-disjoint 2-cycles, by the argument before, we know that $\overleftrightarrow{K}_n - M_0$ is minimally strong subgraph $(2, n-2)$-connected. For the case that $\overleftrightarrow{K}_n[M]$ is isomorphic to (a) or (b) in Fig. 4, we have that $\overleftrightarrow{K}_n - M_0$ is a proper subgraph of $\overleftrightarrow{K}_n - M$, so $D = \overleftrightarrow{K}_n - M$ must not be minimally strong subgraph $(2, n-2)$-connected, this also produces a contradiction. Hence, the claim holds.

Claim 4 *If* $|M| > 3$, *then* $\overleftrightarrow{K}_n[M]$ *is a union of* $\lfloor n/2 \rfloor$ *vertex-disjoint* 2*-cycles.*

Proof of Claim 4 Suppose that D is minimally strong subgraph $(2, n-2)$-connected, but $\overleftrightarrow{K}_n[M]$ is not a union of $\lfloor n/2 \rfloor$ vertex-disjoint 2-cycles.

By Claim 1 and Proposition 4.3, we have that $\overleftrightarrow{K}_n[M]$ does not contain graphs in Fig. 3 as a subgraph. Then $\overleftrightarrow{K}_n[M]$ does not contain a path of length at least three. Hence, the underlying undirected graph of M has at least two connectivity components. By the fact that if M is a 3-cycle, then $\overleftrightarrow{K}_n - M$ is minimally strong subgraph $(2, n-2)$-connected, we conclude that $\overleftrightarrow{K}_n[M]$ does not contain a cycle of length three. By Claim 1, $\overleftrightarrow{K}_n[M]$ does not contain a path of length two. By Proposition 5.4, no pair of arcs in M has a common head or tail. Hence, each connectivity component of $\overleftrightarrow{K}_n[M]$ must be a 2-cycle or an arc. Since D is minimally strong subgraph $(2, n-2)$-connected, no connectivity component of $\overleftrightarrow{K}_n[M]$ is an arc. We have arrived at a contradiction, proving Claim 4.

Hence, if a digraph D is minimally strong subgraph $(2, n-2)$-connected, then $D \cong \overleftrightarrow{K}_n - M$, where $\overleftrightarrow{K}_n[M]$ is a cycle of order three or a union of $\lfloor n/2 \rfloor$ vertex-disjoint 2-cycles.

Now the claimed values of $F(n, 2, n-2)$ and $f(n, 2, n-2)$ can easily be verified.

Let $\mathfrak{F}(n, k, \ell)$ be the set of all minimally strong subgraph (k, ℓ)-connected digraphs with order n. We define

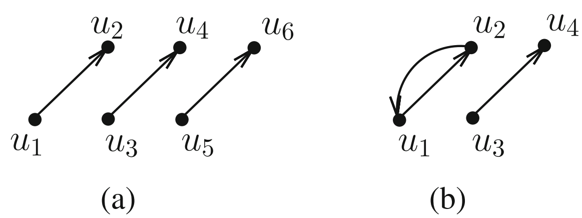

Fig. 4 Two graphs for Claim 3

$$F(n,k,\ell) = \max\{|A(D)| \mid D \in \mathfrak{F}(n,k,\ell)\}$$

and

$$f(n,k,\ell) = \min\{|A(D)| \mid D \in \mathfrak{F}(n,k,\ell)\}.$$

We further define

$$Ex(n,k,\ell) = \{D \mid D \in \mathfrak{F}(n,k,\ell), |A(D)| = F(n,k,\ell)\}$$

and

$$ex(n,k,\ell) = \{D \mid D \in \mathfrak{F}(n,k,\ell), |A(D)| = f(n,k,\ell)\}.$$

Note that Theorem 5.3 implies that $Ex(n,2,n-2) = \{\overleftrightarrow{K_n} - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a directed 3-cycle, and $ex(n,2,n-1) = \{\overleftrightarrow{K_n} - M\}$ where M is an arc set such that $\overleftrightarrow{K}_n[M]$ is a union of $\lfloor n/2 \rfloor$ vertex-disjoint directed 2-cycles.

The following result concerns a sharp lower bound for the parameter $f(n,k,\ell)$.

Theorem 5.5 *For $2 \leq k \leq n$, we have*

$$f(n,k,\ell) \geq n\ell.$$

Moreover, the following assertions hold: (i) If $\ell = 1$, then $f(n,k,\ell) = n$; (ii) If $2 \leq \ell \leq n-1$, then $f(n,n,\ell) = n\ell$ for $k = n \notin \{4,6\}$; (iii) If n is even and $\ell = n-2$, then $f(n,2,\ell) = n\ell$.

Proof By (3), for all digraphs D and $k \geq 2$ we have $\kappa_k(D) \leq \delta^+(D)$ and $\kappa_k(D) \leq \delta^-(D)$. Hence for each D with $\kappa_k(D) = \ell$, we have that $\delta^+(D), \delta^-(D) \geq \ell$, so $|A(D)| \geq n\ell$ and then $f(n,k,\ell) \geq n\ell$.

For the case that $\ell = 1$, let D be a dicycle $\overrightarrow{C_n}$. Clearly, D is minimally strong subgraph (k, 1)-connected, and we know $|A(D)| = n$, so $f(n,k,1) = n$.

For the case that $k = n \notin \{4,6\}$ and $2 \leq \ell \leq n-1$, let $D \cong \overleftrightarrow{K_n}$. By Theorem 2.2, D can be decomposed into $n-1$ Hamiltonian cycles $H_i (1 \leq i \leq n-1)$. Let D_ℓ be the spanning subgraph of D with arc sets $A(D_\ell) = \bigcup_{1 \leq i \leq \ell} A(H_i)$. Clearly, we have $\kappa_n(D_\ell) \geq \ell$ for $2 \leq \ell \leq n-1$. Furthermore, by (3), we have $\kappa_n(D_\ell) \leq \ell$ since the in-degree and out-degree of each vertex in D_ℓ are both ℓ. Hence, $\kappa_n(D_\ell) = \ell$ for $2 \leq \ell \leq n-1$. For any $e \in A(D_\ell)$, we have $\delta^+(D_\ell - e) = \delta^-(D_\ell - e) = \ell - 1$, so $\kappa_n(D_\ell - e) \leq \ell - 1$ by (3). Thus, D_ℓ is minimally strong subgraph (n,ℓ)-connected. As $|A(D_\ell)| = n\ell$, we have $f(n,n,\ell) \leq n\ell$. From the lower bound that $f(n,k,\ell) \geq n\ell$, we have $f(n,n,\ell) = n\ell$ for the case that $2 \leq \ell \leq n-1, n \notin \{4,6\}$.

Part (iii) follows directly from Theorem 5.3. □

To prove two upper bounds on the number of arcs in a minimally strong subgraph (k,ℓ)-connected digraph, we will use the following result from [2].

Theorem 5.6 *Every strong digraph D on n vertices has a strong spanning subgraph H with at most $2n-2$ arcs and equality holds only if H is a symmetric digraph whose underlying undirected graph is a tree.*

Proposition 5.7 *We have (i)* $F(n, n, \ell) \leq 2\ell(n-1)$*; (ii) For every* k $(2 \leq k \leq n)$, $F(n, k, 1) = 2(n-1)$ *and Ex(n, k, 1) consists of symmetric digraphs whose underlying undirected graphs are trees.*

Proof (i) Let $D = (V, A)$ be a minimally strong subgraph (n, ℓ)-connected digraph, and let $D_1, \ldots, D_\ell$ be arc-disjoint strong spanning subgraphs of D. Since D is minimally strong subgraph (n, ℓ)-connected and $D_1, \ldots, D_\ell$ are pairwise arc-disjoint, $|A| = \sum_{i=1}^{\ell} |A(D_i)|$. Thus, by Theorem 5.6, $|A| \leq 2\ell(n-1)$.

(ii) In the proof of Proposition 5.2 we showed that a digraph D is strong if and only if $\kappa_k(D) \geq 1$. Now let $\kappa_k(D) \geq 1$ and a digraph D has a minimal number of arcs. By Theorem 5.6, we have that $|A(D)| \leq 2(n-1)$ and if $D \in Ex(n, k, 1)$ then $|A(D)| = 2(n-1)$ and D is a symmetric digraph whose underlying undirected graph is a tree. □

We now study the minimally strong subgraph (k, ℓ)-arc-connected digraphs. By Proposition 4.8 and Theorem 4.5, we have the following result.

Proposition 5.8 *The following assertions hold:*

(i) A digraph D is minimally strong subgraph
(k, 1) -arc-connected if and only if D is minimally strong digraph;
(ii) Let $2 \leq k \leq n$. If $k \notin \{4, 6\}$, or, $k \in \{4, 6\}$ and $k < n$, then a digraph D is minimally strong subgraph $(k, n-1)$-arc-connected if and only if $D \cong \overleftrightarrow{K}_n$.

The following result characterizes minimally strong subgraph $(2, n-2)$-arc-connected digraphs. This characterization is different from the characterization of minimally strong subgraph $(2, n-2)$-connected digraphs obtained in Theorem 5.3.

Theorem 5.9 *A digraph D is minimally strong subgraph* $(2, n-2)$*-arc-connected if and only if D is a digraph obtained from the complete digraph* $\overleftrightarrow{K}_n$ *by deleting an arc set M such that* $\overleftrightarrow{K}_n[M]$ *is a union of vertex-disjoint cycles which cover all but at most one vertex of* $\overleftrightarrow{K}_n$.

Proof Let D be a digraph obtained from the complete digraph $\overleftrightarrow{K}_n$ by deleting an arc set M such that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overleftrightarrow{K}_n$. To prove the theorem it suffices to show that (a) D is minimally strong subgraph $(2, n-2)$-arc-connected, that is, $\lambda_2(D) \geq n-2$ but for any arc $e \in A(D)$, $\lambda_2(D-e) \leq n-3$, and (b) if a digraph H minimally strong subgraph $(2, n-2)$-arc-connected then it must be constructed from $\overleftrightarrow{K}_n$ as the digraph D above. Thus, the remainder of the proof has two parts.

Part (a). We just consider the case that $\overleftrightarrow{K}_n[M]$ is a union of vertex-disjoint cycles which cover all vertices of $\overleftrightarrow{K}_n$, since the argument for the other case is similar. For any $e \in A(\overleftrightarrow{K}_n) \setminus M$, we know e must be adjacent to at least one element of M, so $\lambda_2(D-e) \leq \min\{\delta^+(D-e), \delta^-(D-e)\} = n-3$ by (3). Hence, it suffices to show that $\lambda_2(D) = n-2$ in the following. We clearly have that $\lambda_2(D) \leq n-2$ by (3), so we will show that for $S = \{x, y\} \subseteq V(D)$, there are at least $n-2$ arc-disjoint S-strong subgraphs in D.

Case 1. x and y belong to distinct cycles of $\overleftrightarrow{K}_n[M]$. We just consider the case that the lengths of these two cycles are both at least three, since the arguments for the other cases are similar. Assume that u_1x, xu_2 belong to one cycle, and u_3y, yu_4 belong to the other cycle. Note that $u_1u_2, u_3u_4 \in A(D)$ since the lengths of these two cycles are both at least three.

Let D_1 be the 2-cycle xyx; let D_2 be the subgraph of D with vertex set $\{x, y, u_1, u_2\}$ and arc set $\{xu_1, u_1u_2, u_2x, yu_2, u_2y\}$; let D_3 be the subgraph of D with vertex set $\{x, y, u_3, u_4\}$ and arc set $\{yu_3, u_3u_4, u_4y, xu_3, u_3x\}$; let D_4 be the subgraph of D with vertex set $\{x, y, u_1, u_4\}$ and arc set $\{xu_4, u_4x, yu_1, u_1y, u_1u_4, u_4u_1\}$; for each vertex $u \in V(D) \setminus \{x, y, u_1, u_2, u_3, u_4\}$, let D_u be a subgraph of D with vertex set $\{u, x, y\}$ and arc set $\{ux, xu, uy, yu\}$. It is not hard to check that these $n-2$ S-strong subgraphs are arc-disjoint.

Case 2. x and y belong to the same cycle, say $u_1u_2\cdots u_tu_1$, of $\overleftrightarrow{K}_n[M]$. We just consider the case that the length of this cycle is at least three, since the argument for the remaining case is simpler.

Subcase 2.1. x and y are adjacent in the cycle. Without loss of generality, let $x = u_1, y = u_2$. Let D_1 be the subgraph of D with vertex set $\{x, y, u_3\}$ and arc set $\{yx, xu_3, u_3y\}$; let D_2 be the subgraph of D with vertex set $\{x, y, u_3, u_t\}$ and arc set $\{u_3x, xu_t, u_tu_3, u_ty, yu_t\}$; for each vertex $u \in V(D) \setminus \{x, y, u_3, u_t\}$, let D_u be a subgraph of D with vertex set $\{u, x, y\}$ and arc set $\{ux, xu, uy, yu\}$. It is not hard to check that these $n-2$ S-strong subgraphs are arc-disjoint.

Subcase 2.2. x and y are nonadjacent in the cycle. Without loss of generality, let $x = u_1, y = u_3$. Let D_1 be the 2-cycle xyx; let D_2 be the subgraph of D with vertex set $\{x, y, u_2, u_t\}$ and arc set $\{yu_2, u_2x, xu_t, u_ty\}$; for each vertex $u \in V(D) \setminus \{x, y, u_2, u_t\}$, let D_u be a subgraph of D with vertex set $\{u, x, y\}$ and arc set $\{ux, xu, uy, yu\}$. It is not hard to check that these $n-2$ S-strong subgraphs are arc-disjoint.

Part (b). Let H be minimally strong subgraph $(2, n-2)$-arc-connected. By Lemma 4.4, we have that $H \ncong \overleftrightarrow{K}_n$, that is, H can be obtained from a complete digraph $\overleftrightarrow{K}_n$ by deleting a nonempty arc set M. To end our argument, we need the following claim. Let us start from a simple yet useful observation, which follows by Inequality (3) □

Proposition 5.10 *No pair of arcs in M has a common head or tail.*

Thus, $\overleftrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles or paths, otherwise, there are two arcs of M such that they have a common head or tail, a contradiction with Proposition 5.10.

Claim 1 $\overleftrightarrow{K}_n[M]$ *does not contain a path of order at least two.*

Proof of Claim 1 Let $M' \supseteq M$ be a set of arcs obtained from M by adding some arcs from $\overleftrightarrow{K}_n$ such that the digraph $\overleftrightarrow{K}_n[M']$ contains no path of order at least two. Note that $\overleftrightarrow{K}_n[M']$ is a supergraph of $\overleftrightarrow{K}_n[M]$ and is a union of vertex-disjoint cycles which cover all but at most one vertex of $\overleftrightarrow{K}_n$. By Part (a), we have that $\lambda_2(\overleftrightarrow{K}_n[M']) = n-2$, so $\overleftrightarrow{K}_n[M]$ is not minimally strong subgraph $(2, n-2)$-arc-connected, a contradiction.

It follows from Claim 1 and its proof that $\overleftrightarrow{K}_n[M]$ must be a union of vertex-disjoint cycles which cover all but at most one vertex of $\overleftrightarrow{K}_n$, which completes the proof of Part (b).

6 Discussion

Corollaries 3.5 and 3.7 shed some light on the complexity of deciding, for fixed $k, \ell \geq 2$, whether $\lambda_k(D) \geq \ell$ for semicomplete and symmetric digraphs D. However, it is unclear what is the complexity above for every fixed $k, \ell \geq 2$. If Conjecture 1 is correct, then the $\lambda_k(D) \geq \ell$ problem can be solved in polynomial time for semicomplete digraphs. However, Conjecture 1 seems to be very difficult. It was proved in [13] that for fixed $k, \ell \geq 2$ the problem of deciding whether $\kappa_k(D) \geq \ell$ is polynomial-time solvable for both semicomplete and symmetric digraphs, but it appears that the approaches to prove the two results cannot be used for $\lambda_k(D)$. Some well-known results such as the fact that the hamiltonicity problem is NP-complete for undirected 3-regular graphs, indicate that the $\lambda_k(D) \geq \ell$ problem for symmetric digraphs may be NP-complete, too.

One of the most interesting results of this paper is the characterization of minimally strong subgraph $(2, n-2)$-connected digraphs. As a simple consequence of the characterization, we can determine the values of $f(n, 2, n-2)$ and $F(n, 2, n-2)$. It would be interesting to determine $f(n, k, n-2)$ and $F(n, k, n-2)$ for every value of $k \geq 3$. (Obtaining characterizations of all $(k, n-2)$-connected digraphs for $k \geq 3$ seems a very difficult problem.) It would also be interesting to find a sharp upper bound for $F(n, k, \ell)$ for all $k \geq 2$ and $\ell \geq 2$.

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