On the number of t-ary trees with a given path length

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Abstract

We show that the number of t-ary trees with path length equal to p is $\exp(h(t^{-1})t\log t \frac{p}{\log p}(1+o(1)))$, where $h(x) = -x\log x - (1-x)\log(1-x)$ is the binary entropy function. Besides its intrinsic combinatorial interest, the question recently arose in the context of information theory, where the number of t-ary trees with path length p estimates the number of universal types, or, equivalently, the number of different possible Lempel-Ziv'78 dictionaries for sequences of length p over an alphabet of size t.

Key words binary trees; t-ary trees; path length; universal types

1 Introduction

Path length is an important global parameter of a tree that arises in various computational contexts (cf. [7, Sec. 2.3.4.5]). Although the distribution of path lengths among trees with a given number of nodes has been studied, the problem of estimating their distribution by path length alone has remained open. The question recently arose in an information-theoretic context, in connection with the notion of *universal* type [10, 9], based on the incremental parsing of Ziv and Lempel (LZ78) [14]. When applied to a *t*-ary sequence, the LZ78 parsing produces a dictionary of strings that is best represented by a *t*-ary tree whose path length corresponds to the length of the sequence. Two sequences of the same length are said to be of the same universal type if they yield the same *t*-ary parsing tree. Sequences of the same universal type are, in a sense, statistically indistinguishable, as the variational distance between their empirical probability distributions of any finite order vanishes in the limit [9, 10]. Universal types generalize the notion underlying the classical *method of types*, which has lead to important theoretical results in information theory [3]. Of great interest in this context is the estimation of the number of different types for sequences of

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a given length. For universal types, this translates to the number of different LZ78 dictionaries for t-ary sequences of a given length, or, equivalently, the number of t-ary trees with a given path length, which is the subject of this paper.

First, we present some definitions and formalize the problem. Fix an integer $t \ge 2$. A *t*-ary tree *T* is defined recursively as either being empty or consisting of a root node *r* and the nodes of *t* disjoint, ordered, *t*-ary (sub-) trees T_1, T_2, \ldots, T_t , any number of which may be empty (cf. [7, Sec. 2.3.4.5]). When T_i is not empty, we say that there is an *edge* from *r* to the root *r'* of T_i , and that *r'* is a *child* of *r*. The total number of nodes of *T* is zero if *T* is empty, or $n_T = 1 + \sum_{i=1}^t n_{T_i}$ otherwise. A node of *T* is called a *leaf* if it has no children. The *depth* of a node $v \in T$ is defined as the number of edges traversed to get from the root *r* to *v*. We denote by $D_j^{(T)}$, $j \ge 0$, the number of nodes at depth *j* in *T*. The sequence $\{D_j^{(T)}\}$ is called the *profile* of *T*; we consider only finite trees, so $\{D_j^{(T)}\}$ has finite support. The *path length* of a non-empty tree *T*, denoted by p_T , is the sum of the depths of all the nodes in *T*, namely

$$p_T = \sum_{j \ge 1} j D_j^{(T)}$$

The subscript T in n_T and p_T will be omitted in the sequel when the tree being discussed is clear from the context. We call a *t*-ary tree with n nodes a [t, n] tree. A [t, n] tree with path length equal to p will be called a [t, n, p] tree, and a *t*-ary tree with path length equal to p and an unspecified number of nodes will be referred to as a $[t, \cdot, p]$ tree.

Let $C_t(n)$ denote the number of [t, n] trees, and $L_t(p)$ the number of $[t, \cdot, p]$ trees. It is well known [7, p. 589] that

$$C_t(n) = \frac{1}{(t-1)n+1} \begin{pmatrix} tn \\ n \end{pmatrix}, \quad n \ge 0, \, t > 1.$$
 (1)

In the binary case (t = 2), these are the well known *Catalan numbers* that arise in many combinatorial contexts. The determination of $L_t(p)$, on the other hand, has remained elusive, even for t = 2. Consider the bivariate generating function B(w, z)defined so that the coefficient of $w^p z^n$ in B(w, z) counts the number of [2, n, p] trees. B(w, z) satisfies the functional equation [7, p. 595]

$$zB(w, wz)^2 = B(w, z) - 1.$$

However, deriving the generating function, B(w, 1), of the numbers $L_2(p)$ from this equation appears quite challenging. Nevertheless, the equation and others of similar structure have been studied in the literature. In particular, the limiting distribution of the path length for a given number of nodes is related to the area under a Brownian excursion [11, 13, 12], which is also known as an *Airy distribution*. This distribution occurs in many combinatorial problems of theoretical and practical interest (cf. [4] and references therein). These studies, however, have not yielded explicit asymptotic estimates for the numbers $L_t(p)$. Let $h(x) = -x \log x - (1-x) \log(1-x)$ denote the binary entropy function.¹The main result of this paper is the following asymptotic estimate of $L_t(p)$.

Theorem 1 Let
$$\alpha = h(t^{-1}) t \log t$$
. Then, $L_t(p) = \exp\left(\frac{\alpha p}{\log p}(1+o(1))\right)$.

The theorem is derived by proving matching upper and lower bounds on $\log L_t(p)$. The proof is presented in Section 2.

We remark that Knessl and Szpankowski [6] have recently applied the WKB heuristic [1] to obtain an asymptotic expansion of $\log L_2(p)$ using tools of complex analysis. The heuristic makes certain assumptions on the form of asymptotic expansions, and is often considered a practically effective albeit non-rigorous method. The main term in the expansion of [6] is consistent with Theorem 1 for t = 2. The proofs in this paper, presented in the next section, are based mostly on simple combinatorial arguments.

2 Proof of the main result

In the following lemma, we list some elementary properties of t-ary trees that will be referred to in the proof of Theorem 1. For a discussion of these properties, see [7, Sec. 2.3.4.5].²

Lemma 1 (i) Let ℓ be a positive integer, and let T be a [t, n, p] tree achieving minimal path length among all t-ary trees with ℓ leaves. Then,

$$n = \ell + \left\lceil \frac{\ell - 1}{t - 1} \right\rceil \,. \tag{2}$$

Define

$$m = \lceil \log_t \ell \rceil,\tag{3}$$

and

$$\ell_1 = \ell - \left\lfloor \frac{t^m - \ell}{t - 1} \right\rfloor \,. \tag{4}$$

Then, the profile of T is given by

$$D_{j}^{(T)} = \begin{cases} t^{j}, & 0 \le j \le m - 1, \\ \ell_{1}, & j = m, \\ 0, & j > m, \end{cases}$$
(5)

In particular, all the leaves of T are either at depth m or m - 1.

 $^{^{1}}$ Unless a base is explicitly specified, exp and log denote, respectively, the exponential and logarithm functions with respect to an arbitrary base that remains consistent throughout the paper.

²A slight change of terminology is required: nodes of t-ary trees in our terminology correspond to *internal* nodes of *extended* t-ary trees in [7].

(ii) A[t, n, p] tree with minimal path length satisfies

$$p = p_{\min} = \left(n + \frac{1}{t-1}\right)\mu - \frac{t(t^{\mu} - 1)}{(t-1)^2} = n\log_t n - O(n),\tag{6}$$

where $\mu = m$ whenever $n \not\equiv 2 \mod t$, or $\mu = m+1$ otherwise, with m defined in (3) for the number of leaves, ℓ , of the tree. In particular, the tree of (i) satisfies (6) with $\mu = m$.

(iii) The number of nodes of a [t, n, p] tree satisfies

$$n \le \frac{p}{\log_t p - O(\log \log p)} = \frac{p}{\log_t p} (1 + o(1)).$$
(7)

(iv) The maximal path length of a [t, n] tree is achieved by a tree in which each internal node has exactly one child (hence, there is exactly one leaf in the tree). The path length of such a tree is

$$p_{\max} = \frac{n(n-1)}{2}$$
 (8)

(v) There is a [t, n, p] tree for each p in the range $p_{\min} \leq p \leq p_{\max}$.

Proof. Items (i),(ii), and (iv) follow immediately from the discussion in [7, Sec. 2.3.4.5]. For convenience in the proof of Theorem 1, we characterize, in Item (i), trees with minimal path length for a given number of *leaves*, while the discussion in [7] does so for trees with a given number of *nodes*. The two characterizations coincide, except for values of n such that $n \equiv 2 \mod t$, which never occur in (2). In that case, a tree with n - 1 nodes would have the same number of leaves and a shorter path length. A tree that has minimal path length for its number of leaves, on the other hand, always has minimal path length also for its number of nodes (given in (2)).

Item (iii) follows from (ii) by solving for n in an equation of the form $p = n \log_t n - O(n)$. Solutions of equations of this form are related to the *Lambert W* function, a detailed discussion of which can be found in [2].

To prove the claim of Item (v), consider a [t, n, p] tree T such that $D_j^{(T)} > 1$ for some integer j. Let j_T be the largest such integer for the tree T. It follows from these assumptions that T must have nodes u and v at depth j_T , such that u is a leaf, $v \neq u$, and v has at most one child. Thus, we can transform T by deleting uand adding a child to v, and obtain a [t, n, p + 1] tree. Starting with a $[t, n, p_{\min}]$ tree, the transformation can be applied repeatedly to obtain a sequence of trees with consecutive values of p, as long as the transformed tree has at least two leaves. When this condition ceases to hold, we have the tree of Item (iv), which has path length p_{\max} . We will also rely on the following estimate of $C_t(n)$ derived from (1) using Stirling's approximation (see, e.g., [8, Ch. 10]). For positive real numbers c_1 and c_2 , which depend on t but not on n, we have

$$c_1 n^{-\frac{3}{2}} \exp\left(h(t^{-1})t\,n\right) \le C_t(n) \le c_2 n^{-\frac{3}{2}} \exp\left(h(t^{-1})t\,n\right).$$
 (9)

Proof of Theorem 1. We recall that $\alpha = h(t^{-1}) t \log t$.

(a) Upper bound: $\log L_t(p) \le \frac{\alpha p}{\log p} (1 + o(1)).$

Let n_p denote the maximum number of nodes of any tree with path length equal to p. Clearly, we have

$$L_t(p) \le \sum_{n=1}^{n_p} C_t(n) \le n_p C_t(n_p)$$

and thus, by (9), we obtain

$$\log L_t(p) \leq \log n_p + \log C_t(n_p) \leq h(t^{-1})t n_p - \frac{1}{2}\log n_p + O(1)$$

= $\frac{\alpha}{\log t} n_p - \frac{1}{2}\log n_p + O(1).$ (10)

The claimed upper bound on $\log L_t(p)$ follows from (10) by applying Lemma 1(iii) with $n=n_p$. The asymptotic error term o(1) in the upper bound is, by (7), of the form $O(\log \log p / \log p)$.

(b) Lower bound: $\log L_t(p) \ge \frac{\alpha p}{\log p} (1 + o(1)).$

We prove the lower bound by constructing a sufficiently large class of $[t, \cdot, p]$ trees.

Let ℓ be a positive integer, $\ell > 2$. We start with a *t*-ary tree *T* with ℓ leaves and shortest possible path length, as characterized in Lemma 1(i). Let *q* be the integer satisfying

$$C_t(q-1) < \ell - 1 \le C_t(q),$$
 (11)

and let $\tau_1, \tau_2, \ldots, \tau_{\ell-1}$ be the first $\ell-1$ distinct [t, q] trees, when [t, q] trees are arranged in non-decreasing order of path length. Additionally, let τ_F be a tree with $\beta_F q$ nodes, for some positive constant β_F to be specified later. Finally, let π be a permutation on $\{1, 2, \ldots, \ell-1\}$. We construct a tree T_{π} by attaching the trees $\tau_1, \tau_2, \ldots, \tau_{\ell-1}$ and τ_F to the leaves of T, so that the *i*-th leaf (taken in some fixed order) becomes the root of a copy of $\tau_{\pi(i)}, 1 \leq i < \ell$. The tree τ_F , in turn, is attached to the last leaf of T, which is assumed to be at (the maximal) depth m. The construction is illustrated in Figure 1.

Next, we compute the path length, p, of T_{π} . By Lemma 1(i), all the leaves of T are either at depth $m = \lceil \log_t \ell \rceil$ or at depth m-1. Assume τ_i , $1 \le i \le \ell-1$, is attached to a leaf of depth $m-1+\epsilon_i$, $\epsilon_i \in \{0,1\}$, of T. Also, let ν_i denote the path length of τ_i , $1 \le i \le \ell-1$, and ν_F the path length of τ_F . The contribution of τ_i (excluding its root) to p is

$$p_i = \sum_{j \ge 1} (m - 1 + \epsilon_i + j) D_j^{(\tau_i)} = (m - 1 + \epsilon_i) \sum_{j \ge 1} D_j^{(\tau_i)} + \sum_{j \ge 1} j D_j^{(\tau_i)}$$

= $(m - 1 + \epsilon_i)(q - 1) + \nu_i$,

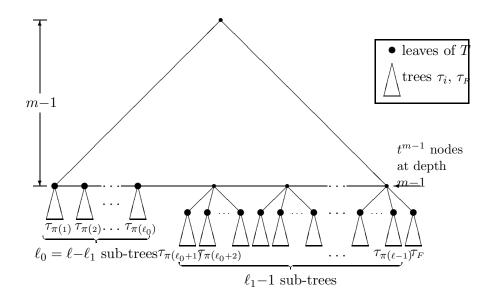


Figure 1: Tree T_{π}

Similarly, the contribution of τ_F to p is

$$p_F = m(\beta_F q - 1) + \nu_F \,.$$

Considering also the contribution of T according to its profile (5), we obtain

$$p = \sum_{i=1}^{\ell-1} (m-1+\epsilon_i)(q-1) + \sum_{i=1}^{\ell-1} \nu_i + m(\beta_F q - 1) + \nu_F + \sum_{j=1}^{m-1} jt^j + \ell_1 m.$$
(12)

Further, observing that $\sum_{i=1}^{\ell-1} \epsilon_i = \ell_1$, and defining $\overline{\nu} = (\ell-1)^{-1} \sum_{i=1}^{\ell-1} \nu_i$, we obtain

$$p = \left((\ell - 1)(m - 1) + \ell_1\right)(q - 1) + (\ell - 1)\overline{\nu} + m(\beta_F q - 1) + \nu_F + \sum_{j=1}^{m-1} jt^j + \ell_1 m \,. \tag{13}$$

Recall that the trees τ_i were selected preferring shorter path lengths, so their average path length $\overline{\nu}$ is at most as large as the average path length of all [t,q] trees. The latter average is known to be $O(q^{3/2})$ (this follows from the results of [5]; see also [7, Sec. 2.3.4.5] for t = 2). Observe also that, from the definition of q in (11), using (9) and (3), and recalling that $\alpha = h(t^{-1})t \log t$, we obtain

$$q = \frac{\log^2 t}{\alpha} m + O(\log m). \tag{14}$$

Recalling now that $n_{\tau_F} = \beta_F q$, and, hence, $\nu_F = O(q^2)$, it follows, after standard algebraic manipulations, that (13) can be rewritten as

$$p = \frac{\log^2 t}{\alpha} m^2 \ell + O(m^{3/2} \ell).$$
 (15)

It also follows from (13) that p is independent of the choice of permutation π . Moreover, by construction, each permutation π defines a different tree T_{π} , and, therefore, we have

$$L_t(p) \ge (\ell - 1)!$$
. (16)

From (16), using Stirling's approximation, applying (3) and (15), and simplifying, we can write

$$\frac{\log L_t(p)}{p} \geq \frac{\log(\ell-1)!}{p} = \frac{\ell \log \ell - O(\ell)}{p} = \frac{\ell m \log t - O(\ell)}{\alpha^{-1} (\log^2 t) m^2 \ell + O(m^{3/2}\ell)} = \frac{\alpha (1 - O(m^{-1}))}{m \log t \left(1 + O(m^{-\frac{1}{2}})\right)}.$$
 (17)

Taking logarithms on both sides of (15), and applying (3), we can write $m \log t = \log p - O(\log m)$. Substituting for $m \log t$ in (17), and simplifying asymptotic expressions, we obtain

$$\frac{\log L_t(p)}{p} \ge \frac{\alpha}{\log p} (1 - o(1)),\tag{18}$$

from which the desired lower bound follows. The o(1) term in (18) is $O(m^{-\frac{1}{2}}) = O((\log p)^{-\frac{1}{2}}).$

The above construction yields large classes of trees of path length p for a sparse sequence of values of p, controlled by the parameter ℓ . Next, we show how the gaps in the sparse sequence can be filled, yielding constructions, and validating the lower bound, for all (sufficiently large) integer values of p. In the following discussion, when we wish to emphasize the dependency of m, ℓ_1 , q, and p on ℓ , we will use the notations $m(\ell), \ell_1(\ell), q(\ell)$, and $p(\ell)$, respectively. Also, for any such function $f(\ell)$, we denote by Δf the difference $f(\ell+1) - f(\ell)$, with the value of ℓ being implied by the context. We start by estimating Δp .

Assume first that ℓ is such that $\Delta q = 0$ and $\Delta m = 0$. Then, substituting $\ell + 1$ for ℓ in (13), and subtracting the original equation, we obtain

$$\Delta p = (m - 1 + \Delta \ell_1)(q - 1) + \nu_\ell + \Delta \ell_1 \, m. \tag{19}$$

It follows from (4) that, with m fixed, we have $0 \leq \Delta \ell_1 \leq 2$. Also, by (8), we have $\nu_{\ell} < \frac{1}{2}q^2$. Hence, recalling (14), it follows from (19) that

$$\Delta p < \left(\frac{\alpha}{\log^2 t} + \frac{1}{2}\right)q^2 + O(q\log q). \tag{20}$$

Notice that, in (13), with all other parameters of the construction staying fixed, any increment in ν_F produces an identical change in p. By Lemma 1(v), by an appropriate evolution of τ_F , we can make ν_F assume any value in the range $(\nu_F)_{\min} \leq \nu_F \leq (\nu_F)_{\max}$, where $(\nu_F)_{\min} = O(\beta_F q \log q)$, and $(\nu_F)_{\max} = \frac{1}{2}\beta_F q(\beta_F q - 1)$. Choosing $\beta_F > \sqrt{2\alpha(\lg t)^{-2} + 1}$, this range of ν_F will make p span the gap between $p(\ell)$ and

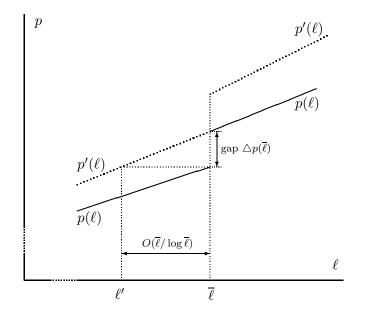


Figure 2: Bridging the gap in *q*-breaks

 $p(\ell + 1)$ as estimated in (20), for all sufficiently large ℓ satisfying the conditions of this case. Still, the variation in the value of p is asymptotically negligible and does not affect the validity of (18).

If $\Delta m = 1$, we must have $\ell = \ell_1(\ell) = t^m$, and $\ell_1(\ell+1) = 2$. In this case, using (13) again, we obtain

$$\Delta p = (\ell m + 2)(q - 1) + \beta_F q - 1 + \nu_\ell + m\ell + 2(m + 1) - ((\ell - 1)(m - 1) + \ell)(q - 1) - \ell m = (m + 1)(q + 1) + \nu_\ell + \beta_F q - 1,$$

which admits the same asymptotic upper bound as Δp in (20). Thus, the gap between $p(\ell)$ and $p(\ell+1)$ is filled also in this case by tuning the structure of τ_F .

The above method cannot be applied directly when $\Delta q = 1$. We call a value of ℓ such that $q(\ell + 1) = q(\ell) + 1$ a *q*-break. At a *q*-break, Δp is exponential in q, and a tree τ_F of polynomial size cannot compensate for such a gap. However, we observe that the construction of T_{π} , and its asymptotic analysis in (13)–(18) would also be valid if we chose q' = q + 1, instead of q, as the size of the trees τ_i . This choice would produce a different sequence of path length values $p'(\ell)$, which, when substituted for p, would also satisfy (18) and would validate the lower bound of the theorem. It follows from (13) that $p'(\ell) > p(\ell)$. Equivalently, for any given (sufficiently large) value ℓ , there exists an integer $\ell' < \ell$ such that $p'(\ell') \leq p(\ell) \leq p'(\ell' + 1)$.

Consider a q-break $\overline{\ell}$. To construct large classes of trees for all values of p, proceed as follows (refer to Figure 2): use the original sequence of values $p(\ell)$, filling the gaps as described above, until $\ell = \overline{\ell}$. At that point, find the largest integer ℓ' such that $p'(\ell') \leq p(\overline{\ell})$, and "backtrack" to $\ell = \ell'$. Continue with the sequence $p'(\ell)$, $\ell = \ell', \ell' + 1, \ldots$, filling the gaps accordingly. Notice that $q'(\ell)$ to the left of $\overline{\ell}$ is the same as $q(\ell)$ to the right of that point. Thus, $p'(\ell)$ continues "smoothly" (i.e., with gaps Δp as in (20)) into $p(\ell)$ at $\ell = \overline{\ell}$. The process now rejoins the sequence $p(\ell)$ as before, until the next q-break point. By (13), since the function $m(\ell)$ remains the same for both p and p', we have, asymptotically,

$$\ell' \approx (1 - 1/q)\overline{\ell} \approx \overline{\ell} - c_3\overline{\ell}/\log\overline{\ell},$$

for some positive constant c_3 . Thus, for sufficiently large $\overline{\ell}$, although the difference between ℓ' and $\overline{\ell}$ is negligible with respect to $\overline{\ell}$, ℓ' is guaranteed to fall properly between *q*-breaks, and the number of sequence points $p'(\ell)$ used between ℓ' and $\overline{\ell}$ is unbounded.

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References

- C. BENDER AND S. ORSZAG, Advanced Mathematical Methods for Scientists and Engineers, Mc-Graw Hill, 1978.
- [2] R. M. CORLESS, G. H. GONNET, D. E. G. HARE, D. J. JEFFREY, AND D. E. KNUTH, On the Lambert W function, Adv. Comput. Math., 5 (1996), pp. 329–359.
- [3] I. CSISZÁR, The method of types, IEEE Trans. Inf. Theory, 44 (1998), pp. 2505–2523.
- [4] P. FLAJOLET AND G. LOUCHARD, Analytic variations on the Airy distribution, Algorithmica, 31 (2001), pp. 361–377.
- [5] P. FLAJOLET AND A. M. ODLYZKO, The average height of binary trees and other simple trees, J. Comput. Syst. Sci., 25 (1982), pp. 171–213.
- [6] C. KNESSL AND W. SZPANKOWSKI, Enumeration of binary trees, Lempel-Ziv78 parsings, and universal types, in Proc. of the Second Workshop on Analytic Algorithmics and Combinatorics (ANALCO05), Vancouver, 2005.
- [7] D. E. KNUTH, The Art of Computer Programming. Fundamental Algorithms, vol. 1, Addison-Wesley, Reading, MA, third ed., 1997.
- [8] F. J. MACWILLIAMS AND N. J. A. SLOANE, The Theory of Error Correcting Codes, North-Holland Publishing Co., Amsterdam, 1983.
- [9] G. SEROUSSI, On universal types, IEEE Trans. Inf. Theory, 52 (2006), pp. 171–189.

- [10] —, Universal types and simulation of individual sequences, in LATIN 2004: Theoretical Informatics, M. Farach-Colton, ed., vol. LNCS 2976, Berlin, 2004, Springer-Verlag, pp. 312–321.
- [11] L. TAKÁCS, A Bernoulli excursion and its various applications, Adv. Appl. Prob., 23 (1991), pp. 557–585.
- [12] —, Conditional limit theorems for branching processes, J. Applied Mathematics and Stochastic Analysis, 4 (1991), pp. 263–292.
- [13] _____, On a probability problem connected with railway traffic, J. Applied Mathematics and Stochastic Analysis, 4 (1991), pp. 1–27.
- [14] J. ZIV AND A. LEMPEL, Compression of individual sequences via variable-rate coding, IEEE Trans. Inf. Theory, 24 (1978), pp. 530–536.