# The Polychromatic Number of Small Subsets of the Integers Modulo $n$ 

Emelie Curl, John Goldwasser, Joe Sampson, Michael Young

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#### Abstract

If $S$ is a subset of an abelian group $G$, the polychromatic number of $S$ in $G$ is the largest integer $k$ so that there is a $k$-coloring of the elements of $G$ such that every translate of $S$ in $G$ gets all $k$ colors. We determine the polychromatic number of all sets of size 2 or 3 in the group of integers mod n .


Keywords polychromatic coloring, abelian group, group tiling, complement set
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## 1 Introduction

Throughout this paper $G$ will denote an arbitrary abelian group. Given $S \subseteq G, a \in G$, $a+S=\{a+s \mid s \in S\}$. Any set of the form $a+S$ is called a translate of $S$. A $k$-coloring of the elements of $G$ is $S$-polychromatic if every translate of $S$ contains an element of each of the $k$ colors. The polychromatic number of $S$ in $G$, denoted $p_{G}(S)$, is the largest number of colors such that there exists an $S$-polychromatic coloring of $G$. The notation $p(S)$ is used when $G$ is the set of integers, $\mathbb{Z}$, and $p_{n}(S)$ is used when $G=\mathbb{Z}_{n}$, the group of integers $\bmod n$. In this paper, $p_{n}(S)$ is determined for all $n \geq 3$ and $|S|=2$ or 3 . The techniques used may be useful in determining $p_{n}(S)$ for larger sets $S$ and for other coloring problems.

The notions of polychromatic colorings and polychromatic number for sets in abelian groups can be extended. If $G$ is any structure and $H$ is a family of substructures then a $k$-coloring of $G$ is $H$-polychromatic if every member of $H$ gets all $k$ colors, and the polychromatic number $p_{G}(H)$ of $H$ in $G$ is the largest $k$ such that there is an $H$-polychromatic coloring with k colors. In this paper, $G$ is $\mathbb{Z}_{n}$ and $H$ is the family of all translates of a subset S. Alon et.al. [1], Bialostocki [4], Offner [9, and Goldwasser et.al. [6] considered the case when $G$ is an $n$-cube and $H$ is the family of all sub- $d$-cubes for some fixed $d \leq n$. Axenovich et. al. [2] considered the case where $G$ is the complete graph on $n$ vertices and $H$ is the family of all perfect matchings or Hamiltonian cycles or 2-factors.

If $S$ and $T$ are subsets of an abelian group $G$, we say $T$ is a blocking set for $S$ if $G \backslash T$ contains no translate of $S$. Blocking sets are of interest in extremal combinatorics, because if $T$ is a minimum size blocking set for $S$ then $G \backslash T$ is a maximum size subset of $G$ with no
translate of $S$, so is the solution to a Turán-type problem. It is well known ([3, [10]) that $T$ is a complement set for $S$ if and only if $-T$ is a blocking set for $S$. Clearly each color class in an $S$-polychromatic coloring is a blocking set for $S$.

In [3], Axenovich et. al. considered the situation when $G$ is the group of integers and $H$ is the family of all translations of a set $S$ of 4 integers. They showed that the polychromatic number of any set $S$ of 4 integers in $\mathbb{Z}$ is at least 3, by finding a particular value of $n$ such that $3 \leq p_{n}(S)$. That implies that any set $S$ of size 4 has a blocking set in $\mathbb{Z}$ of density at most $1 / 3$, proving a conjecture of Newman about densities of complement sets.

Whereas in [3] it was shown that for each set $S$ of integers of size 4, there exists an integer $n$ such that $3 \leq p_{n}(S)$, such an inequality does not hold for all $S$ and $n$. For example, if $S=\{0,1,3,6\}$ and $n=11$, then $p_{n}(S)=2$. It would be difficult to determine $p_{n}(S)$ for all values of $n$ and all sets $S$ of size 4, but in this paper these values are determined for all sets $S$ of size 3 .

Example 1.1. Let $S=\{0, a, b\}$ be a subset of $\mathbb{Z}_{n}$ where $n$ is divisible by $3, a \equiv 1(\bmod 3)$, and $b \equiv 2(\bmod 3)$. Then $p_{n}(S)=3$ as the coloring $R B Y R B Y \ldots$ is obviously $S$-polychromatic.
Example 1.2. If $S=\{0,1,3\}$ and $n=7$ then $p_{n}(S)=1$.


Figure 1: Fano plane and an incidence matrix

Consider the above figure and note that the $7 \times 7$ circulant matrix is an incidence matrix for the Fano plane. It is well known (and it is easy to check) that in any 2 -coloring of the vertices of the Fano plane there is a monochromatic edge, which implies there is no $S$-polychromatic 2 -coloring, so $p_{7}(S)=1$.

The main result of this paper is that examples 1.1 and 1.2 are essentially the only examples of sets $S$ of size three such that $p_{n}(S)$ is not equal to 2 .

## 2 Simplifying assumptions and the main theorem

The polychromatic number of a set $S$ in $\mathbb{Z}_{n}$ is unchanged under certain operations involving translation, multiplication, and scaling. If $|S|=3$ we can use those operations to convert a set $S$ to a set $S^{\prime}$ which has the same polychromatic number, and has one of two specific forms.

Lemma 2.1. If $1 \leq d, t, n \in \mathbb{Z}, S=\left\{a_{1}, a_{2}, \ldots a_{t}\right\} \subseteq \mathbb{Z}_{n}$, and $S^{\prime}=\left\{d a_{1}, d a_{2}, \ldots d a_{t}\right\}$, then $p_{d n}\left(S^{\prime}\right)=p_{n}(S)$.

Proof. Any $S$-polychormatic coloring of $\mathbb{Z}_{n}$ can clearly be copied on the subgroup $\langle d\rangle$ of $\mathbb{Z}_{d n}$, and then duplicated on all the cosets of $\langle d\rangle$, to get an $S^{\prime}$-polychromatic coloring of $\mathbb{Z}_{d n}$. Going the other way, in any $S^{\prime}$-polychromatic coloring of $\mathbb{Z}_{d n}$, the restricted coloring on $\langle d\rangle$ can be copied on $\mathbb{Z}_{n}$ to get an $S$-polychromatic coloring.

Hence we can simply divide out a common factor of $n$ and the elements of $S$ without changing the polychromatic number. Since we can also take any translation of $S$ without changing the polychromatic number, from now on we will assume that every set $S$ of size 3 in $\mathbb{Z}_{n}$ has the form $S=\{0, a, b\}$ where $\operatorname{gcd}(a, b, n)=1$.

Lemma 2.2. Let $1 \leq d, t, n \in \mathbb{Z}$ such that $d<n$ and $\operatorname{gcd}(d, n)=1$. If $S^{\prime}=\left\{d a_{1}, d a_{2}, \ldots d a_{t}\right\}$ and $S=\left\{a_{1}, a_{2}, \ldots a_{t}\right\}$, then $p_{n}(S)=p_{n}\left(S^{\prime}\right)$.

Proof. If $\chi^{\prime}$ is $S^{\prime}$-polychromatic, the coloring $\chi$ defined by $\chi(y)=\chi^{\prime}(d y)$ is clearly $S$-polychromatic. This argument can be reversed since $d$ is invertible in $\mathbb{Z}_{n}$.

Definition 2.3. If $S=\left\{a_{1}, a_{2}, \ldots, a_{t}\right\} \subseteq \mathbb{Z}_{n}$ and $S^{\prime}=\left\{d a_{1}+c, d a_{2}+c, \ldots, d a_{t}+c\right\}$, where $c, d \in \mathbb{Z}_{n}$ and $\operatorname{gcd}(d, n)=1$, then we say that $S$ and $S^{\prime}$ are equivalent sets in $\mathbb{Z}_{n}$.

Thus, Lemma 2.2 says that equivalent sets in $\mathbb{Z}_{n}$ have the same polychromatic number.
Lemma 2.4. For all $b \in \mathbb{Z}_{n}$ with $3 \leq n$ there exists $b^{\prime} \in \mathbb{Z}_{n}$ so that $b^{\prime} \leq\left\lceil\frac{n}{2}\right\rceil$ and $p(\{0,1, b\})=$ $p\left(\left\{0,1, b^{\prime}\right\}\right)$.

Proof. Since, $n-1$ is always relatively prime to $n$ for $3 \leq n, p_{n}(S)=p_{n}(-S)$ for all $S \subseteq \mathbb{Z}_{n}$ by Lemma 2.2. If $\left\lceil\frac{n}{2}\right\rceil<b$, then let $b^{\prime}=n-b+1 \leq \frac{n}{2}$. Therefore, $p(\{0,1, b\})=p(\{-1,0,-b\})=$ $p(\{0,1,-b+1\})=p(\{0,1, n-b+1\})$.

Proposition 2.5. Let $S=\{0, a, b\} \subseteq \mathbb{Z}_{n}$ where $\operatorname{gcd}(a, b, n)=1$. Then at least one of the following occurs.
i. $S$ is equivalent to a set $S^{\prime}=\left\{0,1, b^{\prime}\right\}$ where $b^{\prime} \leq\left\lceil\frac{n}{2}\right\rceil$.
ii. $\operatorname{gcd}(a, n) \neq 1, \operatorname{gcd}(b, n) \neq 1, a \notin\langle b\rangle$ and $b \notin\langle a\rangle$.

Proof. If $\operatorname{gcd}(a, n)=1$ then $a$ is invertible in $\mathbb{Z}_{n}$, so $S$ is equivalent to a set $\{0,1, c\}$, for some $c\left(d=a^{-1}\right.$ in Definition 2.3), and then to $S^{\prime}$ by Lemma 2.4. Similarly if $\operatorname{gcd}(b, n)=1$. Now suppose neither $\operatorname{gcd}(a, n)$ nor $\operatorname{gcd}(b, n)$ is equal to 1 . If $b$ is a multiple of $a$ then, since $\operatorname{gcd}(a, b, n)=1, \operatorname{gcd}(a, n)$ must equal 1 , a contradiction, so $b$ is not a multiple of $a$. Similarly, $a$ is not a multiple of $b$.

We remark that if Case $i i$ occurs and $\operatorname{gcd}(b-a, n)=1$, then Case $i$ also occurs. However, in our proof we just need that at least one of them occurs. We will treat Case $i$ in Section 5 and Case $i i$ in Section 6. The following theorem is the main result of this paper.

Theorem 2.6. Let $S=\{0, a, b\} \subseteq \mathbb{Z}_{n}$ and $\operatorname{gcd}(a, b, n)=1$, then

$$
p_{n}(S)= \begin{cases}3 & \text { if } 3 \mid n \text { and } a \text { and } b \text { are in different nonzero } \bmod 3 \text { congruence classes } \\ 1 & \text { if } n=7 \text { and }\{0, a, b\} \text { is equivalent to }\{0,1,3\} \\ 2 & \text { otherwise. }\end{cases}
$$

If we do not make the assumption that $\operatorname{gcd}(a, b, n)=1$, then we get the following theorem, which is clearly equivalent to Theorem 2.6:

Theorem 2.7. If $3 \leq n, a, b \in \mathbb{Z}_{n}$, and $a \neq b$, then

$$
p_{n}(\{0, a, b\})= \begin{cases}3 \quad \text { if } n \equiv 0 \bmod 3^{j+1}, a=3^{j} m_{a}, b=3^{j} m_{b} \\ \quad m_{a}, m_{b} \not \equiv 0 \bmod 3, \text { and } m_{a}+m_{b} \equiv 0 \bmod 3 \\ 1 \quad \text { if } n \equiv 0 \bmod 7,|\langle a\rangle|=7, \text { and } b=3 a \text { or } 5 a \\ 2 \quad \text { otherwise }\end{cases}
$$

## 3 Sets of size 2

For the following proposition we assume without loss of generality that 0 is in the chosen subset of $\mathbb{Z}_{n}$.

Proposition 3.1. If $S=\{0, b\} \subseteq \mathbb{Z}_{n}$ where $\operatorname{gcd}(b, n)=1$ then

$$
p_{n}(S)= \begin{cases}1 & \text { if }|\langle b\rangle| \text { is odd } \\ 2 & \text { if }|\langle b\rangle| \text { is even }\end{cases}
$$

Proof. Clearly there will be an $S$-polychromatic 2-coloring of the multiples of $b$ if and only if $|\langle b\rangle|$ is even.

## 4 Sets that tile

Given a set $S \subseteq G$ where $G$ is an abelian group, a set $T \subseteq G$ is a complement set for $S$ if $S+T=G$. $S$ tiles $G$ by translation if $T$ is a complement set for $S$ and if $s_{1}, s_{2} \in S$, $t_{1}, t_{2} \in T$, and $s_{1}+t_{1}=s_{2}+t_{2}$ implies $s_{1}=s_{2}$ and $t_{1}=t_{2}$. The notation $S \oplus T$ is used when $S$ tiles $G$ by translation. Without loss of generality, $0 \in S, T$ for all of the following arguments.

Newman [8] proved necessary and sufficient conditions for a finite set $S$ to tile $\mathbb{Z}$ if $|S|$ is a power of a prime.

Theorem 4.1. [8] Let $S=\left\{s_{1}, \ldots, s_{k}\right\}$ be distinct integers with $|S|=p^{\alpha}$ where $p$ is prime and $\alpha$ is a positive integer. For $1 \leq i<j \leq k$ let $p^{e_{i j}}$ be the highest power of $p$ that divides $s_{i}-s_{j}$. Then $S$ tiles $\mathbb{Z}$ if and only if $\left|\left\{e_{i j}: 1 \leq i<j \leq k\right\}\right| \leq \alpha$.

The characterization of sets $S$ of size 3 such that $p_{n}(S)=3$ (Theorem 2.6 and Proposition 4.4) follows immediately from Newman's theorem (Theorem 4.1). When commenting on this theorem in [8] Newman says: "Surely the special case [when $|S|=3$ ] deserves to have a completely trivial proof - but we have not been able to find one."

If there is an $S$-polychromatic $k$-coloring of $\mathbb{Z}_{n}$, then clearly there is an $S$-polychromatic $k$-coloring of $\mathbb{Z}$ with period $n$. If there is an $S$-polychromatic $k$-coloring of $\mathbb{Z}$ for a finite set $S$, then there is an $S$-polychromatic $k$-coloring of $\mathbb{Z}_{n}$ for some $n$. To see this, let $d$ equal the largest difference between two elements in $S$. If $\chi$ is an $S$-polychromatic $k$-coloring of $\mathbb{Z}$, there are only $k^{(d+1)}$ possibilities for the coloring on $d+1$ consecutive integers, so two such strings must be identical. If $n$ is the difference between the first integers in these two strings, then we can "wrap around" the coloring $\chi$ to get an $S$-polychromatic $k$-coloring of $\mathbb{Z}_{n}$.

Suppose $S=\{0, a, b\}$ and $\chi$ is an $S$-polychromatic 3 -coloring of $\mathbb{Z}$. By the above remark there exists an $S$-polychromatic 3 -coloring of $\mathbb{Z}_{n}$ for some $n$. By Proposition 4.4, $a$ and $b$ are in different nonzero mod 3 congruence classes, which fulfills Newman's wish to have a simple proof of his theorem for the special case when $|S|=3$.

Later Coven and Meyerowitz [5] gave necessary and sufficient conditions for $S$ to tile $\mathbb{Z}$ when $|S|=p_{1}^{\alpha_{1}} p_{2}^{\alpha_{2}}$, where $p_{1}$ and $p_{2}$ are primes. The following characterization of tiling by translation in an abelian group was obtained in [3].

Theorem 4.2. [3] Let $G$ be an abelian group and $S$ a finite subset of $G$. $S$ tiles $G$ by translation if and only if $p(S)=|S|$. Moreover, if $\chi$ is an $S$-polychromatic coloring of $G$ with $|S|$ colors and $T$ is a color class of $\chi$, then $S \oplus T=G$.
Lemma 4.3. Suppose $S=\{0, a, b\}$ where $\operatorname{gcd}(a, b, n)=1, S \oplus T=\mathbb{Z}_{n}$ and $0 \in T$. If $x \in T$, then $x+\langle a+b\rangle \subseteq T$.

Proof. Note that because $S \oplus T=\mathbb{Z}_{n}$, every element of $\mathbb{Z}_{n}$ belongs to exactly one of the sets $T, a+T, b+T$.

Suppose $x \in T$. If $x+a+b \in b+T$, then $x+a \in T$. However, $x+a \in a+T$. If $x+a+b \in a+T$, then $x+b \in T$. However, $x+b \in b+T$. Hence $x+a+b \in T$ and, repeating the argument, $x+\langle a+b\rangle \subseteq T$.

Proposition 4.4. Let $S=\{0, a, b\} \subseteq \mathbb{Z}_{n}$ where $\operatorname{gcd}(a, b, n)=1$. Then $p_{n}(S)=3$ if and only if $3 \mid n$ and $a$ and $b$ are in different nonzero $\bmod 3$ congruence classes.

Proof. If $3 \mid n$ and $a$ and $b$ are in different nonzero mod 3 congruence classes then clearly the alternating coloring $R B Y R B Y \ldots$ is polychromatic, so $p_{n}(S)=3$. Conversely, suppose $p_{n}(S)=3$. Hence, by Theorem 4.2, $S$ tiles $\mathbb{Z}_{n}$.

Let $T \subseteq \mathbb{Z}_{n}$ such that $\mathbb{Z}_{n}=\{0, a, b\} \oplus T$ and $0 \in T \subseteq \mathbb{Z}_{n}$. Therefore, $n=3|T|$ which implies $n \equiv 0 \bmod 3$. By Lemma 4.3, for any $x \in T$, the coset $x+\langle a+b\rangle$ is a subset of $T$, so $T$ is the disjoint untion of cosets of $\langle a+b\rangle$. Therefore, there is some integer $q$ such that $q|\langle a+b\rangle|=|T|=\frac{n}{3}$. Also, $|\langle a+b\rangle|=\frac{n}{\operatorname{gcd}(a+b, n)}$. Thus, $q \frac{n}{g c d(a+b, n)}=\frac{n}{3}$, which implies $3 q=\operatorname{gcd}(a+b, n)$. Hence, $3 \mid(a+b)$. Since 3 cannot divide both $a$ and $b$, it follows that $a$ and $b$ are in different nonzero mod 3 congruence classes.

## 5 Subsets of the form $\{0,1, b\}$

As shown in Proposition 2.5, every set $S$ of size 3 is equivalent to a set $S^{\prime}$ with two possible forms. In this section we will consider case $i$ of Proposition 2.5, that $S^{\prime}$ contains 0 and 1.

Lemma 5.1. If $n$ is odd, $5 \leq n$, and $n \neq 7$, then there exists a $\{0,1,3\}$-polychromatic coloring of $\mathbb{Z}_{n}$ with two colors.

Proof. It is easy to check that each integer greater than 3, except 7, is the sum of an even number of 2 s and 3 s . We color $\mathbb{Z}_{n}$ by alternating colors of strings of 2 or 3 consecutive elements with the same color. Of course there must be an even number of strings. For example, $9=2+2+2+3$, so the coloring would be $R R B B R R B B B ; 11=2+3+3+3$, so the coloring would be $R R B B B R R R B B B$. Clearly any translate of S hits two consecutive strings, so gets both colors.

As will be seen in the proof of Theorem 5.3, it is easy to show that $p_{n}(\{0,1, b\}) \geq 2$ if $b$ or $n$ is even. The following lemma takes care of the more difficult case.

Lemma 5.2. Let $9 \leq n, b$ and $n$ both be odd, and $S=\{0,1, b\} \subset \mathbb{Z}_{n}$. There exists an $S$-polychromatic coloring of $\mathbb{Z}_{n}$ with two colors.

Proof. It can be assumed that $5 \leq b \leq\left\lceil\frac{n}{2}\right\rceil$, by Lemma 2.4 and 5.1, and $n=m(b-2)+r$, with $0 \leq r \leq b-3$. Since $2\left(\left\lceil\frac{n}{2}\right\rceil-2\right)+r \leq n-3+r \leq n, m$ is at least 2 .

Let $x \in \mathbb{Z}_{n}$ and $x \equiv y \bmod (b-2)$ such that $0 \leq y \leq b-3$. If $r=0$, then define $\chi_{0}: \mathbb{Z}_{n} \rightarrow\{R, B\}$ such that

$$
\chi_{0}(x)= \begin{cases}R & \text { if } y=0 \\ R & \text { if } y \text { is odd } \\ B & \text { if } y \text { is even and } y>0\end{cases}
$$

If $\chi_{0}(x)=\chi_{0}(x+1)$, then $x \equiv 0 \bmod b-2$ and $\chi_{0}(x+b)=B$. This means that $\chi_{0}(x) \neq$ $\chi_{0}(x+1)$ or $\chi_{0}(x) \neq \chi_{0}(x+b)$. Therefore, every translate of $S=\{x, x+1, x+b\}$ will contain two colors under $\chi_{0}$.

Throughout the remainder of the proof each of the colorings that are constructed will use $\chi_{0}$ to assign colors to at least the first $(m-1)(b-2)$ elements of $\mathbb{Z}_{n}$.

If $r=1$, then define $\chi_{1}: \mathbb{Z}_{n} \rightarrow\{R, B\}$ such that

$$
\chi_{1}(x)= \begin{cases}\chi_{0}(x) & \text { if } x \leq n-b \\ R & \text { if } x \text { is even and } n-b<x<n-1 \\ B & \text { if } x \text { is odd and } n-b<x<n-1 \\ B & \text { if } x=n-1\end{cases}
$$

In $\chi_{1}$, the two translates that are not colored completely by $\chi_{0}$ and don't have $\chi_{1}(x) \neq$ $\chi_{1}(x+1)$ are $\{n-b, n-b+1,0\}$ and $\{n-2, n-1, b-2\}$. In both cases, the nonconsecutive element of the translate is the other color.

If $r=2$, then define $\chi_{2}: \mathbb{Z}_{n} \rightarrow\{R, B\}$ such that

$$
\chi_{2}(x)= \begin{cases}\chi_{0}(x) & \text { if } x \leq n-b-1 \\ R & \text { if } x=n-b \\ B & \text { if } x=n-b+1 \\ R & \text { if } x \text { is odd and } n-b+1<x \\ B & \text { if } x \text { is even and } n-b+1<x\end{cases}
$$

In $\chi_{2}$, the only translate that is not colored completely by $\chi_{0}$ and doesn't have $\chi_{2}(x) \neq$ $\chi_{2}(x+1)$ is $\{n-b+1, n-b+2,1\}$; however, $\chi_{2}(n-b+1) \neq \chi_{2}(1)$.

If $r=3$, then define $\chi_{3}: \mathbb{Z}_{n} \rightarrow\{R, B\}$ such that

$$
\chi_{3}(x)= \begin{cases}\chi_{0}(x) & \text { if } x \leq n-b-2 \\ R & \text { if } x=n-b-1 \\ R & \text { if } x \text { is even and } n-b-1<x<n-1 \\ B & \text { if } x \text { is odd and } n-b-1<x<n-1 \\ B & \text { if } x=n-1\end{cases}
$$

In $\chi_{3}$, the two translates that are not colored completely by $\chi_{0}$ and don't have $\chi_{3}(x) \neq$ $\chi_{3}(x+1)$ are $\{n-b-1, n-b, n-1\}$ and $\{n-2, n-1, b-2\}$. In both cases, the nonconsecutive element of the translate is the other color.

Assume $4 \leq r$. An $S$-polychromatic coloring, $\chi_{4}: \mathbb{Z}_{n} \rightarrow\{R, B\}$, will be constructed. Define $\chi_{4}(x)=\chi_{0}(x)$ for $x \leq n-b-r+4, \chi_{4}(n-r+2)=B$ and $\chi_{4}(n-1)=B$. So each translate with $0 \leq x \leq n-b-r+3$ contains both colors. The two translates with $n-2 \leq x$ also contain both colors even though $\chi_{4}(n-2)$ has not been defined unless $r=4$. This means there is an option for assigning a color to $n-2$. Therefore, $\chi_{4}(x)$ can be defined, and will be defined, such that for $n-b+2 \leq x \leq n-2$ the assigned colors alternate while keeping $\chi_{4}(n-r+2)=B$. This means that each translate with $n-b+2 \leq x$ contains both colors.

If $r=4$, then $n-b=n-b-r+4$ has already been assigned the color $B$ and the translate $\{n-b, n-b+1,0\}$ contains both colors. By defining $\chi_{4}(n-b+1)$ to be $B$, the translate $\{n-b+1, n-b+2,1\}$ contains both colors and $\chi_{4}$ is an $S$-polychromatic coloring.

For $r \neq 4$, consider the translates $\{n-b, n-b+1,0\}$ and $\{n-b+1, n-b+2,1\}$. If $\chi_{4}(n-b+2)=B$, then $n-b+1$ has an option since $\chi_{4}(1)=R$. If $\chi_{4}(n-b+2)=R$, then $\chi_{4}(n-b+1)$ must be defined as $B$ and $n-b$ has an option since $\chi_{4}(0)=R$. Therefore, there will be an option for assigning a color to $n-b$ or $n-b+1$. This allows for $\chi_{4}(x)$ to be defined for $n-b-r+5 \leq x \leq n-b+1$ such that the colors alernate while keeping $\chi_{4}(n-b-r+4) \neq \chi_{4}(n-b-r+5)$. Thus, $\chi_{4}$ is an $S$-polychromatic coloring.

Theorem 5.3. Let $3 \leq n$ and $S=\{0,1, b\} \subseteq \mathbb{Z}_{n}$. If $n \neq 7$ or $b \neq 3$ or 5 , there is an $S$-polychromatic coloring of $\mathbb{Z}_{n}$ with two colors.

Proof. If $n$ is even then alternating colors $R B R B \ldots$ is clearly an $S$-polychromatic coloring. If $n$ is odd and $b$ is even then the coloring $R R B R B R B R \ldots$ which has one repeated color,
and otherwise alternates colors, is $S$-polychromatic. If $n$ and $b$ are both odd then an $S$-polychromatic 2 -coloring exists by Lemmas 5.1 and 5.2, except in the exceptional case when $n=7$.

## 6 Subsets not equivalent to $\{0,1, b\}$

Consider the $s \times t$ matrix

$$
M=\left[\begin{array}{cccc}
x_{00} & x_{01} & \ldots & x_{0(t-1)} \\
x_{10} & x_{11} & \ldots & x_{1(t-1)} \\
\vdots & \vdots & \ddots & \vdots \\
x_{(s-1) 0} & x_{(s-1) 1} & \ldots & x_{(s-1)(t-1)} .
\end{array}\right]
$$

An ell - tile of $M$ is a subset of entries of $M$ consisting of entries of a $2 \times 2$ submatrix without the lower right entry:

| $x_{i j}$ | $x_{i(j+1)}$ |
| :---: | :---: |
| $x_{(i+1) j}$ |  |

The indices are read $\bmod s$ and $\bmod t$, so ell-tiles are allowed to 'wrap around' ( $i=s-1$ or $j=t-1$ ). An ell-tile $2-$ coloring of $M$ is a coloring of the entries of $M$ with two colors such that both colors appear in every ell-tile of $M$.

Lemma 6.1. If $2 \leq s, t$, then every $s \times t$ matrix has an ell - tile 2 -coloring.
Proof. If $s$ is even, then define $\chi$ such that

$$
\chi\left(x_{i j}\right)= \begin{cases}R & \text { if } i \equiv 0 \bmod 2 \\ B & \text { if } i \equiv 1 \bmod 2\end{cases}
$$

Also, a similar coloring that alternates the colors of the columns works when $t$ is even.
If $s$ and $t$ are both odd, then define $\chi$ such that

$$
\chi\left(x_{i j}\right)= \begin{cases}R & \text { if } i \equiv j \bmod 2 \text { and }(i, j) \neq(0, t-1),(s-1,0) \\ B & \text { otherwise } .\end{cases}
$$

If $s$ and $t$ are both odd, then a "checker-board" coloring would assign the same color, say $R$, to all four corner entries, and the ell-tile with entries $x_{s-1, t-1}, x_{0, t-1}$, and $x_{s-1,0}$ would be monochromatic. The coloring $\chi$ avoids this problem by changing the color of entries $x_{0, t-1}$ and $x_{s-1,0}$ from $R$ to $B$, without creating any other monochromatic ell-tiles (just changing the color of one of them would suffice as well).

The goal now is to create matrices with elements from $\mathbb{Z}_{n}$ such that all of the translates of $S$ correspond to ell-tiles. The matrices then can be colored by using Lemma 6.1, which will create $S$-polychromatic colorings.

Lemma 6.2. Let $S=\{0, a, b\} \subseteq \mathbb{Z}_{n}$, where $\operatorname{gcd}(a, b, n)=1$ but $\operatorname{gcd}(a, n)$ and $\operatorname{gcd}(b, n)$ are both greater than 1. Then $p_{n}(S) \geq 2$.

Proof. If $n$ is even then either $a$ or $b$ is odd, so the alternating coloring $R B R B R B \ldots$ is polychromatic, so we can assume $n$ is odd. Let $s=\operatorname{gcd}(a, n), t=\operatorname{gcd}(b, n)$, and $M=\left[m_{i j}\right]$ be the $\frac{n}{s} \times \frac{n}{t}$ matrix with entries in $\mathbb{Z}_{n}$ where $m_{i j}=a i+b j, 0 \leq i \leq \frac{n}{s}-1,0 \leq j \leq \frac{n}{t}-1$. Note that $|\langle a\rangle|=\frac{n}{s},|\langle b\rangle|=\frac{n}{t}$, and $\operatorname{gcd}(s, t)=1$.

If $m_{i j}=m_{i^{\prime} j^{\prime}}$ with $0 \leq i^{\prime} \leq i \leq t-1$ and $0 \leq j, j^{\prime} \leq \frac{n}{t}-1$, then $a\left(i-i^{\prime}\right)=b\left(j^{\prime}-j\right)$. Therefore, $t \mid a\left(i-i^{\prime}\right)$. This means $t \mid\left(i-i^{\prime}\right)$ since $\operatorname{gcd}(a, b)=1$, which implies $i=i^{\prime}$ because $0 \leq i-i^{\prime}<t$. Since $a\left(i-i^{\prime}\right)=b\left(j^{\prime}-j\right)$ it is also the case that $j=j^{\prime}$. Therefore, each element of $\mathbb{Z}_{n}$ will be an entry somewhere in the first $t$ rows of $M$. In fact, the first $t$ rows of $M$ are just the $t$ cosets of $\langle b\rangle$ in $\mathbb{Z}_{n}$.

Now let $M^{\prime}$ be the $\frac{n}{s t} \times \frac{n}{s t}$ block matrix created by partitioning $M$ into $t \times s$ blocks. Let $A_{i, j}$ be the $i j$ th block of $M^{\prime}$. Note that $A_{i+1, j}=A_{i, j}+a t$ and $A_{i, j+1}=A_{i, j}+b s$ and $|\langle a t\rangle|=|\langle b s\rangle|=\frac{n}{s t}$, so the matrix $A_{i, j}+k(b s)$ appears as a block in the $i$ th row of $M^{\prime}$ for each integer $k$. Furthermore, $a=p s$ for some $p \in \mathbb{Z}$ and $b q \equiv t \bmod n$ for some $q \in \mathbb{Z}$ since $t=\operatorname{gcd}(b, n)$. Therefore, $A_{i+1, j}=A_{i, j}+(p q) b s$, so $A_{i+1, j}$ is equal to some block in the $i$ th row of $M^{\prime}$.

This means that the $(i+1)$ st row of $M^{\prime}$ is the $i$ th row of $M^{\prime}$ shifted by $p q$, for all $i$. So coloring each matrix in $M^{\prime}$ with the same ell-tile 2-coloring from Lemma 6.1 will ensure that $M$ is a well-defined ell-tile 2-coloring. It is well-defined since every element is colored and each time an element appears it recieves the same color. It is an ell-tile 2-coloring because it is periodic using an ell-tile 2-coloring that 'wraps around'. This yields an $S$-polychromatic coloring of $\mathbb{Z}_{n}$ with two colors.

Here is an example of how to get the coloring of $\mathbb{Z}_{105}$ when $a=18$ and $b=25$. The
matrix $M$ is

| 0 | 25 | 50 | 75 | 100 | 20 | 45 | 70 | 95 | 15 | 40 | 65 | 90 | 10 | 35 | 60 | 85 | 5 | 30 | 55 | 80 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 18 | 43 | 68 | 93 | 13 | 38 | 63 | 88 | 8 | 33 | 58 | 83 | 3 | 28 | 53 | 78 | 103 | 23 | 48 | 73 | 98 |
| 36 | 61 | 86 | 6 | 31 | 56 | 81 | 1 | 26 | 51 | 76 | 101 | 21 | 46 | 71 | 96 | 16 | 41 | 66 | 91 | 11 |
| 54 | 79 | 104 | 24 | 49 | 74 | 99 | 19 | 44 | 69 | 94 | 14 | 39 | 64 | 89 | 9 | 34 | 59 | 84 | 4 | 29 |
| 72 | 97 | 17 | 42 | 67 | 92 | 12 | 37 | 62 | 87 | 7 | 32 | 57 | 82 | 2 | 27 | 52 | 77 | 102 | 22 | 47 |
| 90 | 10 | 35 | 60 | 85 | 5 | 30 | 55 | 80 | 0 | 25 | 50 | 75 | 100 | 20 | 45 | 70 | 95 | 15 | 40 | 65 |
| 3 | 28 | 53 | 78 | 103 | 23 | 48 | 73 | 98 | 18 | 43 | 68 | 93 | 13 | 38 | 63 | 88 | 8 | 33 | 58 | 83 |
| 21 | 46 | 71 | 96 | 16 | 41 | 66 | 91 | 11 | 36 | 61 | 86 | 6 | 31 | 56 | 81 | 1 | 26 | 51 | 76 | 101 |
| 39 | 64 | 89 | 9 | 34 | 59 | 84 | 4 | 29 | 54 | 79 | 104 | 24 | 49 | 74 | 99 | 19 | 44 | 69 | 94 | 14 |
| 57 | 82 | 2 | 27 | 52 | 77 | 102 | 22 | 47 | 72 | 97 | 17 | 42 | 67 | 92 | 12 | 37 | 62 | 87 | 7 | 32 |
| 75 | 100 | 20 | 45 | 70 | 95 | 15 | 40 | 65 | 90 | 10 | 35 | 60 | 85 | 5 | 30 | 55 | 80 | 0 | 25 | 50 |
| 93 | 13 | 38 | 63 | 88 | 8 | 33 | 58 | 83 | 3 | 28 | 53 | 78 | 103 | 23 | 48 | 73 | 98 | 18 | 43 | 68 |
| 6 | 31 | 56 | 81 | 1 | 26 | 51 | 76 | 101 | 21 | 46 | 71 | 96 | 16 | 41 | 66 | 91 | 11 | 36 | 61 | 86 |
| 24 | 49 | 74 | 99 | 19 | 44 | 69 | 94 | 14 | 39 | 64 | 89 | 9 | 34 | 59 | 84 | 4 | 29 | 54 | 79 | 104 |
| 42 | 67 | 92 | 12 | 37 | 62 | 87 | 7 | 32 | 57 | 82 | 2 | 27 | 52 | 77 | 102 | 22 | 47 | 72 | 97 | 17 |
| 60 | 85 | 5 | 30 | 55 | 80 | 0 | 25 | 50 | 75 | 100 | 20 | 45 | 70 | 95 | 15 | 40 | 65 | 90 | 10 | 35 |
| 78 | 103 | 23 | 48 | 73 | 98 | 18 | 43 | 68 | 93 | 13 | 38 | 63 | 88 | 8 | 33 | 58 | 83 | 3 | 28 | 53 |
| 96 | 16 | 41 | 66 | 91 | 11 | 36 | 61 | 86 | 6 | 31 | 56 | 81 | 1 | 26 | 51 | 76 | 101 | 21 | 46 | 71 |
| 9 | 34 | 59 | 84 | 4 | 29 | 54 | 79 | 104 | 24 | 49 | 74 | 99 | 19 | 44 | 69 | 94 | 14 | 39 | 64 | 89 |
| 27 | 52 | 77 | 102 | 22 | 47 | 72 | 97 | 17 | 42 | 67 | 92 | 12 | 37 | 62 | 87 | 7 | 32 | 57 | 82 | 2 |
| 45 | 70 | 95 | 15 | 40 | 65 | 90 | 10 | 35 | 60 | 85 | 5 | 30 | 55 | 80 | 0 | 25 | 50 | 75 | 100 | 20 |
| 63 | 88 | 8 | 33 | 58 | 83 | 3 | 28 | 53 | 78 | 103 | 23 | 48 | 73 | 98 | 18 | 43 | 68 | 93 | 13 | 38 |
| 81 | 1 | 26 | 51 | 76 | 101 | 21 | 46 | 71 | 96 | 16 | 41 | 66 | 91 | 11 | 36 | 61 | 86 | 6 | 31 | 56 |
| 99 | 19 | 44 | 69 | 94 | 14 | 39 | 64 | 89 | 9 | 34 | 59 | 84 | 4 | 29 | 54 | 79 | 104 | 24 | 49 | 74 |
| 12 | 37 | 62 | 87 | 7 | 32 | 57 | 82 | 2 | 27 | 52 | 77 | 102 | 22 | 47 | 72 | 97 | 17 | 42 | 67 | 92 |
| 30 | 55 | 80 | 0 | 25 | 50 | 75 | 100 | 20 | 45 | 70 | 95 | 15 | 40 | 65 | 90 | 10 | 35 | 60 | 85 | 5 |
| 48 | 73 | 98 | 18 | 43 | 68 | 93 | 13 | 38 | 63 | 88 | 8 | 33 | 58 | 83 | 3 | 28 | 53 | 78 | 103 | 23 |
| 66 | 91 | 11 | 36 | 61 | 86 | 6 | 31 | 56 | 81 | 1 | 26 | 51 | 76 | 101 | 21 | 46 | 71 | 96 | 16 | 41 |
| 84 | 4 | 29 | 54 | 79 | 104 | 24 | 49 | 74 | 99 | 19 | 44 | 69 | 94 | 14 | 39 | 64 | 89 | 9 | 34 | 59 |
| 102 | 22 | 47 | 72 | 97 | 17 | 42 | 67 | 92 | 12 | 37 | 62 | 87 | 7 | 32 | 57 | 82 | 2 | 27 | 52 | 77 |
| 15 | 40 | 65 | 90 | 10 | 35 | 60 | 85 | 5 | 30 | 55 | 80 | 0 | 25 | 50 | 75 | 100 | 20 | 45 | 70 | 95 |
| 33 | 58 | 83 | 3 | 28 | 53 | 78 | 103 | 23 | 48 | 73 | 98 | 18 | 43 | 68 | 93 | 13 | 38 | 63 | 88 | 8 |
| 51 | 76 | 101 | 21 | 46 | 71 | 96 | 16 | 41 | 66 | 91 | 11 | 36 | 61 | 86 | 6 | 31 | 56 | 81 | 1 | 26 |
| 69 | 94 | 14 | 39 | 64 | 89 | 9 | 34 | 59 | 84 | 4 | 29 | 54 | 79 | 104 | 24 | 49 | 74 | 99 | 19 | 44 |
| 87 | 7 | 32 | 57 | 82 | 2 | 27 | 52 | 77 | 102 | 22 | 47 | 72 | 97 | 17 | 42 | 67 | 92 | 12 | 37 | 62 |

Therefore,

$$
M^{\prime}=\left[\begin{array}{lllllll}
A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} & A_{0,5} & A_{0,6} \\
A_{0,4} & A_{0,5} & A_{0,6} & A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} \\
A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} & A_{0,5} & A_{0,6} & A_{0,0} \\
A_{0,5} & A_{0,6} & A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} \\
A_{0,2} & A_{0,3} & A_{0,4} & A_{0,5} & A_{0,6} & A_{0,0} & A_{0,1} \\
A_{0,6} & A_{0,0} & A_{0,1} & A_{0,2} & A_{0,3} & A_{0,4} & A_{0,5} \\
A_{0,3} & A_{0,4} & A_{0,5} & A_{0,6} & A_{0,0} & A_{0,1} & A_{0,2}
\end{array}\right] .
$$

Now Lemma 6.1 can be used to color each $A_{i, j}$ in the following way:

$$
\chi\left(A_{i, j}\right)=\left[\begin{array}{lll}
R & B & B \\
B & R & B \\
R & B & R \\
B & R & B \\
B & B & R
\end{array}\right] .
$$

Of course only the first row of $M^{\prime}$ is needed to get the coloring of $\mathbb{Z}_{n}$.
Finally, we give a proof of Theorem 2.6.

Proof. By Proposition 2.5 we have either Case $i$, where $S$ is equivalent to a set of the form $\{0,1, b\}$, or Case $i i$, where $\operatorname{gcd}(a, n)$ and $\operatorname{gcd}(b, n)$ are both greater than 1. Proposition 4.4 characterizes the sets $S$ for which $p_{n}(S)=3$, and Theorem 5.3 shows show that $p_{n}(S)=2$ for all other sets $S$ in Case $i$, except when $n=7$ and $b=3$. Then Lemma 6.2 takes care of Case $i i$.

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