The Polychromatic Number of Small Subsets of the Integers Modulo n

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Abstract

If S is a subset of an abelian group G, the *polychromatic number* of S in G is the largest integer k so that there is a k-coloring of the elements of G such that every translate of S in G gets all k colors. We determine the polychromatic number of all sets of size 2 or 3 in the group of integers mod n.

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1 Introduction

Throughout this paper G will denote an arbitrary abelian group. Given $S \subseteq G$, $a \in G$, $a + S = \{a + s | s \in S\}$. Any set of the form a + S is called a *translate* of S. A k-coloring of the elements of G is S-polychromatic if every translate of S contains an element of each of the k colors. The polychromatic number of S in G, denoted $p_G(S)$, is the largest number of colors such that there exists an S-polychromatic coloring of G. The notation p(S) is used when G is the set of integers, \mathbb{Z} , and $p_n(S)$ is used when $G = \mathbb{Z}_n$, the group of integers mod n. In this paper, $p_n(S)$ is determined for all $n \geq 3$ and |S| = 2 or 3. The techniques used may be useful in determining $p_n(S)$ for larger sets S and for other coloring problems.

The notions of polychromatic colorings and polychromatic number for sets in abelian groups can be extended. If G is any structure and H is a family of substructures then a k-coloring of G is H-polychromatic if every member of H gets all k colors, and the polychromatic number $p_G(H)$ of H in G is the largest k such that there is an H-polychromatic coloring with k colors. In this paper, G is \mathbb{Z}_n and H is the family of all translates of a subset S. Alon et.al. [1], Bialostocki [4], Offner [9], and Goldwasser et.al. [6] considered the case when G is an n-cube and H is the family of all sub-d-cubes for some fixed $d \leq n$. Axenovich et. al. [2] considered the case where G is the complete graph on n vertices and H is the family of all perfect matchings or Hamiltonian cycles or 2-factors.

If S and T are subsets of an abelian group G, we say T is a blocking set for S if $G \setminus T$ contains no translate of S. Blocking sets are of interest in extremal combinatorics, because if T is a minimum size blocking set for S then $G \setminus T$ is a maximum size subset of G with no

translate of S, so is the solution to a Turán-type problem. It is well known ([3],[10]) that T is a complement set for S if and only if -T is a blocking set for S. Clearly each color class in an S-polychromatic coloring is a blocking set for S.

In [3], Axenovich et. al. considered the situation when G is the group of integers and H is the family of all translations of a set S of 4 integers. They showed that the polychromatic number of any set S of 4 integers in \mathbb{Z} is at least 3, by finding a particular value of n such that $3 \leq p_n(S)$. That implies that any set S of size 4 has a blocking set in \mathbb{Z} of density at most 1/3, proving a conjecture of Newman about densities of complement sets.

Whereas in [3] it was shown that for each set S of integers of size 4, there exists an integer n such that $3 \leq p_n(S)$, such an inequality does not hold for all S and n. For example, if $S = \{0, 1, 3, 6\}$ and n = 11, then $p_n(S) = 2$. It would be difficult to determine $p_n(S)$ for all values of n and all sets S of size 4, but in this paper these values are determined for all sets S of size 3.

Example 1.1. Let $S = \{0, a, b\}$ be a subset of \mathbb{Z}_n where *n* is divisible by 3, $a \equiv 1 \pmod{3}$, and $b \equiv 2 \pmod{3}$. Then $p_n(S) = 3$ as the coloring *RBYRBY*... is obviously *S*-polychromatic.

Example 1.2. If $S = \{0, 1, 3\}$ and n = 7 then $p_n(S) = 1$.



Figure 1: Fano plane and an incidence matrix

Consider the above figure and note that the 7×7 circulant matrix is an incidence matrix for the Fano plane. It is well known (and it is easy to check) that in any 2-coloring of the vertices of the Fano plane there is a monochromatic edge, which implies there is no S-polychromatic 2-coloring, so $p_7(S) = 1$.

The main result of this paper is that examples 1.1 and 1.2 are essentially the only examples of sets S of size three such that $p_n(S)$ is not equal to 2.

2 Simplifying assumptions and the main theorem

The polychromatic number of a set S in \mathbb{Z}_n is unchanged under certain operations involving translation, multiplication, and scaling. If |S| = 3 we can use those operations to convert a set S to a set S' which has the same polychromatic number, and has one of two specific forms.

Lemma 2.1. If $1 \le d, t, n \in \mathbb{Z}$, $S = \{a_1, a_2, \dots a_t\} \subseteq \mathbb{Z}_n$, and $S' = \{da_1, da_2, \dots da_t\}$, then $p_{dn}(S') = p_n(S)$.

Proof. Any S-polychormatic coloring of \mathbb{Z}_n can clearly be copied on the subgroup $\langle d \rangle$ of \mathbb{Z}_{dn} , and then duplicated on all the cosets of $\langle d \rangle$, to get an S'-polychromatic coloring of \mathbb{Z}_{dn} . Going the other way, in any S'-polychromatic coloring of \mathbb{Z}_{dn} , the restricted coloring on $\langle d \rangle$ can be copied on \mathbb{Z}_n to get an S-polychromatic coloring. \Box

Hence we can simply divide out a common factor of n and the elements of S without changing the polychromatic number. Since we can also take any translation of S without changing the polychromatic number, from now on we will assume that every set S of size 3 in \mathbb{Z}_n has the form $S = \{0, a, b\}$ where gcd(a, b, n) = 1.

Lemma 2.2. Let $1 \le d, t, n \in \mathbb{Z}$ such that d < n and gcd(d, n) = 1. If $S' = \{da_1, da_2, \dots, da_t\}$ and $S = \{a_1, a_2, \dots, a_t\}$, then $p_n(S) = p_n(S')$.

Proof. If χ' is S'-polychromatic, the coloring χ defined by $\chi(y) = \chi'(dy)$ is clearly S-polychromatic. This argument can be reversed since d is invertible in \mathbb{Z}_n .

Definition 2.3. If $S = \{a_1, a_2, \ldots, a_t\} \subseteq \mathbb{Z}_n$ and $S' = \{da_1 + c, da_2 + c, \ldots, da_t + c\}$, where $c, d \in \mathbb{Z}_n$ and gcd(d, n) = 1, then we say that S and S' are *equivalent* sets in \mathbb{Z}_n .

Thus, Lemma 2.2 says that equivalent sets in \mathbb{Z}_n have the same polychromatic number.

Lemma 2.4. For all $b \in \mathbb{Z}_n$ with $3 \le n$ there exists $b' \in \mathbb{Z}_n$ so that $b' \le \lceil \frac{n}{2} \rceil$ and $p(\{0, 1, b\}) = p(\{0, 1, b'\})$.

Proof. Since, n-1 is always relatively prime to n for $3 \le n$, $p_n(S) = p_n(-S)$ for all $S \subseteq \mathbb{Z}_n$ by Lemma 2.2. If $\lceil \frac{n}{2} \rceil < b$, then let $b' = n-b+1 \le \frac{n}{2}$. Therefore, $p(\{0,1,b\}) = p(\{-1,0,-b\}) = p(\{0,1,-b+1\}) = p(\{0,1,n-b+1\})$.

Proposition 2.5. Let $S = \{0, a, b\} \subseteq \mathbb{Z}_n$ where gcd(a, b, n) = 1. Then at least one of the following occurs.

- i. S is equivalent to a set $S' = \{0, 1, b'\}$ where $b' \leq \lceil \frac{n}{2} \rceil$.
- *ii.* $gcd(a, n) \neq 1$, $gcd(b, n) \neq 1$, $a \notin \langle b \rangle$ and $b \notin \langle a \rangle$.

Proof. If gcd(a, n) = 1 then *a* is invertible in \mathbb{Z}_n , so *S* is equivalent to a set $\{0, 1, c\}$, for some c $(d = a^{-1}$ in Definition 2.3), and then to *S'* by Lemma 2.4. Similarly if gcd(b, n) = 1. Now suppose neither gcd(a, n) nor gcd(b, n) is equal to 1. If *b* is a multiple of *a* then, since gcd(a, b, n) = 1, gcd(a, n) must equal 1, a contradiction, so *b* is not a multiple of *a*. Similarly, *a* is not a multiple of *b*.

We remark that if Case ii occurs and gcd(b-a, n) = 1, then Case i also occurs. However, in our proof we just need that at least one of them occurs. We will treat Case i in Section 5 and Case ii in Section 6. The following theorem is the main result of this paper.

Theorem 2.6. Let $S = \{0, a, b\} \subseteq \mathbb{Z}_n$ and gcd(a, b, n) = 1, then

$$p_n(S) = \begin{cases} 3 & \text{if } 3|n \text{ and } a \text{ and } b \text{ are in different nonzero mod } 3 \text{ congruence classes} \\ 1 & \text{if } n = 7 \text{ and } \{0, a, b\} \text{ is equivalent to } \{0, 1, 3\} \\ 2 & \text{otherwise.} \end{cases}$$

If we do not make the assumption that gcd(a, b, n) = 1, then we get the following theorem, which is clearly equivalent to Theorem 2.6:

Theorem 2.7. If $3 \le n$, $a, b \in \mathbb{Z}_n$, and $a \ne b$, then

$$p_n(\{0, a, b\}) = \begin{cases} 3 & \text{if } n \equiv 0 \mod 3^{j+1}, a = 3^j m_a, b = 3^j m_b, \\ m_a, m_b \not\equiv 0 \mod 3, \text{ and } m_a + m_b \equiv 0 \mod 3 \\ 1 & \text{if } n \equiv 0 \mod 7, |\langle a \rangle| = 7, \text{ and } b = 3a \text{ or } 5a \\ 2 & \text{otherwise.} \end{cases}$$

3 Sets of size 2

For the following proposition we assume without loss of generality that 0 is in the chosen subset of \mathbb{Z}_n .

Proposition 3.1. If $S = \{0, b\} \subseteq \mathbb{Z}_n$ where gcd(b, n) = 1 then

$$p_n(S) = \begin{cases} 1 & \text{if } |\langle b \rangle| \text{ is odd} \\ 2 & \text{if } |\langle b \rangle| \text{ is even.} \end{cases}$$

Proof. Clearly there will be an S-polychromatic 2-coloring of the multiples of b if and only if $|\langle b \rangle|$ is even.

4 Sets that tile

Given a set $S \subseteq G$ where G is an abelian group, a set $T \subseteq G$ is a *complement set* for S if S + T = G. S tiles G by translation if T is a complement set for S and if $s_1, s_2 \in S$, $t_1, t_2 \in T$, and $s_1 + t_1 = s_2 + t_2$ implies $s_1 = s_2$ and $t_1 = t_2$. The notation $S \oplus T$ is used when S tiles G by translation. Without loss of generality, $0 \in S, T$ for all of the following arguments.

Newman [8] proved necessary and sufficient conditions for a finite set S to tile \mathbb{Z} if |S| is a power of a prime.

Theorem 4.1. [8] Let $S = \{s_1, \ldots, s_k\}$ be distinct integers with $|S| = p^{\alpha}$ where p is prime and α is a positive integer. For $1 \le i < j \le k$ let $p^{e_{ij}}$ be the highest power of p that divides $s_i - s_j$. Then S tiles \mathbb{Z} if and only if $|\{e_{ij} : 1 \le i < j \le k\}| \le \alpha$. The characterization of sets S of size 3 such that $p_n(S) = 3$ (Theorem 2.6 and Proposition 4.4) follows immediately from Newman's theorem (Theorem 4.1). When commenting on this theorem in [8] Newman says: "Surely the special case [when |S| = 3] deserves to have a completely trivial proof - but we have not been able to find one."

If there is an S-polychromatic k-coloring of \mathbb{Z}_n , then clearly there is an S-polychromatic k-coloring of \mathbb{Z} with period n. If there is an S-polychromatic k-coloring of \mathbb{Z} for a finite set S, then there is an S-polychromatic k-coloring of \mathbb{Z}_n for some n. To see this, let d equal the largest difference between two elements in S. If χ is an S-polychromatic k-coloring of \mathbb{Z} , there are only $k^{(d+1)}$ possibilities for the coloring on d + 1 consecutive integers, so two such strings must be identical. If n is the difference between the first integers in these two strings, then we can "wrap around" the coloring χ to get an S-polychromatic k-coloring of \mathbb{Z}_n .

Suppose $S = \{0, a, b\}$ and χ is an *S*-polychromatic 3-coloring of \mathbb{Z} . By the above remark there exists an *S*-polychromatic 3-coloring of \mathbb{Z}_n for some *n*. By Proposition 4.4, *a* and *b* are in different nonzero mod 3 congruence classes, which fulfills Newman's wish to have a simple proof of his theorem for the special case when |S| = 3.

Later Coven and Meyerowitz [5] gave necessary and sufficient conditions for S to tile \mathbb{Z} when $|S| = p_1^{\alpha_1} p_2^{\alpha_2}$, where p_1 and p_2 are primes. The following characterization of tiling by translation in an abelian group was obtained in [3].

Theorem 4.2. [3] Let G be an abelian group and S a finite subset of G. S tiles G by translation if and only if p(S) = |S|. Moreover, if χ is an S-polychromatic coloring of G with |S| colors and T is a color class of χ , then $S \oplus T = G$.

Lemma 4.3. Suppose $S = \{0, a, b\}$ where gcd(a, b, n) = 1, $S \oplus T = \mathbb{Z}_n$ and $0 \in T$. If $x \in T$, then $x + \langle a + b \rangle \subseteq T$.

Proof. Note that because $S \oplus T = \mathbb{Z}_n$, every element of \mathbb{Z}_n belongs to exactly one of the sets T, a + T, b + T.

Suppose $x \in T$. If $x + a + b \in b + T$, then $x + a \in T$. However, $x + a \in a + T$. If $x + a + b \in a + T$, then $x + b \in T$. However, $x + b \in b + T$. Hence $x + a + b \in T$ and, repeating the argument, $x + \langle a + b \rangle \subseteq T$.

Proposition 4.4. Let $S = \{0, a, b\} \subseteq \mathbb{Z}_n$ where gcd(a, b, n) = 1. Then $p_n(S) = 3$ if and only if 3|n and a and b are in different nonzero mod 3 congruence classes.

Proof. If 3|n and a and b are in different nonzero mod 3 congruence classes then clearly the alternating coloring RBYRBY... is polychromatic, so $p_n(S) = 3$. Conversely, suppose $p_n(S) = 3$. Hence, by Theorem 4.2, S tiles \mathbb{Z}_n .

Let $T \subseteq \mathbb{Z}_n$ such that $\mathbb{Z}_n = \{0, a, b\} \oplus T$ and $0 \in T \subseteq \mathbb{Z}_n$. Therefore, n = 3|T| which implies $n \equiv 0 \mod 3$. By Lemma 4.3, for any $x \in T$, the coset $x + \langle a + b \rangle$ is a subset of T, so T is the disjoint untion of cosets of $\langle a + b \rangle$. Therefore, there is some integer q such that $q|\langle a + b \rangle| = |T| = \frac{n}{3}$. Also, $|\langle a + b \rangle| = \frac{n}{gcd(a+b,n)}$. Thus, $q\frac{n}{gcd(a+b,n)} = \frac{n}{3}$, which implies 3q = gcd(a + b, n). Hence, 3|(a + b). Since 3 cannot divide both a and b, it follows that aand b are in different nonzero mod 3 congruence classes. \Box

5 Subsets of the form $\{0, 1, b\}$

As shown in Proposition 2.5, every set S of size 3 is equivalent to a set S' with two possible forms. In this section we will consider case i of Proposition 2.5, that S' contains 0 and 1.

Lemma 5.1. If n is odd, $5 \le n$, and $n \ne 7$, then there exists a $\{0, 1, 3\}$ -polychromatic coloring of \mathbb{Z}_n with two colors.

Proof. It is easy to check that each integer greater than 3, except 7, is the sum of an even number of 2s and 3s. We color \mathbb{Z}_n by alternating colors of strings of 2 or 3 consecutive elements with the same color. Of course there must be an even number of strings. For example, 9=2+2+2+3, so the coloring would be *RRBBRRBBB*; 11=2+3+3+3, so the coloring would be *RRBBBRRRBBB*. Clearly any translate of S hits two consecutive strings, so gets both colors.

As will be seen in the proof of Theorem 5.3, it is easy to show that $p_n(\{0, 1, b\}) \ge 2$ if b or n is even. The following lemma takes care of the more difficult case.

Lemma 5.2. Let $9 \leq n$, b and n both be odd, and $S = \{0, 1, b\} \subset \mathbb{Z}_n$. There exists an S-polychromatic coloring of \mathbb{Z}_n with two colors.

Proof. It can be assumed that $5 \le b \le \lceil \frac{n}{2} \rceil$, by Lemma 2.4 and 5.1, and n = m(b-2) + r, with $0 \le r \le b-3$. Since $2(\lceil \frac{n}{2} \rceil - 2) + r \le n - 3 + r \le n$, m is at least 2.

Let $x \in \mathbb{Z}_n$ and $x \equiv y \mod (b-2)$ such that $0 \leq y \leq b-3$. If r = 0, then define $\chi_0 : \mathbb{Z}_n \to \{R, B\}$ such that

$$\chi_0(x) = \begin{cases} R & \text{if } y = 0\\ R & \text{if } y \text{ is odd}\\ B & \text{if } y \text{ is even and } y > 0. \end{cases}$$

If $\chi_0(x) = \chi_0(x+1)$, then $x \equiv 0 \mod b - 2$ and $\chi_0(x+b) = B$. This means that $\chi_0(x) \neq \chi_0(x+1)$ or $\chi_0(x) \neq \chi_0(x+b)$. Therefore, every translate of $S = \{x, x+1, x+b\}$ will contain two colors under χ_0 .

Throughout the remainder of the proof each of the colorings that are constructed will use χ_0 to assign colors to at least the first (m-1)(b-2) elements of \mathbb{Z}_n .

If r = 1, then define $\chi_1 : \mathbb{Z}_n \to \{R, B\}$ such that

$$\chi_1(x) = \begin{cases} \chi_0(x) & \text{if } x \le n - b \\ R & \text{if } x \text{ is even and } n - b < x < n - 1 \\ B & \text{if } x \text{ is odd and } n - b < x < n - 1 \\ B & \text{if } x = n - 1. \end{cases}$$

In χ_1 , the two translates that are not colored completely by χ_0 and don't have $\chi_1(x) \neq \chi_1(x+1)$ are $\{n-b, n-b+1, 0\}$ and $\{n-2, n-1, b-2\}$. In both cases, the nonconsecutive element of the translate is the other color.

If r = 2, then define $\chi_2 : \mathbb{Z}_n \to \{R, B\}$ such that

$$\chi_2(x) = \begin{cases} \chi_0(x) & \text{if } x \le n - b - 1 \\ R & \text{if } x = n - b \\ B & \text{if } x = n - b + 1 \\ R & \text{if } x \text{ is odd and } n - b + 1 < x \\ B & \text{if } x \text{ is even and } n - b + 1 < x. \end{cases}$$

In χ_2 , the only translate that is not colored completely by χ_0 and doesn't have $\chi_2(x) \neq \chi_2(x+1)$ is $\{n-b+1, n-b+2, 1\}$; however, $\chi_2(n-b+1) \neq \chi_2(1)$.

If r = 3, then define $\chi_3 : \mathbb{Z}_n \to \{R, B\}$ such that

$$\chi_3(x) = \begin{cases} \chi_0(x) & \text{if } x \le n - b - 2\\ R & \text{if } x = n - b - 1\\ R & \text{if } x \text{ is even and } n - b - 1 < x < n - 1\\ B & \text{if } x \text{ is odd and } n - b - 1 < x < n - 1\\ B & \text{if } x = n - 1. \end{cases}$$

In χ_3 , the two translates that are not colored completely by χ_0 and don't have $\chi_3(x) \neq \chi_3(x+1)$ are $\{n-b-1, n-b, n-1\}$ and $\{n-2, n-1, b-2\}$. In both cases, the nonconsecutive element of the translate is the other color.

Assume $4 \leq r$. An S-polychromatic coloring, $\chi_4 : \mathbb{Z}_n \to \{R, B\}$, will be constructed. Define $\chi_4(x) = \chi_0(x)$ for $x \leq n - b - r + 4$, $\chi_4(n - r + 2) = B$ and $\chi_4(n - 1) = B$. So each translate with $0 \leq x \leq n - b - r + 3$ contains both colors. The two translates with $n - 2 \leq x$ also contain both colors even though $\chi_4(n - 2)$ has not been defined unless r = 4. This means there is an *option* for assigning a color to n - 2. Therefore, $\chi_4(x)$ can be defined, and will be defined, such that for $n - b + 2 \leq x \leq n - 2$ the assigned colors alternate while keeping $\chi_4(n - r + 2) = B$. This means that each translate with $n - b + 2 \leq x$ contains both colors.

If r = 4, then n-b = n-b-r+4 has already been assigned the color B and the translate $\{n-b, n-b+1, 0\}$ contains both colors. By defining $\chi_4(n-b+1)$ to be B, the translate $\{n-b+1, n-b+2, 1\}$ contains both colors and χ_4 is an S-polychromatic coloring.

For $r \neq 4$, consider the translates $\{n - b, n - b + 1, 0\}$ and $\{n - b + 1, n - b + 2, 1\}$. If $\chi_4(n - b + 2) = B$, then n - b + 1 has an option since $\chi_4(1) = R$. If $\chi_4(n - b + 2) = R$, then $\chi_4(n - b + 1)$ must be defined as B and n - b has an option since $\chi_4(0) = R$. Therefore, there will be an option for assigning a color to n - b or n - b + 1. This allows for $\chi_4(x)$ to be defined for $n - b - r + 5 \leq x \leq n - b + 1$ such that the colors alernate while keeping $\chi_4(n - b - r + 4) \neq \chi_4(n - b - r + 5)$. Thus, χ_4 is an S-polychromatic coloring.

Theorem 5.3. Let $3 \leq n$ and $S = \{0, 1, b\} \subseteq \mathbb{Z}_n$. If $n \neq 7$ or $b \neq 3$ or 5, there is an S-polychromatic coloring of \mathbb{Z}_n with two colors.

Proof. If n is even then alternating colors RBRB... is clearly an S-polychromatic coloring. If n is odd and b is even then the coloring RRBRBRBR... which has one repeated color,

and otherwise alternates colors, is S-polychromatic. If n and b are both odd then an S-polychromatic 2-coloring exists by Lemmas 5.1 and 5.2, except in the exceptional case when n = 7.

6 Subsets not equivalent to $\{0, 1, b\}$

Consider the $s \times t$ matrix

$$M = \begin{vmatrix} x_{00} & x_{01} & \dots & x_{0(t-1)} \\ x_{10} & x_{11} & \dots & x_{1(t-1)} \\ \vdots & \vdots & \ddots & \vdots \\ x_{(s-1)0} & x_{(s-1)1} & \dots & x_{(s-1)(t-1)}. \end{vmatrix}$$

An *ell* - *tile* of M is a subset of entries of M consisting of entries of a 2×2 submatrix without the lower right entry:

$$\begin{array}{|c|c|c|c|}\hline x_{ij} & x_{i(j+1)} \\\hline x_{(i+1)j} & \hline \end{array}.$$

The indices are read mod s and mod t, so ell-tiles are allowed to 'wrap around' (i = s - 1) or j = t - 1). An *ell-tile 2-coloring* of M is a coloring of the entries of M with two colors such that both colors appear in every ell-tile of M.

Lemma 6.1. If $2 \leq s, t$, then every $s \times t$ matrix has an ell - tile 2-coloring.

Proof. If s is even, then define χ such that

$$\chi(x_{ij}) = \begin{cases} R & \text{if } i \equiv 0 \mod 2\\ B & \text{if } i \equiv 1 \mod 2. \end{cases}$$

Also, a similar coloring that alternates the colors of the columns works when t is even.

If s and t are both odd, then define χ such that

$$\chi(x_{ij}) = \begin{cases} R & \text{if } i \equiv j \mod 2 \text{ and } (i,j) \neq (0,t-1), (s-1,0) \\ B & \text{otherwise.} \end{cases}$$

If s and t are both odd, then a "checker-board" coloring would assign the same color, say R, to all four corner entries, and the ell-tile with entries $x_{s-1,t-1}$, $x_{0,t-1}$, and $x_{s-1,0}$ would be monochromatic. The coloring χ avoids this problem by changing the color of entries $x_{0,t-1}$ and $x_{s-1,0}$ from R to B, without creating any other monochromatic ell-tiles (just changing the color of one of them would suffice as well).

The goal now is to create matrices with elements from \mathbb{Z}_n such that all of the translates of S correspond to ell-tiles. The matrices then can be colored by using Lemma 6.1, which will create S-polychromatic colorings. **Lemma 6.2.** Let $S = \{0, a, b\} \subseteq \mathbb{Z}_n$, where gcd(a, b, n) = 1 but gcd(a, n) and gcd(b, n) are both greater than 1. Then $p_n(S) \ge 2$.

Proof. If n is even then either a or b is odd, so the alternating coloring $RBRBRB \dots$ is polychromatic, so we can assume n is odd. Let s = gcd(a, n), t = gcd(b, n), and $M = [m_{ij}]$ be the $\frac{n}{s} \times \frac{n}{t}$ matrix with entries in \mathbb{Z}_n where $m_{ij} = ai + bj$, $0 \le i \le \frac{n}{s} - 1$, $0 \le j \le \frac{n}{t} - 1$. Note that $|\langle a \rangle| = \frac{n}{s}$, $|\langle b \rangle| = \frac{n}{t}$, and gcd(s, t) = 1.

If $m_{ij} = m_{i'j'}$ with $0 \le i' \le i \le t - 1$ and $0 \le j, j' \le \frac{n}{t} - 1$, then a(i - i') = b(j' - j). Therefore, t|a(i - i'). This means t|(i - i') since gcd(a, b) = 1, which implies i = i' because $0 \le i - i' < t$. Since a(i - i') = b(j' - j) it is also the case that j = j'. Therefore, each element of \mathbb{Z}_n will be an entry somewhere in the first t rows of M. In fact, the first t rows of M are just the t cosets of $\langle b \rangle$ in \mathbb{Z}_n .

Now let M' be the $\frac{n}{st} \times \frac{n}{st}$ block matrix created by partitioning M into $t \times s$ blocks. Let $A_{i,j}$ be the *ij*th block of M'. Note that $A_{i+1,j} = A_{i,j} + at$ and $A_{i,j+1} = A_{i,j} + bs$ and $|\langle at \rangle| = |\langle bs \rangle| = \frac{n}{st}$, so the matrix $A_{i,j} + k(bs)$ appears as a block in the *i*th row of M' for each integer k. Furthermore, a = ps for some $p \in \mathbb{Z}$ and $bq \equiv t \mod n$ for some $q \in \mathbb{Z}$ since t = gcd(b, n). Therefore, $A_{i+1,j} = A_{i,j} + (pq)bs$, so $A_{i+1,j}$ is equal to some block in the *i*th row of M'.

This means that the (i + 1)st row of M' is the *i*th row of M' shifted by pq, for all *i*. So coloring each matrix in M' with the same ell-tile 2-coloring from Lemma 6.1 will ensure that M is a well-defined ell-tile 2-coloring. It is well-defined since every element is colored and each time an element appears it receives the same color. It is an ell-tile 2-coloring because it is periodic using an ell-tile 2-coloring that 'wraps around'. This yields an S-polychromatic coloring of \mathbb{Z}_n with two colors.

Here is an example of how to get the coloring of \mathbb{Z}_{105} when a = 18 and b = 25. The

matrix M is

Γ 0	25	50	75	100	20	45	70	95	15	40	65	90	10	35	60	85	5	30	55	80 -
18	43	68	93	13	38	63	88	8	- 33	58	83	3	28	53	78	103	23	48	73	98
36	61	86	6	31	56	81	1	26	51	76	101	21	46	71	96	16	41	66	91	11
54	79	104	24	49	74	99	19	44	69	94	14	39	64	89	9	34	59	84	4	29
72	97	17	42	67	92	12	37	62	87	7	32	57	82	2	27	52	77	102	22	47
90	10	35	60	85	5	30	55	80	0	25	50	75	100	20	45	70	95	15	40	65
3	28	53	78	103	23	48	73	98	18	43	68	93	13	38	63	88	8	- 33	58	83
21	46	71	96	16	41	66	91	11	36	61	86	6	31	56	81	1	26	51	76	101
39	64	89	9	34	59	84	4	29	54	79	104	24	49	74	99	19	44	69	94	14
57	82	2	27	52	77	102	22	47	72	97	17	42	67	92	12	37	62	87	7	32
75	100	20	45	70	95	15	40	65	90	10	35	60	85	5	30	55	80	0	25	50
93	13	38	63	88	8	- 33	58	83	3	28	53	78	103	23	48	73	98	18	43	68
6	31	56	81	1	26	51	76	101	21	46	71	96	16	41	66	91	11	36	61	86
24	49	74	99	19	44	69	94	14	39	64	89	9	34	59	84	4	29	54	79	104
42	67	92	12	37	62	87	7	32	57	82	2	27	52	77	102	22	47	72	97	17
60	85	5	30	55	80	0	25	50	75	100	20	45	70	95	15	40	65	90	10	35
78	103	23	48	73	98	18	43	68	93	13	38	63	88	8	33	58	83	3	28	53
96	16	41	66	91	11	36	61	86	6	31	56	81	1	26	51	76	101	21	46	71
9	34	59	84	4	29	54	79	104	24	49	74	99	19	44	69	94	14	39	64	89
27	52	77	102	22	47	72	97	17	42	67	92	12	37	62	87	7	32	57	82	2
45	70	95	15	40	65	90	10	35	60	85	5	30	55	80	0	25	50	75	100	20
63	88	8	33	58	83	3	28	53	78	103	23	48	73	98	18	43	68	93	13	38
81	1	26	51	76	101	21	46	71	96	16	41	66	91	11	36	61	86	6	31	56
99	19	44	69	94	14	39	64	89	9	34	59	84	4	29	54	79	104	24	49	74
12	37	62	87	7	32	57	82	2	27	52	77	102	22	47	72	97	17	42	67	92
30	55	80	0	25	50	75	100	20	45	70	95	15	40	65	90	10	35	60	85	5
48	73	98	18	43	68	93	13	38	63	88	8	33	58	83	3	28	53	78	103	23
66	91	11	36	61	86	6	31	56	81	1	26	51	76	101	21	46	71	96	16	41
84	4	29	54	79	104	24	49	74	99	19	44	69	94	14	39	64	89	9	34	59
102	22	47	72	97	17	42	67	92	12	37	62	87	7	32	57	82	2	27	52	77
15	40	65	90	10	35	60	85	5	30	55	80	0	25	50	75	100	20	45	70	95
33	58	83	3	28	53	78	103	23	48	73	98	18	43	68	93	13	38	63	88	8
51	76	101	21	46	71	96	16	41	66	91	11	36	61	86	6	31	56	81	1	26
69	94	14	39	64	89	9	34	59	84	4	29	54	79	104	24	49	74	99	19	44
L 87	7	32	57	82	2	27	52	77	102	22	47	72	97	17	42	67	92	12	37	62

.

Therefore,

$$M' = \begin{bmatrix} A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} & A_{0,5} & A_{0,6} \\ A_{0,4} & A_{0,5} & A_{0,6} & A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} \\ A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} & A_{0,5} & A_{0,6} & A_{0,0} \\ A_{0,5} & A_{0,6} & A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} \\ A_{0,2} & A_{0,3} & A_{0,4} & A_{0,5} & A_{0,6} & A_{0,0} & A_{0,1} \\ A_{0,6} & A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} & A_{0,5} \\ A_{0,3} & A_{0,4} & A_{0,5} & A_{0,6} & A_{0,0} & A_{0,1} & A_{0,2} \end{bmatrix}.$$

Now Lemma 6.1 can be used to color each $A_{i,j}$ in the following way:

$$\chi(A_{i,j}) = \begin{bmatrix} R & B & B \\ B & R & B \\ R & B & R \\ B & R & B \\ B & B & R \end{bmatrix}.$$

Of course only the first row of M' is needed to get the coloring of \mathbb{Z}_n . Finally, we give a proof of Theorem 2.6. *Proof.* By Proposition 2.5 we have either Case i, where S is equivalent to a set of the form $\{0, 1, b\}$, or Case ii, where gcd(a, n) and gcd(b, n) are both greater than 1. Proposition 4.4 characterizes the sets S for which $p_n(S) = 3$, and Theorem 5.3 shows show that $p_n(S) = 2$ for all other sets S in Case i, except when n = 7 and b = 3. Then Lemma 6.2 takes care of Case ii.

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