

# Minimum Manhattan network problem in normed planes with polygonal balls: a factor 2.5 approximation algorithm

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## Abstract

Let  $\mathcal{B}$  be a centrally symmetric convex polygon of  $\mathbb{R}^2$  and  $\|\mathbf{p} - \mathbf{q}\|$  be the distance between two points  $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$  in the normed plane whose unit ball is  $\mathcal{B}$ . For a set  $T$  of  $n$  points (terminals) in  $\mathbb{R}^2$ , a  $\mathcal{B}$ -Manhattan network on  $T$  is a network  $N(T) = (V, E)$  with the property that its edges are parallel to the directions of  $\mathcal{B}$  and for every pair of terminals  $\mathbf{t}_i$  and  $\mathbf{t}_j$ , the network  $N(T)$  contains a shortest  $\mathcal{B}$ -path between them, i.e., a path of length  $\|\mathbf{t}_i - \mathbf{t}_j\|$ . A *minimum  $\mathcal{B}$ -Manhattan network* on  $T$  is a  $\mathcal{B}$ -Manhattan network of minimum possible length. The problem of finding minimum  $\mathcal{B}$ -Manhattan networks has been introduced by Gudmundsson, Levcopoulos, and Narasimhan (APPROX'99) in the case when the unit ball  $\mathcal{B}$  is a square (and hence the distance  $\|\mathbf{p} - \mathbf{q}\|$  is the  $l_1$  or the  $l_\infty$ -distance between  $\mathbf{p}$  and  $\mathbf{q}$ ) and it has been shown recently by Chin, Guo, and Sun [6] to be strongly NP-complete. Several approximation algorithms (with factors 8, 4, 3, and 2) for the minimum Manhattan problem are known. In this paper, we propose a factor 2.5 approximation algorithm for the minimum  $\mathcal{B}$ -Manhattan network problem. The algorithm employs a simplified version of the strip-staircase decomposition proposed in our paper [5] and subsequently used in other factor 2 approximation algorithms for the minimum Manhattan problem.

*Keywords.* Normed plane, distance, geometric network design, Manhattan network, approximation algorithms.

## 1 Introduction

### 1.1 Normed planes

Given a compact, centrally symmetric, convex set  $\mathcal{B}$  in the plane  $\mathbb{R}^2$ , one can define a norm  $\|\cdot\| := \|\cdot\|_{\mathcal{B}} : \mathbb{R}^2 \rightarrow \mathbb{R}^+$  by setting  $\|\mathbf{v}\| = \lambda$ , where  $\mathbf{v} = \lambda\mathbf{u}$  and  $\mathbf{u}$  is a unit vector belonging to the boundary of  $\mathcal{B}$ . We can then define a metric  $d := d_{\mathcal{B}}$  on  $\mathbb{R}^2$  by setting  $d(\mathbf{p}, \mathbf{q}) = \|\mathbf{p} - \mathbf{q}\|$ . The resulting metric space  $(\mathbb{R}^2, d_{\mathcal{B}})$  is called a *normed* (or *Minkowski*) plane with unit disk (gauge)  $\mathcal{B}$  [2, 24]. In this paper, we consider normed planes in which the unit ball  $\mathcal{B}$  is a centrally symmetric convex polygon (i.e., a *zonotope*) of  $\mathbb{R}^2$ . We denote by  $\mathbf{b}_0, \dots, \mathbf{b}_{2m-1}$  the  $2m$  vertices of  $\mathcal{B}$  (in counterclockwise order around the circle) as well as the  $2m$  unit vectors

that define these vertices. By central symmetry of  $\mathcal{B}$ ,  $\mathbf{b}_k = -\mathbf{b}_{k+m}$  for  $k = 0, \dots, m-1$ . A *legal  $k$ -segment* of  $(\mathbb{R}^2, d_{\mathcal{B}})$  is a segment  $\mathbf{pq}$  lying on a line parallel to the line passing via  $\mathbf{b}_k$  and  $\mathbf{b}_{k+m}$ . A *legal path*  $\pi(\mathbf{p}, \mathbf{q})$  between two points  $\mathbf{p}, \mathbf{q}$  of  $\mathbb{R}^2$  is any path connecting  $\mathbf{p}$  and  $\mathbf{q}$  in which all edges are legal segments. The length of  $\pi(\mathbf{p}, \mathbf{q})$  is the sum of lengths of its edges. A *shortest  $\mathcal{B}$ -path* between  $\mathbf{p}$  and  $\mathbf{q}$  is a legal  $(\mathbf{p}, \mathbf{q})$ -path of minimum length. The best known example of normed planes with polygonal unit balls is the  $l_1$ -plane (also called the rectilinear plane) with norm  $\|\mathbf{v}\| = |x(\mathbf{v})| + |y(\mathbf{v})|$ . The unit ball of the  $l_1$ -plane is a square whose two diagonals lie on the  $x$ -axis and  $y$ -axis, respectively. The  $l_1$ -distance between two points  $\mathbf{p}$  and  $\mathbf{q}$  is  $d(\mathbf{p}, \mathbf{q}) := \|\mathbf{p} - \mathbf{q}\|_1 = |x(\mathbf{p}) - x(\mathbf{q})| + |y(\mathbf{p}) - y(\mathbf{q})|$ . The legal paths of the rectilinear plane are the paths consisting of horizontal and vertical segments, i.e., rectilinear paths. Another important particular case of polygonal norms is that of  $\lambda$ -norms (alias uniform norms) [4, 3] for which the unit ball  $\mathcal{B}$  is a regular polygon.

## 1.2 Minimum Manhattan and $\mathcal{B}$ -Manhattan network problems

A *rectilinear network*  $N = (V, E)$  in  $\mathbb{R}^2$  consists of a finite set  $V$  of points and horizontal and vertical segments connecting pairs of points of  $V$ . The *length* of  $N$  is the sum of lengths of its edges. Given a finite set  $T$  of points in the plane, a *Manhattan network* [15] on  $T$  is a rectilinear network  $N(T) = (V, E)$  such that  $T \subseteq V$  and for every pair of points in  $T$ , the network  $N(T)$  contains a shortest rectilinear path between them. A *minimum Manhattan network* on  $T$  is a Manhattan network of minimum possible length and the Minimum Manhattan Network problem (*MMN problem*) is to find such a network.

More generally, given a zonotope  $\mathcal{B} \subset \mathbb{R}^2$ , a  $\mathcal{B}$ -network  $N = (V, E)$  consists of a finite set  $V$  of points and legal segments connecting pairs of points of  $V$  (the edges of  $N$ ). The *length*  $l(N)$  of  $N$  is the sum of lengths of its edges. Given a set  $T = \{\mathbf{t}_1, \dots, \mathbf{t}_n\}$  of  $n$  points (called *terminals*), a  *$\mathcal{B}$ -Manhattan network* on  $T$  is a  $\mathcal{B}$ -network  $N(T) = (V, E)$  such that  $T \subseteq V$  and for every pair of terminals in  $T$ , the network  $N(T)$  contains a shortest  $\mathcal{B}$ -path between them (see Fig. 1). A *minimum  $\mathcal{B}$ -Manhattan network* on  $T$  is a  $\mathcal{B}$ -Manhattan network of minimum possible length and the Minimum  $\mathcal{B}$ -Manhattan Network problem ( *$\mathcal{B}$ -MMN problem*) is to find such a network. Fig. 3 illustrates the evolution of a minimum  $\mathcal{B}$ -Manhattan network defined on the same set of terminals when the number of directions in the unit ball  $\mathcal{B}$  is increasing (the directions of the unit ball are indicated at the upper left corner of each figure).

## 1.3 Known results

The minimum Manhattan network problem has been introduced by Gudmundsson, Levcopoulos, and Narasimhan [15]. Gudmundsson et al. [15] proposed an  $O(n^3)$ -time 4-approximation algorithm, and an  $O(n \log n)$ -time 8-approximation algorithm. They also conjectured that there exists a 2-approximation algorithm for this problem and asked if this problem is NP-complete. Quite recently, Chin, Guo, and Sun [6] solved this last open question from [15] and established that indeed the minimum Manhattan network problem is strongly NP-complete. Kato, Imai, and Asano [17] presented a 2-approximation algorithm, however, their correctness proof is incomplete (see [1]). Following [17], Benkert, Wolff, Shirabe, and Widmann

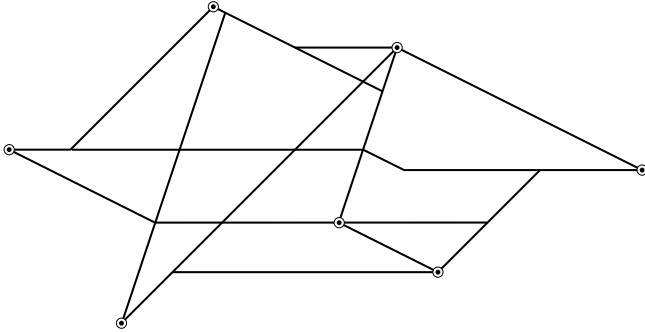


Figure 1: A  $\mathcal{B}$ -Manhattan network in the normed plane whose unit ball is depicted in Fig. 4

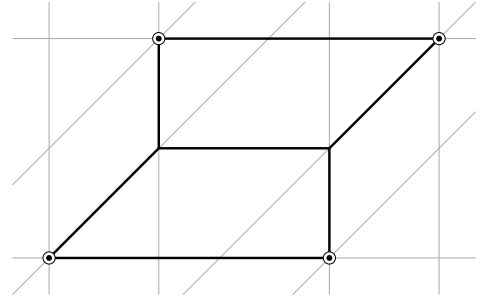


Figure 2: The unique optimal solution for this instance does not belong to the grid  $\Gamma$  (the unit ball  $\mathcal{B}$  is a hexagon)

[1] described an  $O(n \log n)$ -time 3-approximation algorithm and presented a mixed-integer programming formulation of the MMN problem. Nouioua [20] and later Fuchs and Schulze [12] presented two simple  $O(n \log n)$ -time 3-approximation algorithms. The first correct 2-approximation algorithm (thus solving the first open question from [15]) was presented by Chepoi, Nouioua, and Vaxès [5]. The algorithm is based on a strip-staircase decomposition of the problem and uses a rounding method applied to the optimal solution of the flow based linear program described in [1]. In his PhD thesis, Nouioua [20] described a  $O(n \log n)$ -time 2-approximation algorithm based on the primal-dual method from linear programming and the strip-staircase decomposition. In 2008, Guo, Sun, and Zhu [13, 14] presented two combinatorial factor 2 approximation algorithms, one with complexity  $O(n^2)$  and another with complexity  $O(n \log n)$  (see also the PhD thesis [22] of Schulze for yet another  $O(n \log n)$ -time 2-approximation algorithm). Finally, Seibert and Unger [21] announced a 1.5-approximation algorithm, however the conference format of their paper does not permit to understand the description of the algorithm and to check its claimed performance guarantee (a counterexample that an important intermediate step of their algorithm is incorrect was given in [12, 22]). Quite surprisingly, despite a considerable number of prior work on minimum Manhattan network problem, no previous paper, to our knowledge, consider its generalization to normed planes.

Gudmundsson et al. [15] introduced the minimum Manhattan networks in connection with the construction of sparse geometric spanners. Given a set  $T$  of  $n$  points in a normed plane and a real number  $t \geq 1$ , a geometric network  $N$  is a  $t$ -spanner for  $T$  if for each pair of points  $\mathbf{p}, \mathbf{q} \in T$ , there exists a  $(\mathbf{p}, \mathbf{q})$ -path in  $N$  of length at most  $t$  times the distance  $\|\mathbf{p} - \mathbf{q}\|$  between  $\mathbf{p}$  and  $\mathbf{q}$ . In the Euclidian plane and more generally, for normed planes with round balls, the line segment is the unique shortest path between two endpoints, and therefore the unique 1-spanner of  $T$  is the complete graph on  $T$ . On the other hand, if the unit ball of the norm is a polygon, the points are connected by several shortest  $\mathcal{B}$ -paths, therefore the problem of finding the sparsest 1-spanner becomes non trivial. In this connection, minimum  $\mathcal{B}$ -Manhattan networks are precisely the optimal 1-spanners. Sparse geometric spanners have

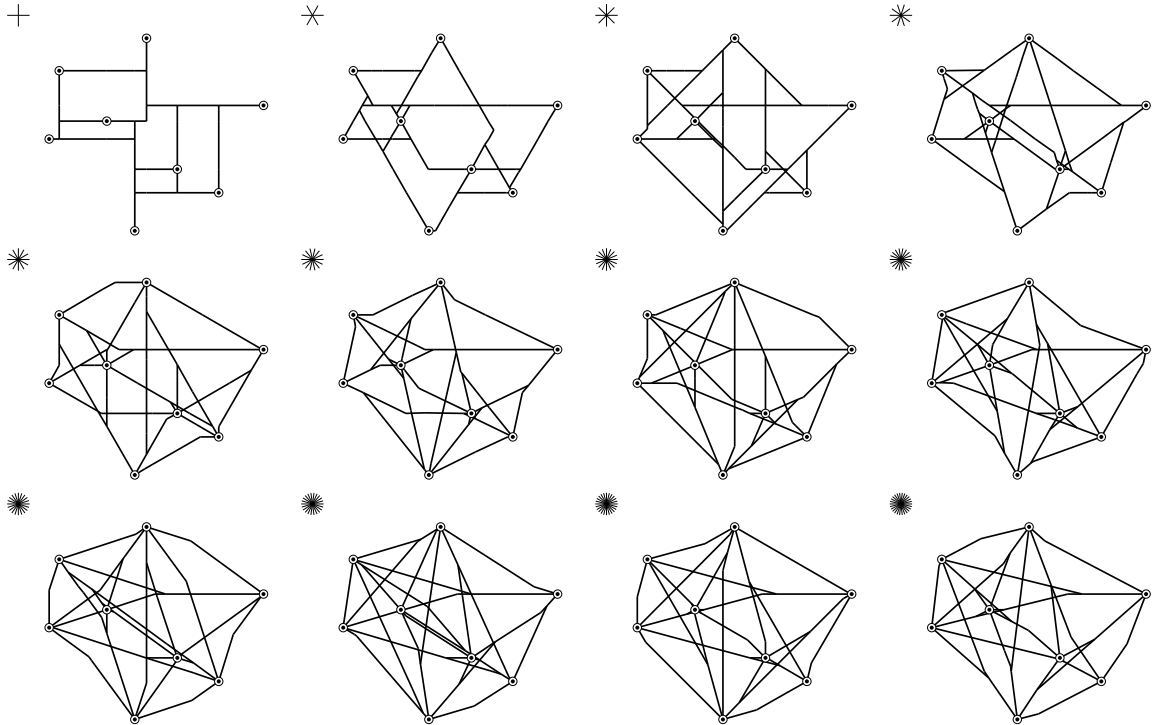


Figure 3: “Morphing” a minimum  $\mathcal{B}$ -Manhattan network

applications in VLSI circuit design, network design, distributed algorithms and other areas, see for example the survey of [10] and the book [18]. Lam, Alexandersson, and Pachter [19] suggested to use minimum Manhattan networks to design efficient search spaces for pair hidden Markov model (PHMM) alignment algorithms.

Algorithms for solving different distance problems in normed spaces with polygonal and polyhedral balls were proposed by Widmayer, Wu, and Wang [25] (for more references and a systematic study of such problems, see the book by Fink and Wood [11]). There is also an extensive bibliography on facility location problems in normed spaces with polyhedral balls, see for example [9, 23]. Finally, the minimum Steiner tree problem in the normed planes was a subject of intensive investigations, both from structural and algorithmic points of view; [4, 3, 8] is just a short sample of papers on the subject.

## 2 Preliminaries

### 2.1 Definitions, notations, auxiliary results

We continue by setting some basic definitions, notations, and known results. Let  $\mathcal{B}$  be a zonotope of  $\mathbb{R}^2$  with  $2m$  vertices  $\mathbf{b}_0, \dots, \mathbf{b}_{2m-1}$  having its center of symmetry at the origin of coordinates (see Fig. 4 for an example). The segment  $s_k := \mathbf{b}_k \mathbf{b}_{k+1(\text{mod } 2m)}$  is a *side* of  $\mathcal{B}$ . We will call the line  $\ell_i$  passing via the points  $\mathbf{b}_k$  and  $\mathbf{b}_{k+m}$  an *extremal line* of  $\mathcal{B}$ .

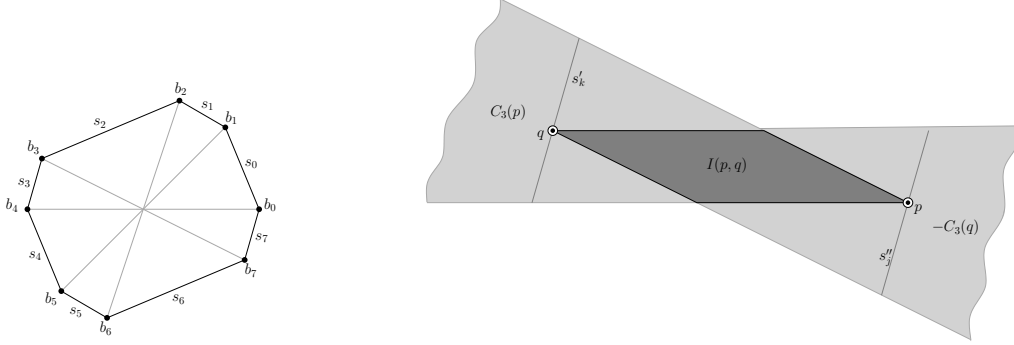


Figure 4: A unit ball  $\mathcal{B}$  and an interval  $I(\mathbf{p}, \mathbf{q})$

Two consecutive extremal lines  $\ell_k$  and  $\ell_{k+1}$  defines two opposite *elementary k-cones*  $C_k$  and  $C_{k+m} = -C_k$  containing the sides  $s_k$  and  $s_{k+m}$ , respectively. We extend this terminology, and call *elementary k-cones with apex  $\mathbf{v}$*  the cones  $C_k(\mathbf{v}) = C_k + \mathbf{v}$  and  $-C_k(\mathbf{v}) = C_{k+m} + \mathbf{v}$  obtained by translating the cones  $C_k$  and  $C_{k+m}$  by the vector  $\mathbf{v}$ . We will call a pair of consecutive lines  $D_k = \{\ell_k, \ell_{k+1}\}$  a *direction* of the normed plane. Denote by  $\mathcal{B}(\mathbf{v}, r) = r \cdot \mathcal{B} + \mathbf{v}$  the ball of radius  $r$  centered at the point  $\mathbf{v}$ .

Let  $I(\mathbf{p}, \mathbf{q}) = \{\mathbf{z} \in \mathbb{R}^2 : d(\mathbf{p}, \mathbf{q}) = d(\mathbf{p}, \mathbf{z}) + d(\mathbf{z}, \mathbf{q})\}$  be the *interval* between  $\mathbf{p}$  and  $\mathbf{q}$ . The inclusion  $\mathbf{pq} \subseteq I(\mathbf{p}, \mathbf{q})$  holds for all normed spaces. If  $\mathcal{B}$  is round, then  $\mathbf{pq} = I(\mathbf{p}, \mathbf{q})$ , i.e., the shortest path between  $\mathbf{p}$  and  $\mathbf{q}$  is unique. Otherwise,  $I(\mathbf{p}, \mathbf{q})$  may host a continuous set of shortest paths. The intervals  $I(\mathbf{p}, \mathbf{q})$  in a normed plane (and, more generally, in a normed space) can be constructed in the following pretty way, described, for example, in the book [2]. If  $\mathbf{pq}$  is a legal segment, then  $\mathbf{pq}$  is the unique shortest path between  $\mathbf{p}$  and  $\mathbf{q}$ , whence  $I(\mathbf{p}, \mathbf{q}) = \mathbf{pq}$ . Otherwise, set  $r = d(\mathbf{p}, \mathbf{q})$ . Let  $s'_k$  be the side of the ball  $\mathcal{B}(\mathbf{p}, r)$  containing the point  $\mathbf{q}$  and let  $s''_j$  be the side of the ball  $\mathcal{B}(\mathbf{q}, r)$  containing the point  $\mathbf{p}$ . Notice that these sides are well-defined, otherwise  $\mathbf{q}$  is a vertex of  $\mathcal{B}(\mathbf{p}, r)$  and  $\mathbf{pq}$  is a legal segment. The segments  $s'_k$  and  $s''_j$  are parallel, thus  $|k - j| = m$ , say  $k \leq m$  and  $j = k + m$ . Then  $I(\mathbf{p}, \mathbf{q})$  is the intersection of the elementary cones  $C_k(\mathbf{p})$  and  $C_{k+m}(\mathbf{q}) = -C_k(\mathbf{q})$  (see Fig. 4 for an illustration):

**Lemma 2.1** [2]  $I(\mathbf{p}, \mathbf{q}) = C_k(\mathbf{p}) \cap (-C_k(\mathbf{q}))$ .

An immediate consequence of this result is the following characterization of shortest  $\mathcal{B}$ -paths between two points  $\mathbf{p}$  and  $\mathbf{q}$ .

**Lemma 2.2** *If  $\mathbf{pq}$  is a legal segment, then  $\mathbf{pq}$  is the unique shortest  $\mathcal{B}$ -path. Otherwise, if  $I(\mathbf{p}, \mathbf{q}) = C_k(\mathbf{p}) \cap (-C_k(\mathbf{q}))$ , then any shortest  $\mathcal{B}$ -path  $\pi(\mathbf{p}, \mathbf{q})$  between  $\mathbf{p}$  and  $\mathbf{q}$  has only  $k$ -segments and  $(k + 1)$ -segments as edges. Moreover,  $\pi(\mathbf{p}, \mathbf{q})$  is a shortest  $\mathcal{B}$ -path if and only*

if it is monotone with respect to  $\ell_k$  and  $\ell_{k+1}$ , i.e., the intersection of  $\pi(\mathbf{p}, \mathbf{q})$  with any line  $\ell$  parallel to the lines  $\ell_k, \ell_{k+1}$  is empty, a point, or a (legal) segment.

**Proof.** The first statement immediately follows from Lemma 2.1. Suppose that  $\mathbf{pq}$  is not a legal segment and  $I(\mathbf{p}, \mathbf{q}) = C_k(\mathbf{p}) \cap C_{k+m}(\mathbf{q})$ . Let  $\mathbf{uv}$  be the first edge on a shortest path  $\pi(\mathbf{p}, \mathbf{q})$  from  $\mathbf{p}$  to  $\mathbf{q}$  which is neither a  $k$ -segment nor a  $(k+1)$ -segment. Since  $\mathbf{u} \in I(\mathbf{p}, \mathbf{q}) = C_k(\mathbf{p}) \cap C_{k+m}(\mathbf{q})$ , the point  $\mathbf{q}$  belongs to the cone  $C_k(\mathbf{u})$  and the point  $\mathbf{u}$  belongs to the cone  $C_{k+m}(\mathbf{q})$ , whence  $I(\mathbf{u}, \mathbf{q}) = C_k(\mathbf{u}) \cap C_{k+m}(\mathbf{q})$ . Obviously, the point  $\mathbf{v}$  belongs to  $I(\mathbf{u}, \mathbf{q})$ . However, by the choice of the segment  $\mathbf{uv}$  and the fact that  $\ell_k$  and  $\ell_{k+1}$  are consecutive lines that forms a direction, the point  $\mathbf{v}$  cannot belong  $C_k(\mathbf{u})$ , a contradiction. This shows that any shortest legal path  $\pi(\mathbf{p}, \mathbf{q})$  between  $\mathbf{p}$  and  $\mathbf{q}$  has only  $k$ - and  $(k+1)$ -segments as edges. Additionally, the intersection of  $\pi(\mathbf{p}, \mathbf{q})$  with any line  $\ell$  parallel to  $\ell_k$  or  $\ell_{k+1}$  is empty, a point, or a (legal) segment. Indeed, pick any two points in this intersection. Since the legal segment defined by these points is the unique shortest path between them, it must also belong to the intersection of  $\pi(\mathbf{p}, \mathbf{q})$  with  $\ell$ . Conversely, consider a monotone path  $\pi(\mathbf{p}, \mathbf{q})$  between  $\mathbf{p}$  and  $\mathbf{q}$ , namely suppose that the intersection of  $\pi(\mathbf{p}, \mathbf{q})$  with any line  $\ell$  parallel to the lines  $\ell_k$  or  $\ell_{k+1}$  is empty, a point, or a (legal) segment). We proceed by induction on the number of edges of  $\pi(\mathbf{p}, \mathbf{q})$ . The monotonicity of  $\pi(\mathbf{p}, \mathbf{q})$  implies that  $\pi(\mathbf{p}, \mathbf{q})$  lies entirely in the interval  $I(\mathbf{p}, \mathbf{q})$ . In particular, the neighbor  $\mathbf{u}$  of  $\mathbf{p}$  in  $\pi(\mathbf{p}, \mathbf{q})$  belongs to  $I(\mathbf{p}, \mathbf{q})$ . The subpath  $\pi(\mathbf{u}, \mathbf{q})$  of  $\pi(\mathbf{p}, \mathbf{q})$  between  $\mathbf{u}$  and  $\mathbf{q}$  is monotone, therefore by induction assumption,  $\pi(\mathbf{u}, \mathbf{q})$  is a shortest path between  $\mathbf{u}$  and  $\mathbf{q}$ . Since  $\mathbf{pu}$  is a legal segment and  $\mathbf{u} \in I(\mathbf{p}, \mathbf{q})$ , we immediately conclude that  $\pi(\mathbf{p}, \mathbf{q})$  is also a shortest path between  $\mathbf{p}$  and  $\mathbf{q}$ .  $\square$

We continue with some notions and notations about the  $\mathcal{B}$ -MMN problem. Denote by  $\text{OPT}(T)$  the length of a minimum  $\mathcal{B}$ -Manhattan network for a set of terminals  $T$ . For a direction  $D_k = \{\ell_k, \ell_{k+1}\}$ , denote by  $F_k$  the set of all pairs  $\{i, j\}$  (or pairs of terminals  $\{\mathbf{t}_i, \mathbf{t}_j\}$ ) such that any shortest  $\mathcal{B}$ -path between  $\mathbf{t}_i$  and  $\mathbf{t}_j$  uses only  $k$ -segments and  $(k+1)$ -segments. Equivalently, by Lemma 2.2,  $F_k$  consists of all pairs of terminals which belong to two opposite elementary cones  $C_k(\mathbf{v})$  and  $-C_k(\mathbf{v})$  with common apex. For each direction  $D_k$  and the set of pairs  $F_k$ , we formulate an auxiliary problem which we call *Minimum 1-Directional Manhattan Network problem* (or *1-DMMN( $F_k$ ) problem*): find a network  $N_k^{\text{opt}}(T)$  of minimum possible length such that every edge of  $N_k^{\text{opt}}(T)$  is an  $k$ -segment or an  $(k+1)$ -segment and any pair  $\{\mathbf{t}_i, \mathbf{t}_j\}$  of  $F_k$  is connected in  $N_k^{\text{opt}}(T)$  by a shortest  $\mathcal{B}$ -path. We denote its length by  $\text{OPT}_k(T)$ . We continue by adapting to 1-DMMN the notion of a generating set introduced in [17] for MMN problem: a *generating set* for  $F_k$  is a subset  $F$  of  $F_k$  with the property that a  $\mathcal{B}$ -Manhattan network containing shortest  $\mathcal{B}$ -paths for all pairs in  $F$  is a 1-Directional Manhattan network for  $F_k$ .

## 2.2 Our approach

Let  $N^*(T)$  be a minimum  $\mathcal{B}$ -Manhattan network, i.e., a  $\mathcal{B}$ -Manhattan network of total length  $l(N^*(T)) = \text{OPT}(T)$ . For each direction  $D_k$ , let  $N_k^*(T)$  be the set of  $k$ -segments and  $(k+1)$ -segments of  $N^*(T)$ . The network  $N_k^*(T)$  is an admissible solution for 1-DMMN( $F_k$ ), thus

the length  $l(N_k^*(T))$  of  $N_k^*(T)$  is at least  $\text{OPT}_k(T)$ . Any  $k$ -segment of  $N^*(T)$  belongs to two one-directional networks  $N_k^*(T)$  and  $N_{k-1}^*(T)$ . Vice-versa, if  $N_k(T), k = 0, \dots, m-1$ , are admissible solutions for the 1-DMMN( $F_k$ ) problems, since  $\bigcup_{k=0}^{m-1} F_k = T \times T$ , the network  $N(T) = \bigcup_{k=0}^{m-1} N_k(T)$  is a  $\mathcal{B}$ -Manhattan network. Moreover, if each  $N_k(T)$  is an  $\alpha$ -approximation for respective 1-DMMN problem, then the network  $N(T)$  is a  $2\alpha$ -approximation for the minimum  $\mathcal{B}$ -Manhattan network problem. Therefore, to obtain a factor 2.5-approximation for  $\mathcal{B}$ -MMN, we need to provide a 1.25-approximation for the 1-DMMN problem. The remaining part of our paper describe such a combinatorial algorithm. The 1-DMMN problem is easier and less restricted than the  $\mathcal{B}$ -MMN problem because we have to connect with shortest paths only the pairs of terminals of the set  $F_k$  corresponding to one direction  $D_k$ , while in case of the MMN problem the set  $T \times T$  of all pairs is partitioned into two sets corresponding to the two directions of the  $l_1$ -plane. For our purposes, we will adapt the strip-staircase decomposition of [5], by considering only the strips and the staircases which “are oriented in direction  $D_k$ ”.

### 3 One-directional strips and staircases

In the next two sections, we assume that  $D_k = \{l_k, l_{k+1}\}$  is a fixed but arbitrary direction of the normed plane. We recall the definitions of vertical and horizontal strips and staircases introduced in [5]. Then we consider only those of them which correspond to pairs of terminals from the set  $F_k$ , which we call one-directional strips and staircases. We formulate several properties of one-directional strips and staircases and we prove those of them which do not hold for usual strips and staircases.

Denote by  $L_k$  and  $L_{k+1}$  the set of all lines passing via the terminals of  $T$  and parallel to the extremal lines  $\ell_k$  and  $\ell_{k+1}$ , respectively. Let  $\Gamma_k$  be the grid defined by the lines of  $L_k$  and  $L_{k+1}$ . The following lemma can be proved in the same way as for rectilinear Steiner trees or Manhattan networks (quite surprisingly, this is not longer true for the  $\mathcal{B}$ -MMN problem: Fig. 2 presents an instance of  $\mathcal{B}$ -MMN for which the unique optimal solution does not belong to the grid  $\Gamma := \bigcup_{k=0}^{m-1} \Gamma_k$ ):

**Lemma 3.1** *There exists a minimum 1-Directional Manhattan Network for  $F_k$  contained in the grid  $\Gamma_k$ .*

For two terminals  $\mathbf{t}_i, \mathbf{t}_{i'}$ , set  $R_{i,i'} := I(\mathbf{t}_i, \mathbf{t}_{i'})$ . A pair  $\mathbf{t}_i, \mathbf{t}_{i'}$  defines a  $k$ -strip  $R_{i,i'}$  if either (i) (degenerated strip)  $\mathbf{t}_i$  and  $\mathbf{t}_{i'}$  are consecutive terminals belonging to the same line of  $L_k$  or (ii)  $\mathbf{t}_i$  and  $\mathbf{t}_{i'}$  belong to two consecutive lines of  $L_k$  and the intersection of  $R_{i,i'}$  with any degenerated  $k$ -strip is either empty or one of the terminals  $\mathbf{t}_i$  or  $\mathbf{t}_{i'}$ ; see Fig. 6 of [5]. The two  $k$ -segments of  $R_{i,i'}$  are called the *sides* of  $R_{i,i'}$ . The  $(k+1)$ -strips and their sides are defined analogously (with respect to  $L_{k+1}$ ). With some abuse of language, we will call the  $k$ -strips *horizontal* and the  $(k+1)$ -strips *vertical*. If a pair  $\{\mathbf{t}_i, \mathbf{t}_{i'}\}$  defining a horizontal or a vertical strip  $R_{i,i'}$  belongs to the set  $F_k$ , then we say that  $R_{i,i'}$  is a *one-directional strip* or a *1-strip*, for short. Denote by  $F'_k$  the set of all pairs of  $F_k$  defining one-directional strips.



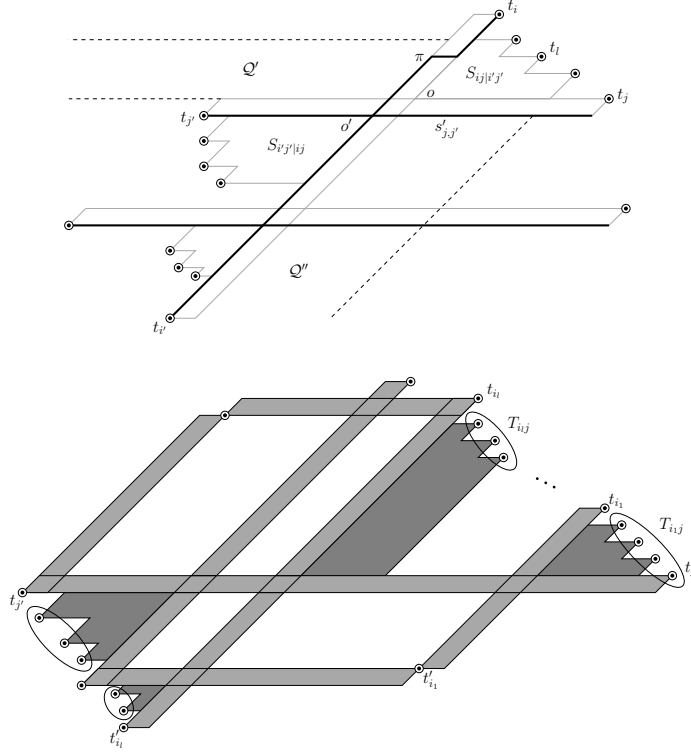


Figure 5: Strips, staircases, and completion

**Lemma 3.2** *If  $R_{i,i'}$  and  $R_{j,j'}$  are two horizontal 1-strips or two vertical 1-strips, then  $R_{i,i'} \cap R_{j,j'} = \emptyset$  if  $\{i, i'\} \cap \{j, j'\} = \emptyset$  and  $R_{i,i'} \cap R_{j,j'} = \{t_i\}$  if  $\{i, i'\} \cap \{j, j'\} = \{i\}$ .*

**Proof.** From the definition follows that if  $R_{i,i'}$  and  $R_{j,j'}$  are both degenerated or one is degenerated and another one not, then they are either disjoint or intersect in a single terminal. If  $R_{i,i'}$  and  $R_{j,j'}$  are both non-degenerated and intersect, then from the definition immediately follows that the intersection is one point or a segment belonging to their sides. However, if  $R_{i,i'}$  and  $R_{j,j'}$  intersects in a segment, then one can easily see that at least one of  $R_{i,i'}$  and  $R_{j,j'}$  cannot be a 1-strip.  $\square$

We say that a vertical 1-strip  $R_{i,i'}$  and a horizontal 1-strip  $R_{j,j'}$  (degenerated or not) form a *crossing configuration* if they intersect (and therefore cross each other).

**Lemma 3.3** *If  $R_{i,i'}$  and  $R_{j,j'}$  form a crossing configuration, then from the shortest  $\mathcal{B}$ -paths between  $t_i$  and  $t_{i'}$  and between  $t_j$  and  $t_{j'}$  one can derive shortest  $\mathcal{B}$ -paths connecting  $t_i, t_{j'}$  and  $t_{i'}, t_j$ , respectively.*

For a crossing configuration defined by the 1-strips  $R_{i,i'}$ ,  $R_{j,j'}$ , denote by  $\mathbf{o}$  and  $\mathbf{o}'$  the two opposite corners of the parallelogram  $R_{i,i'} \cap R_{j,j'}$ , such that the cones  $C_k(\mathbf{o})$  and  $-C_k(\mathbf{o}')$  do not intersect the interiors of  $R_{i,i'}$  and  $R_{j,j'}$ . Additionally, suppose without loss of generality,



that  $\mathbf{t}_i$  and  $\mathbf{t}_j$  belong to the cone  $C_k(\mathbf{o})$ , while  $\mathbf{t}_{i'}$  and  $\mathbf{t}_{j'}$  belong to the cone  $-C_k(\mathbf{o}')$ . Denote by  $T_{i,j}$  the set of all terminals  $\mathbf{t}_l \in (T \setminus \{\mathbf{t}_i, \mathbf{t}_j\}) \cap C_k(\mathbf{o})$  such that  $(-C_k(\mathbf{t}_l)) \setminus (-C_k(\mathbf{o}))$  does not contain any terminal except  $\mathbf{t}_l$ . Denote by  $\mathcal{S}_{i,j|i',j'}$  the region of  $C_k(\mathbf{o})$  which is the union of the intervals  $I(\mathbf{t}_l, \mathbf{o})$ ,  $\mathbf{t}_l \in T_{i,j}$ , and call this polygon an *one-directional staircase* or a *1-staircase*, for short; see Fig. 5 and Figures 7,8 of [5] for an illustration. Note that  $\mathcal{S}_{i,j|i',j'}$  is bounded by the 1-strips  $R_{i,i'}$  and  $R_{j,j'}$  and a legal path between  $\mathbf{t}_i$  and  $\mathbf{t}_j$  passing via all terminals of  $T_{i,j}$  and consisting of  $k$ -segments and  $(k+1)$ -segments. The point  $\mathbf{o}$  is called the *origin* and  $R_{i,i'}$  and  $R_{j,j'}$  are called the *basis* of this staircase. Since  $I(\mathbf{t}_l, \mathbf{o}) \subset (-C_k(\mathbf{t}_l)) \setminus (-C_k(\mathbf{o}))$  for all  $\mathbf{t}_l \in T_{i,j}$ ,  $I(\mathbf{t}_l, \mathbf{o}) \cap T = \{\mathbf{t}_l\}$  and therefore  $\mathcal{S}_{i,j|i',j'} \cap T = T_{i,j}$ . For the same reason, there are no terminals of  $T$  located in the regions  $\mathcal{Q}'$  and  $\mathcal{Q}''$  depicted in Fig. 5 ( $\mathcal{Q}'$  is the region comprised between the leftmost side of  $R_{i,i'}$ , the highest side of  $R_{j,j'}$ , and the line of  $L_k$  passing via the highest terminal of  $T_{i,j}$ , while  $\mathcal{Q}''$  is the region comprised between the rightmost side of  $R_{i,i'}$ , the lowest side of  $R_{j,j'}$ , and the line of  $L_{k+1}$  passing via the rightmost terminal of  $T_{i,j}$ ). Analogously one can define the set  $T_{i',j'}$  and the staircase  $\mathcal{S}_{i',j'|i,j}$  with origin  $\mathbf{o}'$  and basis  $R_{i,i'}$  and  $R_{j,j'}$ .

**Lemma 3.4** *If a 1-strip  $R_{l,l'}$  intersects a 1-staircase  $\mathcal{S}_{i',j'|i,j}$  and  $R_{l,l'}$  is different from the 1-strips  $R_{i,i'}$  and  $R_{j,j'}$ , then  $R_{l,l'} \cap \mathcal{S}_{i',j'|i,j}$  is a single terminal.*

**Proof.** If a 1-strip  $R_{l,l'}$  traverses a staircase  $\mathcal{S}_{i',j'|i,j}$ , then one of the terminals  $\mathbf{t}_l, \mathbf{t}_{l'}$  must be located in one of the regions  $\mathcal{Q}'$  and  $\mathcal{Q}''$ , which is impossible because  $(\mathcal{Q}' \cup \mathcal{Q}'') \cap T = \emptyset$ . Thus, if  $R_{l,l'}$  and  $\mathcal{S}_{i',j'|i,j}$  intersect more than in one point, then they intersect in a segment  $s$  which belongs to one side of  $R_{l,l'}$  and to the boundary of  $\mathcal{S}_{i',j'|i,j}$ . If say the 1-strip  $R_{l,l'}$  is horizontal, then necessarily  $s$  is a part of the lowest side of  $R_{l,l'}$  and of the highest horizontal side of  $\mathcal{S}_{i',j'|i,j}$ . Let  $\mathbf{t}$  be the highest terminal of  $T_{i,j}$ . Then either  $\mathbf{t}$  belongs to  $R_{l,l'}$  and is different from  $\mathbf{t}_l, \mathbf{t}_{l'}$ , contrary to the assumption that  $R_{l,l'}$  is a strip, or  $\mathbf{t}$  together with the lowest terminal  $\mathbf{t}_{l'}$  of  $R_{l,l'}$  define a degenerated strip with  $\mathbf{t}_{l'}$  belonging to  $\mathcal{Q}'$ , contrary to the assumption that  $\mathcal{Q}' \cap T = \emptyset$ .  $\square$

**Lemma 3.5** *Two 1-staircases either are disjoint or intersect only in common terminals.*

**Proof.** From the definition of a staircase follows that the interiors of two staircases are disjoint (for a short formal proof of this see [5]). Therefore two staircases may intersect only on the boundary. In this case, the intersection is either a subset of terminals of both staircases or a single edge. In the second case, one of the two staircases necessarily is not a 1-staircase with respect to the chosen direction.  $\square$

Let  $F_k''$  be the set of all pairs  $\{\mathbf{t}_{j'}, \mathbf{t}_l\}$  such that there exists a 1-staircase  $\mathcal{S}_{i,j|i',j'}$  with  $\mathbf{t}_l$  belonging to the set  $T_{i,j}$ . The proof of the following essential result is identical to the proof of Lemma 3.2 of [5] and therefore is omitted.

**Lemma 3.6**  *$F := F_k' \cup F_k''$  is a generating set for  $F_k$ .*

## 4 The algorithm

We continue with the description of our factor 1.25 approximation algorithm for 1-DMMN problem. Let  $F_k^h$  and  $F_k^v$  denote the pairs of  $F'_k$  defining horizontal and vertical 1-strips, respectively. Let  $S_1^h$  and  $S_2^h$  be the networks consisting of lower sides and respectively upper sides of the horizontal 1-strips of  $F_k^h$ . Analogously, let  $S_1^v$  and  $S_2^v$  be the networks consisting of rightmost sides and respectively leftmost sides of the vertical 1-strips of  $F_k^v$ . The algorithm completes optimally each of the networks  $S_1^h, S_2^h, S_1^v$ , and  $S_2^v$ , and from the set of four completions  $N_1^h, N_2^h, N_1^v, N_2^v$ , the algorithm returns the shortest one, which we will denote by  $N_k(T)$  (in this respect, our algorithm has some similarity with the approach of Benkert et al. [1]). We will describe now the optimal completion  $N_1^h$  for the network  $S_1^h$ , the three other networks are completed in the same way (up to symmetry).

An *optimal completion* of  $S_1^h$  is a subnetwork  $N_1^h$  of  $\Gamma_k$  extending  $S_1^h$  ( $S_1^h \subseteq N_1^h$ ) of smallest total length such that any pair of terminals of  $F$  can be connected in  $N_1^h$  by a shortest path. By Lemma 3.6, to solve the completion problem for  $S_1^h$ , it suffices to (i) select a shortest path  $\pi(\mathbf{t}_i, \mathbf{t}_{i'})$  of  $\Gamma_k$  between each pair  $\mathbf{t}_i, \mathbf{t}_{i'}$  defining a vertical 1-strip  $R_{i,i'}$ , (ii) for each horizontal 1-strip  $R_{j,j'}$  find a shortest path  $\pi(\mathbf{t}_j, \mathbf{t}_{j'})$  between  $\mathbf{t}_j$  and  $\mathbf{t}_{j'}$  subject to the condition that the lowest side  $s'_{j,j'}$  of  $R_{j,j'}$  is already available, (iii) for each staircase  $S_{i,j|i',j'}$  whose sides are  $R_{i,i'}$  and  $R_{j,j'}$  select shortest paths from the terminals of  $T_{i,j}$  to the terminal  $\mathbf{t}_{j'}$  subject to the condition that the lowest side  $s'_{j,j'}$  of  $R_{j,j'}$  is already available. We need to minimize the total length of the resulting network  $N_1^h$  over all vertical 1-strips and all 1-staircases. To solve the issue (ii) for a horizontal 1-strip  $R_{j,j'}$ , we consider the rightmost 1-staircase  $S_{i,j|i',j'}$  having  $R_{j,j'}$  as a basis, set  $T_{i,j} := T_{i,j} \cup \{\mathbf{t}_j\}$ , and solve for this staircase the issue (iii) for the extended set of terminals. For all other 1-staircases  $S_{i,j|i',j'}$  and  $S_{i',j'|i,j}$  having  $R_{j,j'}$  as a basis, we will solve only the issue (iii) for  $T_{i,j}$  and  $T_{i',j'}$ , respectively.

To deal with (iii), for each vertical 1-strip  $R_{i,i'}$ , we pick each shortest path  $\pi$  of  $\Gamma_k$  between  $\mathbf{t}_i$  and  $\mathbf{t}_{i'}$ , include it in the current completion, and solve (iii) for all 1-staircases having  $R_{i,i'}$  as a vertical base and taking into account that  $\pi$  is already present. We have to connect the terminals of  $T_{i,j}$  by shortest paths of  $\Gamma_k$  of least total length to the terminal  $\mathbf{t}_{j'}$  subject to the condition that the union  $\pi \cup s'_{j,j'}$  is already available; see Fig. 5. For a fixed path  $\pi$ , this task can be done by dynamic programming in  $O(|T_{i,j}|^3)$  time. For this, notice that in an optimal solution (a) either the highest terminal of  $T_{i,j}$  is connected by a vertical segment to  $s'_{j,j'}$ , or (b) the lowest terminal of  $T_{i,j}$  is connected by a horizontal segment to  $\pi$ , or (c)  $T_{i,j}$  contains two consecutive (in the staircase) terminals  $\mathbf{t}_l, \mathbf{t}_{l+1}$ , such that  $\mathbf{t}_l$  is connected to  $\pi$  by a horizontal segment and  $\mathbf{t}_{l+1}$  is connected to  $s'_{j,j'}$  by a vertical segment. In each of the three cases and subsequent recursive calls, we are lead to solve subproblems of the following type: given a set  $T'$  of consecutive terminals of  $T_{i,j}$ , the path  $\pi$  and a horizontal segment  $s'$ , connect to  $\mathbf{t}_{j'}$  the terminals of  $T'$  by shortest paths of least total length if the union  $\pi \cup s'$  is available. We define by  $C_{i,i'}^\pi$  the optimal completion obtained by solving by dynamic programming those problems for all staircases having  $R_{i,i'}$  as a vertical basis (note that  $\pi \subseteq C_{i,i'}^\pi$  however  $S_1^h \cap C_{i,i'}^\pi = \emptyset$ ). For each vertical 1-strip  $R_{i,i'}$ , the completion algorithm returns the partial completion  $C_{i,i'}^{opt}$  of least total length, i.e.,  $C_{i,i'}^{opt}$  is the smallest completion of

the form  $C_{i,i'}^\pi$  taken over all  $O(n)$  shortest paths  $\pi$  running between  $\mathbf{t}_i$  and  $\mathbf{t}_{i'}$  in  $\Gamma_k$ . Finally, let  $N_1^h$  be the union of all  $C_{i,i'}^{opt}$  over all vertical 1-strips  $R_{i,i'}$  and  $S_1^h$ . The pseudocode of the completion algorithm is presented below (the total complexity of this algorithm is  $O(n^3)$ ).

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**Algorithm 1** Optimal completion( $S_1^h$ )

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1:  $N_1^h \leftarrow S_1^h$ 
2: for each vertical 1-strip  $R_{i,i'}$  do
3:   for each shortest path  $\pi$  of  $\Gamma_k$  connecting the terminals  $\mathbf{t}_i$  and  $\mathbf{t}_{i'}$  do
4:     compute the partial completion  $C_{i,i'}^\pi$  in the following way:
5:      $C_{i,i'}^\pi \leftarrow \pi \setminus S_1^h$ 
6:     for each 1-staircase  $S_{i,j|i',j'}$  and each 1-staircase  $S_{i',j'|i,j}$  do
7:       if  $S_{i,j|i',j'}$  is the rightmost staircase having  $R_{j,j'}$  as a base, then set  $T_{i,j} \leftarrow T_{i,j} \cup \{\mathbf{t}_j\}$ 
8:       compute by dynamic programming the subset  $C$  of edges of  $\Gamma_k$  of least total length
       such that  $C \cup (\pi \cup s'_{j,j'})$  contains a shortest path of  $\Gamma_k$  from each terminal of  $T_{i,j}$ 
       to  $\mathbf{t}_{j'}$  or from each terminal of  $T_{i',j'}$  to  $\mathbf{t}_j$ 
9:        $C_{i,i'}^\pi \leftarrow C_{i,i'}^\pi \cup C$ 
10:    end for
11:  end for
12:  let  $C_{i,i'}^{opt}$  be the partial completion of least total length, i.e,  $C_{i,i'}^{opt}$  is the smallest comple-
  tion  $C_{i,i'}^\pi$  over all shortest paths  $\pi$  between  $\mathbf{t}_i$  and  $\mathbf{t}_{i'}$ 
13:   $N_1^h \leftarrow N_1^h \cup C_{i,i'}^{opt}$ 
14: end for
15: return  $N_1^h$ 

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**Lemma 4.1** *The network  $N_1^h$  returned by the algorithm Optimal completion is an optimal completion for  $S_1^h$ .*

**Proof.** We described above how to compute for each 1-staircase  $S_{i,j|i',j'}$  a subset  $C$  of edges of  $\Gamma_k$  of minimum total length such that  $C \cup (\pi \cup s'_{j,j'})$  contains a shortest path of  $\Gamma_k$  from each terminal of  $T_{i,j}$  to  $\mathbf{t}_{j'}$ . This standard dynamical programming approach explores all possible solutions and therefore achieves optimality for this problem. Next, we assert that, for each vertical 1-strip  $R_{i,i'}$ , the subset of edges  $C_{i,i'}^{opt}$  computed by our algorithm, is an optimal completion of  $S_1^h$  for the strip  $R_{i,i'}$  and the staircases having  $R_{i,i'}$  as vertical bases. Indeed, our algorithm considers every possible shortest path  $\pi$  of  $\Gamma_k$  between  $\mathbf{t}_i$  and  $\mathbf{t}_{i'}$ . Once the path  $\pi$  is fixed, the subproblems related to distinct staircases become independent and can be solved optimally by dynamic programming. The problems arising from distinct vertical 1-strips are also disjoint and independent (according to Lemmas 3.4 and 3.5). Therefore the solution  $N_1^h$  obtained by combining the optimal solutions  $C_{i,i'}^{opt}$  of every vertical 1-strip  $R_{i,i'}$  is an optimal completion of  $S_1^h$ .

It remains to show that to obtain a completion satisfying the conditions (i),(ii), and (iii), it suffices for each horizontal 1-strip  $R_{j,j'}$  to add  $\mathbf{t}_j$  to the set  $T_{i,j}$  of terminals of the rightmost

staircase  $S_{i,j|i',j'}$  having  $R_{j,j'}$  as a basis and to solve (iii) for this extended set of terminals. Indeed, in any completion any shortest path between  $\mathbf{t}_j$  and  $\mathbf{t}_{j'}$  necessarily makes a vertical switch either before arriving at the origin  $\mathbf{o}$  of  $S_{i,j|i',j'}$  or this path traverses the vertical basis of this staircase. Since the completion contains a shortest path connecting the terminals of the vertical basis of  $S_{i,j|i',j'}$ , combining these two paths, we can derive a shortest path between  $\mathbf{t}_j$  and  $\mathbf{t}'_j$  which turns in  $R_{i,i'} \cap R_{j,j'}$ . As a result, we conclude that at least one shortest path between  $\mathbf{t}_j$  and  $\mathbf{t}_{j'}$  passes via  $\mathbf{o}'$ . This shows that indeed it suffices to take into account the condition (ii) only for each rightmost staircase.  $\square$

**Lemma 4.2** *The network  $N_k(T)$  is an admissible solution for the problem 1-DMMN( $F_k$ ).*

**Proof.** By Lemma 4.1,  $N_1^h$  is a completion of  $S_1^h$  and thus contains a shortest path between every pairs of vertices from  $F$ . By symmetry, we get the same result for  $N_2^h$ ,  $N_1^v$  and  $N_2^v$ . Since  $N_k(T)$  is one of these networks, by Lemma 3.6, it is admissible solution for the problem 1-DMMN( $F_k$ ).  $\square$

## 5 Approximation ratio and complexity

In this section, we will prove the following main result:

**Theorem 5.1** *The network  $N_k(T)$  is a factor 1.25 approximation for 1-DMMN( $F_k$ ) problem for  $k = 0, \dots, m-1$ . The network  $N(T) := \bigcup_{k=0}^{m-1} N_k(T)$  is a factor 2.5 approximation for the  $\mathcal{B}$ -MMN problem and can be constructed in  $O(mn^3)$  time.*

*Proof of Theorem 5.1.* First we prove the first assertion of the theorem. Let  $\Lambda_h = l(S_1^h) = l(S_2^h)$  and  $\Lambda_v = l(S_1^v) = l(S_2^v)$ . Further, we suppose that  $\Lambda_h \leq \Lambda_v$ . Assume  $N_k^{opt}$  be an optimal 1-restricted Manhattan network for  $F_k$ . Let  $M$  be a subnetwork of  $N_k^{opt} \cap (S_1^h \cup S_2^h)$  of minimum total length which completed with some vertical edges of  $N_k^{opt}$  contains a shortest path between each pair of terminals defining a horizontal 1-strip of  $F_k$ . Such  $M$  exists because the network  $N_k^{opt} \cap (S_1^h \cup S_2^h)$  already satisfies this requirement. Further, we assume that  $l(M \cap S_1^h) \geq l(M \cap S_2^h)$ .

**Lemma 5.2**  $l(M) = \Lambda_h$ .

**Proof.** By Lemma 3.2, two horizontal 1-strips either are disjoint or intersect only in common terminals, thus any horizontal 1-strip  $R_{i,i'}$  contributes to  $M$  separately from other horizontal 1-strips. Since the terminals  $\mathbf{t}_i$  and  $\mathbf{t}'_i$  defining  $R_{i,i'}$  are connected in  $N_k^{opt}$  by a shortest path consisting of two horizontal segments of total length equal to the length of a side of  $R_{i,i'}$  and a vertical switch between these segments, from the optimality choice of  $M$  we conclude that the contribution of  $R_{i,i'}$  to  $M$  is precisely the length of one of its sides.  $\square$

**Lemma 5.3**  $l(M \cap S_1^h) \geq 0.5 l(M)$ .

**Proof.** The proof follows from the assumption  $l(M \cap S_1^h) \geq l(M \cap S_2^h)$  and the fact that  $M \cap S_1^h$  and  $M \cap S_2^h$  form a partition of  $M$ .  $\square$

**Lemma 5.4**  $l(S_1^h \setminus M) \leq 0.25 l(N_k^{opt})$ .

**Proof.** Since  $l(S_1^h \setminus M) = l(S_1^h) - l(M \cap S_1^h)$ , by Lemma 5.2 and 5.3 we get  $l(S_1^h \setminus M) \leq 0.5 l(M) = 0.5 \Lambda_h \leq 0.25 l(N_k^{opt})$ . The last inequality follows from  $l(N_k^{opt}) \geq \Lambda_h + \Lambda_v$  (a consequence of Lemma 3.2) and the assumption  $\Lambda_h \leq \Lambda_v$ .  $\square$

Now, we complete the proof of Theorem 5.1. Note that

$$l(N_k(T)) \leq l(S_1^h) = l(S_1^h \cup N_1^h) \leq l(S_1^h \cup N_k^{opt}) = l(S_1^h \setminus N_k^{opt}) + l(N_k^{opt}) \leq 1.25 l(N_k^{opt}).$$

The first inequality follows from the choice of  $N_k(T)$  as the shortest network among the four completions  $N_1^h, N_2^h, N_1^v, N_2^v$ . The second inequality follows from Lemma 4.1 and the fact that  $N_k^{opt}$  (and therefore  $S_1^h \cup N_k^{opt}$ ) is an admissible solution for the completion problem for  $S_1^h$ . Finally, the last inequality follows from Lemma 5.4 by noticing that  $M \subseteq N_k^{opt}$  and thus  $l(S_1^h \setminus N_k^{opt}) \leq l(S_1^h \setminus M) \leq 0.25 l(N_k^{opt})$ . This concludes the proof of the first assertion of Theorem 5.1.

Let  $N^*(T)$  be a minimum  $\mathcal{B}$ -Manhattan network. For each direction  $D_k$ , let  $N_k^*(T)$  be the set of  $k$ -segments and  $(k+1)$ -segments of  $N^*(T)$ . By Lemma 2.2 the network  $N_k^*(T)$  is an admissible solution for 1-DMMN( $F_k$ ) problem, thus  $l(N_k^*(T)) \geq \text{OPT}_k(T)$ . Any  $k$ -segment of  $N^*(T)$  belongs to exactly two one-directional networks  $N_k^*(T)$  and  $N_{k-1}^*(T)$ , we conclude that  $\sum_{k=0}^{m-1} \text{OPT}_k(T) \leq \sum_{k=0}^{m-1} l(N_k^*(T)) \leq 2 l(N^*(T)) = 2 \text{OPT}(T)$ . The first assertion of Theorem 5.1 implies that  $l(N_k(T)) \leq 1.25 l(N_k^{opt}) = 1.25 \text{OPT}_k(T)$  for all  $k = 0, \dots, m-1$ , hence

$$l(N(T)) \leq \sum_{k=0}^{m-1} l(N_k(T)) \leq 1.25 \sum_{k=0}^{m-1} \text{OPT}_k(T) \leq 2.5 \text{OPT}(T).$$

This concludes the proof that the approximation factor of  $N(T) := \bigcup_{k=0}^{m-1} N_k(T)$  is 2.5.

To finish the proof of Theorem 5.1, it remains to analyze the complexity of the algorithm. First, we use a straightforward analysis to establish a  $O(mn^4)$  bound on its running time. Then we show that this bound can be reduced to  $O(mn^3)$  by using a more advanced implementation. The time complexity of **Optimal completion**( $S_1^h$ ) is dominated by the execution of the dynamic programming algorithm that computes an optimal completion for each staircase  $S_{i,j|i',j'}$ . The staircase  $S_{i,j|i',j'}$  is processed  $O(|T_{i,j}|)$  times (once for each shortest  $(\mathbf{t}_i, \mathbf{t}_{i'})$ -path in  $\Gamma_k$ ) using a  $O(|T_{i,j}|^3)$ -time dynamic programming algorithm (each of the  $O(|T_{i,j}|^2)$  entries of the dynamic programming table is computed in time  $O(|T_{i,j}|)$ ). Therefore, each staircase  $S_{i,j|i',j'}$  contributes  $O(|T_{i,j}|^4)$  to the execution of the algorithm **Optimal completion**( $S_1^h$ ). Since each terminal belongs to at most two staircases, the overall complexity of **Optimal completion**( $S_1^h$ ) is  $O(n^4)$ . This algorithm is processed to compute four optimal completions for each of the  $m$  directions. Therefore the total complexity of our 2.5-approximation algorithm for the  $\mathcal{B}$ -MMN problem is  $O(mn^4)$ .

The following simple idea allows to reduce the contribution of each staircase  $S_{i,j|i',j'}$  to  $O(|T_{i,j}|^3)$  instead of  $O(|T_{i,j}|^4)$ , leading to a total complexity of  $O(mn^3)$ . First, note that among all  $O(|T_{i,j}|^2)$  subproblems, whose optima are stored in the dynamic programming table, only  $O(|T_{i,j}|)$  are affected by the choice of the  $\pi$  (those are the subproblems containing the highest and the rightmost terminal of  $T_{i,j}$ ). Therefore, instead of solving each of  $O(|T_{i,j}|^2)$  subproblems  $O(|T_{i,j}|)$  times, we solve the subproblems not affected by the choice of  $\pi$  only once. Now, consider the number of subproblems obtained by taking into account the choice of  $\pi$ , then it is easy to verify that the total number of subproblems encountered is not  $O(|T_{i,j}|^3)$  but only  $O(|T_{i,j}|^2)$ . Since each entry of the dynamic programming table is computed in time  $O(|T_{i,j}|)$ , we obtain a contribution of  $O(|T_{i,j}|^3)$  for each staircase  $S_{i,j|i',j'}$ , and thus a total complexity of  $O(mn^3)$ .  $\square$

## 6 Conclusion

In this paper, we presented a combinatorial factor 2.5 approximation algorithm for NP-hard minimum Manhattan network problem in normed planes with polygonal unit balls (the  $\mathcal{B}$ -MMN problem). Its complexity is  $O(mn^3)$ , where  $n$  is the number of terminals and  $2m$  is the number of extremal points of the unit ball  $\mathcal{B}$ . Any  $\mathcal{B}$ -Manhattan network  $N(T)$  can be decomposed into  $m$  subnetworks, one for each direction of the normed plane. Each such subnetwork  $N_k(T)$  ensures the existence of shortest paths between the pairs of terminals for which all legal paths use only  $k$ - and  $(k+1)$ -segments. We presented a factor 1.25  $O(n^3)$  algorithm for computing one-directional Manhattan networks, which lead to a factor 2.5 algorithm for minimum  $\mathcal{B}$ -Manhattan network problem. One of the open questions is whether *the one-directional Manhattan network problem is NP-complete?* Another open question is *designing a factor 2 approximation algorithm for  $\mathcal{B}$ -MMN*, thus meeting the current best approximation factor for the classical MMN problem. Notice that polynomial time algorithm for 1-DMMN problem will directly lead to a factor 2 approximation for  $\mathcal{B}$ -MMN.

Notice some similarity between the 1-DMMN problem and the oriented minASS problem investigated in relationship with the minimum stabbing box problem [16], alias the minimum arborally satisfied superset problem (minASS) [7]. In the *minASS problem*, given a set of  $n$  terminals  $T \subset \mathbb{R}^2$ , one need to add a minimum number of points  $S$  such that for any pair  $\mathbf{t}_i, \mathbf{t}_j \in T \cup S$ , either  $\mathbf{t}_i \mathbf{t}_j$  is a horizontal or a vertical segment, or the (axis-parallel) rectangle  $R_{i,j}$  spanned by  $\mathbf{t}_i, \mathbf{t}_j$  contains a third point of  $T \cup S$ . The *oriented minASS problem* is analogous to minASS problem except that the above requirement holds only for pairs  $\mathbf{t}_i, \mathbf{t}_j \in T \cup S$  such that  $\{i, j\} \in F_0$ , i.e.,  $\mathbf{t}_i$  and  $\mathbf{t}_j$  lie in the first and the third quadrants of the plane with the same origin. The authors of [7] presented a polynomial primal-dual algorithm for oriented minASS problem, however, in contrast to  $\mathcal{B}$ -MMN problem, solving oriented minASS problems for pairs of  $F_0$  and  $F_1$  (where  $F_0 \cup F_1 = (T \cup S) \times (T \cup S)$ ) does not lead to an admissible solution and thus to a constant factor approximation for minASS (which, as we have shown before, is the case for 1-DMMN and  $\mathcal{B}$ -MMN problems).



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