Towards Optimal and Expressive Kernelization for d-Hitting Set^{*}

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Abstract

d-HITTING SET is the NP-hard problem of selecting at most k vertices of a hypergraph so that each hyperedge, all of which have cardinality at most d, contains at least one selected vertex. The applications of d-HIT-TING SET are, for example, fault diagnosis, automatic program verification, and the noise-minimizing assignment of frequencies to radio transmitters.

We show a linear-time algorithm that transforms an instance of d-HIT-TING SET into an equivalent instance comprising at most $O(k^d)$ hyperedges and vertices. In terms of parameterized complexity, this is a *problem kernel*. Our kernelization algorithm is based on speeding up the wellknown approach of finding and shrinking *sunflowers* in hypergraphs, which yields problem kernels with structural properties that we condense into the concept of *expressive kernelization*.

We conduct experiments to show that our kernelization algorithm can kernelize instances with more than 10^7 hyperedges in less than five minutes.

Finally, we show that the number of vertices in the problem kernel can be further reduced to $O(k^{d-1})$ with additional $O(k^{1.5d})$ processing time by nontrivially combining the sunflower technique with *d*-HITTING SET problem kernels due to Abu-Khzam and Moser.

1 Introduction

Many problems, like the examples given below, can be modeled as the NP-hard *d*-HITTING SET problem:

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d-Hitting Set

Input: A hypergraph H = (V, E) with hyperedges whose cardinality is bounded from above by a constant d, and a natural number k.

Question: Is there a hitting set $S \subseteq V$ with $|S| \leq k$ and $\forall e \in E \colon e \cap S \neq \emptyset$?

Problems that can be modeled as d-HITTING SET arise, among others, in the following fields.

Construction of Golomb rulers. A *Golomb ruler* of length n is a subset of marks $R \subseteq [n]$ such that no pair of marks in R has the same distance as another pair. The task of finding shortest Golomb rulers with a fixed number of marks or Golomb rulers of fixed length with a maximum number of marks arises, among others, in radio frequency allocation [13]. Sorge et al. [37] showed how to construct Golomb rulers using 4-HITTING SET. That is, d = 4.

Fault diagnosis. The task is to detect faulty components of a malfunctioning system. To this end, those sets of components are mapped to hyperedges of a hypergraph that are known to contain at least one broken component [9, 34]. By the principle of Occam's razor, a small hitting set is then a likely explanation of the malfunction. In this application, d is the maximum number of components that a wrong observation depends on.

Program verification. O'Callahan and Choi [32] used *d*-HITTING SET in order to automatically detect bugs in parallel Java programs while aiming for a small slowdown of the program monitored at execution time. In their experiments, $d \leq 10$ was sufficient to debug complex software suites. In most cases, even $d \leq 5$ sufficed. Remarkably, in this application, one is interested in the question whether a hypergraph allows for a hitting set of size at most k = d, that is, both k and d are small.

The described problems have in common that a large number of "conflicts" (the possibly $O(n^d)$ hyperedges in a *d*-HITTING SET instance) is caused by a small number of elements (the hitting set S), whose removal or repair could fix a broken system or establish a useful property.

A powerful tool to attack NP-hard problems like *d*-HITTING SET is problem kernelization [20, 26]—a form of provably efficient and effective data reduction. We show how to compute a problem kernel with $O(k^d)$ hyperedges for *d*-HITTING SET in linear time. We experimentally evaluate our kernelization algorithm on 4-HITTING SET instances arising in the construction of Golomb rulers with a maximum number of marks and see that instances with more than 10^7 hyperedges are kernelizable in less than five minutes.

Known results. HITTING SET is W[2]-complete with respect to the parameter k when the cardinality of the hyperedges is unbounded [18, Theorem 7.14]. Hence, unless FPT = W[2], it has no problem kernel. Dell and van Melkebeek [10] showed that the existence of a problem kernel with $O(k^{d-\varepsilon})$ hyperedges for any $\varepsilon > 0$ for *d*-HITTING SET implies a collapse of the polynomial-time hierarchy. Therefore, *d*-HITTING SET is assumed not to admit problem kernels with $O(k^{d-\varepsilon})$ hyperedges. For the same reason, *d*-HITTING SET presumably has no polynomial-size problem kernels if *d* is *not* constant.

Various problem kernels for d-HITTING SET have been developed [1, 8, 15, 18, 25, 28, 30, 31]. Niedermeier and Rossmanith [30] showed a problem kernel for 3-HITTING SET of size $O(k^3)$. They implicitly claimed that a polynomial-size problem kernel for d-HITTING SET is computable in linear time, without giving a proof for the running time. Nishimura et al. [31] claimed that a problem kernel with $O(k^{d-1})$ vertices is computable in $O(k(n+m)+k^d)$ time, which, however. does not always yield correct problem kernels [1]. Damaschke [8] focused on developing small problem kernels for *d*-HITTING SET and other problems with the focus on preserving all minimal solutions of size at most k (so-called full kernels). Fafianie and Kratsch [15] presented a so-called streaming kernelization for *d*-HITTING SET, which reads every hyperedge in the input hypergraph at most once and has logarithmic memory usage for fixed k. Abu-Khzam [1] showed a problem kernel with $O(k^{d-1})$ vertices for d-HITTING SET, thus proving the previously claimed result of Nishimura et al. [31] on the number of vertices in the problem kernel. Moser [28, Section 7.3] built upon the work of Abu-Khzam [1] to show a problem kernel for d-HITTING SET that also comprises $O(k^{d-1})$ vertices but, in contrast to the problem kernel of Abu-Khzam [1], yields a subgraph of the input hypergraph. The problem kernels of Abu-Khzam [1] and Moser [28] comprise $\Omega(k^{2d-2})$ hyperedges in the worst case.¹

Several exponential-time algorithms for HITTING SET exist and aim to decrease the exponential dependence of the running time on the number of input vertices [36], on the number of input hyperedges [16], and on the size of the sought hitting set [17]. Also exponential-time approximation stepped into the field of interest [6], since, in polynomial time, d-HITTING SET appears to be hard to approximate within a factor of better than d [24].

Our results. We show that a problem kernel for *d*-HITTING SET with $O(k^d)$ hyperedges and vertices is computable in linear time. Thereby, we prove the previously claimed result by Niedermeier and Rossmanith [30] and complement recent results in improving the efficiency of kernelization algorithms [5, 15, 21, 22, 33].

Our problem kernel has useful structural properties that ensure the interpretability of the problem kernel. We condense these properties into the concept of *expressive kernelization*. Moreover, in the sense that a problem kernel with $O(k^{d-\varepsilon})$ hyperedges for some $\varepsilon > 0$ would lead to a collapse of the polynomialtime hierarchy, the size of our problem kernel is optimal.

We implement our kernelization algorithm and evaluate its applicability to the problem of constructing Golomb rulers with a maximum number of marks

¹Although not directly analyzed in the works of Abu-Khzam [1] and Moser [28], this can be seen as follows: the kernel comprises vertices of a set W of "weakly related" hyperedges and an independent set I. In the worst case, $|W| = k^{d-1}$, $|I| = dk^{d-1}$, and each hyperedge in Whas d subsets of size d-1. Each such subset can constitute a hyperedge with each vertex in Iand the kernel has $\Omega(k^{2d-2})$ hyperedges.

and find instances with more than 10^7 hyperedges to be kernelizable in less than five minutes.

Finally, using ideas of Abu-Khzam [1] and Moser [28], we show that the number of vertices can be further reduced to $O(k^{d-1})$ with an additional amount of $O(k^{1.5d})$ time. By merging these techniques, we can compute in $O(n + m + k^{1.5d})$ time a problem kernel comprising $O(k^d)$ hyperedges and $O(k^{d-1})$ vertices.

Preliminaries. A hypergraph H = (V, E) consists of a set of vertices V and a set of (hyper)edges E, where each hyperedge in E is a subset of V. We use n := |V| and m := |E|. In a *d*-uniform hypergraph every edge has cardinality exactly d. A 2-uniform hypergraph is a graph. A hypergraph G = (V', E') is a subgraph of its supergraph H if $V' \subseteq V$ and $E' \subseteq E$. A set $S \subseteq V$ intersecting every set in E is a hitting set. A parameterized problem is a subset $L \subseteq \Sigma^* \times$ \mathbb{N} [12, 18, 29]. A problem kernel for a parameterized problem L is a polynomialtime algorithm that, given an instance (I, k), computes an instance (I', k') such that $|I'| + k' \leq f(k)$ and $(I', k') \in L \iff (I, k) \in L$. Herein, the function f is called the size of the problem kernel and depends only on k.

Paper outline. We start by giving a concept of expressive kernelization in Section 2.

Then, we present an expressive linear-time kernelization algorithm for *d*-HIT-TING SET in Section 3, which we evaluate experimentally on hypergraphs occurring in the computation of optimal Golomb rulers in Section 4.

Finally, we show how the number of vertices can be reduced to $O(k^d)$ in additional $O(k^{1.5k})$ time in Section 5. Since the resulting problem kernel is not expressive, we have not implemented it.

2 Expressive kernelization

The core component of our linear-time kernelization algorithm for d-HITTING SET is an algorithm to find and shrink *sunflowers* in linear time. Sunflowers are special constellations of hyperedges that Erdős and Rado [14] discovered to appear in any sufficiently large hypergraph and their use in kernelization algorithms for d-HITTING SET is a standard technique [18, 19, 25]. They are defined as follows and illustrated in Figure 1.

Definition 1. A sunflower in a hypergraph H = (V, E) is a set of petals $P \subseteq E$ such that each pair of sets in P intersects in exactly the same set $C \subseteq V$, which is called the *core* (possibly, $C = \emptyset$). The *size* of the sunflower is |P|.

The approach of finding and shrinking sunflowers yields problem kernels that contain more structural information than the formal definition of problem kernels requires. Specifically, sunflowers help computing problem kernels that have the following three properties, which we henceforth require to be guaranteed by *expressive* problem kernels for d-HITTING SET and that we will describe in more detail in the following.

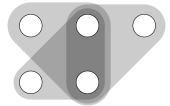


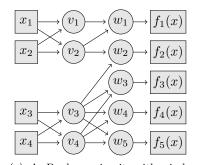
Figure 1: A sunflower with three petals and two core elements.

Definition 2. A kernelization algorithm for *d*-HITTING SET is *expressive* if, given an instance (H, k), it outputs an instance (H', k') such that

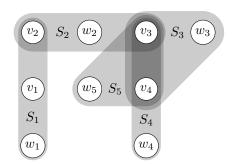
- i) H' is a subgraph of H,
- ii) any vertex set of size at most k is a minimal hitting set for H if and only if it is a minimal hitting set for H', and
- iii) it outputs a certificate for (H', k') being a yes-instance if and only if (H, k) is.

Interpretability of the problem kernel. The kernelization algorithm should output a subgraph of the input hypergraph. Kernelization algorithms for *d*-HITTING SET with this explicit goal have been developed by Moser [28] and Kratsch [25], since newly introduced hyperedges or vertices in the problem kernel might not be interpretable in the context of the original problem modeled as *d*-HITTING SET. Kratsch [25] exploited this property to show polynomial-size problem kernels for a large class of problems formalizable as *d*-HITTING SET. In some scenarios, as pointed out by Abu-Khzam and Fernau [2], it is even desirable that the kernelization algorithm outputs an *induced* subgraph of the input hypergraph. However, our problem kernel for *d*-HITTING SET will not satisfy this requirement.

Interpretability of solutions. Any vertex set of size at most k should be a minimal hitting set for the resulting problem kernel if and only if it is a minimal hitting set for the original instance. If the input instance and the problem kernel allow for exactly the same minimal hitting sets of size at most k, the problem kernel retains enough information for interpreting solutions and finding alternative solutions without having to consider the input hypergraph. This property has been exploited by Fomin et al. [19] as an important building block in a polynomial-size problem kernel for a problem that cannot easily be modeled as d-HITTING SET for constant d. As pointed out by Fomin et al. [19], this property is stronger than those guaranteed by the *full kernels* introduced by Damaschke [8]: full kernels contain all minimal hitting sets of size at most k for the input hypergraph, but not necessarily the information whether a hitting set is minimal.



(a) A Boolean circuit with circle nodes representing gates and square nodes representing input and output nodes.



(b) Sets containing at least one faulty gate, found by the analysis of the circuit.

Figure 2: Illustrations for Example 1.

Certifying data reduction. Similarly to how certifying algorithms provide a certificate for the correctness of their output [27], an expressive kernelization algorithm should provide a certificate for the correctness of the executed data reduction. Ideally, the proof that a certificate indeed certifies the correctness should be easily understandable, so that a human can easily verify the executed data reduction to be correct without having to trust on the correctness of algorithms and their implementations. A sunflower P with k + 1 petals in a d-HITTING SET instance fulfills this requirement: every hitting set S of size at most k contains an element of the core C of P, since otherwise S cannot contain an element of each of the k + 1 petals. Thus, any additional hyperedge in the hypergraph that contains C already contains an element of S; it is redundant and may be removed. The sunflower P is a certificate for this being correct.

Example 1. Sunflowers not only certify the correctness of data reduction, but also lead the way to alternative solutions. We illustrate this using an example of *d*-HITTING SET in a fault diagnosis context.

Figure 2(a) represents a Boolean circuit. It gets as input a 4-bit string $x = x_1 \dots x_4$ and outputs a 5-bit string $f(x) = f_1(x) \dots f_5(x)$. The nodes drawn as circles represent Boolean gates, which output some bit depending on their two input bits. They might, for example, represent the logical operators \wedge or \vee . Assume that all output bits of f(x) are observed to be the opposite of what would have been expected by the designer of the circuit. We want to identify broken gates. For each wrong output bit $f_i(x)$, we obtain a set S_i of gates for which we know that at least one is broken because $f_i(x)$ is wrong. That is, S_i contains precisely those gates that have a directed path to $f_i(x)$ in the graph shown in Figure 2(a). We obtain the sets S_1, \dots, S_5 illustrated in Figure 2(b).

The sets S_1 and S_4 are disjoint. Therefore, the wrong output is not explainable by only one broken gate. Hence, we assume that there are *two* broken gates and search for a hitting set of size k = 2 in the hypergraph with the vertices $v_1, \ldots, v_4, w_1, \ldots, w_5$ and hyperedges S_1, \ldots, S_5 . The set $\{S_3, S_4, S_5\}$ is a sunflower of size k + 1 = 3 with core $\{v_3, v_4\}$. Therefore, the functionality of gate v_3 and v_4 must be checked. If, in contrast to our expectations, both gates v_3 and v_4 turn out to be working correctly, the sunflower shows not only that at least three gates are broken, but also shows which gates have to be checked for malfunctions next: w_3, w_4 , and w_5 .

There are few expressive kernelization algorithms for *d*-HITTING SET in the literature. For example, the algorithm of Abu-Khzam [1] does not yield a subgraph of the input hypergraph. Thus, it does not satisfy Definition 2(i), which has been "fixed" by Moser [28]. However, both problem kernels may discard minimal solutions of size at most k and, thus, do not satisfy Definition 2(ii). Damaschke [8] designed a problem kernel that retains all minimal solutions of size at most k. However, it is not certifying and thus does not satisfy Definition 2(ii).

We are aware of only one expressive kernelization algorithm for d-HITTING SET: this is the problem kernel shown by Kratsch [25], which is also used by Fomin et al. [19]. This is precisely the algorithm we will improve to run in linear time.

3 A linear-time kernelization algorithm

This section shows a linear-time computable problem kernel for d-HITTING SET comprising $O(k^d)$ hyperedges. That is, we show that a hypergraph H can be transformed in linear time into a hypergraph G such that G has $O(k^d)$ hyperedges and allows for a hitting set of size k if and only H does. In Section 5, we show how to shrink the number of vertices to $O(k^{d-1})$.

Theorem 1. d-HITTING SET allows for an expressive problem kernel with $d! \cdot d^{d+1} \cdot (k+1)^d$ hyperedges and d times as many vertices that is computable in $O(d \cdot n + 2^d d \cdot m)$ time.

We prove Theorem 1 with the help of the sunflower lemma of Erdős and Rado [14], who showed that every sufficiently large hypergraph contains a sunflower with k + 2 petals: if we shrink all of these sunflowers, it follows that the resulting hypergraph will be small. Kernelization algorithms based on this strategy, like those of Flum and Grohe [18] and Kratsch [25] usually proceed along the lines of repeatedly

- finding a sunflower of size k + 2 in the input hypergraph and
- deleting redundant petals until no more sunflowers of size k + 2 exist.

This approach has the drawback of finding only one sunflower at a time and restarting the process from the beginning.

In contrast, to prove Theorem 1, we construct a subgraph G of a given hypergraph H not by hyperedge deletion, but by a bottom-up approach that allows us to "grow" many sunflowers in G simultaneously, stopping "growing sunflowers" when they become too large. Algorithm 1 repeatedly (after some initialization work in lines 1–6) in line 9 copies a hyperedge e from H to the initially empty G unless we find in line 8 that e contains the core C of a sunflower of size k + 1 in G. We maintain the number of petals found for a core C in petals[C]. If we find that a hyperedge e can be added to a sunflower with core C in line 11, then we increment petals[C] in line 12 and mark the vertices in $e \setminus C$ as "used" for the core C in line 13. This information is maintained by setting "used[C][v] \leftarrow true." In this way, vertices in $e \setminus C$ are not considered again for finding petals for the core C in line 11, therefore ensuring that petals later found for the core C intersect e only in C.

Algorithm 1: Linear-Time kernelization for <i>d</i> -HITTING SET	
Input : A hypergraph $H = (V, E)$ and a natural number k.	
Output : A hypergraph $G = (V', E')$ with $ E' \in O(k^d)$.	
1 $E' \leftarrow \emptyset;$	
2 foreach $e \in E$ do	// Initialization for each hyperedge
3 foreach $C \subseteq e$ do	// Initialization for all possible cores of sunflowers
4 $petals[C] \leftarrow 0;$	// No petals found for sunflower with core C yet
5 foreach $v \in e$ do	
$6 \qquad \qquad \mathbf{used}[C][v] \leftarrow \mathbf{fa}$	lse; // No vertex v is in a petal of a
	// sunflower with core C yet
7 foreach $e \in E$ do	
$\mathbf{s} \mathbf{if} \ \forall C \subseteq e \colon \text{petals}[C] \leq k \ \mathbf{then}$	
9 $E' \leftarrow E' \cup \{e\};$	
10 foreach $C \subseteq e$ do	// Consider all possible cores for the petal e
11 if $\forall v \in e \setminus C$: used $[C][v]$ = false then	
12petals[C] \leftarrow petals[C] + 1;13foreach $v \in e \setminus C$ do used[C][v] \leftarrow true;	
13 foreach $v \in e \setminus C$ do used $[C][v] \leftarrow$ true;	
14 $V' := \bigcup_{e \in E'} e;$	
15 return $(V', E');$	

By storing in petals[C] a list of found petals, the algorithm can output the discovered sunflowers without any increase in running time. Thus, it gives certificates for the correctness of the executed data reduction.

It is important to note that, as illustrated in Figure 3, the value in petals [C] is not necessarily the size of the largest possible sunflower with core C, but depends on the order in that Algorithm 1 processes the hyperedges of the input hypergraph. Computing the size of a largest sunflower with core C is, for $C = \emptyset$, the problem of computing a maximum matching in a hypergraph, which is NP-hard [23].

Towards proving Theorem 1, we now proceed as follows. Section 3.1 shows that Algorithm 1 is correct and expressive. Section 3.2 shows that the hypergraph output by Algorithm 1 contains $O(k^d)$ hyperedges. Finally, Section 3.3 shows that Algorithm 1 runs in linear time.

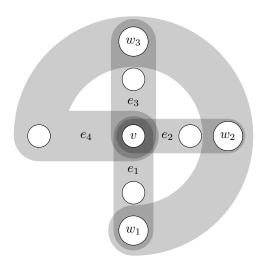


Figure 3: The result of applying Algorithm 1 depends on the order in that it processes the hyperedges of the input hypergraph H. If applied for k = 2, then Algorithm 1 will not add the hyperedge e_4 to the output hypergraph Gif it before added e_1 , e_2 , and e_3 , since it discovers e_4 to contain the core $\{v\}$ of the sunflower $\{e_1, e_2, e_3\}$ with k + 1 = 3 petals. However, if it first adds e_4 to G, then it marks the vertices w_1 , w_2 , and w_3 as used for the sunflower with core $\{v\}$. Thus, none of the hyperedges e_1 , e_2 , and e_3 is recognized as a petal for a sunflower with core $\{v\}$ and all shown hyperedges are added to the output hypergraph.

3.1 Correctness

On our way to proving that *d*-HITTING SET has an expressive linear-time computable problem kernel with $O(k^d)$ hyperedges and thus proving Theorem 1, we now prove the correctness and expressiveness of Algorithm 1. This, together with a proof for the size of the output hypergraph and a proof of the running time of Algorithm 1, will provide a proof of Theorem 1.

Proposition 1. Let G be the hypergraph returned by Algorithm 1 when given a hypergraph H and an integer k. Then,

- i) any hitting set S of size at most k for G is a hitting set for H and
- ii) any minimal hitting set S of size at most k for H is a hitting set for G.

Moreover, G is a subgraph of H and any subset S of at most k vertices of H is a minimal hitting set for H if and only if it is a minimal hitting set for G.

Proof. By construction of G from H in Algorithm 1, it is clear that G is a subgraph of H. We now show that it is sufficient to prove (i) and (ii) to also conclude the last statement of Proposition 1: let S be a minimal hitting set of size at most k for H. By (ii), it is a hitting set for G. Assume, towards a contradiction, that S is not a *minimal* hitting set for G. Then, there is a hitting set $S' \subsetneq S$ for G. However, by (i), $S' \subsetneq S$ is also a hitting set for H. This contradicts S being a minimal hitting set for G. By (ii), S is a hitting set for H. Assume, towards a contradiction, that S is not a *minimal* hitting set for H. Then, there is a minimal hitting set S of size at most k for G. By (i), S is a hitting set for H. Assume, towards a contradiction, that S is not a *minimal* hitting set for H. Then, there is a minimal hitting set $S' \subsetneq S$ for G. This contradicts S being a minimal hitting set for G. By (ii), $S' \subsetneq S$ is also a hitting set for H. Then, there is a minimal hitting set $S' \subseteq S$ for H. However, by (ii), $S' \subseteq S$ is also a hitting set for G. This contradicts S being a minimal hitting set for G. It remains to prove (i) and (ii).

(i) Let S be a hitting set of size at most k for G. Obviously, all hyperedges that H and G have in common are hit in H by S. We show that every hyperedge e in H that is not in G is also hit. If e is in H but not in G, then adding e to G in line 9 of Algorithm 1 has been skipped because the condition in line 8 is false. That is, petals $[C] \ge k + 1$ for some $C \subseteq e$. Consequently, for this particular C, a sunflower P with k + 1 petals and core C exists in G, since we only increment petals[C] in line 12 if we find a suitable additional petal for the sunflower with core C in line 11. Note that $C \neq \emptyset$ because, otherwise, k + 1 pairwise disjoint hyperedges would exist in G, contradicting our assumption that S is a hitting set of size k for G. Since $|S| \le k$, we have $S \cap C \neq \emptyset$. Therefore, since $C \subseteq e$, the hyperedge e is hit by S also in H.

(ii) Let S be a minimal hitting set of size at most k for H = (V, E). The set $S' := S \cap V'$ is a hitting set for G = (V', E') with $S' \subseteq S$: the set S contains an element of every hyperedge in E and, since $E' \subseteq E$ and $V' = \bigcup_{e \in E'} e$, the set S' contains an element of every hyperedge in E'. By (i), S' is also a hitting set for H. Since $S' \subseteq S$ and we required S to be a minimal hitting set of size at most k for H, we have that S' = S and, thus, that S is a hitting set for G. \Box

3.2 Problem kernel size

Having shown that Algorithm 1 is correct, we now show that the hypergraph output by Algorithm 1 contains $O(k^d)$ hyperedges. To prove Theorem 1, it then remains to prove that Algorithm 1 runs in linear time.

In order to show an upper bound on the size of the hypergraph output by Algorithm 1, we exploit an upper bound on the size of the sunflowers in the output hypergraph:

Lemma 1. Let G be the hypergraph output by Algorithm 1 applied to a hypergraph H and a natural number k. Every sunflower P in G with core $C \notin P$ has size at most d(k + 1).

Proof. Let P be a sunflower in G with core $C \notin P$. Then, $|P| \leq d(k+1)$ follows from the following two observations:

(i) Every petal $e \in P$ present in G is copied from H in line 9 of Algorithm 1. Consequently, every petal $e \in P$ contains a vertex v satisfying used[C][v] = true: if this condition is violated in line 11, then line 13 applies "used $[C][v] \leftarrow$ true" to all vertices $v \in e \setminus C$.

(ii) Whenever petals[C] is incremented by one in line 12, then, in line 13, "used $[C][v] \leftarrow \text{true}$ " is applied to the at most d vertices $v \in e$. Thus, since petals[C] never exceeds k+1, at most d(k+1) vertices v satisfy used[C][v] = true. Moreover, since, by line 13, no $v \in C$ satisfies used[C][v] = true and the petals in P pairwise intersect only in C, it follows that at most d(k+1) petals in Pcontain vertices satisfying used[C][v] = true. \Box

Having shown an upper bound on the size of the sunflowers in the hypergraph output by Algorithm 1, we now show that the output hypergraph contains $O(k^d)$ hyperedges. To this end, in a way similar to Flum and Grohe [18, Lemma 9.7], we show the following refined version of Erdős and Rado [14]'s sunflower lemma. Herein, recall that a hypergraph is ℓ -uniform if and only if every hyperedge has cardinality exactly ℓ .

Lemma 2. Let H be an ℓ -uniform hypergraph and $b, c \in \mathbb{N}$ with $b \leq \ell$ such that every pair of hyperedges in H intersects in at most $\ell - b$ vertices.

If H contains more than $\ell! c^{\ell+1-b}$ hyperedges, then H contains a sunflower with more than c petals.

For b = 1, we obtain the sunflower lemma stated by Flum and Grohe [18]. For b = 2, we will exploit it in Section 5 to reduce the number of vertices in the output hypergraph.

Proof. We prove the lemma by induction on ℓ . As base case, consider $\ell = b$. For $\ell = b$, all hyperedges in H are pairwise disjoint. Hence, if H has more than $\ell! c^{\ell+1-b}$ hyperedges, then these form a sunflower with empty core and more than $\ell! c^{\ell+1-b} = \ell! c \geq c$ petals. That is, the lemma holds for $\ell = b$.

Now, assume that the lemma holds for some $\ell \geq b$. It remains to prove that it holds for $\ell + 1$. Let M be a maximal set of pairwise disjoint hyperedges of the $(\ell + 1)$ -uniform hypergraph H := (V, E). If |M| > c, then the lemma holds because M is a sunflower with empty core. Otherwise, for $N := \bigcup_{e \in M} e$, it holds that $|N| \le (\ell + 1)c$ and some vertex $w \in N$ is contained in a set E_w of more than

$$\frac{|E|}{|N|} \ge \frac{(\ell+1)!c^{\ell+2-b}}{(\ell+1)c} = \ell!c^{\ell+1-b} \text{ hyperedges.}$$

The hypergraph H_w that contains for each hyperedge $e \in E_w$ the hyperedge $e \setminus \{w\}$ is an ℓ -uniform hypergraph and, by induction hypothesis, contains a sunflower P with more than c petals. Adding w to each of the petals of Pyields a sunflower P' with more than c petals in H. \Box

By combining Lemma 1 with Lemma 2, we can easily show that the hypergraph output by Algorithm 1 contains $O(k^d)$ hyperedges. Since we have already shown in Proposition 1 that the algorithm is correct, it thereafter only remains to show that Algorithm 1 runs in linear time in order to complete the proof of Theorem 1.

Proposition 2. The hypergraph G returned by Algorithm 1 on input H and k contains at most $d! \cdot d^{d+1} \cdot (k+1)^d$ hyperedges and d times as many vertices.

Proof. Obviously, G has at most d times as many vertices as hyperedges, since the vertex set of G is constructed as the union of its hyperedges in line 14 of Algorithm 1.

To bound the number of hyperedges, consider, for $1 \leq \ell \leq d$, the ℓ -uniform hypergraph $G_{\ell} = (V_{\ell}, E_{\ell})$ comprising only the hyperedges of size ℓ of G. If Ghad more than $d! \cdot d^{d+1} \cdot (k+1)^d$ hyperedges, then, for some $\ell \leq d$, G_{ℓ} would have more than $d! \cdot d^d \cdot (k+1)^d$ hyperedges. Lemma 2 with b = 1 and c = d(k+1)states that, if G_{ℓ} had more than $\ell! \cdot d^{\ell} \cdot (k+1)^{\ell}$ hyperedges, then G_{ℓ} would contain a sunflower P with core C and more than d(k+1) petals. Obviously, $C \notin P$, since all petals have cardinality ℓ . Moreover, this sunflower would also exist in the supergraph G of G_{ℓ} , contradicting Lemma 1.

3.3 Running time

Since Proposition 1 has shown that Algorithm 1 is correct and Proposition 2 has shown that the output hypergraph contains $O(k^d)$ hyperedges, to prove Theorem 1, it remains to show that Algorithm 1 runs in linear time. In order to implement the algorithm efficiently, we need data structures that allow us to quickly look up the values petals [C] and used [C] for some vertex set C of size at most d.

The usual approach to realize table look-ups for subsets of some universe of size γ in $O(\gamma)$ time is representing the subsets as bitstrings of length γ and looking up these in a trie [3, Section 5.3]. However, here, our universe is the set of vertices of the input hypergraph and, thus, has size n. Hence, this method would yield table look-ups in $\Theta(n)$ time, which is too slow to prove that Algorithm 1 runs in linear time. For this reason, we will not represent vertex subsets of size at most d as bitstrings, but uniquely represent them as sorted sequences of at most d integers. Then, we will exploit the following lemma.

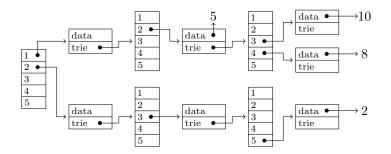


Figure 4: A trie that associates integer values with sequences of integers in $\{1, \ldots, 5\}$. That is, each node of the trie is an array of size five. With (1, 2) the trie associates 5, with (1, 2, 3) it associates 10, with (1, 2, 4) it associates 8, and, finally, with (2, 3, 5) it associates 2.

Lemma 3. Let L be a list of sequences of length at most d of integers in [n].

In $O(d \cdot n + d \cdot |L|)$ time, we can compute an associative array A[] such that, for each sequence s in L, accessing the value A[s] and storing a value to A[s]works in O(d) time.

Proof. We use a trie to associate values with sequences in L. However, the trie will be too large to initialize it fully in linear time. We have to show that we can create the trie so that, for a look-up of a value for any sequence s in L, no uninitialized memory cells are read.

We define a *trie* as a size-*n* array, of which each cell contains a pointer to a structure consisting of two more pointers: one of them points to data, the other one to another trie. This is illustrated in Figure 4. A look-up of the value associated with a sequence $s = (s_1, \ldots, s_d)$ in a trie T_1 then works in O(d) time as follows: for $i \in [d-1]$, we get the trie T_{i+1} pointed to by $T_i[s_i]$. Then, from $T_d[s_d]$, we get a pointer to the data associated with s.

In the creation of the trie to associate values with the sequences in L, we face a problem: we do not have enough time to initialize all cells of all arrays that implement the inner nodes of the trie: this would take $\Theta(n)$ time per node and, as seen in Figure 4, the number of nodes required in the trie can be more than |L|. This is a problem since, when creating the trie, we do not know whether an array cell already contains a pointer to a trie node of the next level or whether we have to create such a pointer with a corresponding new node. We have to make sure that we only follow pointers of initialized cells and that we do not overwrite previously correctly set up pointers since, otherwise, information in subtries will be lost. We achieve this as follows:

The input list L contains sequences of length at most d of integers in [n]. Hence, we can sort L lexicographically in $O(d \cdot (n + |L|)) = O(d \cdot n + d \cdot |L|)$ time using radix sort [7, Section 8.3]. We construct the trie by iterating over L once. For each sequence p in L, we find in O(d) time the first position i in which the sequence p differs from its predecessor sequence in the lexicographically sorted list L. This tells us that we already created all nodes on the path from the trie's root node to the leaf corresponding to s up to a depth of i. Pointers up to this depth i are valid and may not be overwritten. Nodes and pointers beyond this depth have to be newly created.

Using Lemma 3, we can finally prove that Algorithm 1 runs in linear time. Note that, together with Propositions 1 and 2, Proposition 3 completes the proof of Theorem 1.

Proposition 3. Algorithm 1 can be implemented to run in $O(d \cdot n + 2^d d \cdot m)$ time.

Proof. We first describe how Lemma 3 helps us efficiently implementing the associative arrays petals[] and used[] required by Algorithm 1. To this end, we assume that every vertex is represented as an integer in [n] and that every hyperedge is represented as a sequence sorted by increasing vertex numbers, which we call *sorted hyperedge*. We can initially sort each hyperedge of H in $O(m \cdot d \log d)$ total time. Note that, on hyperedges represented as sorted sequences, the set subtraction operation needed in line 11 can be executed in O(d) time such that the resulting set is again sorted [3, Section 4.4]. Moreover, we can generate all subsets of a sorted set such that the resulting subsets are sorted. Hence, we may assume to always deal with sorted hyperedges as a unique representation of hyperedges.

We now apply Lemma 3. Observe that Algorithm 1 looks up petals[C] and used[C] only for sets $C \subseteq e$ for some hyperedge e. Thus, from the set of sorted hyperedges, in $O(2^d d \cdot m)$ time, we compute a length- $(2^d \cdot m)$ list L of all possible sets $C \subseteq e$ for all hyperedges e and use this list in Lemma 3 to create the associative arrays petals[] and used[] in $O(d \cdot n + d \cdot |L|) = O(d \cdot n + 2^d d \cdot m)$ time.

Now, we can implement lines 1–6 of Algorithm 1 to run in $O(d \cdot n + 2^d d \cdot m)$ time, observing that the loop in line 5 can be implemented to run in O(d)-time, since only one look-up to used[C] is needed to obtain a pointer to an array in which, then, O(d) values are set.

The for-loop in line 7 iterates m times. Its body works in $O(2^d d)$ time: obviously, this time bound holds for lines 8 and 9; it remains to show that the body of the for-loop in line 10 works in O(d) time. This is easy to see if one considers that, in lines 11 and 13, one only has to do one look-up to used[C] to find a pointer to an array that holds the values for the at most d vertices of a hyperedge. Also line 14 works in linear time by first initializing all entries of an array vertices[] of size n to "false" and then, for each output hyperedge eand each vertex $v \in e$, setting "vertices $[v] \leftarrow$ true" in O(d) time. Afterward, we can build the vertex set V' of the output hypergraph G using the vertices v for which vertices[v] = true. This takes $O(n + d \cdot m)$ time. \Box

4 Experimental evaluation

This section experimentally evaluates the linear-time kernelization algorithm from Section 3. We demonstrate to which size our algorithm can process instances within five minutes.

Implementation details. Our implementation of Algorithm 1 comprises about 700 lines of C++ and is freely available.² The experiments were run on a computer with a 3.6 GHz Intel Xeon processor and 64 GB RAM under Linux 3.2.0, where the C++ source code has been compiled using the GNU C++ compiler in version 4.7.2 and using the highest optimization level (-O3).

Given a hypergraph H = (V, E), our implementation of Algorithm 1 checks for each hyperedge $e \in E$ with $\ell := |e|$ independently whether it is a *large* hyperedge $(2^{\ell} > m)$ or a small hyperedge $(2^{\ell} \le m)$. For a small hyperedge e, Algorithm 1 chooses to consider all subsets $C \in e$ as possible cores of sunflowers in line 8. For a large hyperedge e, all subsets $e \cap e'$ for any $e' \in E$ are considered as possible cores instead. Hence, our implementation chooses the variant which promises the lower running time for each hyperedge independently.

Additionally to discarding hyperedges that contain some core of a sunflower of size k + 1, our implementation of Algorithm 1 also makes sure that the output hypergraph contains no pair of hyperedges such that one is a superset of the other. To this end, the implementation initially sorts all hyperedges by increasing cardinality in O(d + m) time using counting sort [7, Section 8.2]. Moreover, after adding a hyperedge e to the output hypergraph, the implementation sets petals[e] to k + 1: in this way, the algorithm will not add hyperedges to the output hypergraph that are supersets of already added hyperedges.

As data structures to hold the values used[C] and petals[C] used by Algorithm 1 to associate values with sets C of size at most d, we implemented the following variants.

- By malloc trie, we refer to the associative array created in Lemma 3. It is implemented as a trie whose nodes are allocated as uninitialized arrays in constant time using the C-routine malloc. It guarantees O(d) look-up time and, as described in the proof of Proposition 3, $O(d \cdot n + 2^d d \cdot m)$ creation time. However, $\Omega(n \cdot m)$ random access memory cells may need to be reserved by the program in the worst case, although at most $O(d \cdot n + 2^d d \cdot m)$ memory cells are actually accessed.
- By calloc trie, we refer to a trie whose nodes are allocated as arrays preinitialized by zero using the C-routine calloc. This makes the intricate initialization by Lemma 3 unnecessary. However, the running time of acquiring a zero-initialized array and its actual memory usage may vary depending on the implementation of the routine by the used C library. For naïve implementations of calloc, creation time and memory usage of the calloc tree could be $\Omega(n \cdot m)$ in the worst case. However, we can still guarantee O(d) look-up time.
- By hash table, we refer to an associative array implemented using the data structure unordered_map provided in the C++11 Standard Template Library. At most $O(2^d \cdot m)$ values are stored in the hash table. According to the C++11 reference, storage and look-up work in $O(2^d \cdot m)$ time in the

²http://fpt.akt.tu-berlin.de/hslinkern/

worst case, but in O(d) time in the average case, where O(d) time accounts for computing the hash value of a hyperedge of cardinality d.

By balanced tree, we refer to an associative array implemented using the map data structure provided by the C++11 Standard Template Library. According to the C++11 reference, it is usually implemented as a balanced binary tree. Since, in the worst case, $O(2^d \cdot m)$ values are stored in the tree, the C++11 reference guarantees $O(d + \log m)$ time for storage and look-up. Its memory requirements are $O(2^d m)$.

Data. We execute our experiments on hypergraphs generated from the GOLOMB SUBRULER problem: one gets as input a set $R \subseteq \mathbb{N}$ and wants to remove at most k numbers ("marks") from R such that the result is a Golomb ruler, that is, no pair of remaining marks has the same distance as another pair. The applications of Golomb rulers lie, among others, in radio frequency allocation [13]. Optimum solutions for GOLOMB SUBRULER are only known for R = [n] with $n \leq 553$ at the current time.³

From a GOLOMB SUBRULER instance, we obtain a conflict hypergraph as follows: the vertex set is R, and for each $a, b, c, d \in R$, create a hyperedge $\{a, b, c, d\}$ if |a - b| = |c - d|. Asking for a hitting set of size k in this conflict hypergraph is equivalent to GOLOMB SUBRULER [37]. As shown by Sorge et al. [37], the class of conflict hypergraphs for R = [n] has n vertices and $\Theta(n^3)$ hyperedges, their cardinality being three or four. Our data set consists of the conflict hypergraphs for GOLOMB SUBRULER instances R = [n] with $100 \le n \le 600$, which yields conflict hypergraphs with 10^5 to $2 \cdot 10^7$ hyperedges. Since, in this way, we obtain a whole family of growing hypergraphs, this data set is well-suited to show the running time and memory scalability of Algorithm 1.

Experimental setup. Algorithm 1 requires as input not only a hypergraph H but also an upper bound k on the size of a sought hitting set. We choose as k an upper bound on the size of a *minimum* hitting set, so that the kernelization algorithm will not output small trivial no-instances and so that the computed problem kernel will retain all minimum hitting sets.

To obtain this upper bound for the 4-HITTING SET instances that we obtain from GOLOMB SUBRULER, we exploit that, for all $n \leq 4.2 \cdot 10^9$, Dimitromanolakis [11, Theorem 6.1] verified that there is a Golomb ruler $R \subseteq [n]$ with strictly more than \sqrt{n} marks. Hence, in our experiments with $n \leq 600$, a conflict hypergraph of a GOLOMB SUBRULER instance R = [n] has a hitting set of size at most $k := \lfloor n - \sqrt{n} \rfloor$. We use this k to compute problem kernels for 4-HITTING SET. Figure 5 shows this upper bound together with a lower bound.

Experimental results. In all plots to be shown, each point has been obtained from a single run of our algorithm; the running times and memory usage are not averaged in any way.

³http://blogs.distributed.net/2014/02/

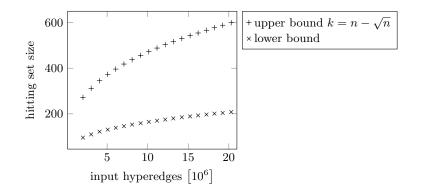


Figure 5: Upper and lower bounds for the minimum hitting set sizes for the data set obtained from the GOLOMB SUBRULER problem. The number of vertices n in the input instances is omitted, as it almost coincides with the upper bound $k = n - \sqrt{n}$ of the hitting set size. The lower bound was obtained from a maximal set of pairwise disjoint hyperedges.

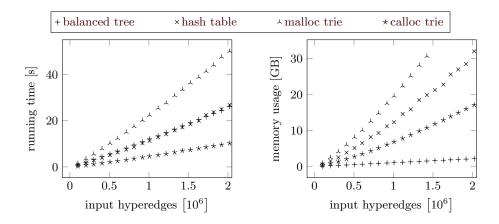


Figure 6: Performance of Algorithm 1 on conflict hypergraphs of the GOLOMB SUBRULER problem with at most $2 \cdot 10^6$ hyperedges.

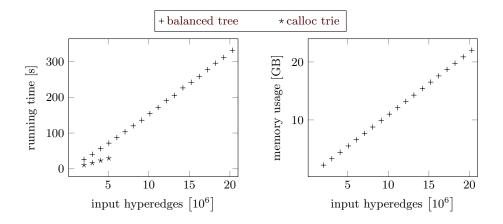


Figure 7: Performance of Algorithm 1 on conflict hypergraphs of the GOLOMB SUBRULER problem with at most $20 \cdot 10^6$ hyperedges.

Figure 6 shows the performance of our kernelization algorithm on conflict hypergraphs of the GOLOMB SUBRULER problem of size up to $2 \cdot 10^6$ hyperedges. On larger instances, the implementations based on malloc tries, calloc tries, and hash tables hit the 32 GB memory limit of the valgrind memory measuring tool. One can observe that the implementation using the malloc trie is the slowest. This is due to the complicated initialization procedure required by Lemma 3. The fastest implementation is the variant using the calloc trie, which is the same as the malloc trie implementation except that we skip the intricate initialization of the trie using Lemma 3. Unsurprisingly, the memory usage of the balanced tree implementation is the lowest, as it grows linearly with the number of stored elements. Surprisingly, the hash table implementation of the GNU C++ compiler consumes even more memory than our calloc trie.

Since the malloc trie, calloc trie, and hash table reach the 32 GB memory limit of the valgrind memory measurement tool between $2 \cdot 10^6$ and $5 \cdot 10^6$ input hyperedges, we made ongoing experiments only with the balanced tree implementation. Thus, unfortunately, we were unable to see how the running time of our fastest implementation—using calloc tries—scales to larger instances. Figure 7 shows that the implementation using the balanced tree solves 4-HITTING SET instances on conflict hypergraphs of GOLOMB SUBRULER with $20 \cdot 10^6$ hyperedges in less than five minutes and does not even hit the 32GB memory limit of the valgrind memory measurement tool.

Effect of data reduction. As shown in Figure 8, the kernelization algorithm removed between $20 \cdot 10^3$ and $60 \cdot 10^3$ hyperedges from the input instances. Thus, although we observed the algorithm to handle large input instances well, the observed data reduction effect is rather limited. This is partly due to the lack of a better upper bound k for the size of the sought hitting set: we observed that the input 4-HITTING SET instances obtained from GOLOMB SUBRULER have

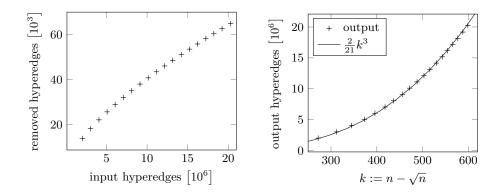


Figure 8: Size of the resulting problem kernel when Algorithm 1 is applied to conflict hypergraph of the GOLOMB SUBRULER problem.

roughly $1/12 \cdot n^3$ hyperedges. This, for any $n \ge 0$, is already below the upper bound of $4! \cdot 4^5 \cdot (k+1)^4$ with $k = \lfloor n - \sqrt{n} \rfloor$ given by Theorem 1 on the problem kernel size for 4-HITTING SET.

The limited effect of the data reduction on 4-HITTING SET instances obtained from GOLOMB SUBRULER is also due to the fact that they are nearly 4-uniform: this prevents hyperedges from getting deleted for being supersets of smaller hyperedges. The presence of smaller hyperedges can significantly influence the output instance size. As an extreme example, consider a hypergraph containing $\binom{n}{2}$ hyperedges of cardinality two. Then, the output problem kernel will contain $O(k^2)$ output hyperedges, regardless of how many more input hyperedges of cardinality 100 there are. Such phenomena are not captured in the theoretical upper bound on the problem kernel size given by Theorem 1, which is based on the analysis of uniform hypergraphs.

As shown in Figure 8, when measuring the size of the problem kernels in k, we observe that the resulting problem kernels contain about $2/21 \cdot k^3$ hyperedges. Thus, our empirically measured problem kernel size is lower than the upper bound of $3k^3 + 3k^2$ hyperedges that Sorge et al. [37] have proven using data reduction rules specifically designed for GOLOMB SUBRULER. Moreover, our problem kernel is computable in linear time, while the problem kernel of Sorge et al. [37] takes O(k(n + m)) time. Both problem kernels require the conflict hypergraph as input.

Summary. The calloc trie implementation of Algorithm 1 is superior when enough memory is available, since it is the fastest variant if the C++ environment at hand implements the allocation of zero-initialized memory using calloc efficiently. In all other cases, the balanced tree implementation of Algorithm 1 yields a good compromise between scalability with respect to running time and memory usage.

One can observe that the data reduction effect on nearly uniform hypergraphs,

like those from GOLOMB SUBRULER, is rather limited. On the other hand, problem kernels for d-HITTING SET can be small even for high values of d if the input hypergraph is less uniform.

5 Reducing the number of vertices in $O(k^{1.5d})$ additional time

This section combines the linear-time computable problem kernel from Section 3 with techniques of Abu-Khzam [1] and Moser [28, Section 7.3]. This will yield a problem kernel for *d*-HITTING SET with $O(k^d)$ hyperedges and $O(k^{d-1})$ vertices in $O(n + m + k^{1.5d})$ time. Towards this problem kernel, Section 5.1 first briefly sketches the running-time bottleneck of the kernelization idea of Abu-Khzam [1], which is also a bottleneck in the algorithm of Moser [28]. Then, Section 5.2 describes our improvements.

5.1 The approaches of Abu-Khzam and Moser

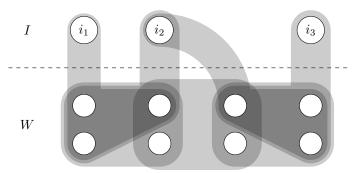
Abu-Khzam [1] has shown a problem kernel for *d*-HITTING SET that comprises $O(k^{d-1})$ vertices. Moser [28, Section 7.3] built upon the work of Abu-Khzam [1] to show a problem kernel for *d*-HITTING SET that also comprises $O(k^{d-1})$ vertices but that, in contrast to the kernelization algorithm of Abu-Khzam [1], yields a subgraph of the input hypergraph.

The approach of Abu-Khzam [1] and Moser [28] is as follows. Given a hypergraph H and a natural number k, Abu-Khzam [1] first computes a maximal weakly related set W:

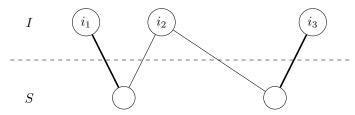
Definition 3 (Abu-Khzam [1]). A set W of hyperedges is *weakly related* if every pair of hyperedges in W intersects in at most d-2 vertices.

Whether a given hyperedge e can be added to a weakly related set W, Abu-Khzam [1] checks in O(d|W|) time. After adding a hyperedge e to W, he applies data reduction to W in $O(2^d|W|\log|W|)$ time that ensures $|W| \leq k^{d-1}$. Hence, since |W| never exceeds k^{d-1} , Abu-Khzam [1] can compute the maximal weakly related set W in $O(2^d \cdot k^{d-1} \cdot (d-1) \log k \cdot m)$ time.

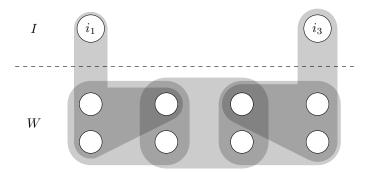
Since $|W| \leq k^{d-1}$, it remains to bound the size of the set I of vertices not contained in hyperedges of W. This is achieved by the following steps, which are illustrated in Figure 9. The set I is an *independent set*, that is, I contains no pair of vertices occurring in the same hyperedge [1]. A bipartite graph $B = (I \uplus S, E')$ is constructed from the input hypergraph H = (V, E), where $S := \{e \subseteq V \mid \exists v \in I : \exists w \in W : e \subseteq w, \{v\} \cup e \in E\}$ and $E' := \{\{v, e\} \mid v \in I, e \in S, \{v\} \cup e \in E\}$. Whereas Abu-Khzam [1] shrinks the size of I using so-called *crown reductions*, Moser [28, Lemma 7.16] shows that it is sufficient to compute a maximum matching in B and to remove unmatched vertices in Itogether with the hyperedges containing them from the input hypergraph. The bound of the number of vertices in the problem kernel is thus $O(k^{d-1})$, since $|W| \leq k^{d-1}$, and, therefore, $|I| \leq |S| \leq d|W| \leq dk^{d-1}$.



(a) Input hypergraph. The hyperedges fully below the dashed line are a maximal weakly related set of hyperedges. The vertices above the dashed line are the independent set I.



(b) The resulting bipartite graph ${\cal B}$ with the thick edges being a maximum matching.



(c) The resulting hypergraph with the unmatched vertex i_2 and its incident hyperedges removed.

Figure 9: Illustration of the kernelization of Moser [28, Lemma 7.16] using 4-HITTING SET.

5.2 Our improvements

We now discuss our running time improvements over the kernelization algorithms of Abu-Khzam [1] and Moser [28].

Given a hypergraph H and a natural number k, we first compute our problem kernel in O(n + m) time, leaving $O(k^d)$ hyperedges in H. Afterward, we aim for applying the ideas of Abu-Khzam [1] and Moser [28] to reduce the number of vertices to $O(k^{d-1})$. However, as discussed in Section 5.1, the computation of a maximal weakly related set on our reduced instance already takes $O(2^d \cdot k^{d-1} \cdot (d-1) \log k \cdot m) = O(k^{2d-1} \log k)$ time. We improve the running time of this step in order to show the following theorem.

Theorem 2. d-HITTING SET has a problem kernel with $d! \cdot d^{d+1} \cdot (k+1)^d$ hyperedges and $2 \cdot d! \cdot d^{d+1} \cdot (k+1)^{d-1}$ vertices computable in $O(d \cdot n + 2^d d \cdot m + (d! \cdot d^{d+1} \cdot (k+1)^d)^{1.5})$ time.

Note that the problem kernel resulting from Theorem 2 will no longer be expressive in the sense of Section 2. For example, not every minimal hitting set of size at most k for the input hypergraph will be a minimal hitting set for the problem kernel. This can be observed in Figure 9, where the vertex i_2 might be contained in a minimal hitting set of the input hypergraph and is absent in the output hypergraph.

To prove Theorem 2, we compute a maximal weakly related set W in linear time and show that our problem kernel already ensures $|W| \in O(k^{d-1})$ and thus, that further data reduction on W is unnecessary. To compute a maximal weakly related set in linear time, we employ Algorithm 2.

After some initialization work in lines 1–5, Algorithm 2 in lines 6–11 adds a hyperedge e to the weakly related set W if none of the subsets $C \subseteq e$ with |C| = d-1 is a subset of a set previously added to W. The information whether C is some subset of a hyperedge previously added to W is saved in intersection[C]. Note that, in line 11, Algorithm 2 also sets "intersection[$e \setminus C$] \leftarrow true" and thus saves which vertices are parts of hyperedges added to W. We will use this later to quickly reduce the number of vertices not contained in hyperedges in W.

Lemma 4. Given a hypergraph H, a maximal weakly related set is computable in $O(d \cdot n + d^2 \cdot m)$ time.

Proof. First, observe that the set W returned in line 12 of Algorithm 2 when applied to H = (V, E) is indeed weakly related: let $w_1 \neq w_2 \in E$ intersect in more than d-2 vertices and assume that w_1 is added to W in line 8. Let $C := w_1 \cap w_2$. Obviously, |C| = d - 1. Hence, when w_1 is added to W, we apply "intersection $[C] \leftarrow$ true" in line 10. Therefore, when $e = w_2$ is considered in line 6, the condition in line 7 does not hold, which implies that w_2 is not added to W in line 8. In the same way it follows that each hyperedge is added to W if it does not intersect any hyperedge of W in more than d-2 vertices. Therefore, W is maximal.

Now, Algorithm 2 works as follows. We use Lemma 3 to look up values in intersection [] in O(d) time. To this end, like in the proof of Proposition 3, we

Algorithm 2: Computation of a maximal weakly related set

Input: Hypergraph H = (V, E), natural number k. **Output**: Maximal weakly related set W. 1 $W \leftarrow \emptyset;$ 2 foreach $e \in E$ do // Initialization for each hyperedge foreach $C \subseteq e, |C| = d - 1$ do 3 intersection $[C] \leftarrow$ false; // No hyperedges in W contain C yet. $\mathbf{4}$ intersection $[e \setminus C] \leftarrow$ false; // The vertex in $e \setminus C$ is not in W yet. 5 6 foreach $e \in E$ do if $\forall C \subseteq e, |C| = d - 1$: intersection [C] = false then 7 $W \leftarrow W \cup \{e\};$ 8 foreach $C \subseteq e, |C| = d - 1$ do 9 $\mathbf{10}$ intersection $[C] \leftarrow \text{true};$ 11 intersection $[e \setminus C] \leftarrow$ true; 12 return W;

represent vertex subsets of size at most d as sorted sequences of length at most d. Thus, we first sort each hyperedge of H in $O(m \cdot d \log d)$ total time. To apply Lemma 3 to create the associative array intersection[], we need a list L of all values that we are going to store values for. As L, we use the list that, for each hyperedge e of H and each vertex $v \in e$, contains $e \setminus \{v\}$ and $\{v\}$. Of course, $e \setminus \{v\}$ can be computed in O(d) time from e so that $e \setminus \{v\}$ is sorted. It follows that L contains at most $2d \cdot m$ elements and is computable in $O(d^2m)$ time. Hence, by Lemma 3, we can build the associative array intersection[] in $O(dn + d^2 \cdot m)$ time and looking up values in intersection[] works in O(d) time for elements of L.

Now, the initialization in lines 1–5 works in $O(d^2 \cdot m)$ time. Finally, for every hyperedge, the body of the for-loop in line 6 can be executed in $O(d^2)$ time by doing O(d)-time look-ups for each of the $2d \cdot m$ sets.

We can now prove Theorem 2 by showing how to compute a problem kernel with $O(k^{d-1})$ vertices in $O(n + m + k^{1.5d})$ time.

Proof of Theorem 2. It is shown in Theorem 1 that d-HITTING SET has a problem kernel with $d! \cdot d^{d+1} \cdot (k+1)^d$ hyperedges that is computable in $O(dn+2^d d \cdot m)$ time. It remains to show that, in additional $O(d! \cdot d^{d+1} \cdot (k+1)^d)^{1.5}$ time, the number of vertices of a hypergraph H output by Algorithm 1 can be reduced to $2 \cdot d! \cdot d^{d+1} \cdot (k+1)^{d-1}$. To this end, we follow the approaches of Abu-Khzam [1] and Moser [28] as discussed in Section 5.1 and as illustrated in Figure 9.

First, we compute a maximal weakly related set W in H in $O(d \cdot n + d^2 \cdot m)$ time using Algorithm 2. We show that $|W| \leq d! \cdot d^d \cdot (k+1)^{d-1}$. To this end, consider the hypergraph $H_{\ell} := (V, W_{\ell})$ for $1 \leq \ell \leq d$, where W_{ℓ} is the set of cardinality- ℓ hyperedges in W. Since H has been output by Algorithm 1, we know that, by Lemma 1, H_{ℓ} has no sunflowers with more than d(k+1) petals. Moreover, since every pair of hyperedges in W intersects in at most d-2 vertices, also each pair of hyperedges of H_{ℓ} intersects in at most d-2 vertices. Hence, by Lemma 2 with b = 2 and c = d(k+1), we know that H_{ℓ} for $\ell \geq 2$ has at most $\ell! d^{\ell-1}(k+1)^{\ell-1}$ hyperedges. Moreover, H_1 contains at most d(k+1) hyperedges, as they form a sunflower with empty core. Therefore, $|W| \leq d! \cdot d^d \cdot (k+1)^{d-1}$.

Next, we construct a bipartite graph $B = (I \uplus S, E')$ from the input hypergraph H = (V, E), where

- i) I is the set of vertices in V not contained in any hyperedge in W, which is an independent set [1],
- ii) $S := \{ e \subseteq V \mid \exists v \in I : \exists w \in W : e \subseteq w, \{v\} \cup e \in E \}, \text{ and }$
- iii) $E' := \{\{v, e\} \mid v \in I, e \in S, \{v\} \cup e \in E\}.$

This can be done in $O(d^2 \cdot m)$ time by exploiting the information stored in the associative array intersection[] computed by Algorithm 2: for each $e \in E$ with |e| = d and each $v \in e$, add $\{v, e \setminus \{v\}\}$ to the graph B if and only if intersection $[e \setminus \{v\}]$ = true and intersection $[\{v\}]$ = false. In this case, it follows that e can be partitioned into

- i) a subset $e \setminus \{v\}$ of a hyperedge of W, since intersection $[e \setminus \{v\}] =$ true, and
- ii) the vertex v, which is not contained in any hyperedge in W, since we have intersection $[\{v\}] =$ false, and, hence, is contained in I.

Thus, e clearly satisfies the definition of E'. Observe that the graph B constructed in this way contains at most |E| = m edges. It remains to shrink I so that it contains at most |S| vertices. Then, the number of vertices in the output hypergraph will be at most $d|W|+|I| \leq d|W|+|S| \leq 2d|W| = 2 \cdot d! \cdot d^{d+1} \cdot (k+1)^{d-1}$. This, as shown by Moser [28, Section 7.3], is achieved by computing a maximum matching in B and deleting from H the unmatched vertices in I and the hyperedges containing them. To analyze the running time of computing the maximum matching, recall that the number of edges in B is at most $m \leq d! \cdot d^{d+1} \cdot (k+1)^d$ and that the number |I|+|S| of vertices is at most twice as much. Hence, a maximum matching in B can be computed in $O(\sqrt{|I \uplus S|} \cdot |E'|) = O(d! \cdot d^{d+1} \cdot (k+1)^d)^{1.5}$ time using the algorithm of Hopcroft and Karp [35, Theorem 16.4]. \Box

6 Conclusion

We have given an understanding of expressive kernelization for *d*-HITTING SET and have shown, as earlier claimed by Niedermeier and Rossmanith [30], that a problem kernel for *d*-HITTING SET with $O(k^d)$ hyperedges and vertices can be computed in linear time. Using the linear-time computable problem kernel for *d*-HITTING SET, we have improved the worst-case running times of the $O(k^{d-1})$ -vertex problem kernels by Abu-Khzam [1] and Moser [28].

Our experiments have shown that the kernelization algorithm runs efficiently, yet the observed data reduction effect on the nearly uniform hypergraphs occurring in the construction of Golomb rulers was limited.

An interesting question is whether a problem kernel with $O(k^{d-1})$ vertices and $O(k^d)$ hyperedges for *d*-HITTING SET can be computed in linear time. Answering this question would merge the best known results for problem kernels for *d*-HITTING SET. However, to date, all $O(k^{d-1})$ -vertex problem kernels for *d*-HITTING SET that we are aware of, that is, the problem kernels by Abu-Khzam [1] and Moser [28], involve the computation of maximum matchings. This seems to be difficult to avoid this bottleneck.

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