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ORIGINAL PAPER

Paired Domination in Trees

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Abstract

A set S of vertices in a graph G is a paired dominating set if every vertex of G is adjacent to a vertex in S and the subgraph induced by S contains a perfect matching (not necessarily as an induced subgraph). The paired domination number, $\gamma_{\mathrm{pr}}(G)$, of G is the minimum cardinality of a paired dominating set of G. In this paper, we show that if T is a tree of order at least 2, then $\gamma_{\mathrm{pr}}(T) \le 2\alpha(T) - \varphi(T)$ where $\alpha(T)$ is the independence number and $\varphi(T)$ is the P_3-packing number. We present a tight upper bound on the paired domination number of a tree T in terms of its maximum degree Δ. For $\Delta \ge 1$, we show that if T is a tree of order n with maximum degree Δ, then $\gamma_{\mathrm{pr}}(T) \le \left(\frac{5\Delta-4}{8\Delta-4}\right)n + \frac{1}{2}n_1(T) + \frac{1}{4}n_2(T) - \left(\frac{\Delta-2}{4\Delta-2}\right)$, where $n_1(T)$ and $n_2(T)$ denote the number of vertices of degree 1 and 2, respectively, in T. Further, we show that this bound is tight for all $\Delta \ge 3$. As a consequence of this result, if T is a tree of order $n \ge 2$, then $\gamma_{\mathrm{pr}}(T) \le \frac{5}{8}n + \frac{1}{2}n_1(T) + \frac{1}{4}n_2(T)$, and this bound is asymptotically best possible.

Keywords Paired domination · Trees · Independence number

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1 Introduction

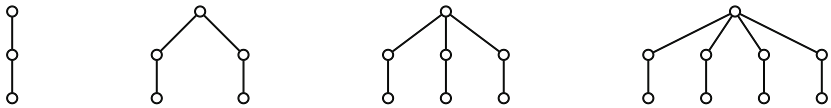
A *dominating set* of a graph G is a set $S \subseteq V(G)$ such that every vertex of $V(G) \setminus S$ is adjacent to some vertex in S. A *paired dominating set*, abbreviated PD-set, of an isolate-free graph G is a dominating set S of G with the additional property that the subgraph $G[S]$ induced by S contains a perfect matching M (not necessarily induced). With respect to the matching M, two vertices joined by an edge of M are *paired* and are called *partners* in S. The *paired domination number*, $\gamma_{\rm pr}(G)$, of G is the minimum cardinality of a PD-set of G. We call a PD-set of G of cardinality $\gamma_{\rm pr}(G)$ a $\gamma_{\rm pr}$*-set of* G. We note that the paired domination number $\gamma_{\rm pr}(G)$ is an even integer. For a recent survey on paired domination in graphs, we refer the reader to the book chapter [3].

We in general follow the graph theory notation in [5]. In particular, we denote the *degree* of a vertex v in a graph G by $d_G(v)$. A vertex of degree 0 is called an *isolated vertex*, and a graph is *isolate-free* if it contains no isolated vertex. The maximum (minimum) degree among the vertices of G is denoted by $\Delta(G)$ ($\delta(G)$, respectively). A *leaf* of a tree T is a vertex of degree 1 in T, and a *support vertex* of T is a vertex with a leaf neighbor.

The *distance* $d(u, v)$ between two vertices u and v in a connected graph G, equals the minimum length of a (u, v)-path in G from u to v. A shortest, or minimum length, path between two vertices u and v is called a (u, v)-*geodesic*. A *geodesic* is any shortest path in a graph. The *diameter* diam(G) of G is the maximum distance among all pairs of vertices in G. A *diametral path* in G is a geodesic which has length equal to diameter of G.

A *rooted tree* T distinguishes one vertex r called the *root*. For each vertex $v \neq r$ of T, the *parent* of v is the neighbor of v on the unique (r, v)-path, while a *child* of v is any other neighbor of v. A *descendant* of v is a vertex $u \neq v$ such that the unique (r, u)-path contains v. We let $D(v)$ denote the set of descendants of v, and we define $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at v is the subtree of T induced by $D[v]$, and is denoted by T_v.

The *independence number* $\alpha(G)$ of a graph G is the maximum cardinality of an independent set of vertices in G. For $k \geq 1$ an integer, we use the standard notation $[k] = \{1, \ldots, k\}$.



(a) $S(K_{1,1})$ (b) $S(K_{1,2})$ (c) $S(K_{1,3})$ (d) $S(K_{1,4})$

Fig. 1 The subdivided stars $S(K_{1,1})$, $S(K_{1,2})$, $S(K_{1,3})$, and $S(K_{1,4})$

For $r \geq 1$ a *subdivided star* $S(K_{1,r})$ is the tree of order $2r+1$ obtained from a star $K_{1,r}$ by subdividing every edge exactly once. For example, the subdivided stars $S(K_{1,1})$, $S(K_{1,2})$, $S(K_{1,3})$, and $S(K_{1,4})$ are shown in Figs. 1a,b,c,d.

2 Known results in trees

The paired domination number of a path P_n on $n \geq 2$ vertices is essentially one-half its order.

Observation 1 For $n \geq 2$, we have $\gamma_{\mathrm{pr}}(P_n) = 2\lceil \frac{n}{4} \rceil$.

Every support vertex in a tree T is contained in every PD-set of T. Further we note that if every PD-set in T contains an independent set I of vertices, then in order to pair the vertices of I with (distinct) vertices in the PD-set of T, we have $\gamma_{\mathrm{pr}}(T) \geq 2|I|$. For example, if T is a subdivided star $S(K_{1,r})$ for some $r \geq 2$, then T has order $n = 2r+1$ and the set of r support vertices in T form an independent set and belong to every PD-set of T, implying that $\gamma_{\mathrm{pr}}(T) \geq 2r$. However, we can pair each support vertex with its leaf neighbor to form a PD-set of T, implying that $\gamma_{\mathrm{pr}}(T) \leq 2r$. Consequently, $\gamma_{\mathrm{pr}}(T) = 2r$. We state this formally as follows.

Observation 2 If T is a subdivided star of order n, then $\gamma_{\mathrm{pr}}(T) = n-1$.

In 1998 Haynes and Slater [4] obtained the following upper bound on the paired domination number of a tree of order at least 3.

Theorem 3 ([4]) *If T is a tree of order $n \geq 3$, then $\gamma_{\mathrm{pr}}(G) \leq n-1$ with equality if and only if T is the path P_3 or a subdivided star $S(K_{1,r})$ for $r \geq 2$.*

Subsequent to the 1998 result of Theorem 3, several authors presented improved bounds on the paired domination number of a tree. We mention, for example, the 2004 paper by Chellali and Haynes [1], the 2006 paper by Raczek [6] and the 2014 paper by Dehgardi, Sheikholeslami and Khodkar [2]. In this paper, we present tight upper bounds on the paired domination number of a tree in terms of its order, maximum degree, and number of vertices of degree 1 and 2. We also present tight upper bounds on the paired domination number of a tree in terms of its independence number.

3 Main Results

In view of Observation 1, it is only of interest to determine upper bounds on the paired domination number of a tree with maximum degree at least 3. In this paper, we present a stronger result than the trivial upper bound of Theorem 3.

In order to state our first result, we define a P_3*-packing* in a tree T as a collection of vertex disjoint paths P_3 (on three vertices) each of which contains at least one leaf of the original tree T. Further, we define the P_3*-packing number* in T, denoted $\varphi(T)$, as the maximum cardinality of a P_3-packing in T. We are now in a position to state the following upper bound on the paired domination of a tree in terms of its independence number. We present a proof of Theorem 4 in Sect. 4.

Theorem 4 *If T is a tree of order at least2, then $\gamma_{\mathrm{pr}}(T) \leq 2\alpha(T) - \varphi(T)$, and this bound is tight.*

The natural consequence of the definition of a P_3*-packing* is its extension to the set of subdivided stars in trees. For this purpose, let T be a tree of maximum degree Δ where $\Delta \geq 3$. We define a *subdivided star set* of T as a set of vertex disjoint subdivided stars each of which is a subgraph of T. Further, the number of leaves of each such subdivided star belongs to the set $\{2, \ldots, \Delta - 1\}$, and every leaf from a subdivided star in the set is a leaf of T. More formally, a set $\mathcal{P} = \{T_1, \ldots, T_p\}$ is a *subdivided star set* of T if the following holds.

- T_i is a subdivided star $S(K_{1,n_i})$ where $2 \leq n_i \leq \Delta - 1$ for every $i \in [p]$.
- Every leaf of T_i is a leaf of T for all $i \in [p]$.
- $V(T_i) \cap V(T_j) = \emptyset$ for $1 \leq i < j \leq p$.

Further, if $\mathcal{P} = \emptyset$, we define $\xi_{\mathcal{P}}(T) = 0$, and if $\mathcal{P} \neq \emptyset$, we define

$$\xi_{\mathcal{P}}(T) = \sum_{i=1}^{p}(n_i - 1) \text{ and } \Phi_\Delta(T) = \max \xi_{\mathcal{P}}(T)$$

where the maximum in the definition of $\Phi_\Delta(T)$ is taken over all subdivided star sets $\mathcal{P}$ in the tree T (which satisfies $\Delta(T) = \Delta \geq 3$). A subdivided star set $\mathcal{P}$ of T satisfying $\Phi_\Delta(T) = \xi_{\mathcal{P}}(T)$ we call an *optimal subdivided star set*. We note that taking $\mathcal{P} = \emptyset$, we have $\xi_{\mathcal{P}}(T) = 0$, and so $\Phi_\Delta(T) \geq 0$.

To illustrate this definition, let T be the tree of maximum degree $\Delta(T) = 6$ (here $\Delta = 6$) shown in Fig. 2. Let T_i be the subtree of T induced by the vertex v_i, the support vertices of v_i, and the leaves at distance 2 from v_i. We note that $T_i \cong S(K_{1,i+1})$ for $i \in [4]$. The set $\mathcal{P} = \{T_1, T_2, T_3, T_4\}$ is a subdivided star set satisfying $\xi_{\mathcal{P}}(T) = 1 + 2 + 3 + 4 = 10$, and so $\Phi_6(T) \geq 10$. From the structure of the tree T we can readily deduce that $\Phi_6(T) \leq 10$. Consequently, $\Phi_6(T) = 10$.

Let $n_1(T)$ and $n_2(T)$ denote the number of vertices of degree 1 and 2, respectively, in a tree T, and let $n_{\geq 3}(T)$ denote the number of vertices of degree at least 3 in T. We note that if T is a tree of order $n \geq 3$, then $n = n_1(T) + n_2(T) + n_{\geq 3}(T)$. We are now in a position to state our second main result, a proof of which we present in Sect. 5.

Theorem 5 *For $\Delta \geq 1$, if T is a tree of order$n$ with maximum degree $\Delta(T) = \Delta$, then*

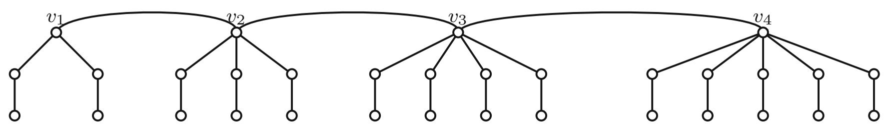


Fig. 2 A tree T with $\Delta(T) = 6$ and $\Phi_6(T) = 10$

$$4\gamma_{\mathrm{pr}}(T) \le 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T), \tag{1}$$

and this bound is tight for all $\Delta \ge 3$.

We next present the following upper bound on the paired domination of a tree, a proof of which is presented in Sect. 6.

Theorem 6 *For $\Delta \ge 1$, if T is a tree of order n with maximum degree $\Delta(T) = \Delta$, then*

$$\gamma_{\mathrm{pr}}(T) \le \left(\frac{5\Delta - 4}{8\Delta - 4}\right)n + \frac{1}{2}n_1(T) + \frac{1}{4}n_2(T) - \left(\frac{\Delta - 2}{4\Delta - 2}\right). \tag{2}$$

As an immediate consequence of Theorem 6, we have the following upper bound on the paired domination number of a tree.

Corollary 7 *If T is a tree of order $n \ge 2$, then*

$$\gamma_{\mathrm{pr}}(T) \le \frac{5}{8}n + \frac{1}{2}n_1(T) + \frac{1}{4}n_2(T), \tag{3}$$

and this bound is asymptotically best possible.

4 Proof of Theorem 4

In this section we give a proof of Theorem 4. Recall its statement.

Theorem 4. *If T is a tree of order at least 2, then $\gamma_{\mathrm{pr}}(T) \le 2\alpha(T) - \varphi(T)$, and this bound is tight.*

Proof We proceed by induction on the order $n \ge 2$ of a tree T. If $n = 2$, then $T = P_2$, and $\gamma_{\mathrm{pr}}(T) = 2$, $\alpha(T) = 1$ and $\varphi(T) = 0$, and so $\gamma_{\mathrm{pr}}(T) = 2\alpha(T) - \varphi(T)$. This establish the base case. Let $n \ge 3$ and assume that if T' is a tree of order n' where $2 \le n' < n$, then $\gamma_{\mathrm{pr}}(T') \le 2\alpha(T') - \varphi(T')$. Let T be a tree of order n.

Suppose that T contains a strong support vertex v, and so v has at least two leaf neighbors in T. Let u_1 and u_2 be two leaf neighbors of v, and let $T' = T - u_1$. We can choose a maximum independent set in a tree to contain all its leaves, implying that $\alpha(T) = \alpha(T') + 1$. Further, we note that if $\mathcal{P}$ is a maximum P_3-packing in T, then either there is a path $P' \in \mathcal{P}$ that contains the vertex u_1, in which case $\mathcal{P} \setminus \{P'\}$ is a P_3-packing in T', or no path in $\mathcal{P}$ contain the vertex u_1, in which case $\mathcal{P}$ is a P_3-packing in T'. Thus, $\varphi(T') \ge |\mathcal{P}| - 1 = \varphi(T) - 1$. Every PD-set of T' contains the support vertex v, implying that $\gamma_{\mathrm{pr}}(T) \le \gamma_{\mathrm{pr}}(T')$. Applying the inductive hypothesis to T', we therefore have $\gamma_{\mathrm{pr}}(T) \le \gamma_{\mathrm{pr}}(T') \le 2\alpha(T') - \varphi(T') \le 2(\alpha(T) - 1) - (\varphi(T) - 1) < 2\alpha(T) - \varphi(T)$. Hence, we may assume that T contains no strong support vertex, that is, every support vertex in T has exactly one leaf neighbor.

Since T has order $n \ge 3$, our earlier assumptions imply that the tree T is not a star, and so $\mathrm{diam(T)} \ge 3$. Further our assumptions imply that if $\mathrm{diam(T)} = 3$, then $T = P_4$. In this case, $\gamma_{\mathrm{pr}}(T) = 2$, $\alpha(T) = 2$ and $\varphi(T) = 1$, and so $\gamma_{\mathrm{pr}}(T) < 2\alpha(T) - \varphi(T)$. Hence, we may assume that $\mathrm{diam(T)} \ge 4$, for otherwise the desired result follows. Let $P: v_0v_1v_2\ldots v_d$ be a longest path in T, and so $d = \mathrm{diam(T)} \geq 4$. We now root the tree T at the vertex $r = v_d$. Since every support vertex in T has exactly one leaf neighbor, we note that $d_T(v_1) = 2$. We proceed further with the following series of claims.

claim 1 If $d_T(v_2) \geq 3$, then $\gamma_{\mathrm{pr}}(T) \leq 2\alpha(T) - \varphi(T)$.

Proof Suppose that $d_T(v_2) \geq 3$. Suppose firstly that the vertex v_2 is a support vertex with (unique) leaf neighbor u_1. Let $T' = T - u_1$. We can choose a γ_{pr}-set of T' to contain the vertices v_1 and v_2, implying that $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T')$. Every independent set in T' is an independent set in T, implying that $\alpha(T) \geq \alpha(T')$. We can choose a maximum P_3-packing $\mathcal{P}$ in T so that it contains the path $P' \in \mathcal{P}$ where $P' : v_0v_1v_2$. The set $\mathcal{P}$ is a P_3-packing in T', and so $\varphi(T') \geq |\mathcal{P}| = \varphi(T)$. Therefore applying the inductive hypothesis to the tree T', we have $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') \leq 2\alpha(T') - \varphi(T') \leq 2\alpha(T) - \varphi(T)$. Hence, we may assume that v_2 is not a support vertex in T, and so every child of v_2 is a support vertex of degree 2 in T.

By supposition, $d_T(v_2) \geq 3$. Let w_1 be a child of v_2 different from v_1, and let w_0 be the child of w_1. We consider the tree $T' = T - \{w_0, w_1\}$. In this case, we note that $\alpha(T) = \alpha(T') + 1$. Every γ_{pr}-set of T' can be extended to a PD-set of T by adding to it the vertices w_0 and w_1, and so $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') + 2$. We can choose a maximum P_3-packing $\mathcal{P}$ in T so that it contains the path $P' \in \mathcal{P}$ where $P' : v_0v_1v_2$. The set $\mathcal{P}$ is a P_3-packing in T', and so $\varphi(T') \geq |\mathcal{P}| = \varphi(T)$. Therefore applying the inductive hypothesis to the tree T', we have $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') + 2 \leq 2\alpha(T') - \varphi(T') + 2 \leq 2(\alpha(T) - 1) - \varphi(T) + 2 = 2\alpha(T) - \varphi(T)$. □

By Claim 1, we may assume that $d_T(v_2) = 2$, for otherwise the desired result follows. More generally, we may assume that every vertex at distance $d - 2$ from the root $r = v_d$ of the rooted tree T has degree equal to 2.

claim 2 If $d_T(v_3) = 2$, then $\gamma_{\mathrm{pr}}(T) \leq 2\alpha(T) - \varphi(T)$.

Proof Suppose that $d_T(v_3) = 2$. If $T \cong P_5$, then the inequality holds. Thus, we may further assume that $T \ncong P_5$. In this case, we consider the tree $T' = T - \{v_0, v_1, v_2, v_3\}$. Every independent set in T' can be extended to an independent set in T by adding to it the vertices v_0 and v_2, and so $\alpha(T) \geq \alpha(T') + 2$. Every γ_{pr}-set of T' can be extended to a PD-set of T by adding to it the vertices v_1 and v_2, and so $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') + 2$. We can choose a maximum P_3-packing $\mathcal{P}$ in T so that it contains the path $P' \in \mathcal{P}$ where $P' : v_0v_1v_2$. The set $\mathcal{P} \setminus \{P'\}$ is a P_3-packing in T', and so $\varphi(T') \geq |\mathcal{P}| - 1 = \varphi(T) - 1$. Therefore, applying the inductive hypothesis to the tree T', we have $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') + 2 \leq 2\alpha(T') - \varphi(T') + 2 \leq 2(\alpha(T) - 2) - (\varphi(T) - 1) + 2 < 2\alpha(T) - \varphi(T)$. □

claim 3 If v_3 is a support vertex, then $\gamma_{\mathrm{pr}}(T) \leq 2\alpha(T) - \varphi(T)$.

Proof Suppose that the vertex v_3 has a leaf neighbor u_2. In this case, we consider the tree $T' = T - \{v_0, v_1, v_2\}$. We can choose a maximum independent set of T' to contain the leaf u_2. Such a maximum independent set can be extended to an independent set of T by adding to it the vertices v_0 and v_2, and so $\alpha(T) \geq \alpha(T') + 2$. Every γ_{pr}-set of T' can be extended to a PD-set of T by adding to it the vertices v_1 and v_2, and so $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') + 2$. We can choose a maximum P_3-packing $\mathcal{P}$ in T so that it contains the path $P' \in \mathcal{P}$ where $P' : v_0v_1v_2$. The set $\mathcal{P} \setminus \{P'\}$ is a P_3-packing in T', and so $\varphi(T') \geq |\mathcal{P}| - 1 = \varphi(T) - 1$. Therefore applying the inductive hypothesis to the tree T', we have $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') + 2 \leq 2\alpha(T') - \varphi(T') + 2 \leq 2(\alpha(T) - 2) - (\varphi(T) - 1) + 2 < 2\alpha(T) - \varphi(T)$. □

claim 4 If the vertex v_3 has a descendant at distance 3 that is different from v_0, then $\gamma_{\mathrm{pr}}(T) \leq 2\alpha(T) - \varphi(T)$.

Proof Suppose that the vertex v_3 has a descendant w_0 at distance 3 that is different from v_0. Let $w_0w_1w_2v_3$ be the path from w_0 to the vertex v_3. By our earlier assumptions, the vertex w_0 is a leaf and $d_T(w_1) = d_T(w_2) = 2$. We now consider the tree $T' = T - \{v_0, v_1, v_2\}$. We can choose a maximum independent set of T' to contain the vertices w_0 and w_2. Such a maximum independent set can be extended to an independent set of T by adding to it the vertices v_0 and v_2, and so $\alpha(T) \geq \alpha(T') + 2$. Every γ_{pr}-set of T' can be extended to a PD-set of T by adding to it the vertices v_1 and v_2, and so $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') + 2$. We can choose a maximum P_3-packing $\mathcal{P}$ in T so that it contains the path $P' \in \mathcal{P}$ where $P' : v_0v_1v_2$. The set $\mathcal{P} \setminus \{P'\}$ is a P_3-packing in T', and so $\varphi(T') \geq |\mathcal{P}| - 1 = \varphi(T) - 1$. Therefore applying the inductive hypothesis to the tree T', we have $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') + 2 \leq 2\alpha(T') - \varphi(T') + 2 \leq 2(\alpha(T) - 2) - (\varphi(T) - 1) + 2 < 2\alpha(T) - \varphi(T)$. □

By Claim 2, 3 and 4, we may assume that $d_T(v_3) \geq 3$ and that every child of v_3 different from v_2 is a support vertex of degree 2 in T. Let w_2 be an arbitrary child of v_3 different from v_2, and let w_1 be the child of w_2. Let ℓ be the number of children of v_3. By assumption, $\ell \geq 2$ and every leaf in T_{v_3} different from v_0 is at distance 2 from v_3, where T_{v_3} is the maximal subtree rooted at v_3. Thus, T_{v_3} is obtained from a star $K_{1,\ell}$ by subdividing $\ell - 1$ edges once and subdividing the remaining edge of the star twice, and so T_{v_3} has order $2\ell + 2$. Let T' be the tree obtained from T by deleting the vertex v_3 and all descendants of v_3, that is, $T' = T - V(T_{v_3})$. By our earlier assumptions, the tree T' has order at least 3.

Every independent set in T' can be extended to an independent set in T by adding to it the vertex v_2 and the ℓ leaves of T_{v_3}, and so $\alpha(T) \geq \alpha(T') + \ell + 1$. Every γ_{pr}-set of T' can be extended to a PD-set of T by adding to it 2ℓ vertices from the tree T_{v_3}, and so $\gamma_{\mathrm{pr}}(T) \leq \gamma_{\mathrm{pr}}(T') + 2\ell$. We can choose a maximum P_3-packing $\mathcal{P}$ in T so that it contains the paths $P' : v_0v_1v_2$ and $Q' : w_1w_2v_3$. The set $\mathcal{P} \setminus \{P', Q'\}$ is a P_3-packing in T', and so $\varphi(T') \geq |\mathcal{P}| - 2 = \varphi(T) - 2$. Therefore applying the inductive hypothesis to the tree T', we have

$$\begin{aligned}\gamma_{\mathrm{pr}}(T) &\le \gamma_{\mathrm{pr}}(T') + 2\ell \\ &\le 2\alpha(T') - \varphi(T') + 2\ell \\ &\le 2(\alpha(T) - \ell - 1) - (\varphi(T) - 2) + 2\ell \\ &= 2\alpha(T) - \varphi(T).\end{aligned}$$

This completes the proof of the upper bound.

That the upper bound in Theorem 4 is sharp may be seen as follows. For an even $k \ge 2$, let $T_1, T_2, \ldots, T_k$ be vertex disjoint subdivided stars, that is, $T_i = S(K_{1,n_i})$ where $n_i \ge 1$. If $n_i \ge 2$, then let v_i denote the central vertex (of degree i) of the subdivided star T_i, while if $n_i = 1$, then let v_i be one of the two leaves of $T_i \cong P_3$. Let $T = T_k(n_1, \ldots, n_k)$ be the tree obtained from the disjoint union of the trees $T_1, T_2, \ldots, T_k$ by adding the edges $v_i v_{i+1}$ for all $i \in [k-1]$, and so $v_1 v_2 \ldots v_k$ is a path in T. The resulting tree T satisfies $\gamma_{\mathrm{pr}}(T) = 2\alpha(T) - \varphi(T)$ noting that

$$\gamma_{\mathrm{pr}}(T) = \sum_{i=1}^{k} 2n_i,\ \alpha(T) = \frac{1}{2}k + \sum_{i=1}^{k} n_i \text{ and } \varphi(T) = k.$$

In the special case when $n_i = 1$ for all $i \in [k]$, the tree $T = T_k(n_1, \ldots, n_k)$ is the 2-corona of a path P_k, that is, $T = P_k \circ P_2$ is obtained from a path P_k by attaching a path of length 2 to each vertex of P_k so that the resulting paths are vertex-disjoint. In this case, $\gamma_{\mathrm{pr}}(T) = 2k$, $\alpha(T) = \frac{3}{2}k$ and $\varphi(T) = k$, and so $\gamma_{\mathrm{pr}}(T) = 2\alpha(T) - \varphi(T)$. For example, the 2-corona $T = P_6 \circ P_2$ of a path P_6 is illustrated in Fig. 3.

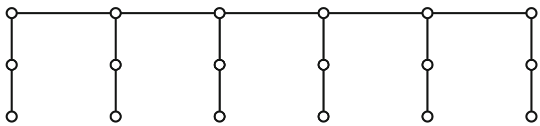
When $k = 4$ and $n_1 = 5$, $n_2 = n_3 = 4$ and $n_4 = 6$, the tree $T = T_k(n_1, \ldots, n_k)$, for example, is illustrated in Fig. 4. For this example, $\gamma_{\mathrm{pr}}(T) = 38$, $\alpha(T) = 21$ and $\varphi(T) = 4$, and so $\gamma_{\mathrm{pr}}(T) = 2\alpha(T) - \varphi(T)$.

5 Proof of Theorem 5

In this section we give a proof of Theorem 5. Recall its statement.

Theorem 5. *For $\Delta \ge 1$, if T is a tree of order n with maximum degree $\Delta(T) = \Delta$, then*

Fig. 3 The 2-corona $P_6 \circ P_2$ of a path P_6



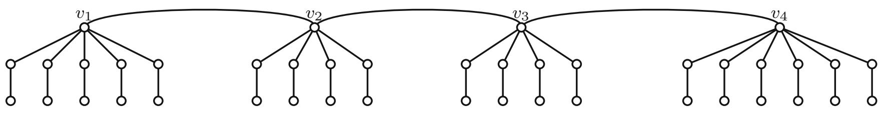


Fig. 4 The tree $T = T_4(5, 4, 4, 6)$

$$4\gamma_{\rm pr}(T) \le 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T),$$

and this bound is tight for all $\Delta \ge 3$.

Proof For a tree T of order n with maximum degree $\Delta(T) = \Delta$ where $\Delta \ge 1$, we define the *weight* of T by

$$\mathrm{w(T) = 2n + 2n_1(T) + n_2(T)} + \Phi_\Delta(\mathrm{T}).$$

We prove by induction on $n + \Delta$ that $4\gamma_{\rm pr}(T) \le \mathrm{w(T)}$. If $\Delta = 1$, then $T = K_2$ and $\gamma_{\rm pr}(T) = 2$, $n = n_1(T) = 2$, and $n_2(T) = \Phi_\Delta(T) = 0$, and so $4\gamma_{\rm pr}(T) = 8 = \mathrm{w(T)}$. If $\Delta = 2$, then T is a path P_n, where $n \ge 3$. In this case, $\mathrm{w(T) = 3n + 2} + \Phi_\Delta(\mathrm{T})$. If $n = 5$, then $\Phi_\Delta(T) = 1$, while if $n \ne 5$, then $\Phi_\Delta(T) = 0$. By Observation 1, we therefore have that $4\gamma_{\rm pr}(T) < \mathrm{w(T)}$. Hence, we may assume in what follows that $\Delta \ge 3$, for otherwise the desired result is immediate.

Since $\Delta(T) = \Delta$, we note that $n \ge \Delta + 1$, and so the smallest value of $n + \Delta$ is $2\Delta + 1$. If $n + \Delta = 2\Delta + 1$, then $n = \Delta + 1$ and T is a star $K_{1,\Delta}$. In this case, $\gamma_{\rm pr}(T) = 2$, $n_1(T) = \Delta$, $n_2(T) = 0$, and $\Phi_\Delta(T) = 0$, and so $4\gamma_{\rm pr}(T) = 8 < 4\Delta + 2 = \mathrm{w(T)}$. This establishes the base cases. Let $n \ge \Delta + 2$ where $\Delta \ge 3$, and assume that if T' is a tree of order n' and maximum degree $\Delta(T') = \Delta'$ where $n' \le n$ and $\Delta' \le \Delta$ satisfying $n' + \Delta' < n + \Delta$, then $4\gamma_{\rm pr}(T') \le \mathrm{w(T')}$. Let $\Delta \ge 3$ and let T be a tree of order n with $\Delta(T) = \Delta$. We proceed further with the following claim.

claim 1 If T contains a support vertex with at least two leaf neighbors, then $4\gamma_{\rm pr}(T) \le \mathrm{w(T)}$.

Proof Suppose that there is a vertex v in T with at least two leaf neighbors, say v_1 and v_2. Let S be a $\gamma_{\rm pr}$-set of T. At most one of v_1 and v_2 belongs to the set S. Renaming v_1 and v_2 if necessary, we may assume that $v_1 \notin S$. We now consider the tree $T' = T - v_1$. The set S is a PD-set of T', and so $\gamma_{\rm pr}(T') \le |S| = \gamma_{\rm pr}(T)$. Every PD-set of T' contains the support vertex v, implying that $\gamma_{\rm pr}(T) \le \gamma_{\rm pr}(T')$. Consequently, $\gamma_{\rm pr}(T') = \gamma_{\rm pr}(T)$. Let T' have order n' with maximum degree $\Delta(T') = \Delta'$. We note that $n' = n - 1$, $n_1(T') = n_1(T) - 1$, $n_2(T') \le n_2(T) + 1$ and $\Delta' \le \Delta$. Every subdivided star set of T' is a subdivided star set of T, implying that $\Phi'_\Delta(T') \le \Phi_\Delta(T)$. These observations imply that

$$\begin{aligned}\mathrm{w(T) - w(T')} &= 2(n - n') + 2(n_1(T) - n_1(T')) \\ &\quad + (n_2(T) - n_2(T')) + (\Phi_\Delta(T) - \Phi'_\Delta(T')) \\ &\ge 2 + 2 - 1 + 0 = 3,\end{aligned}$$

and so $\mathrm{w(T) \ge w(T') + 3}$. Applying the inductive hypothesis to the tree T', we have

$$4\gamma_{\rm pr}(T) = 4\gamma_{\rm pr}(T') \le \mathrm{w(T') \le w(T) - 3 < w(T)}.$$

This completes the proof of Claim 1. □

By Claim 1, we may assume that every support vertex of T has exactly one leaf neighbor, for otherwise the desired inequality, namely $4\gamma_{\mathrm{pr}}(T) \le \mathrm{w(T)}$ holds. Recall that $n \ge \Delta + 2$, and so $\mathrm{diam(T)} \ge 3$. Let $P : v_0 v_1 \ldots v_d$ be a diametral path in T, and so v_1 and v_d are two vertices at maximum distance apart in T and $d = \mathrm{diam(T)} \ge 3$. The vertices v_1 and v_{d-1} are support vertices in T. By Claim 1 and the maximality of the path P, both v_1 and v_{d-1} have degree 2 in T with v_0 and v_d, respectively, as their unique leaf neighbors.

If $d = 3$, then $T = P_4$, contradicting the fact that $\Delta(T) = \Delta \ge 3$. If $d = 4$, then T is a subdivided star $S(K_{1,\Delta})$ obtained from a star $K_{1,\Delta}$ by subdividing every edge exactly once. In this case, $\gamma_{\mathrm{pr}}(T) = 2\Delta = n - 1$. Moreover, $n_1(T) = n_2(T) = \Delta$ and $\Phi_\Delta(T) = \Delta - 2$. Thus,

$$\mathrm{w(T)} = 2(2\Delta + 1) + 2\Delta + \Delta + (\Delta - 2) = 8\Delta = 4\gamma_{\mathrm{pr}}(\mathrm{T}),$$

which yields equality in the desired bound. Hence, we may assume that $d \ge 5$. We now root the tree T at the vertex v_d. By Claim 1, at most one child of the vertex v_2 is a leaf. Further, by the maximality if the path P, every child of v_2 that is not a leaf is a support vertex of degree 2 in T. Let ℓ be the number of children of v_2 that are not leaves. We note that $1 \le \ell \le \Delta - 1$ and that each child of v_2 that is not a leaf is a support vertex of degree 2. If v_2 has a leaf neighbor, then let $\ell_0 = 1$, while if v_2 is not a support vertex, let $\ell_0 = 0$.

claim 2 If $d_T(v_3) \ge 3$, then $4\gamma_{\mathrm{pr}}(T) \le \mathrm{w(T)}$.

Proof Suppose that $d_T(v_3) \ge 3$. In this case, we consider the tree T' obtained from T by deleting the vertex v_2 and all descendants of v_2, that is, $T' = T - V(T_{v_2})$ where T_{v_2} is the maximal subtree rooted at v_2. Let T' have order n' with maximum degree $\Delta(T') = \Delta'$. We note that $n' = n - 2\ell - \ell_0 - 1$, $n_1(T') = n_1(T) - \ell - \ell_0$, $n_2(T') \le n_2(T) - \ell + 1$ and $\Delta' \le \Delta$. Every optimal subdivided star set $\mathcal{P}'$ of T' is a subdivided star set of T. Thus if $\ell = 1$, then $\Phi_\Delta(T) \ge \Phi_\Delta(T') = \Phi_\Delta(T') + \ell - 1 = \Phi_\Delta(T')$. If $\ell \ge 2$ and $\ell_0 = 0$, then the maximal subtree T_{v_2} is a subdivided star $S(K_{1,\ell})$ that can be added to the set $\mathcal{P}'$, while if $\ell \ge 2$ and $\ell_0 = 1$, then removing the leaf neighbor of v_2 from the maximal subtree T_{v_2} produces a subdivided star $S(K_{1,\ell})$ that can be added to the set $\mathcal{P}'$, implying that $\Phi_\Delta(T) \ge \Phi_\Delta(T') + \ell - 1$. These observations imply that

$$\begin{aligned}
\mathrm{w(T)} - \mathrm{w(T')} &= 2(n - n') + 2(n_1(T) - n_1(T')) \\
&\quad + (n_2(T) - n_2(T')) + (\Phi_\Delta(T) - \Phi'_\Delta(T')) \\
&\ge 2(2\ell + \ell_0 + 1) + 2(\ell + \ell_0) + (\ell - 1) + (\ell - 1) \\
&= 8\ell + 4\ell_0 \\
&\ge 8\ell,
\end{aligned}$$

and so $\mathrm{w(T)} \ge \mathrm{w(T')} + 8\ell$. Every γ_{pr}-set of T' can be extended to a PD-set of T by adding to it the vertex v_2 and all children of v_2 of degree 2 together with their leaf neighbors, excluding the vertex v_0. In the resulting PD-set of T, we note that v_1 and v_2 are paired, and every child of v_2 different from v_1 is paired with its (unique) child.

Thus, $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2\ell$. Applying the inductive hypothesis to the tree T', we have

$$4\gamma_{pr}(T) = 4(\gamma_{pr}(T') + 2\ell) \le \mathrm{w(T')} + 8\ell \le \mathrm{w(T)}.$$

This completes the proof of Claim 2. □

By Claim 2, we may assume that $d_T(v_3) = 2$, for otherwise the desired inequality holds. By our earlier assumptions, $d = \mathrm{diam(T)} \ge 5$. We consider the tree T' obtained from T by deleting the vertex v_3 and all descendants of v_3, that is, $T' = T - V(T_{v_3})$ where T_{v_3} is the maximal subtree rooted at v_3. Let T' have order n' with maximum degree $\Delta(T') = \Delta'$. We note that $n' \ge 2$ and $1 \le \Delta' \le \Delta$. Further, $n' = n - 2\ell - \ell_0 - 2$, $n_1(T') \le n_1(T) - \ell - \ell_0 + 1$, and $n_2(T') \le n_2(T) - \ell$. Every optimal subdivided star set $\mathcal{P}'$ of T' is a subdivided star set of T. Analogous arguments as in the proof of Claim 2 show that $\Phi_\Delta(T) \ge \Phi_\Delta(T') + \ell - 1$. These observations imply that

$$\begin{aligned}
\mathrm{w(T)} - \mathrm{w(T')} &= 2(n - n') + 2(n_1(T) - n_1(T')) \\
&\quad + (n_2(T) - n_2(T')) + (\Phi_\Delta(T) - \Phi'_\Delta(T')) \\
&\ge 2(2\ell + \ell_0 + 2) + 2(\ell + \ell_0 - 1) + \ell + (\ell - 1) \\
&= 8\ell + 4\ell_0 + 1 > 8\ell,
\end{aligned}$$

and so $\mathrm{w(T)} > \mathrm{w(T')} + 8\ell$. Every γ_{pr}-set of T' can be extended to a PD-set of T by adding to it the vertex v_2 and all children of v_2 of degree 2 together with their leaf neighbors, excluding the vertex v_0. Thus, $\gamma_{pr}(T) \le \gamma_{pr}(T') + 2\ell$. Applying the inductive hypothesis to the tree T', we have

$$4\gamma_{pr}(T) = 4(\gamma_{pr}(T') + 2\ell) \le \mathrm{w(T')} + 8\ell < \mathrm{w(T)}.$$

This completes the proof of Theorem 5. □

That the upper bound in Theorem 5 is sharp may be seen as follows. For $\Delta \ge 3$ and $\ell \ge 1$, let $T_{\Delta,\ell}$ be the tree constructed as follows. Let $T_1 = S(K_{1,\Delta})$, and for $\ell \ge 2$, let $T_2, \ldots, T_\ell$ be $\ell - 1$ vertex disjoint copies of a subdivided star $S(K_{1,\Delta-1})$. Let v_i be the central vertex (of degree Δ) in T_i, and let u_i be an arbitrary neighbor of v_i in T_i for all $i \in [\ell]$. If $\ell = 1$, we define $T_{\Delta,\ell} = T_1$. For $\ell \ge 2$, let $T_{\Delta,\ell}$ be constructed from the disjoint union of the subdivided stars $T_1, \ldots, T_\ell$ by adding the $\ell - 1$ edges $u_i v_{i+1}$ for all $i \in [\ell - 1]$. For example, the tree $T_{5,4}$ is illustrated in Fig. 5. By construction, the tree $T_{\Delta,\ell}$ has maximum degree Δ.

Suppose that $T = T_{\Delta,1}$ for some $\Delta \ge 3$, and so $T = S(K_{1,\Delta})$. In this case, $\gamma_{pr}(T) = 2\Delta$, $n = 2\Delta + 1$, $n_1(T) = n_2(T) = \Delta$, and $\Phi_\Delta(T) = \Delta - 2$. Hence,

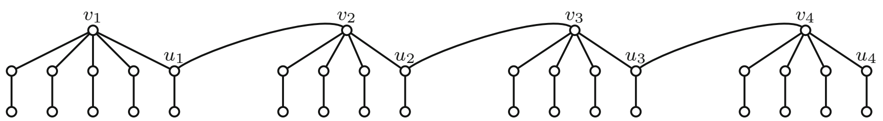


Fig. 5 The tree $T = T_{5,4}$

$4\gamma_{\rm pr}(T) = 8\Delta = 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T)$, and so we have equality in Inequality (2).

Suppose that $T = T_{\Delta,\ell}$ for some $\Delta \geq 3$ and $\ell \geq 2$. The set of $(\Delta-1)\ell+1$ support vertices of T form an independent set, implying that $\gamma_{\rm pr}(T) \geq 2(\Delta-1)\ell+2$. However, we can pair each support vertex with its leaf neighbor to form a PD-set of T, implying that $\gamma_{\rm pr}(T) \leq 2(\Delta-1)\ell+2$. Consequently, $\gamma_{\rm pr}(T) = 2(\Delta-1)\ell+2$. Moreover, $n(T) = 2\Delta\ell - \ell + 2$, $n_1(T) = \Delta\ell - \ell + 1$, $n_2(T) = \Delta\ell - 2\ell + 2$, and $\Phi_\Delta(T) = \ell(\Delta-2)$. Hence, $4\gamma_{\rm pr}(T) = 8(\Delta-1)\ell + 8 = 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T)$, and so we have equality in Inequality (2). We state this formally as follows.

Observation 8 For all integers $\Delta \geq 3$ and $\ell \geq 1$, the tree $T_{\Delta,\ell}$ satisfies equality in Inequality (2).

By Observation 8, the upper bound in Theorem 5 is tight.

6 Proof of Theorem 6

In this section we give a proof of Theorem 6. Recall its statement.

Theorem 6. *For $\Delta \geq 1$, if T is a tree with maximum degree $\Delta(T) = \Delta$, then*

$$\gamma_{\rm pr}(T) \leq \left(\frac{5\Delta-4}{8\Delta-4}\right)n + \frac{1}{2}n_1(T) + \frac{1}{4}n_2(T) - \left(\frac{\Delta-2}{4\Delta-2}\right).$$

Proof Let T be a tree of order n with maximum degree $\Delta \geq 1$. Let $\mathcal{P} = \{T_1, \ldots, T_p\}$ be an optimal subdivided star set in the tree T. Thus, T_i is a subdivided star $S(K_{1,n_i})$ where $2 \leq n_i \leq \Delta - 1$ for every $i \in [p]$. The tree T_i has order $|V(T_i)| = 2n_i + 1$, and so

$$\Phi_\Delta(T) = \sum_{i=1}^{p}(n_i - 1) = \sum_{i=1}^{p}\left(\frac{n_i-1}{2n_i+1}\right)|V(T_i)| \leq \left(\frac{\Delta-2}{2\Delta-1}\right)\sum_{i=1}^{p}|V(T_i)|. \quad (4)$$

Since $\mathcal{P}$ is a subdivided star set, the trees in the set $\mathcal{P}$ are vertex disjoint, implying that

$$\sum_{i=1}^{p}|V(T_i)| \leq n. \quad (5)$$

We consider three cases.

Case 1. $\sum_{i=1}^{p}|V(T_i)| \leq n - 2$. In this case, by Inequalities (1) and (4), we have

$$\begin{aligned}4\gamma_{\mathrm{pr}}(T) &\le 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T)\\ &\le 2n + 2n_1(T) + n_2(T) + \left(\frac{\Delta-2}{2\Delta-1}\right)(n-2)\\ &\le \left(\frac{5\Delta-4}{2\Delta-1}\right)n + 2n_1(T) + n_2(T) - 2\left(\frac{\Delta-2}{2\Delta-1}\right).\end{aligned}$$

Case 2. $\sum_{i=1}^{p}|V(T_i)| = n-1$. In this case, we have

$$n = 1 + \sum_{i=1}^{p}(2n_i+1) = 1 + 3p + 2\sum_{i=1}^{p}(n_i-1) = 2\Phi_\Delta(T) + 3p + 1, \tag{6}$$

and

$$n_1(T) \ge \sum_{i=1}^{p} n_i = \sum_{i=1}^{p}((n_i-1)+1) = \Phi_\Delta(T) + p. \tag{7}$$

Let S be the set of support vertices that belong to the subdivided stars in our optimal subdivided star set $\mathcal{P}$ of T. In this case, the set S can be extended to a PD-set S^* of T by adding to each vertex of S one of its neighbors in such a way as to maximize the pairs of vertices of S that form partners, implying that

$$\gamma_{\mathrm{pr}}(T) \le |S^*| \le 2|S| = \sum_{i=1}^{p} 2n_i = 2\sum_{i=1}^{p}((n_i-1)+1) = 2\Phi_\Delta(T) + 2p. \tag{8}$$

We note that if the set S of support vertices is not an independent set, then we can pair t support vertices as partners in the PD-set S^* for some $t \ge 1$, implying that $\gamma_{\mathrm{pr}}(T) \le |S^*| \le 2(|S|-t)$, and we can improve the inequality in Equality (8). Indeed, the more pairs of support vertices in S that can be paired together as partners in S^*, the smaller the resulting set S^*.

We consider here the case when $\gamma_{\mathrm{pr}}(T)$ is as large as possible, namely when the set S is an independent set, and so $|S^*| = 2|S|$ (the case when $|S^*| < 2|S|$ is simpler to handle). In this case, we note that since at most p edges of T are incident with support vertices of T that belong to one of the subdivided stars in our optimal subdivided star set $\mathcal{P}$, we have

$$n_2(T) \ge \left(\sum_{i=1}^{p} n_i\right) - p = (\Phi_\Delta(T) + p) - p = \Phi_\Delta(T). \tag{9}$$

Hence, by Inequalities (6), (7), (8), and (9), we have

$$4\gamma_{\mathrm{pr}}(T) \le 8\Phi_\Delta(T) + 8p \le 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T) - 2. \tag{10}$$

By Inequalities (4) and (10), we have

$$
\begin{aligned}
4\gamma_{\mathrm{pr}}(T) &\le 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T) - 2 \\
&\le 2n + 2n_1(T) + n_2(T) + \left(\frac{\Delta-2}{2\Delta-1}\right)(n-1) - 2 \\
&\le \left(\frac{5\Delta-4}{2\Delta-1}\right)n + 2n_1(T) + n_2(T) - \left(\frac{5\Delta-4}{2\Delta-1}\right) \\
&< \left(\frac{5\Delta-4}{2\Delta-1}\right)n + 2n_1(T) + n_2(T) - 2\left(\frac{\Delta-2}{2\Delta-1}\right).
\end{aligned}
$$

Case 3. $\sum_{i=1}^{p} |V(T_i)| = n$. In this case, we have

$$
n = \sum_{i=1}^{p}(2n_i + 1) = 2\sum_{i=1}^{p}(n_i - 1) + 3p = 2\Phi_\Delta(T) + 3p. \tag{11}
$$

Inequalities (7) and (8) hold as before. Analogously as in Case 2, we consider here the case when $\gamma_{\mathrm{pr}}(T)$ is as large as possible, namely when the set S is an independent set, and so $|S^*| = 2|S|$ (the case when $|S^*| < 2|S|$ is simpler to handle). In this case, we note that since at most $p-1$ edges of T are incident with support vertices of T that belong to one of the subdivided stars in our optimal subdivided star set $\mathcal{P}$, we have

$$
n_2(T) \ge \left(\sum_{i=1}^{p} n_i\right) - (p-1) = (\Phi_\Delta(T) + p) - (p-1) = \Phi_\Delta(T) + 1. \tag{12}
$$

Hence, by Inequalities (7), (8), (11), and (12), we have

$$
4\gamma_{\mathrm{pr}}(T) \le 8\Phi_\Delta(T) + 8p \le 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T) - 1. \tag{13}
$$

By Inequalities (4) and (13), we have

$$
\begin{aligned}
4\gamma_{\mathrm{pr}}(T) &\le 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T) - 1 \\
&\le 2n + 2n_1(T) + n_2(T) + \left(\frac{\Delta-2}{2\Delta-1}\right)n - 1 \\
&\le \left(\frac{5\Delta-4}{2\Delta-1}\right)n + 2n_1(T) + n_2(T) - 1 \\
&< \left(\frac{5\Delta-4}{2\Delta-1}\right)n + 2n_1(T) + n_2(T) - 2\left(\frac{\Delta-2}{2\Delta-1}\right).
\end{aligned}
$$

In all three cases, the desired Inequality (3) in the statement of the theorem holds. This completes the proof of Theorem 6. $\square$

For $\Delta \ge 3$ and $\ell \ge 1$, let $T_{\Delta,\ell}$ be the tree constructed in Sect. 5. If $T = T_{\Delta,1}$ for some $\Delta \ge 3$, then $T = S(K_{1,\Delta})$, and, by our earlier observations, we have $\gamma_{\mathrm{pr}}(T) = 2\Delta$, $n = n(T) = 2\Delta + 1$, and $n_1(T) = n_2(T) = \Delta$, and we have equality in

Inequality (2). If $T = T_{\Delta,\ell}$ for some $\Delta \geq 3$ and $\ell \geq 2$, then, by our earlier observations, we have $\gamma_{\mathrm{pr}}(T) = 2(\Delta - 1)\ell + 2$, $n = n(T) = 2\Delta\ell - \ell + 2$, $n_1(T) = \Delta\ell - \ell + 1$, and $n_2(T) = \Delta\ell - 2\ell + 2$, and once again we have equality in Inequality (2). We state this formally as follows.

Observation 9 For $\Delta \geq 3$ and $\ell \geq 1$, the tree $T_{\Delta,\ell}$ satisfies equality in Inequality (3).

By Observation 9, the upper bound in Theorem 6 is tight. As a further application of Theorem 5, we have the following upper bound on the paired domination of a tree.

Theorem 10 *For $\Delta \geq 1$, if T is a tree of ordern with maximum degree $\Delta(T) = \Delta$, then*

$$\gamma_{\mathrm{pr}}(T) \leq \frac{1}{2}n + \frac{3}{4}n_1(T) + \frac{1}{4}n_2(T). \tag{14}$$

Proof Let T be a tree of order n with maximum degree $\Delta \geq 1$. We follow the notation employed in the proof of Theorem 6. Since $\mathcal{P}$ is a subdivided star set, the trees in the set $\mathcal{P}$ are vertex disjoint and the leaves of each tree in $\mathcal{P}$ are leaves in the tree T, implying that

$$\Phi_\Delta(T) = \sum_{i=1}^{p}(n_i - 1) = \left(\sum_{i=1}^{p} n_i\right) - p \leq n_1(T) - p. \tag{15}$$

By Inequalities (2), (4) and (15), we have

$$\begin{aligned} 4\gamma_{\mathrm{pr}}(T) &\leq 2n + 2n_1(T) + n_2(T) + \Phi_\Delta(T) \\ &\leq 2n + 2n_1(T) + n_2(T) + (n_1(T) - p) \\ &\leq 2n + 3n_1(T) + n_2(T), \end{aligned}$$

which yields the desired Inequality (14) in the statement of the theorem. □

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Declarations

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