Polynomial-time approximation schemes for subset-connectivity problems in bounded-genus graphs

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Abstract

We present the first polynomial-time approximation schemes (PTASes) for the following subset-connectivity problems in edge-weighted graphs of bounded genus: Steiner tree, low-connectivity survivable-network design, and subset TSP. The schemes run in $\mathcal{O}(n \log n)$ time for graphs embedded on both orientable and nonorientable surfaces. This work generalizes the PTAS frameworks of Borradaile, Klein, and Mathieu (2007) from planar graphs to bounded-genus graphs: any future problems shown to admit the required structure theorem for planar graphs will similarly extend to bounded-genus graphs.

1 Introduction

In many practical scenarios of network design, input graphs have a natural drawing on the sphere or equivalently the plane. In most cases, these embeddings have few crossings, either to avoid digging multiple levels of tunnels for fiber or cable or to avoid building overpasses in road networks. But a few crossings are common, and can easily come in bunches where one tunnel or overpass might carry several links or roads. Thus we naturally arrive at graphs of small (bounded) genus, which is the topic of this work.

We develop a PTAS framework for subset-connectivity problems on edge-weighted graphs of bounded genus. In general, we are given a subset of the nodes, called *terminals*, and the goal is to connect the terminals together with some substructure of the graph by using cost within $1 + \varepsilon$ of the minimum possible cost. Our framework applies to three well-studied problems in this framework. In Steiner Tree, the substructure must be connected, and thus forms a tree. In Subset Tsp, the substructure must be a cycle; to guarantee existence, the cycle may traverse vertices and edges multiple times, but pays for each traversal. In $\{0, 1, 2\}$-edge-connectivity Survivable Network, the substructure must have $\min\{c_x, c_y\}$ edge-disjoint paths connecting vertices x and y, where each $c_x \in \{0, 1, 2\}$; we allow the substructure to include multiple copies of an edge in the graph, but pay for each copy. In particular, if $c_x = 1$ for all terminals x and y, then we obtain the Steiner tree problem; if $c_x = 2$ for all terminals x and y, then we obtain the minimum-cost 2-edge-connected multi-subgraph problem.

Our framework yields the first PTAS for all of these problems in bounded-genus graphs. These PTASes are efficient, running in $\mathcal{O}(f(\varepsilon, g)\, n + h(g)\, n \log n) = \mathcal{O}_{\varepsilon,g}(n \log n)$ time for graphs embedded on orientable surfaces and nonorientable surfaces. (We usually omit the mention of $f(\varepsilon, g)$ and $h(g)$ by assuming ε and g are constant, but we later bound $f(\varepsilon, g)$ as singly exponential in a polynomial in $1/\varepsilon$ and g and $h(g)$ as singly exponential in g.) In contrast, the problems we consider are APX-complete (and constant-factor-approximable) for general graphs.

We build upon the recent PTAS framework of Borradaile, Klein, and Mathieu [5] for subset-connectivity problems on planar graphs. In fact, our result is strictly more general: any problem to which the previous planar-graph framework applies automatically works in our framework as well, resulting in a PTAS for bounded-genus graphs. For example, Borradaile and Klein [4] have recently given a PTAS for the $\{0, 1, 2\}$-edge-connectivity Survivable Network problem using the planar framework. This will imply a similar result in bounded genus graphs. In contrast to the planar-graph framework, our PTASes have the attractive feature that they run correctly on all graphs with the performance degrading with the genus.

Our techniques for attacking bounded-genus graphs include two recent results: decompositions into bounded-treewidth graphs via contractions [8] and fast algorithms for finding the shortest noncontractible cycle [6]. We also use a simplified version of an algorithm for finding a short sequence of loops on a topological surface [11], and sophisticated dynamic programming. Our aim is to prove the following theorem:

Theorem 1. *There exists a* PTAS *for the* Steiner Tree*,* Subset Tsp*, and* $\{0, 1, 2\}$*-edge-connected* Survivable Network *problems in edge-weighted graphs of genus g with running time* $\mathcal{O}(2^{\mathrm{poly}(\varepsilon^{-1}, g)} n + 2^{\mathrm{poly}(g)} n \log n)$.

2 Preliminaries

All graphs $G = (V, E)$ have n vertices, m edges and are undirected with edge lengths (weights). The length of an edge e, subgraph H, and set of subgraphs $\mathcal{H}$ are denoted $\ell(e)$, $\ell(H)$ and $\ell(\mathcal{H})$, respectively. The shortest distance between vertices x and y in a graph G is denoted $\mathrm{dist}_G(x, y)$. The boundary of a graph G embedded in the plane is denoted by ∂G. For an edge $e = uv$, we define the operation of *contracting* e as identifying u and v and removing all loops and duplicate edges.

We use the basic terminology for embeddings as outlined in [20]. In this paper, an embedding refers to a *2-cell embedding*, i.e. a drawing of the vertices and faces of the graph as points and arcs on a surface such that every face is homeomorphic to an open disc. Such an embedding can be described purely combinatorially by specifying a *rotation system*, for the cyclic ordering of edges around vertices of the graph, and a *signature* for each edge of the graph; we use this notion of a *combinatorial embedding*. A combinatorial embedding of a graph G naturally induces such a 2-cell embedding on each subgraph of G. We only consider compact surfaces without boundary. When we refer to a planar embedding, we actually mean an embedding in the 2-sphere. If a surface contains a subset homeomorphic to a Möbius strip, it is *nonorientable*; otherwise it is *orientable*. For a 2-cell embedded graph G with f facial walks, the number $g = 2 + m - n - f$ is called the Euler genus of the surface. The Euler genus is equal to twice the usual genus for orientable surfaces and equals the usual genus for nonorientable surfaces. The *dual* of an embedded graph G is defined as having the set of faces of G as its vertex set and having an edge between two vertices if the corresponding faces of G are adjacent. We denote the dual graph by $G^\star$ and identify each edge of G with its corresponding edge in $G^\star$. A cycle of an embedded graph is *contractible* if it can be continuously deformed to a point; otherwise it is *noncontractible*. The operation of *cutting along a 2-sided cycle* C is essentially: partition the edges adjacent to C into left and right edges and replace C with two copies C_ℓ and C_r, adjacent to the left or right edges, accordingly. The inside of these new cycles is "patched" with two new faces. If the resulting graph is disconnected, the cycle is called *separating*, otherwise *nonseparating*. Cutting along a 1-sided cycle C on nonorientable surfaces is defined similarly, only that C is replaced by one bigger cycle C' that contains every edge of C exactly twice. See [20, pages 105–106] for further technical details.

Next we define the notions related to treewidth as introduced by Robertson and Seymour [22]. A *tree decomposition* of a graph G is a pair (T, χ), where $T = (I, F)$ is a tree and $\chi = \{\chi_i | i \in I\}$ is a family of subsets of $V(G)$, called *bags*, such that

1. every vertex of G appears in some bag of χ;
2. for every edge $e = uv$ of G, there exists a bag that contains both u and v;
3. for every vertex v of G, the set of bags that contain v form a connected subtree T_v of T.

The *width* of a tree decomposition is the maximum size of a bag in χ minus 1. The *treewidth* of a graph G, denoted by $\mathrm{tw}(G)$, is the minimum width over all possible tree decompositions of G.

The input graph is $G_0 = (V_0, E_0)$ and has genus g_0; the terminal set is Q. We assume G_0 is equipped with a combinatorial embedding; such an embedding can be found in linear time, if the genus is known to be fixed, see [19]. Let $\mathcal{P}$ be the considered subset-connectivity problem. In Section 3.2, we show how to find a subgraph $G = (V, E)$ of G_0, so that for $0 \leq \varepsilon \leq 1$ any $(1+\varepsilon)$-approximate solution of $\mathcal{P}$ in G_0 also exists in G. Hence, we may use G instead of G_0 in the rest of the paper. Note that as a subgraph of G_0, G is automatically equipped with a combinatorial embedding.

Let OPT denote the length of a optimal Steiner tree spanning terminals Q. We define $\mathrm{OPT}_{\mathcal{P}}$ to be the length of an optimal solution to problem $\mathcal{P}$. For the problems that we solve, we require that $\mathrm{OPT}_{\mathcal{P}} = \Theta(\mathrm{OPT})$ and in particular that $\mathrm{OPT} \leq \mathrm{OPT}_{\mathcal{P}} \leq \mu\,\mathrm{OPT}$. The constant μ will be used in Section 3.1 and is equal to 2 for both the subset TSP and $\{0, 1, 2\}$-edge-connectivity problems. This requirement is also needed for the planar case; see [4]. Because $\mathrm{OPT}_{\mathcal{P}} \geq \mathrm{OPT}$, upper bounds in terms of OPT hold for all the problems herein. As a result, we can safely drop the $\mathcal{P}$ subscript throughout the paper.

We show how to obtain a $(1 + c\varepsilon)\,\mathrm{OPT}_{\mathcal{P}}$ solution for a fixed constant c. To obtain a $(1 + \varepsilon)\,\mathrm{OPT}_{\mathcal{P}}$ solution, we can simply use $\varepsilon' = \varepsilon/c$ as input to the algorithm.

3 Mortar Graph and Structure Theorem

In [5], Borradaile, Klein and Mathieu developed a PTAS for the Steiner tree problem in planar graphs. The method involves finding a grid-like subgraph called the *mortar graph* that spans the input terminals and has length $\mathcal{O}(\mathrm{OPT})$. The set of feasible Steiner trees is restricted to those that cross between adjacent faces of the mortar graph only at a small number (per face of the mortar graph) of pre-designated vertices called *portals*. A Structure Theorem guarantees the existence of a nearly optimal solution (one that has length at most $(1+\varepsilon)\,\mathrm{OPT}$) in this set. We review the details that are relevant to this work and generalize to genus-g graphs.

Here we define the mortar graph in such a way that generalizes to higher genus graphs. A path P in a graph G is *ε-short in G* if for every pair of vertices x and y on P, the distance from x to y along P is at most $(1 + \varepsilon)$ times the distance from x to y in G: $\mathrm{dist}_P(x, y) \leq (1 + \varepsilon)\,\mathrm{dist}_G(x, y)$. Given a graph G embedded on a surface and a set of terminals Q, a mortar graph is a subgraph of G with the following properties:

Definition 2 (Mortar Graph and Bricks)**.** *Given a graph G embedded on a surface of genus g, a set of terminals Q, and a number $0 < \varepsilon \leq 1$, consider a subgraph* $\mathrm{MG} := \mathrm{MG}(G, Q, \varepsilon)$ *of G spanning Q such that each facial walk of* MG *encloses an area homeomorphic to an open disk. For each face F of* MG*, we construct a* brick *B of G by cutting G along the facial walk ∂F; B is the subgraph of G embedded inside the face, including ∂F. We denote this facial walk as the* mortar boundary *∂B of B. We define the* interior *of B as B without the edges of ∂B. We call* MG *a* mortar graph *if for some constants $\alpha(\varepsilon, g)$ and $\kappa(\varepsilon, g)$ (to be defined later), we have $\ell(\mathrm{MG}) \leq \alpha\,\mathrm{OPT}$ and every brick B satisfies the following properties:*

1. *B is planar.*
2. *The boundary of B is the union of four paths in the clockwise order W, N, E, S.*
3. *Every terminal of Q that is in B is on N or on S.*

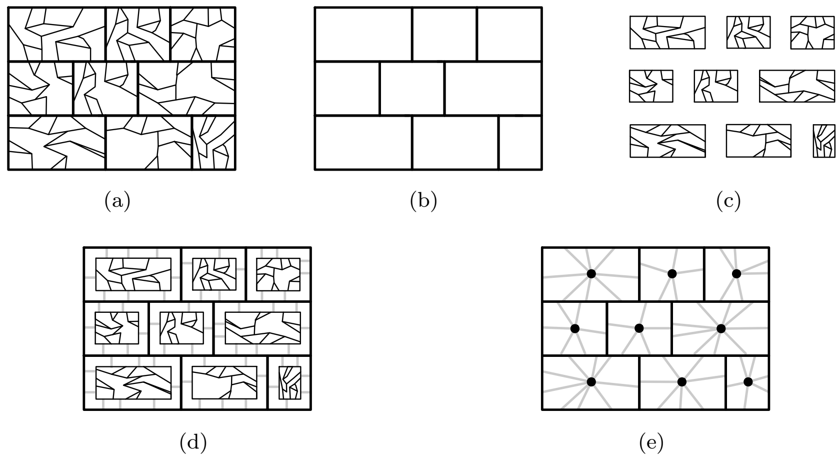


Figure 1: (a) An input graph G with mortar graph MG given by bold edges in (b). (c) The set of bricks corresponding to MG (d) A portal-connected graph, $\mathcal{B}^+(\mathrm{MG}, \theta)$. The portal edges are grey. (e) $\mathcal{B}^+(\mathrm{MG}, \theta)$ with the bricks contracted, resulting in $\mathcal{B}^{\div}(\mathrm{MG}, \theta)$. The dark vertices are brick vertices.

4. *N is 0-short in B, and every proper subpath of S is ε-short in B.*

5. *There exists a number $k \leq \kappa$ and vertices $s_0, s_1, s_2, \ldots, s_k$ ordered from left to right along S such that, for any vertex x of $S[s_i, s_{i+1})$, the distance from x to s_i along S is less than ε times the distance from x to N in B: $\mathrm{dist}_S(x, s_i) < \varepsilon \cdot \mathrm{dist}_B(x, N)$.*

The mortar graph and the set of bricks are illustrated in Figures 1 (a), (b) and (c). Constructing the mortar graph for planar graphs first involves finding a 2-approximate Steiner tree T [18] and cutting open the graph along T creating a new face H and then:

1. Finding shortest paths between certain vertices of H. These paths result in the N and S boundaries of the bricks.

2. Finding shortest paths between vertices of the paths found in Step 1. These paths are called *columns*, do not cross each other, and have a natural order.

3. Taking every κth path found in Step 2. These paths are called *supercolumns* and form the E and W boundaries of the bricks. We sometimes refer to κ as the *spacing* of the supercolumns.

The mortar graph is composed of the edges of T (equivalently, H) and the edges found in Steps 1 and 3. In [5], it is shown that the total length of the mortar graph edges is at most $9\varepsilon^{-1}\,\mathrm{OPT}$. For the purposes of this paper, we bound the length of the mortar graph in terms of $\ell(H)$. The following theorem can be easily deduced from [15] and [5]:

Theorem 3 ([15, 5])**.** *Let $0 < \varepsilon \leq 1$ and G be a planar graph with outer face H containing the terminals Q and such that $\ell(H) \leq \alpha_0 \,\mathrm{OPT}$, for some constant α_0. For $\alpha = (2\alpha_0 + 1)\varepsilon^{-1}$, there is a mortar graph $\mathrm{MG}(G, Q, \varepsilon)$ containing H whose length is at most $\alpha\,\mathrm{OPT}$ and whose supercolumns have length at most $\varepsilon\,\mathrm{OPT}$ with spacing $\kappa = \alpha_0\varepsilon^{-2}(1 + \varepsilon^{-1})$. The mortar graph can be found in $\mathcal{O}(n \log n)$ time.*

3.1 A mortar graph for bounded-genus graphs: Overview

We use Theorem 3 to prove the existence of a mortar graph for genus-g embedded graphs. This section is devoted to proving the following theorem:

Theorem 4. *Let an embedded edge-weighted graph G of Euler genus g, a subset of its vertices Q, an $0 < \varepsilon \leq 1$, and $\mu \geq 1$ be given. For $\alpha = (32\mu g + 9)\varepsilon^{-1}$, there is a mortar graph $\mathrm{MG}(G, Q, \varepsilon)$ of G such that the length of MG is $\leq \alpha\,\mathrm{OPT}$ and the supercolumns of MG have length $\leq \varepsilon\,\mathrm{OPT}$ with spacing $\kappa = (16\mu g + 4)\varepsilon^{-2}(1 + \varepsilon^{-1})$. The mortar graph can be found in $\mathcal{O}(n \log n)$ time.*

Let $G_0 = (V_0, E_0)$ be the input graph of genus g_0 and Q be the terminal set. In a first preprocessing step, we delete a number of unnecessary vertices and edges of G_0 to obtain a graph $G = (V, E)$ of genus $g \leq g_0$ that still contains every $(1 + \varepsilon)$-approximate solution for terminal set Q for all $0 \leq \varepsilon \leq 1$ while fulfilling certain bounds on the length of shortest paths. In the next step, we find a *cut graph* CG of G that contains all terminals and whose length is bounded by a constant times OPT. We cut the graph open along CG, so that it becomes a planar graph with a simple cycle σ as boundary, where the length of σ is twice that of CG. See Figure 2 for an illustration. Afterwards, the remaining steps of building the mortar graph can be the same as in the planar case, by way of Theorem 3.

For an edge $e = vw$ in G_0, we let

$$\mathrm{dist}_{G_0}(r, e) = \min\{\mathrm{dist}_{G_0}(r, v), \mathrm{dist}_{G_0}(r, w)\} + \ell(e)$$

and say that e is *at distance* $\mathrm{dist}_{G_0}(r, e)$ from r. If the root vertex represents a contracted graph H, we use the same terminology with respect to H.

3.2 Preprocessing the input graph

Our first step is to apply the following preprocessing procedure:

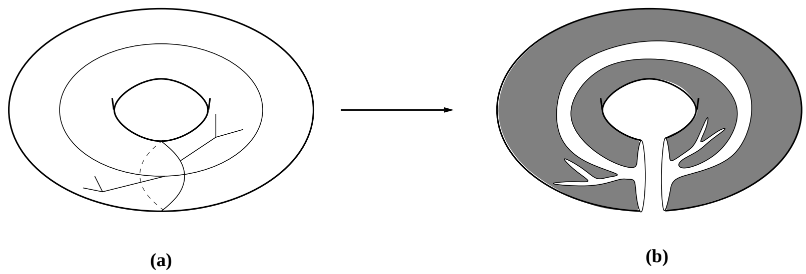


Figure 2: (a) a cut graph of a tree drawn on a torus; (b) the result of cutting the surface open along the cut graph: the shaded area is homeomorphic to a disc and the white area is the additional face of the planarized surface.

Algorithm Preprocess(G_0, Q, μ).

Input. an arbitrary graph G_0, terminals $Q \subseteq V(G_0)$, a constant μ

Output. a preprocessed subgraph of G_0

1. Find a 2-approximate Steiner tree T_0 for Q and contract it to a vertex r.
2. Find a shortest-path tree rooted at r.
3. Delete all vertices v and edges e of G_0 with $\mathrm{dist}_{G_0}(r, v), \mathrm{dist}_{G_0}(r, e) > 2\mu\ell(T_0)$.

Any deleted vertex or edge is at distance $> 2\mu\ell(T_0) > 2\mu\,\mathrm{OPT}$ from any terminal and hence can not be part of a $(1+\varepsilon)$-approximation for any $0 \leq \varepsilon \leq 1$. We call the resulting graph $G = (V, E)$ and henceforth use G instead of G_0 in our algorithm. The preprocessing step can be accomplished in linear time: step 1 using Müller-Hannemann and Tazari's algorithm [23] and step 2 using Henzinger et al.'s algorithm [14]. Trivially, we have

Proposition 5. *All vertices and edges of G are at distance at most $4\mu\,\mathrm{OPT}$ from T_0.*

3.3 Constructing a cut graph

A central fact that we use in this section and also in other parts of our work is the following observation [10]:

Observation 6. *Let G be a planar graph and T a spanning tree of G. Then the set of edges $E(G) - E(T)$ induces a spanning tree $T^\star$ in the dual $G^\star$. If T is a minimum spanning tree of G, then $T^\star$ is a maximum spanning tree of $G^\star$.*

A similar lemma also holds for bounded-genus graphs: if T is a (minimum) spanning tree of G and $T^\star$ a (maximum) spanning tree of $G^\star - E(T)$, then $T^\star$ is a (maximum) spanning tree of $G^\star$ and the size of the set of remaining edges $X := E(G) - E(T) - E(T^\star)$ is g, the Euler genus of G, by Euler's formula.

Eppstein [9] defines such a triple $(T, T^\star, X)$ as a *tree-cotree decomposition* of G and shows that such a decomposition can be found in linear time for graphs on both orientable and nonorientable surfaces.

In order to construct a cut graph, we start again with a 2-approximation T_0 and contract it to a vertex r. Next, we look for a *system of loops* rooted at r: iteratively find short nonseparating cycles through r and cut the graph open along each cycle. Erickson and Whittlesey [11] showed that, for orientable surfaces, taking the *shortest* applicable cycle at each step results in the shortest system of loops through r. They suggest a linear-time algorithm using the tree-cotree decomposition $(T, T^\star, X)$ of Eppstein [9]. Eppstein showed:

Lemma 7 (Lemma 2, [9])**.** *Given a tree-cotree decomposition $(T, T^\star, X)$, the set of elementary cycles $\{loop(T, e) : e \in X\}$ is a cut graph of G where $loop(T, e)$ is the closed walk formed by the paths in T from r to the endpoints of e plus the edge e.*

Eppstein's decomposition also works for nonorientable embeddings. As we only need to bound the length (as opposed to minimizing the length) of our cut graph, we present a simpler algorithm below:

Algorithm Planarize(G_0, Q, μ)**.**
Input. a graph G_0 of genus g, terminals $Q \subseteq V(G_0)$, a constant μ
Output. a preprocessed subgraph $G \subseteq G_0$ and a cutgraph CG of G

1. Apply Preprocess(G_0, Q, μ) and let G be the obtained subgraph.
2. Find a 2-approximate Steiner tree T_0 for Q and contract it to a vertex r.
3. Find a shortest paths tree SPT rooted at r.
4. Uncontract r and set $T_1 = T_0 \cup \mathrm{SPT}$. *($T_1$ is a spanning tree of G)*
5. Find a spanning tree $T_1^\star$ in $G^\star - E(T_1)$. *($T_1^\star$ is a spanning tree of $G^\star$)*
6. Let $X := E(G) - E(T) - E(T^\star)$.
7. Return $\mathrm{CG} := T_0 \cup \{\mathrm{loop}(T_1, e) : e \in X\}$ together with G.

Lemma 8. *The algorithm* Planarize *returns a cut graph* CG *such that cutting G open along* CG *results in a planar graph G_p with a face f_σ whose facial walk σ*

(i) is a simple cycle;

(ii) contains all terminals (some terminals might appear more than once as multiple copies might be created during the cutting process); and

(iii) has length $\ell(\sigma) \leq 2(8\mu g + 2)\,\mathrm{OPT}$.

The algorithm can be implemented in linear time.

Proof. Clearly, $(T_1, T_1^\star, X)$ is tree-cotree decomposition of G and so, by Lemma 7, CG is a cut graph. By Euler's formula, we get that $|X| = g$, the Euler genus of G.

Each edge $e = vw \in X$ completes a (nonseparating, not necessarily simple) closed walk as follows: a shortest path P_1 from T_0 to v, the edge e, a shortest path P_2 from w to T_0 and possibly a path P_3 in T_0. By Proposition 5, we know that e is at distance at most $4\mu\,\mathrm{OPT}$ from T_0 and so, both P_1 and P_2, and at least one of $\{P_1 \cup \{e\}, P_2 \cup \{e\}\}$ have length at most $4\mu\,\mathrm{OPT}$. Hence, we have that $\ell(P_1 \cup \{e\} \cup P_2) \leq 8\mu\,\mathrm{OPT}$. Because there are (exactly) g such cycles in CG, we get that

$$\ell(\mathrm{CG}) \leq g \cdot 8\mu\,\mathrm{OPT} + \ell(T_0) \leq (8\mu g + 2)\,\mathrm{OPT}\,.$$

Since CG is a connected cut graph and $T^\star \cap \mathrm{CG} = \varnothing$, cutting G open along CG results in a connected planar graph with boundary σ. Each edge of CG appears twice in σ and each edge of σ is derived from CG, so $\ell(\sigma) = 2\ell(\mathrm{CG})$ (see Fig. 2).

As mentioned in the previous section, T_0 and SPT can be computed in linear time on bounded-genus graphs [14, 23]. $T_1^\star$ can be obtained, for example, by a simple breadth-first-search in the dual. The remaining steps can also easily be implemented in linear time. □ □

3.4 Proof of Theorem 4

We complete the construction of a mortar graph for genus-g embedded graphs.

Let G_p be the result of planarizing G as guaranteed by Lemma 8. G_p is a planar graph with boundary σ such that σ spans Q and has length $\leq 2(8\mu g + 2)\,\mathrm{OPT}$. Let MG be the mortar graph guaranteed by Theorem 3 as applied to G with σ as its outer face. Every edge of MG corresponds to an edge of G. Let MG' be the subgraph of G composed of edges corresponding to MG. Every face f of MG (other than σ) corresponds to a face f' of MG' and the interior of f' is homeomorphic to a disk on the surface in which G is embedded. It is easy to verify that MG' is indeed a mortar graph of G; and the length bounds specified in the statement of the theorem follow directly from Theorem 3 and the bound on the length of σ. □

3.5 Structure Theorem

Along with the mortar graph, Borradaile et al. [5] define an operation $\mathcal{B}^+$ called *brick-copy* that allows a succinct statement of the Structure Theorem. For each brick B, a subset of θ vertices are selected as *portals* such that the distance along ∂B between any vertex and the closest portal is at most $\ell(\partial B)/\theta$. For every brick B, embed B in the corresponding face of MG and connect every portal of B to the corresponding vertex of MG with a zero-length *portal edge*: this defines $\mathcal{B}^+(\mathrm{MG}, \theta)$. $\mathcal{B}^+(\mathrm{MG}, \theta)$ is illustrated in Figure 1 (d). We denote the set of all portal edges by E_{portal}. The following simple observation, proved in [5] holds also for bounded-genus graphs:

Observation 9 ([5])**.** *If A is a connected subgraph of $\mathcal{B}^+(\mathrm{MG}, \theta)$, then $A - E_{portal}$ is a connected subgraph of G spanning the same vertices of G.*

The following Structure Theorem is the heart of the correctness of the PTASes.

Theorem 10 (Structure Theorem)**.** *Let $\mathcal{P}$ be one of the subset-connectivity problems* Steiner Tree*, $\{0, 1, 2\}$-edge-connectivity* Survivable Network*, or* Subset Tsp*. Let G be an edge-weighted graph embedded on a surface, $Q \subseteq V(G)$ a given set of terminals, and $0 < \varepsilon \leq 1$. Let $\mathrm{MG}(G, Q, \varepsilon)$ be a corresponding mortar graph of weight at most $\alpha\,\mathrm{OPT}$ and supercolumns of weight at most $\varepsilon\,\mathrm{OPT}$ with spacing κ. There exist constants $\beta(\varepsilon, \kappa)$ and $\theta(\alpha, \beta)$ depending polynomially on α and β such that*

$$\mathrm{OPT}_{\mathcal{P}}(\mathcal{B}^+(\mathrm{MG}, \theta), Q) \leq (1 + c\varepsilon)\,\mathrm{OPT}_{\mathcal{P}}(G, Q)\,,$$

where c is an absolute constant. Here $\beta = o(\varepsilon^{-2.5}\kappa)$ for Steiner Tree *and $\{0, 1, 2\}$-edge connectivity* Survivable Network *and $\beta = \mathcal{O}(\kappa)$ for* Subset Tsp*. (Recall that α and κ depend polynomially on ε^{-1} and g by Theorem 4.)*

It is due to our special way of defining and constructing a mortar graph for bounded-genus graphs that this theorem follows immediately as for the planar cases: the crucial point here is that our bricks are always planar – even when the given graph is embedded in a surface of higher genus. The Structure Theorem for Steiner Tree is proved in [5], the case of $\{0, 1, 2\}$-edge-connectivity Survivable Network is studied in [4], and we show that the theorem holds for Subset Tsp in Section 5. Note that for Subset Tsp, it is possible to obtain a singly exponential algorithm by following the spanner construction of Klein [15] after performing the planarizing step (Lemma 8). Our presentation here is chosen to unify the methods for all problems studied.

The Structure Theorem essentially says that there is a constant θ depending polynomially on ε^{-1} such that in finding a near-optimal solution to G, we can restrict our attention to $\mathcal{B}^+(\mathrm{MG}, \theta)$. Whenever we wish to apply our framework to a new problem, it is essential to prove a similar structure theorem for the considered problem.

4 Obtaining PTASes for bounded-genus graphs

We present two methods of obtaining polynomial-time approximation schemes. The first is a generalization of the framework of Klein [15] for planar graphs that is based on finding a *spanner* for a problem, a subgraph containing a nearly optimal solution having length $\mathcal{O}(\mathrm{OPT})$. In Section 4.1 we show how to find such a spanner and in Section 4.2 we generalize Klein's framework to higher genus graphs using the techniques of Demaine et al. [8]. In the second method, dynamic programming is done over the bricks of the mortar graph. This generalizes the framework of Borradaile et al. [5] for planar graphs to higher genus graphs. While both methods result in $\mathcal{O}(n \log n)$ algorithms, the first method is doubly exponential in a polynomial in g and ε^{-1} and the second is singly exponential.

4.1 Spanner for Subset-Connectivity Problems

A spanner is a subgraph of length $\mathcal{O}_{\varepsilon,g}(\mathrm{OPT})$ that contains a $(1+\varepsilon)$-approximate solution. Here we show how to find a spanner for bounded-genus graphs and the subset-connectivity problems considered in this paper. After a mortar graph is computed, the construction is, in fact, exactly the same as in the planar cases, namely:

> For each brick B defined by MG and for each subset X of the portals of B, find the optimal Steiner tree of X in B (using the method of Erickson et al. [12]). The spanner G_{span} is the union of all these trees over all bricks plus the edges of the mortar graph.

To prove the correctness of our spanner theorem for the case of $\{0, 1, 2\}$-edge-connectivity Survivable Network, we need to appeal to the following result of Borradaile and Klein, which we have simplified the statement of here:

Theorem 11 ([4, Theorem 5]). *Consider an instance of the $\{0, 1, 2\}$-edge connectivity problem. There is a feasible solution S to this instance that is a subgraph of $\mathcal{B}^+(MG)$ such that*

- $\ell(S) \leq (1 + c\varepsilon)\,\mathrm{OPT}$ *where c is an absolute constant, and*
- *the intersection of S with any brick B is a set of $O(1)$ trees the set of leaves of which are portals.*

Theorem 12 (Spanner Theorem). *Let G be an edge-weighted graph embedded on a surface of Euler genus g and $Q \subseteq V(G)$ a given set of terminals. There exists a spanner $G_{span} \subseteq G$ such that*

G_{span} is spanning: *G_{span} contains a $(1+c\varepsilon)$-approximate solution to* Steiner Tree, *$\{0, 1, 2\}$-edge-connected* Survivable Network, *and* Subset Tsp*; and*

G_{span} is short: $\ell(G_{span}) \leq f(\varepsilon, g)\,\mathrm{OPT}$*;*

where the function $f(\varepsilon, g)$ is singly exponential in a polynomial in ε^{-1} and g, and c is an absolute constant. The spanner can be found in $\mathcal{O}(n \log n)$ time.

Proof. Given a mortar graph $\mathrm{MG}(G, Q, \varepsilon)$ as guaranteed by Theorem 4, a spanner is constructed as specified above. As in [5], the time to find G_{span} is $\mathcal{O}(n \log n)$. It was proved in [5] that $\ell(G_{\mathrm{span}}) \leq (1 + 2^{\theta+1})\ell(\mathrm{MG})$. Therefore, $\ell(G_{\mathrm{span}}) \leq (1 + 2^{\theta+1})\alpha\,\mathrm{OPT}$ and $f(\varepsilon, g) = (1 + 2^{\theta+1})\alpha$ (recall that α and θ depend polynomially on ε^{-1} and g).

Now we show that G_{span} contains a near-optimal solution to each problem. For Steiner Tree, the proof follows directly from the Structure Theorem: the intersection of a minimal solution in $\mathcal{B}^+(\mathrm{MG}, \theta)$ with a brick B is a forest whose leaves are portals.

For $\{0,1,2\}$-edge-connected Survivable Network, we appeal to Theorem 11: By the Structure Theorem, there is a solution H in $\mathcal{B}^+(\mathrm{MG})$ that has length at most $(1+c\varepsilon)\,\mathrm{OPT}$. For each brick B, let H_B be the intersection of H with B. H_B is the union of trees. Replace each tree with the Steiner tree spanning the same subset as found in the spanner construction. Let H' be the graph resulting from all such replacements: $\ell(H') \le \ell(H) \le (1+c\varepsilon)\,\mathrm{OPT}$. By Observation 9, the edges of $H' - E_{\mathrm{portal}}$ induce a solution to the problem of length at most $(1+c\varepsilon)\,\mathrm{OPT}$.

For Subset Tsp, the proof is similar. By the Structure Theorem, there is a tour T of the terminals Q in $\mathcal{B}^+(\mathrm{MG})$ that has length at most $(1+c\varepsilon)\,\mathrm{OPT}$. For each brick B, let K be a connected component of the intersection of T with B. Because the terminals are in MG and not in B, K is a path between portals of B: replace K with the Steiner tree (i.e. a shortest path) connecting these two portals found in the construction of the spanner[1]. Let T' be the tour resulting from all these replacements: $\ell(T') \le \ell(T) \le (1+c\varepsilon)\,\mathrm{OPT}$. Appealing to Observation 9, the edges of $T' - E_{\mathrm{portal}}$ induce a solution of length at most $(1+c\varepsilon)\,\mathrm{OPT}$. □ □

4.2 PTAS via Spanner

In order to apply the PTAS framework of Klein [16] to bounded-genus graphs, we need the following Contraction Decomposition Theorem due to Demaine et al.:

Theorem 13 ([8, Theorem 1.1])**.** *For a fixed genus g, and any integer $\eta \ge 2$ and for every graph G of Euler genus at most g, the edges of G can be partitioned into η sets such that contracting any one of the sets results in a graph of treewidth at most $\mathcal{O}(g^2 \cdot \eta)$. Furthermore, such a partition can be found in $\mathcal{O}(g^{5/2} n^{3/2} \log n)$ time.*

Recent techniques [6] for finding shortest noncontractible cycles of embedded graphs have improved the above running time to $\mathcal{O}(n \log n)$.[2]

We review the four steps of the framework in our setting:

1. **Spanner Step:** Find a spanner G_{span} of G according to Theorem 12.

2. **Thinning Step:** For $\eta = f(\varepsilon, g)/\varepsilon$ (where $f(\varepsilon, g)$ is the function given in Theorem 12), let $S_1, \ldots, S_\eta$ be the partition of the edges of G_{span} as guaranteed by Theorem 13. Let S^* be the set in the partition with minimum weight: $\ell(S^*) \le \varepsilon\,\mathrm{OPT}$. Let G_{thin} be the graph obtained from G_{span} by contracting the edges of S^*. By Theorem 13, G_{thin} has treewidth at most $\mathcal{O}(g^2 \varepsilon^{-1} f(\varepsilon, g))$.

[1] Note that to construct a spanner for Subset Tsp, we need only shortest paths between pairs of portals.

[2] We would like to thank Jeff Erickson for pointing out in private communication that the algorithm given in [6] works for both orientable and nonorientable surfaces.

3. Dynamic Programming Step: Use dynamic programming (see, e.g. [17]) to find the optimal solution to the problem in G_{thin}.

4. Lifting Step: Convert this solution to a solution in G by incorporating some of the edges of S^*. For Steiner Tree, at most one copy of each edge of S^* is introduced to maintain connectivity [5]. In the case of $\{0,1,2\}$-edge connected Survivable Network, at most two copies of each edge of S^* are required [4]. For Subset Tsp, the method was explained in [15].

Analysis of the running time. By Theorem 12, the spanner step takes $\mathcal{O}_{\varepsilon,g}(n \log n)$ time (with singly exponential dependence on polynomials in g and ε^{-1}). By Theorem 13, thinning takes time $\mathcal{O}(n \log n)$ using [6]. Dynamic programming takes time $2^{\mathcal{O}(g^2\varepsilon^{-1}f(\varepsilon,g))}n$: because $f(\varepsilon, g)$ is singly exponential in polynomials in g and ε^{-1}, this step is doubly exponential in polynomials in g and ε^{-1}. Lifting takes linear time. Hence, the overall running time is $\mathcal{O}(2^{\mathcal{O}(g^2\varepsilon^{-1}f(\varepsilon,g))}n + n \log n)$.

4.3 PTAS via Dynamic Programming over the Bricks

In [5], Borradaile et al. present a PTAS that is singly exponential in a polynomial in ε^{-1} for Steiner Tree in planar graphs. The idea is to incorporate the spanner step into the dynamic programming step and to use a somewhat modified thinning step. To this end, the operator *brick-contraction* $\mathcal{B}^{\div}$ is defined to be the application of the operation $\mathcal{B}^{+}$ followed by contracting each brick to become a single vertex of degree at most θ (see Figure 1(e)). The thinning algorithm decomposes the mortar graph MG into parts of *bounded dual radius* (implying bounded treewidth). Applying $\mathcal{B}^{\div}$ to each part maintains bounded dual radius. The algorithm computes optimal Steiner trees inside the bricks using the method of [12] only at the leaves of the dynamic programming tree, thus eliminating the need of an a-priori constructed spanner. The interaction between subproblems of the dynamic programming is restricted to the portals, of which there are few.

For embedded graphs with genus > 0, the concept of bounded dual radius does not apply in the same way. We deal with treewidth directly and obtain the following algorithm: we apply the Contraction Decomposition Theorem 13 [8] to $\mathcal{B}^{\div}(\text{MG})$ and contract a set of edges $S^\star$ in $\mathcal{B}^{\div}(\text{MG})$. However, we apply a special weight to portal edges so as to prevent them from being included in $S^\star$. Also, in $\mathcal{B}^{\div}(\text{MG})$, we slightly modify the definition of contraction: after contracting an edge, we do not delete parallel portal edges. Because portal edges connect the mortar graph to the bricks, they are not parallel in the graph in which we find a solution via dynamic programming. The details are given below.

Algorithm THINNING(G, MG).
Input. a graph G of fixed genus g, a mortar graph MG of G
Output. a set $S^\star \subseteq E(\mathcal{B}^{\div}(\mathrm{MG}))$,
a tree decomposition (T, χ) of $\mathcal{B}^{\div}(MG)/S^\star$

1. Assign weight $\ell(\partial F)$ to each portal edge in a face F of $\mathcal{B}^{\div}(\mathrm{MG})$.
2. Apply the Contraction Decomposition Theorem 13 to $\mathcal{B}^{\div}(\mathrm{MG})$ with $\eta := 3\theta\alpha\varepsilon^{-1}$ to obtain edge sets $S_1, \ldots, S_\eta$; let $S^\star$ be the set of minimum weight.
3. If $S^\star$ includes a portal edge e of a brick B enclosed in a face F of MG, add ∂F to $S^\star$ and mark B as ignored.
4. Let $\mathrm{MG}_{\mathrm{thin}} := \mathcal{B}^{\div}(\mathrm{MG})/S^\star$ (but do not delete parallel portal edges).
5. Let (T, χ) be a tree decomposition of width $\mathcal{O}(g^2 \cdot \eta)$ of $\mathrm{MG}_{\mathrm{thin}}$.
6. For each vertex b of $\mathrm{MG}_{\mathrm{thin}}$ that represents an unignored contracted brick with portals $\{p_1, \ldots, p_\theta\}$:
   6.1. Replace every occurrence of b in χ with $\{p_1, \ldots, p_\theta\}$;
   6.2. Add a bag $\{b, p_1, \ldots, p_\theta\}$ to χ
   and connect it to a bag containing $\{p_1, \ldots, p_\theta\}$.
7. Reset the weight of the portal edges back to zero.
8. Return (T, χ) and $S^\star$.

Lemma 14. *The algorithm* THINNING(G, MG) *returns a set of edges $S^\star$ and a tree decomposition (T, χ) of $\mathcal{B}^{\div}(\mathrm{MG})/S^\star$, so that*

(i) the treewidth of (T, χ) is at most ξ where $\xi(\varepsilon, g) = \mathcal{O}(g^2\eta\theta) = \mathcal{O}(g^3\varepsilon^{-2}\theta^2)$; in particular, ξ is polynomial in ε^{-1} and g;

(ii) every brick is either

- *marked as ignored, or*
- *none of its portal edges are in $S^\star$; and*

(iii) $\ell(S^\star) \le \varepsilon\,\mathrm{OPT}$.

Proof. We first verify that (T, χ) is indeed a tree decomposition. For a vertex v and a tree decomposition (T', χ'), let T'_v denote the subtree of T' that contains v in all of its bags. Let us denote the tree decomposition of step (5) by (T^0, χ^0). For each brick vertex b and each of its portals p_i, we know that T^0_b is connected and $T^0_{p_i}$ is connected and that these two subtrees intersect; it follows that after the replacement in step (6.2), we have that $T_{p_i} = T^0_b \cup T^0_{p_i}$ is a connected subtree of T and hence, (T, χ) is a correct tree decomposition. Note that Theorem 13 guarantees a tree decomposition of width $\mathcal{O}(g^2\eta)$ if any of $S_1, \ldots, S_\eta$ are contracted; and in step (3), we only add to the set of edges to be contracted. Hence, the treewidth of (T^0, χ^0) is indeed $\mathcal{O}(g^2\eta)$ and with the construction in step (6.1), the size of each bag will be multiplied by a factor of at most θ. This shows the correctness of claim (i). The correctness of claim (ii) is immediate from the construction in step (3). It remains to verify claim (iii).

Let L denote the weight of $\mathcal{B}^{\div}(\mathrm{MG})$ after setting the weights of the portal edges according to step (1) of the algorithm. We have that

$$\begin{aligned} L &\leq \ell(\mathrm{MG}) + \sum_F \ell(\partial F)\theta \leq \alpha\,\mathrm{OPT} + \theta \sum_F \ell(\partial F) \\ &\leq \alpha\,\mathrm{OPT} + \theta \cdot 2\alpha\,\mathrm{OPT} \leq 3\theta\alpha\,\mathrm{OPT}\ . \end{aligned}$$

Hence, the weight of $S^\star$, as selected in step (2), is at most $L/\eta \leq \frac{3\theta\alpha\,\mathrm{OPT}}{3\theta\alpha\varepsilon^{-1}} \leq \varepsilon\,\mathrm{OPT}$. The operation in step (3) does not add to the weight of $S^\star$: if ∂F is added to $S^\star$, the additional weight is subtracted when the corresponding portal-edge weights are set to zero in step (7). □ □

If a brick is "ignored" by Thinning, the boundary of its enclosing mortar graph face is completely added to $S^\star$. Because $S^\star$ can be added to the final solution, every potential connection through that brick can be rerouted through $S^\star$ around the boundary of the brick. The interior of the brick is not needed.

An almost standard dynamic programming algorithm for bounded-treewidth graphs (cf. [1, 17]) can be applied to G_{thin} and (T, χ). However, for the leaves of the tree decomposition that are added in step (6.2) of the Thinning procedure, the cost of a subset of portal edges is calculated as, e.g., the cost of the minimum Steiner tree interconnecting these portals in the corresponding brick. Because the bricks are planar, this cost can be calculated by the algorithm of [12] for Steiner tree or [4] for 2-edge connectivity. Because all the portal edges of this brick are present in this bag (recall that we do not delete parallel portal edges after contractions), all possible solutions restricted to the corresponding brick will be considered. Because the contracted brick vertices only appear in leaves of the dynamic programming tree, the rest of the dynamic programming algorithm can be carried out as in the standard case.

Analysis of the running time. As was shown for the planar Steiner tree PTAS [5], the total time spent in the leaves of the dynamic programming is $\mathcal{O}(4^\theta n)$. The rest of the dynamic programming takes time $\mathcal{O}(2^{\mathcal{O}(\xi)} n)$. The running time of the thinning algorithm is dominated by the Contraction Decomposition Theorem 13 which is $\mathcal{O}_g(n \log n)$ [6]. Hence, the total time is $\mathcal{O}(2^{\mathcal{O}(\xi)} n + 2^{\mathrm{poly}(g)} n \log n)$ for the general case; in particular, this is singly exponential in ε^{-1} and g, as desired. This proves Theorem 1.

5 A Structure Theorem for Subset TSP

Here we prove the Structure Theorem for Subset Tsp. While this theorem can be used to obtain a PTAS for the subset tour problem in planar graphs, a PTAS for this problem [15] predates the mortar graph framework.

Like Steiner Tree and Survivable Network, the Structure Theorem (Section 3.5) must be proved (Section 5.1) for the Subset Tsp problem. To this end, in this section, we state and prove a local structure theorem (Theorem 16, Figure 3). This local structure theorem describes how to replace the intersection of a tour with a brick to reduce the number of times the tour crosses the boundary of the brick: each crossing contributes to the size of the dynamic-programming table. While the intersection of a tour with a brick is quite simple (a set of brick boundary-to-boundary paths), in modifying the tour we must be careful to maintain that our solution is still a tour.

We will use the following lemma due to Arora:

Lemma 15 (Patching Lemma [2]). *Let S be any line segment of length s and π be a closed path that crosses S at least thrice. Then we can break the path in all but two of these places and add to it line segments lying on S of total length at most $3s$ such that π changes into a closed path π' that crosses S at most twice.*

This lemma applies to embedded graphs as well. Note: the patching process connects paths in the tour that end on a common side of S by a subpath of S.

Theorem 16 (TSP Property of Bricks). *Let B be a brick of graph G with boundary $N \cup E \cup S \cup W$ (where E and W are supercolumns). Let T be a tour in G such that T crosses E and W at most 2 times each. Let H be the intersection of T with B. Then there is another subgraph of B, H', such that:*

(T1) H' has at most $\beta(\varepsilon)$ joining vertices with ∂B.

(T2) $\ell(H') \leq (1 + 5\varepsilon)\ell(H)$.

(T3) There is a tour in the edge set $T \setminus H \cup H'$ that spans the vertices in $\partial B \cap T$.

In the above, $\beta(\varepsilon) = \mathcal{O}(\kappa)$.

Proof. Let H be the subgraph of T that is strictly enclosed by B (i.e., H contains no edges of ∂B). We can write H as the union of 3 sets of minimal ∂B-to-∂B paths $\mathcal{P}_{S \vee N} \cup \mathcal{P}_{E \vee W} \cup \mathcal{P}_{S \wedge N}$ where paths in: $\mathcal{P}_{S \vee N}$ either start and end on S or start and end on N; $\mathcal{P}_{E \vee W}$ start on E or W and end on ∂B; $\mathcal{P}_{S \wedge N}$ start on S and end on N. For the constructions below, refer to Figure 3.

Since T crosses E and W at most 4 times, $|\mathcal{P}_{E \vee W}| \leq 4$. Therefore, this set of paths result in at most 8 joining vertices with ∂B.

For each path $P \in \mathcal{P}_{S \vee N}$, let $\widehat{P}$ be the minimal subpath of ∂B that spans P's endpoints. Let $\widehat{\mathcal{P}}_{S \vee N}$ be the resulting set of paths. As N is 0-short and S is ε-short, we have

$$\ell(\widehat{\mathcal{P}}_{S \vee N}) \leq (1 + \varepsilon)\ell(\mathcal{P}_{S \vee N}). \tag{1}$$

Since $\widehat{\mathcal{P}}_{S \vee N}$ are subpaths of ∂B, they have no joining vertices with ∂B. Since paths in $\widehat{\mathcal{P}}_{S \vee N}$ correspond one-to-one with paths in $\mathcal{P}_{S \vee N}$, $T \setminus \mathcal{P}_{S \vee N} \cup \widehat{\mathcal{P}}_{S \vee N}$ is a tour. See Figure 3(a).

It remains to deal with the paths in $\mathcal{P}_{S\wedge N}$. Let $s_0, s_1, s_2, \ldots, s_k$ (where $k \leq \kappa$) be the vertices of S guaranteed by the properties of the bricks (see Definition 2). Let $\mathcal{X}_i$ be the subset of paths of $\mathcal{P}_{S\wedge N}$ that start on $S[s_i, s_{i+1})$, i.e. the vertices between s_i and s_{i+1} including s_i but not s_{i+1}.

If $|\mathcal{X}_i| > 2$, we do the following: Let P_i be the path in $\mathcal{X}_i$ whose endpoint x on S is closest to s_{i+1}. Let Q_i be the subpath of S from s_i to x. By the properties of the bricks, $\ell(Q_i) \leq \varepsilon\ell(P_i)$. Apply the Patching Lemma to the tour T and path Q_i; the new tour, T', crosses Q_i at most twice. However T' may still have many joining vertices with Q_i. Let $\mathcal{Q}_i$ be the subpaths of Q_i that are added to the tour.

Let $\mathcal{L}_i$ be a maximal set of N-to-N paths in $\mathcal{X}_i \cup \mathcal{Q}_i$. $\mathcal{L}_i$ accounts for all but two (corresponding to the two crossings of T' with Q_i) of the joining vertices of T' with Q_i. For each path $L \in \mathcal{L}_i$, let $\widehat{L}$ be the minimal subpath of N that spans L's endpoints and let $\widehat{\mathcal{L}}_i$ be the resulting set of paths. Replacing $\mathcal{L}_i$ with $\widehat{\mathcal{L}}_i$ is still a tour, since the paths have a one-to-one correspondence. However, the resulting tour may no longer span all terminals on Q_i. Adding in two copies of Q_i fixes this. Since N is 0-short, $\ell(\widehat{\mathcal{L}}_i) \leq \ell(\mathcal{L}_i)$.

Let $\widehat{\mathcal{X}}_i = \mathcal{X}_i \cup \mathcal{Q}_i \setminus \mathcal{L}_i \cup \widehat{\mathcal{L}}_i \cup Q_i \cup Q_i$. Replacing $\mathcal{X}_i$ with $\widehat{\mathcal{X}}_i$ is still a tour, as argued above. Since the additional length added is at most 5 copies of Q_i, we have:

$$\ell(\widehat{\mathcal{X}}_i) \leq \ell(\mathcal{X}_i) + 5\ell(Q_i) \leq \ell(\mathcal{X}_i) + 5\varepsilon\ell(P_i) \leq (1 + 5\varepsilon)\ell(\mathcal{X}_i) \tag{2}$$

Since $\mathcal{L}_i$ accounted for all but 2 of the joining vertices of T' with Q_i - so all but 4 of the joining vertices of $\mathcal{X}_i$ with ∂B, and $\widehat{\mathcal{L}}_i$ has no joining vertices with ∂B, $\widehat{\mathcal{X}}_i$ has at most 4 joining vertices with ∂B.

Let $\widehat{\mathcal{P}}_{S\wedge N} = \bigcup_i \widehat{\mathcal{X}}_i$. $\widehat{\mathcal{P}}_{S\wedge N}$ has at most 6κ joining vertices with ∂B and, by Equation (2),

$$\ell(\widehat{\mathcal{P}}_{S\wedge N}) \leq (1 + 5\varepsilon)\ell(\mathcal{P}_{S\wedge N}). \tag{3}$$

Let $\widehat{H}$ be the union of the paths in $\mathcal{P}_{E\vee W}$, $\widehat{\mathcal{P}}_{S\vee N}$ and $\widehat{\mathcal{P}}_{S\wedge N}$. Combining Equations (1) and (3), we find that $\ell(\widehat{H}) \leq (1 + 5\varepsilon)\ell(H)$. By construction, the edges in $T \setminus H \cup \widehat{H}$ contain a tour. $\widehat{H}$ has $6\kappa + 8$ joining vertices with ∂B. ☐ ☐

5.1 Proof of the Structure Theorem for Subset Tsp

Using the TSP Property of Bricks, we prove the Structure Theorem (Section 3.5) for Subset Tsp.

Let T be the optimal tour spanning terminals Q in G. From T we build a tour $\widehat{T}$ spanning Q in $\mathcal{B}^+(MG)$ such that $\ell(\widehat{T}) \leq (1 + c\varepsilon)\ell(T)$.

Let C be a supercolumn. By the Patching Lemma, if T crosses C at least thrice, we can add to T at most three copies of C and create a new tour that crosses C at most twice. Let T_1 be the tour that results from applying the Patching Lemma to each supercolumn. Because the sum of the weights of the supercolumns is at most $\varepsilon\,\mathrm{OPT}$,

$$\ell(T_1) \leq (1 + 3\varepsilon)\ell(T). \tag{4}$$

Let B be a brick of G. Let H be the intersection of T_1 with B. By the construction above, T_1 satisfies the requirements of Theorem 16: let H' be the guaranteed subgraph of B. We replace H with H' in T_1. Let T_2 be the tour resulting from such replacements over all the bricks. Theorem 16 guarantees that

$$\ell(T_2) \leq (1+5\varepsilon)\ell(T_1). \tag{5}$$

Now we map the edges of T_2 to a subgraph of $\mathcal{B}^+(MG)$. Every edge of G has at least one corresponding edge in $\mathcal{B}^+(MG)$. For every edge e of T_2, we select one corresponding edge in $\mathcal{B}^+(MG)$ as follows: if e is an edge of MG select the corresponding mortar edge of $\mathcal{B}^+(MB)$, otherwise select the unique edge corresponding to e in $\mathcal{B}^+(MG)$. (An edge of $\mathcal{B}^+(MG)$ is a *mortar edge* if it is in MG.) This process constructs a subgraph T_3 of $\mathcal{B}^+(MG)$ such that

$$\ell(T_3) = \ell(T_2). \tag{6}$$

Because T_3 is not connected, we connect it via portal and mortar edges. Let V_B be the set of joining vertices of T_3 with ∂B for a brick B of $\mathcal{B}^+(MG)$. For any vertex v on the interior boundary ∂B of a brick, let p_v be the portal on ∂B that is closest to v, let P_v be the shortest v-to-p_v path along ∂B and let P'_v be the corresponding path of mortar edges. Let e be the portal edge corresponding to p_v. Add P_v, P'_v, and e to T_3. Repeat this process for every $v \in V_B$ and for every brick B, building a graph $\widehat{T}$. This completes the definition of $\widehat{T}$. From the construction, $\widehat{T}$ is a tour spanning the terminals Q in $\mathcal{B}^+(MG)$.

Now we analyze the increase in length:

$$\ell(\widehat{T}) \leq \ell(T_3) + \sum_{B\in\mathcal{B}} \sum_{v\in V_B} (\ell(P_v) + \ell(e) + \ell(P'_v)), \tag{7}$$

and we have:

$$\begin{aligned}
\sum_{B\in\mathcal{B}} \sum_{v\in V_B} \ell(P_v) + \ell(e) + \ell(P'_v) &= 2\sum_{B\in\mathcal{B}} \sum_{v\in V_B} \ell(P_v), \text{ because } \ell(\text{portal edges}) = 0 \\
&\leq 2\sum_{B\in\mathcal{B}} \sum_{v\in V_B} \ell(\partial B)/\theta, \text{ by the choice of portals} \\
&\leq 2\sum_{B\in\mathcal{B}} \frac{\beta}{\theta}\ell(\partial B), \text{ by Theorem 16} \\
&\leq 2\frac{\beta}{\theta}2\alpha(\varepsilon^{-1}, g)\,\mathrm{OPT}, \text{ by Theorem 4} \\
&\leq \varepsilon\ \mathrm{OPT}, \text{ for } \theta = 4\varepsilon^{-1}\beta\alpha, \text{ as required.}
\end{aligned}$$

Combining Equations (4), (5), (6) and (7), we obtain $\ell(\widehat{T}) \leq (1+3\varepsilon)(1+5\varepsilon)\ell(T) + \varepsilon\,\mathrm{OPT} \leq (1+c\varepsilon)\,\mathrm{OPT}$. The Structure Theorem is proved for the Subset Tsp problem.

6 Conclusion and Outlook

We presented a framework to obtain PTASes on bounded-genus graphs for subset-connectivity problems, where we are given a graph and a set of terminals and require a certain connectivity among the terminals. Specifically, we obtained the first PTAS for Steiner Tree on bounded-genus graphs running in $\mathcal{O}(n \log n)$-time with a constant that is singly exponential in ε^{-1} and the genus of the graph. Our method is based on the framework of Borradaile et al. [5] for planar graphs; in fact, we generalize their work in the sense that basically any problem that is shown to admit a PTAS on planar graphs using their framework easily generalizes to bounded-genus graphs using the methods presented in this work. In particular, this gives rise to PTASes in bounded-genus graphs for Subset Tsp (Section 5), $\{0, 1, 2\}$-edge-connected Survivable Network [4], and also Steiner Forest [3].

A natural question is to ask what other classes of graphs admit a PTAS for the problems discussed in this work. An important generalization of bounded-genus graphs are proper classes of graphs that are closed under taking *minors*. Such *H-minor-free graphs* have earned much attention in recent years. Many hard optimization problems have been shown to admit PTASes and fixed-parameter algorithms on these classes of graphs; see, e.g., [7, 13]. But subset-connectivity problems, specifically Subset Tsp and Steiner Tree, remain important open problems [13, 8]. Both a spanner theorem and a contraction decomposition theorem are still missing for the H-minor-free case. Very often, results on H-minor-free graphs are first shown for planar graphs, then extended to bounded-genus graphs, and finally obtained for H-minor-free graphs. This is due to the powerful decomposition theorem of Robertson and Seymour [21] that essentially says that every H-minor-free graph can be decomposed into a number of parts that are "almost embeddable" in a bounded-genus surface. We conjecture that our framework extends to H-minor-free graphs via this decomposition theorem. The advantage of our methodology is that handling weighted graphs and subset-type problems are naturally incorporated, and thus it might be possible to combine all the steps for a potential PTAS into a single framework for H-minor-free graphs based on what we presented in this work. Hence, whereas our work is an important step towards this generalization, still a number of hard challenges remain; see also [8] for a further discussion on this matter.

References

[1] Arnborg, S., Proskurowski, A.: Linear time algorithms for NP-hard problems restricted to partial k-trees. Discrete Applied Mathematics **23**(1), 11–24 (1989)

[2] Arora, S.: Approximation schemes for NP-hard geometric optimization problems: a survey. Mathematical Programming **97**(1–2), 43–69 (2003)

[3] Bateni, M., Hajiaghayi, M., Marx, D.: Approximation schemes for steiner forest on planar graphs and graphs of bounded treewidth. In: STOC '10: Proceedings of the 42nd annual ACM Symposium on Theory of Computing, p. to appear. ACM (2010)

[4] Borradaile, G., Klein, P.: The two-edge connectivity survivable network problem in planar graphs. In: ICALP '08: Proceedings of the 35th International Colloquium on Automata, Languages and Programming, *LNCS*, vol. 5125, pp. 485–501. Springer (2008)

[5] Borradaile, G., Klein, P.N., Mathieu, C.: An $O(n \log n)$ approximation scheme for Steiner tree in planar graphs. ACM Transactions on Algorithms **5**(3) (2009)

[6] Cabello, S., Chambers, E.W.: Multiple source shortest paths in a genus g graph. In: SODA '07: Proceedings of the 18th annual ACM-SIAM Symposium on Discrete Algorithms, pp. 89–97. SIAM (2007)

[7] Demaine, E.D., Hajiaghayi, M.: Bidimensionality: new connections between FPT algorithms and PTASs. In: SODA '05: Proceedings of the 16th annual ACM-SIAM Symposium on Discrete Algorithms, pp. 590–601 (2005)

[8] Demaine, E.D., Hajiaghayi, M., Mohar, B.: Approximation algorithms via contraction decomposition. In: SODA '07: Proceedings of the 18th annual ACM-SIAM Symposium on Discrete Algorithms, pp. 278–287. SIAM (2007)

[9] Eppstein, D.: Dynamic generators of topologically embedded graphs. In: SODA '03: Proceedings of the 14th annual ACM-SIAM Symposium on Discrete Algorithms, pp. 599–608. SIAM (2003)

[10] Eppstein, D., Italiano, G., Tamassia, R., Tarjan, R., Westbrook, J., Yung, M.: Maintenance of a minimum spanning forest in a dynamic planar graph. J. Algorithms **13**(1), 33–54 (1992). Special issue for 1st SODA

[11] Erickson, J., Whittlesey, K.: Greedy optimal homotopy and homology generators. In: SODA '05: Proceedings of the 16th annual ACM-SIAM Symposium on Discrete Algorithms, pp. 1038–1046. SIAM (2005)

[12] Erickson, R.E., Monma, C.L., Arthur F. Veinott, J.: Send-and-split method for minimum-concave-cost network flows. Math. Oper. Res. **12**(4), 634–664 (1987)

[13] Grohe, M.: Local tree-width, excluded minors, and approximation algorithms. Combinatorica **23**(4), 613–632 (2003)

[14] Henzinger, M.R., Klein, P.N., Rao, S., Subramanian, S.: Faster shortest-path algorithms for planar graphs. Journal of Computer and System Sciences **55**(1), 3–23 (1997)

[15] Klein, P.N.: A subset spanner for planar graphs, with application to subset TSP. In: STOC '06: Proceedings of the 38th annual ACM Symposium on Theory of Computing, pp. 749–756 (2006)

[16] Klein, P.N.: A linear-time approximation scheme for TSP in undirected planar graphs with edge-weights. SIAM J. Comput. **37**(6), 1926–1952 (2008)

[17] Korach, E., Solel, N.: Linear time algorithm for minimum weight Steiner tree in graphs with bounded treewidth. Manuscript (1990)

[18] Mehlhorn, K.: A faster approximation algorithm for the Steiner problem in graphs. Information Processing Letters **27**, 125–128 (1988)

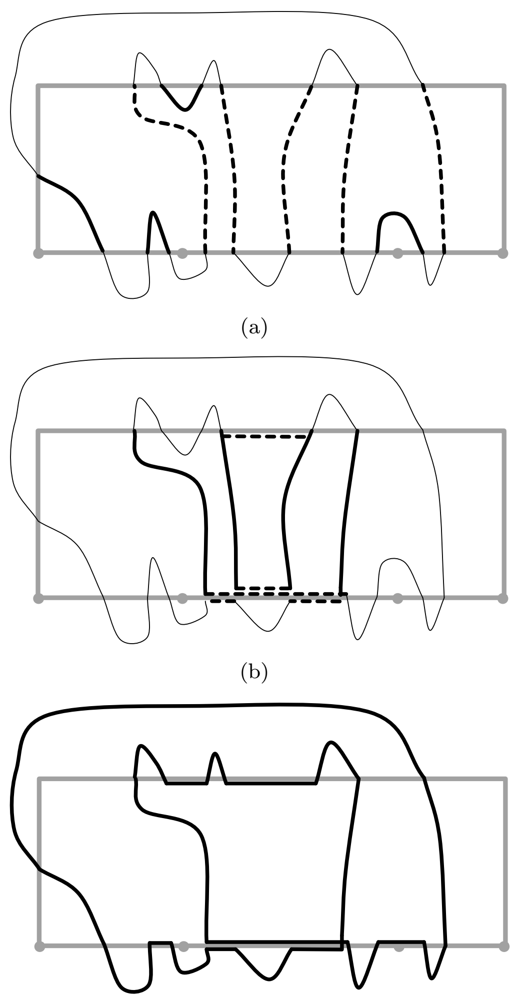
[19] Mohar, B.: A linear time algorithm for embedding graphs in an arbitrary surface. SIAM Journal on Discrete Mathematics **12**(1), 6–26 (1999)

[20] Mohar, B., Thomassen, C.: Graphs on Surfaces. The John Hopkins University Press (2001)

[21] Robertson, N., Seymour, P.: Graph minors. XVI. Excluding a non-planar graph. J. Comb. Theory Ser. B **89**(1), 43–76 (2003)

[22] Robertson, N., Seymour, P.D.: Graph minors. II. Algorithmic aspects of tree-width. Journal of Algorithms **7**(3), 309–322 (1986)

[23] Tazari, S., Müller-Hannemann, M.: Shortest paths in linear time on minor-closed graph classes, with an application to Steiner tree approximation. Discrete Applied Mathematics **157**, 673–684 (2009)



(a)

(b)

(c)

Figure 3: (a) A brick with a tour crossing through it. The bold paths are in H. The bold vertices are $s_0, s_1, s_2, \ldots, s_k$. The dotted paths are in $\mathcal{P}_{S \wedge N}$, the first four of which are in $\mathcal{X}_1$. (b) The patching process introduces the dotted paths on the lower boundary S of the brick and reroutes the tour to cross S twice between s_1 and s_2. The dotted subpath L of the top boundary N of the brick is used to replace the portion of the tour between its endpoints. (c) The tour after the entire construction given by Theorem 16.