

Fixed-parameter tractability of satisfying beyond the number of variables*

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Abstract

We consider a CNF formula F as a multiset of clauses: $F = \{c_1, \dots, c_m\}$. The set of variables of F will be denoted by $V(F)$. Let B_F denote the bipartite graph with partite sets $V(F)$ and F and with an edge between $v \in V(F)$ and $c \in F$ if $v \in c$ or $\bar{v} \in c$. The matching number $\nu(F)$ of F is the size of a maximum matching in B_F . In our main result, we prove that the following parameterization of MAXSAT (denoted by $(\nu(F) + k)$ -SAT) is fixed-parameter tractable: Given a formula F , decide whether we can satisfy at least $\nu(F) + k$ clauses in F , where k is the parameter.

A formula F is called variable-matched if $\nu(F) = |V(F)|$. Let $\delta(F) = |F| - |V(F)|$ and $\delta^*(F) = \max_{F' \subseteq F} \delta(F')$. Our main result implies fixed-parameter tractability of MAXSAT parameterized by $\delta(F)$ for variable-matched formulas F ; this complements related results of Kullmann (2000) and Szeider (2004) for MAXSAT parameterized by $\delta^*(F)$.

To obtain our main result, we reduce $(\nu(F) + k)$ -SAT into the following parameterization of the HITTING SET problem (denoted by $(m - k)$ -HITTING SET): given a collection \mathcal{C} of m subsets of a ground set U of n elements, decide whether there is $X \subseteq U$ such that $C \cap X \neq \emptyset$ for each $C \in \mathcal{C}$ and $|X| \leq m - k$, where k is the parameter. Gutin, Jones and Yeo (2011) proved that $(m - k)$ -HITTING SET is fixed-parameter tractable by obtaining an exponential kernel for the problem. We obtain two algorithms for $(m - k)$ -HITTING SET: a deterministic algorithm of runtime $O((2e)^{2k+O(\log^2 k)}(m+n)^{O(1)})$ and a randomized algorithm of expected runtime $O(8^{k+O(\sqrt{k})}(m+n)^{O(1)})$. Our deterministic algorithm improves an algorithm that follows from the kernelization result of Gutin, Jones and Yeo (2011).

1 Introduction

In this paper we study a parameterization of MAXSAT. We consider a CNF formula F as a multiset of clauses: $F = \{c_1, \dots, c_m\}$. (We allow repetition of clauses.) We assume that no clause contains both a variable and its negation, and no clause is empty. The set of variables of F will be denoted by $V(F)$, and for a clause c , $V(c) = V(\{c\})$. A *truth assignment* is a function $\tau : V(F) \rightarrow \{\text{TRUE}, \text{FALSE}\}$. A truth assignment τ *satisfies* a clause C if there exists $x \in V(F)$ such that $x \in C$ and $\tau(x) = \text{TRUE}$, or $\bar{x} \in C$ and $\tau(x) = \text{FALSE}$. We will denote the number of clauses in F satisfied by τ as $\text{sat}_\tau(F)$ and the maximum value of $\text{sat}_\tau(F)$, over all τ , as $\text{sat}(F)$.

Let B_F denote the bipartite graph with partite sets $V(F)$ and F with an edge between $v \in V(F)$ and $c \in F$ if $v \in V(c)$. The *matching number* $\nu(F)$ of F is the size of a maximum matching in B_F . Clearly, $\text{sat}(F) \geq \nu(F)$ and this lower bound for $\text{sat}(F)$ is tight as there are formulas F for which $\text{sat}(F) = \nu(F)$.

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In this paper we study the following parameterized problem, where the parameterization is above a tight lower bound.

$(\nu(F) + k)$ -SAT

Instance: A CNF formula F and a positive integer α .

Parameter: $k = \alpha - \nu(F)$.

Question: Is $\text{sat}(F) \geq \alpha$?

A natural and well-studied parameter in most optimization problems is the size of the solution. In particular, for MAXSAT, the standard parameterized problem is whether $\text{sat}(F) \geq k$ for a CNF formula F . Using a simple observation that $\text{sat}(F) \geq m/2$ for every CNF formula F on m clauses, Mahajan and Raman [21] showed that this problem is fixed-parameter tractable. The tight bound $\text{sat}(F) \geq m/2$ on $\text{sat}(F)$ means that the problem is interesting only when $k > m/2$, i.e., when the values of k are relatively large. To remedy this situation, Mahajan and Raman introduced, and showed fixed-parameter tractable, a more natural parameterized problem: whether the given CNF formula has an assignment satisfying at least $m/2 + k$ clauses. Since this pioneering paper [21], researchers have studied numerous problems parameterized above tight bounds including a few such parameterizations of MAXSAT [2, 6, 14], all stated in or inspired by Mahajan *et al.* [22]. Like the parameterizations in [2, 6, 14], $(\nu(F) + k)$ -SAT will be proved fixed-parameter tractable, but unlike them, $(\nu(F) + k)$ -SAT will be shown to have no polynomial-size kernel unless $\text{coNP} \subseteq \text{NP/poly}$, which is highly unlikely [4].

In our main result, we show that $(\nu(F) + k)$ -SAT is fixed-parameter tractable by obtaining an algorithm with running time $O((2e)^{2k+O(\log^2 k)}(n+m)^{O(1)})$, where e is the base of the natural logarithm. (We provide basic definitions on parameterized algorithms and complexity, including kernelization, in the next section.) We also develop a randomized algorithm for $(\nu(F) + k)$ -SAT of expected runtime $O(8^{k+O(\sqrt{k})}(m+n)^{O(1)})$.

The *deficiency* $\delta(F)$ of a formula F is $|F| - |V(F)|$; the *maximum deficiency* $\delta^*(F) = \max_{F' \subseteq F} \delta(F')$. A formula F is called *variable-matched* if $\nu(F) = |V(F)|$. Our main result implies fixed-parameter tractability of MAXSAT parameterized by $\delta(F)$ for variable-matched formulas F .

There are two related results: Kullmann [18] obtained an $O(n^{O(\delta^*(F))})$ -time algorithm for solving MAXSAT for formulas F with n variables and Szeider [28] gave an $O(f(\delta^*(F))n^4)$ -time algorithm for the problem, where f is a function depending on $\delta^*(F)$ only. Note that we cannot just drop the condition of being variable-matched from our result and expect a similar algorithm: it is not hard to see that the satisfiability problem remains NP-complete for formulas F with $\delta(F) = 0$.

A formula F is *minimal unsatisfiable* if it is unsatisfiable but $F \setminus c$ is satisfiable for every clause $c \in F$. Papadimitriou and Wolfe [26] showed that recognition of minimal unsatisfiable CNF formulas is complete for the complexity class¹ D^P . Kleine Büning [16] conjectured that for a fixed integer k , it can be decided in polynomial time whether a formula F with $\delta(F) \leq k$ is minimal unsatisfiable. Independently, Kullmann [18] and Fleischner and Szeider [11] (see also [10]) resolved this conjecture by showing that minimal unsatisfiable formulas with n variables and $n+k$ clauses can be recognized in $n^{O(k)}$ time. Later, Szeider [28] showed that the problem is fixed-parameter tractable by obtaining an algorithm of running time $O(2^k n^4)$. Note that Szeider's results follow from his results mentioned in the previous paragraph and the well-known fact that $\delta^*(F) = \delta(F)$ holds for every minimal unsatisfiable formula F . Since every minimal unsatisfiable formula is variable-matched [1], our main result also implies fixed-parameter tractability of recognizing minimal unsatisfiable formula with n variables and $n+k$ clauses, parameterized by k .

¹ D^P is the class of problems that can be considered as the difference of two NP-problems; clearly D^P contains all NP and all co-NP problems

To obtain our main result, we introduce some reduction rules and branching steps and reduce the problem to a parameterized version of HITTING SET, namely, $(m - k)$ -HITTING SET defined below. Let H be a hypergraph. A set $S \subseteq V(H)$ is called a *hitting set* if $e \cap S \neq \emptyset$ for all $e \in E(H)$.

$(m - k)$ -HITTING SET

Instance: A hypergraph H ($n = |V(H)|$, $m = |E(H)|$) and a positive integer k .

Parameter: k .

Question: Does there exist a hitting set $S \subseteq V(H)$ of size $m - k$?

Gutin *et al.* [13] showed that $(m - k)$ -HITTING SET is fixed-parameter tractable by obtaining a kernel for the problem. The kernel result immediately implies a $2^{O(k^2)}(m + n)^{O(1)}$ -time algorithm for the problem. Here we obtain a faster algorithm for this problem that runs in $O((2e)^{2k+O(\log^2 k)}(m + n)^{O(1)})$ time using the color-coding technique. This happens to be the dominating step for solving the $(\nu(F) + k)$ -SAT problem. We also obtain a randomized algorithm for $(m - k)$ -HITTING SET of expected runtime $O(8^{k+O(\sqrt{k})}(m + n)^{O(1)})$. To obtain the randomized algorithm, we reduce $(m - k)$ -HITTING SET into a special case of the SUBGRAPH ISOMORPHISM problem and use a recent randomized algorithm of Fomin *et al.* [9] for SUBGRAPH ISOMORPHISM.

It was shown in [13] that the $(m - k)$ -HITTING SET problem cannot have a kernel whose size is polynomial in k unless $\text{NP} \subseteq \text{coNP/poly}$. In this paper, we give a parameter preserving reduction from this problem to the $(\nu(F) + k)$ -SAT problem, thereby showing that $(\nu(F) + k)$ -SAT problem has no polynomial-size kernel unless $\text{NP} \subseteq \text{coNP/poly}$.

Organization of the rest of the paper. In Section 2, we provide additional terminology and notation and some preliminary results. In Section 3, we give a sequence of polynomial time preprocessing rules on the given input of $(\nu(F) + k)$ -SAT and justify their correctness. In Section 4, we give two simple branching rules and reduce the resulting input to a $(m - k)$ -HITTING SET problem instance. Section 5 gives an improved fixed-parameter algorithm for $(m - k)$ -HITTING SET using color coding. There we also obtain a faster randomized algorithm for $(m - k)$ -HITTING SET. Section 6 summarizes the entire algorithm for the $(\nu(F) + k)$ -SAT problem, shows its correctness and analyzes its running time. Section 7 proves the hardness of kernelization result. Section 8 concludes with some remarks.

2 Additional Terminology, Notation and Preliminaries

Graphs and Hypergraphs. For a subset X of vertices of a graph G , $N_G(X)$ denotes the set of all neighbors of vertices in X . When G is clear from the context, we write $N(X)$ instead of $N_G(X)$. A matching *saturates* all end-vertices of its edges. For a bipartite graph $G = (V_1, V_2; E)$, the classical Hall's matching theorem states that G has a matching that saturates every vertex of V_1 if and only if $|N(X)| \geq |X|$ for every subset X of V_1 . The next lemma follows from Hall's matching theorem: add d vertices to V_2 , each adjacent to every vertex in V_1 .

Lemma 1. *Let $G = (V_1, V_2; E)$ be a bipartite graph, and suppose that for all subsets $X \subseteq V_1$, $|N(X)| \geq |X| - d$ for some $d \geq 0$. Then $\nu(G) \geq |V_1| - d$.*

We say that a bipartite graph $G = (A, B; E)$ is q -*expanding* if for all $A' \subseteq A$, $|N_G(A')| \geq |A'| + q$. Given a matching M , an *alternating path* is a path in which the edges belong alternatively to M and not to M .

A *hypergraph* $H = (V(H), \mathcal{F})$ consists of a nonempty set $V(H)$ of *vertices* and a family \mathcal{F} of nonempty subsets of V called *edges* of H (\mathcal{F} is often denoted $E(H)$). Note that \mathcal{F} may have *parallel* edges, i.e., copies of the same subset of $V(H)$. For any vertex $v \in V(H)$, and any $\mathcal{E} \subseteq \mathcal{F}$,

$\mathcal{E}[v]$ is the set of edges in \mathcal{E} containing v , $N[v]$ is the set of all vertices contained in edges of $\mathcal{F}[v]$, and the *degree* of v is $d(v) = |\mathcal{F}[v]|$. For a subset T of vertices, $\mathcal{F}[T] = \bigcup_{v \in T} \mathcal{F}[v]$.

CNF formulas. For a subset X of the variables of CNF formula F , F_X denotes the subset of F consisting of all clauses c such that $V(c) \cap X \neq \emptyset$. A formula F is called *q-expanding* if $|X| + q \leq |F_X|$ for each $X \subseteq V(F)$. Note that, by Hall's matching theorem, a formula is variable-matched if and only if it is 0-expanding. Clearly, a formula F is *q-expanding* if and only if B_F is *q-expanding*.

For $x \in V(F)$, $n(x)$ and $n(\bar{x})$ denote the number of clauses containing x and the number of clauses containing \bar{x} , respectively.

A function $\pi : U \rightarrow \{\text{TRUE}, \text{FALSE}\}$, where U is a subset of $V(F)$, is called a *partial truth assignment*. A partial truth assignment $\pi : U \rightarrow \{\text{TRUE}, \text{FALSE}\}$ is an *autarky* if π satisfies all clauses of F_U . We have the following:

Lemma 2 ([6]). *Let $\pi : U \rightarrow \{\text{TRUE}, \text{FALSE}\}$ be an autarky for a CNF formula F and let γ be any truth assignment on $V(F) \setminus U$. Then for the combined assignment $\tau := \pi \cup \gamma$, it holds that $\text{sat}_\tau(F) = |F_U| + \text{sat}_\gamma(F \setminus F_U)$. Clearly, τ can be constructed in polynomial time given π and γ .*

Autarkies were first introduced in [23]; they are the subject of much study, see, e.g., [10, 19, 28], and see [17] for an overview.

Treewidth. A *tree decomposition* of an (undirected) graph G is a pair (U, T) where T is a tree whose vertices we will call *nodes* and $U = (\{U_i \mid i \in V(T)\})$ is a collection of subsets of $V(G)$ such that

1. $\bigcup_{i \in V(T)} U_i = V(G)$,
2. for each edge $vw \in E(G)$, there is an $i \in V(T)$ such that $v, w \in U_i$, and
3. for each $v \in V(G)$ the set $\{i : v \in U_i\}$ of nodes forms a subtree of T .

The U_i 's are called *bags*. The *width* of a tree decomposition $(\{U_i : i \in V(T)\}, T)$ equals $\max_{i \in V(T)} \{|U_i| - 1\}$. The *treewidth* of a graph G is the minimum width over all tree decompositions of G . We use notation $\text{tw}(G)$ to denote the treewidth of a graph G .

Parameterized Complexity. A *parameterized problem* is a subset $L \subseteq \Sigma^* \times \mathbb{N}$ over a finite alphabet Σ . The unparameterized version of a parameterized problem L is the language $L^c = \{x\#1^k \mid (x, k) \in L\}$. The problem L is *fixed-parameter tractable* if the membership of an instance (x, k) in $\Sigma^* \times \mathbb{N}$ can be decided in time $f(k)|x|^{O(1)}$, where f is a function of the *parameter* k only [8, 12, 24]. Given a parameterized problem L , a *kernelization* of L is a polynomial-time algorithm that maps an instance (x, k) to an instance (x', k') (the *kernel*) such that (i) $(x, k) \in L$ if and only if $(x', k') \in L$, (ii) $k' \leq g(k)$, and (iii) $|x'| \leq g(k)$ for some function g . We call $g(k)$ the *size* of the kernel. It is well-known [8, 12] that a decidable parameterized problem L is fixed-parameter tractable if and only if it has a kernel. Polynomial-size kernels are of main interest, due to applications [8, 12, 24], but unfortunately not all fixed-parameter problems have such kernels unless $\text{coNP} \subseteq \text{NP}/\text{poly}$, see, e.g., [4, 5, 7].

For a positive integer q , let $[q] = \{1, \dots, q\}$.

3 Preprocessing Rules

In this section we give preprocessing rules and their correctness.

Let F be the given CNF formula on n variables and m clauses with a maximum matching M on B_F , the variable-clause bipartite graph corresponding to F . Let α be a given integer and recall

that our goal is to check whether $\text{sat}(F) \geq \alpha$. For each preprocessing rule below, we let (F', α') be the instance resulting by the application of the rule on (F, α) . We say that a rule is *valid* if (F, α) is a YES instance if and only if (F', α') a YES instance.

Reduction Rule 1. *Let x be a variable such that $n(x) = 0$ (respectively $n(\bar{x}) = 0$). Set $x = \text{FALSE}$ ($x = \text{TRUE}$) and remove all the clauses that contain \bar{x} (x). Reduce α by $n(\bar{x})$ (respectively $n(x)$).*

The proof of the following lemma is immediate.

Lemma 3. *If $n(x) = 0$ (respectively $n(\bar{x}) = 0$) then $\text{sat}(F) = \text{sat}(F') + n(\bar{x})$ (respectively $\text{sat}(F) = \text{sat}(F') + n(x)$), and so Rule 1 is valid.*

Reduction Rule 2. *Let $n(x) = n(\bar{x}) = 1$ and let c' and c'' be the two clauses containing x and \bar{x} , respectively. Let $c^* = (c' - x) \cup (c'' - \bar{x})$ and let F' be obtained from F by deleting c' and c'' and adding the clause c^* . Reduce α by 1.*

Lemma 4. *For F and F' in Reduction Rule 2, $\text{sat}(F) = \text{sat}(F') + 1$, and so Rule 2 is valid.*

Proof. Consider any assignment for F . If it satisfies both c' and c'' , then the same assignment will satisfy c^* . So when restricted to variables of F' , it will satisfy at least $\text{sat}(F) - 1$ clauses of F' . Thus $\text{sat}(F') \geq \text{sat}(F) - 1$ which is equivalent to $\text{sat}(F) \leq \text{sat}(F') + 1$. Similarly if an assignment γ to F' satisfies c^* then at least one of c', c'' is satisfied by γ . Therefore by setting x true if γ satisfies c'' and false otherwise, we can extend γ to an assignment on F that satisfies both of c', c'' . On the other hand, if c^* is not satisfied by γ then neither c' nor c'' is satisfied by γ , and any extension of γ will satisfy exactly one of c', c'' . Therefore in either case $\text{sat}(F) \geq \text{sat}(F') + 1$. We conclude that $\text{sat}(F) = \text{sat}(F') + 1$, as required. \square

Our next reduction rule is based on the following lemma proved in Fleischner *et al.* [10, Lemma 10], Kullmann [19, Lemma 7.7] and Szeider [28, Lemma 9].

Lemma 5. *Let F be a CNF formula. Given a maximum matching in B_F , in time $O(|F|)$ we can find an autarky $\pi : U \rightarrow \{\text{TRUE}, \text{FALSE}\}$ such that $F \setminus F_U$ is 1-expanding.*

Reduction Rule 3. *Find an autarky $\pi : U \rightarrow \{\text{TRUE}, \text{FALSE}\}$ such that $F \setminus F_U$ is 1-expanding. Set $F' = F \setminus F_U$ and reduce α by $|F_U|$.*

The next lemma follows from Lemma 2.

Lemma 6. *For F and F' in Reduction Rule 3, $\text{sat}(F) = \text{sat}(F') + |F_U|$ and so Rule 3 is valid.*

After exhaustive application of Rule 3, we may assume that the resulting formula is 1-expanding. For the next reduction rule, we need the following results.

Theorem 1 (Szeider [28]). *Given a variable-matched formula F , with $|F| = |V(F)| + 1$, we can decide whether F is satisfiable in time $O(|V(F)|^3)$.*

Consider a bipartite graph $G = (A, B; E)$. Recall that a formula F is q -expanding if and only if B_F is q -expanding. From a bipartite graph $G = (A, B; E)$, $x \in A$ and $q \geq 1$, we obtain a bipartite graph G_{qx} , by adding new vertices x_1, \dots, x_q to A and adding edges such that new vertices have exactly the same neighborhood as x , that is, $G_{qx} = (A \cup \{x_1, \dots, x_q\}, B; E \cup \{(x_i, y) : (x, y) \in E\})$. The following result is well known.

Lemma 7. [20, Theorem 1.3.6] *Let $G = (A, B; E)$ be a 0-expanding bipartite graph. Then G is q -expanding if and only if G_{qx} is 0-expanding for all $x \in A$.*

Lemma 8. *Let $G = (A, B; E)$ be a 1-expanding bipartite graph. In polynomial time, we can check whether G is 2-expanding, and if it is not, find a set $S \subseteq A$ such that $|N_G(S)| = |S| + 1$.*

Proof. Let $x \in A$. By Hall's Matching Theorem, G_{2x} is 0-expanding if and only if $\nu(G_{2x}) = |A| + 2$. Since we can check the last condition in polynomial time, by Lemma 7 we can decide whether G is 2-expanding in polynomial time. So, assume that G is not 2-expanding and we know this because G_{2y} is not 0-expanding for some $y \in A$. By Lemma 3(4) in [28], in polynomial time, we can find a set $T \subseteq A \cup \{y_1, y_2\}$ such that $|N_{G_{2y}}(T)| < |T|$. Since G is 1-expanding, $y_1, y_2 \in T$ and $|N_{G_{2y}}(T)| = |T| - 1$. Hence, $|S| + 1 = |N_G(S)|$, where $S = T \setminus \{y_1, y_2\}$. \square

For a formula F and a set $S \subseteq V(F)$, $F[S]$ denotes the formula obtained from F_S by deleting all variables not in S .

Reduction Rule 4. *Let F be a 1-expanding formula and let $B = B_F$. Using Lemma 8, check whether F is 2-expanding. If it is then do not change F , otherwise find a set $S \subseteq V(F)$ with $|N_B(S)| = |S| + 1$. Let M be a matching that saturates S in $B[S \cup N_B(S)]$ (that exists as $B[S \cup N_B(S)]$ is 1-expanding). Use Theorem 1 to decide whether $F[S]$ is satisfiable, and proceed as follows.*

$F[S]$ is satisfiable: *Obtain a new formula F' by removing all clauses in $N_B(S)$ from F . Reduce α by $|N_B(S)|$.*

$F[S]$ is not satisfiable: *Let c' be the clause obtained by deleting all variables in S from $\cup_{c'' \in N_B(S)} c''$. That is, a literal l belongs to c' if and only if it belongs to some clause in $N_B(S)$ and the variable corresponding to l is not in S . Obtain a new formula F' by removing all clauses in $N_B(S)$ from F and adding c' . Reduce α by $|S|$.*

Lemma 9. *For F , F' and S introduced in Rule 4, if $F[S]$ is satisfiable $\text{sat}(F) = \text{sat}(F') + |N_B(S)|$, otherwise $\text{sat}(F) = \text{sat}(F') + |S|$ and thus Rule 4 is valid.*

Proof. We consider two cases.

Case 1: $F[S]$ is satisfiable. Observe that there is an autarky on S and thus by Lemma 2, $\text{sat}(F) = \text{sat}(F') + |N_B(S)|$.

Case 2: $F[S]$ is not satisfiable. Let $F'' = F' \setminus c'$. As any optimal truth assignment to F will satisfy at least $\text{sat}(F) - |N_B(S)|$ clauses of F'' , it follows that $\text{sat}(F) \leq \text{sat}(F'') + |N_B(S)| \leq \text{sat}(F') + |N_B(S)|$.

Let y denote the clause in $N_B(S)$ that is not matched to a variable in S by M . Let S' be the set of variables, and Z the set of clauses, that can be reached from y with an M -alternating path in $B[S \cup N_B(S)]$. We argue now that $Z = N_B(S)$. Since Z is made up of clauses that are reachable in $B[S \cup N_B(S)]$ by an M -alternating path from the single unmatched clause y , $|Z| = |S'| + 1$. It follows that $|N_B(S) \setminus Z| = |S \setminus S'|$, and M matches every clause in $N_B(S) \setminus Z$ with a variable in $S \setminus S'$. Furthermore, $N_B(S \setminus S') \cap Z = \emptyset$ as otherwise the matching partners of some elements of $S \setminus S'$ would have been reachable by an M -alternating path from y , contradicting the definition of $N_B(S)$ and S' . Thus $S \setminus S'$ has an autarky such that $F \setminus F_{S \setminus S'}$ is 1-expanding which would have been detected by Rule 3, hence $S \setminus S' = \emptyset$ and so $S = S'$. That is, all clauses in $N_B(S)$ are reachable from the unmatched clause y by an M -alternating path. We have now shown that $Z = N_B(S)$, as desired.

Suppose that there exists an assignment γ to F' , that satisfies $\text{sat}(F')$ clauses of F' that also satisfies c' . Then there exists a clause $c'' \in N_B(S)$ that is satisfied by γ . As c'' is reachable from y by an M -alternating path, we can modify M to include y and exclude c'' , by taking the symmetric difference of the matching and the M -alternating path from y to c'' . This will give a matching saturating S and $N_B(S) \setminus c''$, and we use this matching to extend the assignment γ to one which satisfies all of $N_B(S) \setminus c''$. We therefore have satisfied all the clauses of $N_B(S)$. Therefore since c' is satisfied in F' but does not appear in F , we have satisfied extra $|N_B(S)| - 1 = |S|$ clauses. Suppose on the other hand that every assignment γ for F' that satisfies $\text{sat}(F')$ clauses does not

satisfy c' . We can use the matching on $B[S \cup N_B(S)]$ to satisfy $|N_B(S)| - 1$ clauses in $N_B(S)$, which would give us an additional $|S|$ clauses in $N_B(S)$. Thus $\text{sat}(F) \geq \text{sat}(F') + |S|$.

As $|N_B(S)| = |S| + 1$, it suffices to show that $\text{sat}(F) < \text{sat}(F') + |N_B(S)|$. Suppose that there exists an assignment γ to F that satisfies $\text{sat}(F') + |N_B(S)|$ clauses, then it must satisfy all the clauses of $N_B(S)$ and $\text{sat}(F')$ clauses of F' . As $F[S]$ is not satisfiable, variables in S alone can not satisfy all of $N_B(S)$. Hence there exists a clause $c'' \in N_B(S)$ such that there is a variable $v \in V(c'') \setminus S$ that satisfies c'' . But then $v \in V(c')$ and hence c' would be satisfiable by γ , a contradiction as γ satisfies $\text{sat}(F')$ clauses of F' . \square

4 Branching Rules and Reduction to $(m - k)$ -Hitting Set

Our algorithm first applies Reduction Rules 1, 2, 3 and 4 exhaustively on (F, α) . Then it applies two branching rules we describe below, in the following order.

Branching on a variable x means that the algorithm constructs two instances of the problem, one by substituting $x = \text{TRUE}$ and simplifying the instance and the other by substituting $x = \text{FALSE}$ and simplifying the instance. Branching on x or y being false means that the algorithm constructs two instances of the problem, one by substituting $x = \text{FALSE}$ and simplifying the instance and the other by substituting $y = \text{FALSE}$ and simplifying the instance. Simplifying an instance is done as follows. For any clause c , if c contains a literal z with $z = \text{TRUE}$, remove c and reduce α by 1. If c contains a literal z with $z = \text{FALSE}$ and c contains other literals, remove z from c . If c consists of the single literal $z = \text{FALSE}$, remove c .

A branching rule is correct if the instance on which it is applied is a YES-instance if and only if the simplified instance of (at least) one of the branches is a YES-instance.

Branching Rule 1. *If $n(x) \geq 2$ and $n(\bar{x}) \geq 2$ then we branch on x .*

Before attempting to apply Branching Rule 2, we apply the following rearranging step: For all variables x such that $n(\bar{x}) = 1$, swap literals x and \bar{x} in all clauses. Clearly, this will not change $\text{sat}(F)$. Observe that now for every variable $n(x) = 1$ and $n(\bar{x}) \geq 2$.

Branching Rule 2. *If there is a clause c such that positive literals $x, y \in c$ then we branch on x being false or y being false.*

Branching Rule 1 is exhaustive and thus its correctness also follows. When we reach Branching Rule 2 for every variable $n(x) = 1$ and $n(\bar{x}) \geq 2$. As $n(x) = 1$ and $n(y) = 1$ we note that c is the only clause containing these literals. Therefore there exists an optimal solution with x or y being false (if they are both true just change one of them to false). Thus, we have the following:

Lemma 10. *Branching Rules 1 and 2 are correct.*

Let (F, α) be the given instance on which Reduction Rules 1, 2, 3 and 4, and Branching Rules 1 and 2 do not apply. Observe that for such an instance F the following holds:

1. For every variable x , $n(x) = 1$ and $n(\bar{x}) \geq 2$.
2. Every clause contains at most one positive literal.

We call a formula F satisfying the above properties *special*. In what follows we describe an algorithm for our problem on special instances. Let $c(x)$ denote the *unique* clause containing positive literal x . We can obtain a matching saturating $V(F)$ in B_F by taking the edge connecting the variable x and the clause $c(x)$. We denote the resulting matching by M_u .

We first describe a transformation that will be helpful in reducing our problem to $(m - k)$ -HITTING SET. Given a formula F we obtain a new formula F' by changing the clauses of F as follows. If there exists some $c(x)$ such that $|c(x)| \geq 2$, do the following. Let $c' = c(x) - x$ (that is,

c' contain the same literals as $c(x)$ except for x) and add c' to all clauses containing the literal \bar{x} . Furthermore remove c' from $c(x)$ (which results in $c(x) = (x)$ and therefore $|c(x)| = 1$).

Next we prove the validity of the above transformation.

Lemma 11. *Let F' be the formula obtained by applying the transformation described above on F . Then $\text{sat}(F') = \text{sat}(F)$ and $\nu(B_F) = \nu(B_{F'})$.*

Proof. We note that the matching M_u remains a matching in $B_{F'}$ and thus $\nu(B_F) = \nu(B_{F'})$. Let γ be any truth assignment to the variables in F (and F') and note that if c' is false under γ then F and F' satisfy exactly the same clauses under γ (as we add and subtract something false to the clauses). So assume that c' is true under γ .

If γ maximizes the number of satisfied clauses in F then clearly we may assume that x is false (as $c(x)$ is true due to c'). Now let γ' be equal to γ except the value of x has been flipped to true. Note that exactly the same clauses are satisfied in F and F' by γ and γ' , respectively. Analogously, if an assignment maximizes the number of satisfied clauses in F' we may assume that x is true and by changing it to false we satisfy equally many clauses in F . Hence, $\text{sat}(F') = \text{sat}(F)$. \square

Given a special instance (F, α) we apply the above transformation repeatedly until no longer possible and obtain an instance (F', α) such that $\text{sat}(F') = \text{sat}(F)$, $\nu(B_F) = \nu(B_{F'})$ and $|c(x)| = 1$ for all $x \in V(F')$. We call such an instance (F', α) *transformed special*. Observe that, it takes polynomial time, to obtain the transformed special instance from a given special instance.

For simplicity of presentation we denote the transformed special instance by (F, α) . Let C^* denote all clauses that are not matched by M_u (and therefore only contain negated literals). We associate a hypergraph H^* with the transformed special instance. Let H^* be the hypergraph with vertex set $V(F)$ and edge set $E^* = \{V(c) \mid c \in C^*\}$.

We now show the following equivalence between $(\nu(F) + k)$ -SAT on transformed special instances and $(m - k)$ -HITTING SET.

Lemma 12. *Let (F, α) be the transformed special instance and H^* be the hypergraph associated with it. Then $\text{sat}(F) \geq \alpha$ if and only if there is a hitting set in H^* of size at most $|E(H^*)| - k$, where $k = \alpha - \nu(F)$.*

Proof. We start with a simple observation about an assignment satisfying the maximum number of clauses of F . There exists an optimal truth assignment to F , such that all clauses in C^* are true. Assume that this is not the case and let γ be an optimal truth assignment satisfying as many clauses from C^* as possible and assume that $c \in C^*$ is not satisfied. Let $\bar{x} \in c$ be an arbitrary literal and note that $\gamma(x) = \text{TRUE}$. However, changing x to false does not decrease the number of satisfied clauses in F and increases the number of satisfied clauses in C^* .

Now we show that $\text{sat}(F) \geq \alpha$ if and only if there is a hitting set in H^* of size at most $|E(H^*)| - k$. Assume that γ is an optimal truth assignment to F , such that all clauses in C^* are true. Let $U \subseteq V(F)$ be all variables that are false in γ and note that U is a hitting set in H^* . Analogously if U' is a hitting set in H^* then by letting all variables in U' be false and all other variables in $V(F)$ be true we get a truth assignment that satisfies $|F| - |U'|$ clauses in F . Therefore if $\tau(H^*)$ is the size of a minimum hitting set in H^* we have $\text{sat}(F) = |F| - \tau(H^*)$. Hence, $\text{sat}(F) = |F| - \tau(H^*) = |V(F)| + |C^*| - \tau(H^*)$ and thus $\text{sat}(F) \geq \alpha$ if and only if $|C^*| - \tau(H^*) \geq k$, which is equivalent to $\tau(H^*) \leq |E(H^*)| - k$. \square

Therefore our problem is fixed-parameter tractable on transformed special instances, by the next theorem that follows from the kernelization result in [13].

Theorem 2. *There exists an algorithm for $(m - k)$ -HITTING SET running in time $2^{O(k^2)} + O((n + m)^{O(1)})$.*

In the next section we give faster algorithms for $(\nu(F) + k)$ -SAT on transformed special instances by giving faster algorithms for $(m - k)$ -HITTING SET.

5 Algorithms for $(m - k)$ -Hitting Set

To obtain faster algorithms for $(m - k)$ -HITTING SET, we utilize the following concept of k -mini-hitting set introduced in [13].

Definition 1. Let $H = (V, \mathcal{F})$ be a hypergraph and k be a nonnegative integer. A k -mini-hitting set is a set $S_{\text{MINI}} \subseteq V$ such that $|S_{\text{MINI}}| \leq k$ and $|\mathcal{F}[S_{\text{MINI}}]| \geq |S_{\text{MINI}}| + k$.

Lemma 13 ([13]). A hypergraph H has a hitting set of size at most $m - k$ if and only if it has a k -mini-hitting set. Moreover, given a k -mini-hitting set S_{MINI} , we can construct a hitting set S with $|S| \leq m - k$ such that $S_{\text{MINI}} \subseteq S$ in polynomial time.

5.1 Deterministic Algorithm

Next we give an algorithm that finds a k -mini-hitting set S_{MINI} if it exists, in time $c^k(m + n)^{O(1)}$, where c is a constant. We first describe a randomized algorithm based on color-coding [3] and then derandomize it using hash functions. Let $\chi : E(H) \rightarrow [q]$ be a function. For a subset $S \subseteq V(H)$, $\chi(S)$ denotes the maximum subset $X \subseteq [q]$ such that for all $i \in X$ there exists an edge $e \in E(H)$ with $\chi(e) = i$ and $e \cap S \neq \emptyset$. A subset $S \subseteq V(H)$ is called a *colorful hitting set* if $\chi(S) = [q]$. We now give a procedure that given a coloring function χ finds a minimum colorful hitting set, if it exists. This algorithm will be useful in obtaining a k -mini-hitting set S_{MINI} .

Lemma 14. Given a hypergraph H and a coloring function $\chi : E(H) \rightarrow [q]$, we can find a minimum colorful hitting set if there exists one in time $O(2^q q(m + n))$.

Proof. We first check whether for every $i \in [q]$, $\chi^{-1}(i) \neq \emptyset$. If for any i we have that $\chi^{-1}(i) = \emptyset$, then we return that there is no colorful hitting set. So we may assume that for all $i \in [q]$, $\chi^{-1}(i) \neq \emptyset$. We will give an algorithm using dynamic programming over subsets of $[q]$. Let γ be an array of size 2^q indexed by the subsets of $[q]$. For a subset $X \subseteq [q]$, let $\gamma[X]$ denote the size of a smallest set $W \subseteq V(H)$ such that $X \subseteq \chi(W)$. We obtain a recurrence for $\gamma[X]$ as follows:

$$\gamma[X] = \begin{cases} \min_{(v \in V(H), \chi(\{v\}) \cap X \neq \emptyset)} \{1 + \gamma[X \setminus \chi(\{v\})]\} & \text{if } |X| \geq 1, \\ 0 & \text{if } X = \emptyset. \end{cases}$$

The correctness of the above recurrence is clear. The algorithm computes $\gamma[[q]]$ by filling the γ in the order of increasing set sizes. Clearly, each cell can be filled in time $O(q(n + m))$ and thus the whole array can be filled in time $O(2^q q(n + m))$. The size of a minimum colorful hitting set is given by $\gamma[[q]]$. We can obtain a minimum colorful hitting set by the routine back-tracking. \square

Now we describe a randomized procedure to obtain a k -mini-hitting set S_{MINI} in a hypergraph H , if there exists one. We do the following for each possible value p of $|S_{\text{MINI}}|$ (that is, for $1 \leq p \leq k$). Color $E(H)$ uniformly at random with colors from $[p + k]$; we denote this random coloring by χ . Assume that there is a k -mini-hitting set S_{MINI} of size p and some $p + k$ edges e_1, \dots, e_{p+k} such that for all $i \in [p + k]$, $e_i \cap S_{\text{MINI}} \neq \emptyset$. The probability that for all $1 \leq i < j \leq p + k$ we have that $\chi(e_i) \neq \chi(e_j)$ is $\frac{(p+k)!}{(p+k)^{p+k}} \geq e^{-(p+k)} \geq e^{-2k}$. Now, using Lemma 14 we can test in time $O(2^{p+k}(p + k)(m + n))$ whether there is a colorful hitting set of size at most p . Thus with probability at least e^{-2k} we can find a S_{MINI} , if there exists one. To boost the probability we repeat the procedure e^{2k} times and thus in time $O((2e)^{2k} 2k(m + n)^{O(1)})$ we find a S_{MINI} , if there exists one, with probability at least $1 - (1 - \frac{1}{e^{2k}})^{e^{2k}} \geq \frac{1}{2}$. If we obtained S_{MINI} then using Lemma 13 we can construct a hitting set of H of size at most $m - k$.

To derandomize the procedure, we need to replace the first step of the procedure where we color the edges of $E(H)$ uniformly at random from the set $[p + k]$ to a deterministic one. This is done by making use of an $(m, p + k, p + k)$ -perfect hash family. An $(m, p + k, p + k)$ -perfect hash family,

\mathcal{H} , is a set of functions from $[m]$ to $[p+k]$ such that for every subset $S \subseteq [m]$ of size $p+k$ there exists a function $f \in \mathcal{H}$ such that f is injective on S . That is, for all $i, j \in S$, $f(i) \neq f(j)$. There exists a construction of an $(m, p+k, p+k)$ -perfect hash family of size $O(e^{p+k} \cdot k^{O(\log k)} \cdot \log m)$ and one can produce this family in time linear in the output size [27]. Using an $(m, p+k, p+k)$ -perfect hash family \mathcal{H} of size at most $O(e^{2k} \cdot k^{O(\log k)} \cdot \log m)$ rather than a random coloring we get the desired deterministic algorithm. To see this, it is enough to observe that if there is a subset $S_{\text{mini}} \subseteq V(H)$ such that $|\mathcal{F}[S_{\text{mini}}]| \geq |S_{\text{mini}}| + k$ then there exists a coloring $f \in \mathcal{H}$ such that the $p+k$ edges e_1, \dots, e_{p+k} that intersect S_{mini} are distinctly colored. So if we generate all colorings from \mathcal{H} we will encounter the desired f . Hence for the given f , when we apply Lemma 14 we get the desired result. This concludes the description. The total time of the derandomized algorithm is $O(k2^{2k}(m+n)e^{2k} \cdot k^{O(\log k)} \cdot \log m) = O((2e)^{2k+O(\log^2 k)}(m+n)^{O(1)})$.

Theorem 3. *There exists an algorithm solving $(m-k)$ -HITTING SET in time $O((2e)^{2k+O(\log^2 k)}(m+n)^{O(1)})$.*

By Theorem 3 and the transformation discussed in Section 4 we have the following theorem.

Theorem 4. *There exists an algorithm solving a transformed special instance of $(\nu(F)+k)$ -SAT in time $O((2e)^{2k+O(\log^2 k)}(m+n)^{O(1)})$.*

5.2 Randomized Algorithm

In this subsection we give a randomized algorithm for $(m-k)$ -HITTING SET running in time $O(8^{k+O(\sqrt{k})}(m+n)^{O(1)})$. However, unlike the algorithm presented in the previous subsection we do not know how to derandomize this algorithm. Essentially, we give a randomized algorithm to find a k -mini-hitting set S_{MINI} in the hypergraph H , if it exists.

Towards this we introduce notions of a star-forest and a bush. We call $K_{1,\ell}$ a *star of size ℓ* ; a vertex of degree ℓ in $K_{1,\ell}$ is a *central vertex* (thus, both vertices in $K_{1,1}$ are central). A *star-forest* is a forest consisting of stars. A star-forest F is said to have *dimension* (a_1, a_2, \dots, a_p) if F has p stars with sizes a_1, a_2, \dots, a_p respectively. Given a star-forest F of dimension (a_1, a_2, \dots, a_p) , we construct a graph, which we call a *bush of dimension* (a_1, a_2, \dots, a_p) , by adding a triangle (x, y, z) and making y adjacent to a central vertex of in every star of F .

For a hypergraph $H = (V, \mathcal{F})$, the *incidence bipartite graph* B_H of H has partite sets V and \mathcal{F} , and there is an edge between $v \in V$ and $e \in \mathcal{F}$ in H if $v \in e$. Given B_H , we construct B_H^* by adding a triangle (x, y, z) and making y adjacent to every vertex in the V . The following lemma relates k -mini-hitting sets to bushes.

Lemma 15. *A hypergraph $H = (V, \mathcal{F})$ has a k -mini-hitting set S_{MINI} if and only if there exists a tuple (a_1, \dots, a_p) such that*

(a) $p \leq k$, $a_i \geq 1$ for all $i \in [p]$, and $\sum_{i=1}^p a_i = p+k$; and

(b) *there exists a subgraph of B_H^* isomorphic to a bush of dimension (a_1, \dots, a_p) .*

Proof. We first prove that the existence of a k -mini-hitting set in H implies the existence of a bush in B_H^* of dimension satisfying (a) and (b). Let $S_{\text{MINI}} = \{w_1, \dots, w_q\}$ be a k -mini-hitting set and let $S_i = \{w_1, \dots, w_i\}$. We know that $q \leq k$ and $|\mathcal{F}[S_{\text{MINI}}]| \geq |S_{\text{MINI}}| + k$. We define $\mathcal{E}_i := \mathcal{F}[S_i] \setminus \mathcal{F}[S_{i-1}]$ for every $i \geq 2$, and $\mathcal{E}_1 := \mathcal{F}[S_1]$. Let $\mathcal{E}_{s_1}, \dots, \mathcal{E}_{s_r}$ be the subsequence of the sequence $\mathcal{E}_1, \dots, \mathcal{E}_q$ consisting only of non-empty sets \mathcal{E}_i , and let $b_j = |\mathcal{E}_{s_j}|$ for each $j \in [r]$. Let p be the least integer from $[r]$ such that $\sum_{i=1}^p b_i \geq k+p$.

Observe that for every $j \in [p]$, the vertex w_{s_j} belongs to each hyperedge of \mathcal{E}_{s_j} . Thus, the bipartite graph B_H contains a star-forest F of dimension (b_1, \dots, b_p) , such that $p \leq k$, $b_j \geq 1$ for all $j \in [p]$, and $c := \sum_{j=1}^p b_j \geq p+k$. Moreover, each star in F has a central vertex in V . By the

minimality of p , we have $\sum_{j=1}^{p-1} b_j < p - 1 + k$ and so $b_p \geq c + 1 - (p + k)$. Thus, the integers a_j defined as follows are positive: $a_j := b_j$ for every $j \in [p - 1]$ and $a_p := b_p - c + (p + k)$. Hence, B_H contains a star-forest F' of dimension (a_1, \dots, a_p) , such that each star in F' has a central vertex in V .

Thus, all central vertices are in V , $p \leq k$, $a_i \geq 1$ for all $i \in [p]$, and $\sum_{i=1}^p a_i = p + k$, which implies that B_H^* contains, as a subgraph, a bush with dimension (a_1, \dots, a_p) satisfying the conditions above.

The construction above relating a k -mini-hitting set of H with the required bush of B_H^* can be easily reversed in the following sense: the existence of a bush of dimension satisfying (a) and (b) in B_H^* implies the existence of a k -mini-hitting set in H . Here the triangle ensures that the central vertices are in V . This completes the proof. \square

Next we describe a fast randomized algorithm for deciding the existence of a k -mini-hitting set using the characterization obtained in Lemma 15. Towards this we will use a fast randomized algorithm for the SUBGRAPH ISOMORPHISM problem. In the SUBGRAPH ISOMORPHISM problem we are given two graphs F and G on k and n vertices, respectively, as an input, and the question is whether there exists a subgraph of G isomorphic to F . Recall that $\text{tw}(G)$ denotes the treewidth of a graph G . We will use the following result.

Theorem 5 (Fomin *et al.*[9]). *Let F and G be two graphs on q and n vertices respectively and $\text{tw}(F) \leq t$. Then, there is a randomized algorithm for the SUBGRAPH ISOMORPHISM problem that runs in expected time $O(2^q(nt)^{t+O(1)})$.*

Let $\mathcal{P}_\ell(s)$ be the set of all *unordered partitions* of an integer s into ℓ parts. Nijenhuis and Wilf [25] designed a polynomial delay generation algorithm for partitions of $\mathcal{P}_\ell(s)$. Let $p(s)$ be the partition function, i.e., the overall number of partitions of s . The asymptotic behavior of $p(s)$ was first evaluated by Hardy and Ramanujan in the paper in which they develop the famous “circle method.”

Theorem 6 (Hardy and Ramanujan [15]). *We have $p(s) \sim e^\pi \sqrt{\frac{2s}{3}} / (4s\sqrt{3})$, as $s \rightarrow \infty$.*

This theorem and the algorithm of Nijenhuis and Wilf [25] imply the following:

Proposition 1. *There is an algorithm of runtime $2^{O(\sqrt{s})}$ for generating all partitions in $\mathcal{P}_\ell(s)$.*

Now we are ready to describe and analyze a fast randomized algorithm for deciding the existence of a k -mini-hitting set in a hypergraph H . By Lemma 15, it suffices to design and analyze a fast randomized algorithm for deciding the existence of a bush in B_H^* of dimension (a_1, \dots, a_p) satisfying conditions (a) and (b) of Lemma 15. Our algorithm starts by building B_H^* . Then it considers all possible values of p one by one ($p \in [k]$) and generates all partitions in $\mathcal{P}_p(p+k)$ using the algorithm of Proposition 1. For each such partition (a_1, \dots, a_p) that satisfies conditions (a) and (b) of Lemma 15, the algorithm of Fomin *et al.*[9] mentioned in Theorem 5 decides whether B_H^* contains a bush of dimension (a_1, \dots, a_p) . If such a bush exists, we output YES and we output NO, otherwise.

To evaluate the runtime of our algorithm, observe that the treewidth of any bush is 2 and any bush in Lemma 15 has at most $3k + 3$ vertices. This observation, the algorithm above, Theorem 5 and Proposition 1 imply the following:

Theorem 7. *There exists a randomized algorithm solving $(m - k)$ -HITTING SET in expected time $O(8^{k+O(\sqrt{k})}(m+n)^{O(1)})$.*

This theorem, in turn, implies the following:

Theorem 8. *There exists a randomized algorithm solving a transformed special instance of $(\nu(F) + k)$ -SAT in expected time $O(8^{k+O(\sqrt{k})}(m+n)^{O(1)})$.*

6 Complete Algorithm, Correctness and Analysis

The complete algorithm for an instance (F, α) of $(\nu(F) + k)$ -SAT is as follows.

Find a maximum matching M on B_F and let $k = \alpha - |M|$. If $k \leq 0$, return YES. Otherwise, apply Reduction Rules 1 to 4, whichever is applicable, in that order and then run the algorithm on the reduced instance and return the answer. If none of the Reduction Rules apply, then apply Branching Rule 1 if possible, to get two instances (F', α') and (F'', α'') . Run the algorithm on both instances; if one of them returns YES, return YES, otherwise return NO. If Branching Rule 1 does not apply then we rearrange the formula and attempt to apply Branching Rule 2 in the same way. Finally if $k > 0$ and none of the reduction or branching rules apply, then we have for all variables x , $n(x) = 1$ and every clause contains at most one positive literal, i.e. (F, α) is a special instance. Then solve the problem by first obtaining the transformed special instance, then the corresponding instance H^* of $(m - k)$ -HITTING SET and solving H^* in time $O((2e)^{2k+O(\log^2 k)}(m+n)^{O(1)})$ as described in Sections 4 and 5.

Correctness of all the preprocessing rules and the branching rules follows from Lemmata 3, 4, 6, 9 and 10.

Analysis of the algorithm. Let (F, α) be the input instance. Let $\mu(F) = \mu = \alpha - \nu(F)$ be the measure. We will first show that our preprocessing rules do not increase this measure. Following this, we will prove a lower bound on the decrease in the measure occurring as a result of the branching, thus allowing us to bound the running time of the algorithm in terms of the measure μ . For each case, we let (F', α') be the instance resulting by the application of the rule or branch. Also let M' be a maximum matching of $B_{F'}$.

Reduction Rule 1: We consider the case when $n(x) = 0$; the other case when $n(\bar{x}) = 0$ is analogous. We know that $\alpha' = \alpha - n(\bar{x})$ and $\nu(F') \geq \nu(F) - n(\bar{x})$ as removing $n(\bar{x})$ clauses can only decrease the matching size by $n(\bar{x})$. This implies that $\mu(F) - \mu(F') = \alpha - \nu(F) - \alpha' + \nu(F') = (\alpha - \alpha') + (\nu(F') - \nu(F)) \geq n(\bar{x}) - n(\bar{x})$. Thus, $\mu(F') \leq \mu(F)$.

Reduction Rule 2: We know that $\alpha' = \alpha - 1$. We show that $\nu(F') \geq \nu(F) - 1$. In this case we remove the clauses c' and c'' and add $c^* = (c' - x) \cup (c'' - \bar{x})$. We can obtain a matching of size $\nu(F) - 1$ in $B_{F'}$ as follows. If at most one of the c' and c'' is the end-point of some matching edge in M then removing that edge gives a matching of size $\nu(F) - 1$ for $B_{F'}$. So let us assume that some edges (a, c') and (b, c'') are in M . Clearly, either $a \neq x$ or $b \neq \bar{x}$. Assume $a \neq x$. Then $M \setminus \{(a, c'), (b, c'')\} \cup \{(a, c^*)\}$ is a matching of size $\nu(F) - 1$ in $B_{F'}$. Thus, we conclude that $\mu(F') \leq \mu(F)$.

Reduction Rule 3: The proof is the same as in the case of Reduction Rule 1.

Reduction Rule 4: The proof that $\mu(F') \leq \mu(F)$ in the case when $F[S]$ is satisfiable is the same as in the case of Reduction Rule 1 and in the case when $F[S]$ is not satisfiable is the same as in the case of Reduction Rule 2.

Branching Rule 1: Consider the case when we set $x = \text{TRUE}$. In this case, $\alpha' = \alpha - n(x)$. Also, since no reduction rules are applicable we have that F is 2-expanding. Hence, $\nu(F) = |V(F)|$. We will show that in (F', α') the matching size will remain at least $\nu(F) - n(x) + 1$ ($= |V(F)| - n(x) + 1 = |V(F')| - n(x) + 2$.) This will imply that $\mu(F') \leq \mu(F) - 1$. By Lemma 1 and the fact that $n(x) - 2 \geq 0$, it suffices to show that in $B' = B_{F'}$, every subset $S \subseteq V(F')$, $|N_{B'}(S)| \geq |S| - (n(x) - 2)$. The only clauses that have been removed by the simplification process after setting $x = \text{TRUE}$ are those where x appears positively and the singleton clauses (\bar{x}) . Hence, the only edges of $G[S \cup N_B[S]]$ that are missing in $N_{B'}(S)$ from $N_B(S)$ are

those corresponding to clauses that contain x as a pure literal and some variable in S . Thus, $|N_{B'}(S)| \geq |S| + 2 - n(x) = |S| - (n(x) - 2)$ (as F is 2-expanding).

The case when we set $x = \text{FALSE}$ is similar to the case when we set $x = \text{TRUE}$. Here, also we can show that $\mu(F') \leq \mu(F) - 1$. Thus, we get two instances, with each instance (F', α') having $\mu(F') \leq \mu(F) - 1$.

Branching Rule 2: The analysis here is the same as for Branching Rule 1 and again we get two instances with $\mu(F') \leq \mu(F) - 1$.

We therefore have a depth-bounded search tree of size of depth at most $\mu = \alpha - \nu(F) = k$, in which any branching splits an instance into two instances. Thus, the search tree has at most 2^k instances. As each reduction and branching rule takes polynomial time, every rule decreases the number of variables, the number of clauses, or the value of μ , and an instance to which none of the rules apply can be solved in time $O((2e)^{2\mu} \mu^{O(\log \mu)} (m+n)^{O(1)})$ (by Theorem 4), we have by induction that any instance can be solved in time

$$O(2 \cdot (2e)^{2(\mu-1)} (\mu-1)^{O(\log(\mu-1))} (m+n)^{O(1)}) = O((2e)^{2\mu} \mu^{O(\log \mu)} (m+n)^{O(1)}).$$

Thus the total running time of the algorithm is at most $O((2e)^{2k+O(\log^2 k)} (n+m)^{O(1)})$. Applying Theorem 8 instead of Theorem 4, we conclude that $(\nu(F) + k)$ -SAT can be solved in expected time $O(8^{k+O(\sqrt{k})} (n+m)^{O(1)})$. Summarizing, we have the following:

Theorem 9. *There are algorithms solving $(\nu(F) + k)$ -SAT in time $O((2e)^{2k+O(\log^2 k)} (n+m)^{O(1)})$ or expected time $O(8^{k+O(\sqrt{k})} (n+m)^{O(1)})$.*

7 Hardness of Kernelization

In this section, we show that $(\nu(F) + k)$ -SAT does not have a polynomial-size kernel, unless $\text{coNP} \subseteq \text{NP/poly}$. To do this, we use the concept of a *polynomial parameter transformation* [5, 7]: Let L and Q be parameterized problems. We say a polynomial time computable function $f : \Sigma^* \times \mathbb{N} \rightarrow \Sigma^* \times \mathbb{N}$ is a *polynomial parameter transformation* from L to Q if there exists a polynomial $p : \mathbb{N} \rightarrow \mathbb{N}$ such that for any $(x, k) \in \Sigma^* \times \mathbb{N}$, $(x, k) \in L$ if and only if $f(x, k) = (x', k') \in Q$, and $k' \leq p(k)$.

Lemma 16. [5, Theorem 3] *Let L and Q be parameterized problems, and suppose that L^c and Q^c are the derived classical problems². Suppose that L^c is NP-complete, and $Q^c \in \text{NP}$. Suppose that f is a polynomial parameter transformation from L to Q . Then, if Q has a polynomial-size kernel, then L has a polynomial-size kernel.*

The proof of the next theorem is similar to the proof of Lemma 12.

Theorem 10. *$(\nu(F) + k)$ -SAT has no polynomial-size kernel, unless $\text{coNP} \subseteq \text{NP/poly}$.*

Proof. By [13, Theorem 3], there is no polynomial-size kernel for the problem of deciding whether a hypergraph H has a hitting set of size $|E(H)| - k$, where k is the parameter unless $\text{coNP} \subseteq \text{NP/poly}$. We prove the theorem by a polynomial parameter reduction from this problem. Then the theorem follows from Lemma 16, as $(\nu(F) + k)$ -SAT is NP-complete.

Given a hypergraph H on n vertices, construct a CNF formula F as follows. Let the variables of F be the vertices of H . For each variable x , let the unit clause (x) be a clause in F . For every edge e in H , let c_e be the clause containing the literal \bar{x} for every $x \in e$. Observe that F is matched, and that H has a hitting set of size $|E(H)| - k$ if and only if $\text{sat}(F) \geq n + k$. \square

²The parameters of L and Q are no longer parameters in L^c and Q^c ; they are part of input.

8 Conclusion

We have shown that for any CNF formula F , it is fixed-parameter tractable to decide if F has a satisfiable subformula containing α clauses, where $\alpha - \nu(F)$ is the parameter. Our result implies fixed-parameter tractability for the problem of deciding satisfiability of F when F is variable-matched and $\delta(F) \leq k$, where k is the parameter. In addition, we show that the problem does not have a polynomial-size kernel unless $\text{coNP} \subseteq \text{NP}/\text{poly}$.

Clearly, parameterizations of MAXSAT above $m/2$ and $\nu(F)$ are “stronger” than the standard parameterization (i.e., when the parameter is the size of the solution). Whilst the two non-standard parameterizations have smaller parameter than the standard one, they are incomparable to each other as for some formulas F , $m/2 < \nu(F)$ (e.g., for variable-matched formulas with $m < 2n$) and for some formulas F , $m/2 > \nu(F)$ (e.g., when $m > 2n$). Recall that Mahajan and Raman [21] proved that MAXSAT parameterized above $m/2$ is fixed-parameter tractable. This result and our main result imply that MAXSAT parameterized above $\max\{m/2, \nu(F)\}$ is fixed-parameter tractable: if $m/2 > \nu(F)$ then apply the algorithm of [21], otherwise apply our algorithm.

If every clause of a formula with m clauses contains exactly two literals then it is well known that we can satisfy at least $3m/4$ clauses. From this, and by applying Reduction Rules 1 and 2, we can get a linear kernel for this version of the $(\nu(F) + k)$ -SAT problem. It would be nice to see whether a linear or a polynomial-size kernel exists for the $(\nu(F) + k)$ -SAT problem if every clause has exactly r literals.

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