# Approximability of Capacitated Network Design 

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#### Abstract

In the capacitated survivable network design problem (Cap-SNDP), we are given an undirected multi-graph where each edge has a capacity and a cost. The goal is to find a minimum cost subset of edges that satisfies a given set of pairwise minimum-cut requirements. Unlike its classical special case of SNDP when all capacities are unit, the approximability of Cap-SNDP is not well understood; even in very restricted settings no known algorithm achieves a $o(m)$ approximation, where $m$ is the number of edges in the graph. In this paper, we obtain several new results and insights into the approximability of Cap-SNDP.

We give an $O(\log n)$ approximation for a special case of Cap-SNDP where the global minimum cut is required to be at least $R$. (Note that this problem generalizes the min-cost $\lambda$-edge-connected subgraph problem, which is the special case of our problem when all capacities are unit.) Our result is based on a rounding of a natural cut-based LP relaxation strengthened with knapsackcover (KC) inequalities. Our technique extends to give a similar approximation for a new network design problem that captures global minimum cut as a special case. We then show that as we move away from global connectivity, even for the single pair case (that is, when only one pair $(s, t)$ has positive connectivity requirement), this strengthened LP has $\Omega(n)$ integrality gap. Furthermore, in directed graphs, we show that single pair Cap-SNDP is $2^{\log ^{1-\delta} n}$-hard to approximate for any fixed constant $\delta>0$.

We also consider a variant of the Cap-SNDP in which multiple copies of an edge can be bought: we give an $O(\log k)$ approximation for this case, where $k$ is the number of vertex pairs with nonzero connectivity requirement. This improves upon the previously known $O\left(\min \left\{k, \log R_{\max }\right\}\right)$ approximation for this problem when the largest minimum-cut requirement, namely $R_{\max }$, is large. On the other hand, we observe that the multiple copy version of Cap-SNDP is $\Omega(\log \log n)$-hard to approximate even for the single-source version of the problem.


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## 1 Introduction

In this paper we consider the capacitated survivable network design problem (Cap-SNDP). The input consists of an undirected $n$-vertex multi-graph $G(V, E)$ and an integer requirement $R_{i j}$ for each unordered pair of nodes $(i, j)$. Each edge $e$ of $G$ has a cost $c(e)$ and an integer capacity $u(e)$. The goal is to find a minimum-cost subgraph $H$ of $G$ such that for each pair of nodes $i, j$ the capacity of the minimum-cut between $i$ and $j$ in $H$ is at least $R_{i j}$. This generalizes the well-known survivable network design problem (SNDP) problem in which all edge capacities are 1. SNDP already captures as special cases a variety of fundamental connectivity problems in combinatorial optimization such as the mincost spanning tree, min-cost Steiner tree and forest, as well as min-cost $\lambda$-edge-connected subgraph; each of these problems has been extensively studied on its own and several of these special cases are NP-hard and APX-hard to approximate. Jain, in an influential paper [15], obtained a 2 -approximation for SNDP via the standard cut-based LP relaxation using the iterated rounding technique.

Although the above mentioned 2-approximation for SNDP has been known since 1998, the approximability of Cap-SNDP has essentially been wide open even in very restricted special cases. Similar to SNDP, Cap-SNDP is motivated by both practial and theoretical considerations. These problems find applications in the design of resilient networks such as in telecommunication infrastructure. In such networks it is often quite common to have equipment with different discrete capacities; this leads naturally to design problems such as Cap-SNDP. At the outset, we mention that a different and somewhat related problem is also referred to by the same name, especially in the operations research literature. In this version the subgraph $H$ has to support simultaneously a flow of $R_{i j}$ between each pair of nodes $(i, j)$; this is more closely related to multicommodity flows and buy-at-bulk network design. Our version is more related to connectivity problems such as SNDP.

As far as we are aware, the version of Cap-SNDP that we study was introduced (in the approximation algorithms literature) by Goemans et al. [14] in conjunction with their work on SNDP. They made several observations on Cap-SNDP: (i) Cap-SNDP reduces to SNDP if all capacities are the same, (ii) there is an $O\left(\min \left(m, R_{\max }\right)\right)$ approximation where $m$ is the number of edges in $G$ and $R_{\max }=\max _{i j} R_{i j}$ is the maximum requirement, and (iii) if multiple copies of an edge are allowed then there is an $O\left(\log R_{\max }\right)$-approximation. We note that in the capacitated case $R_{\max }$ can be exponentially large in $n$, the number of nodes of the graph. Carr et al. [6] observed that the natural cut-based LP relaxation has an unbounded integrality gap even for the graph consisting of only two nodes $s, t$ connected by parallel edges with different capacities. Motivated by this observation and the goal of obtaining improved approximation ratios for Cap-SNDP, 6] strengthened the basic cut-based LP by using knapsack-cover inequalities. (Several subsequent papers in approximation algorithms have fruitfully used these inequalities.) Using these inequalities, [6] obtained a $\beta(G)+1$ approximation for Cap-SNDP where $\beta(G)$ is the maximum cardinality of a bond in the underlying simple graph: a bond is a minimal set of edges that separates some pair of vertices with positive demand. Although $\beta(G)$ could be $\Theta\left(n^{2}\right)$ in general, for certain topologies - for instance, if the underlying graph is a line or a cycle - this gives constant factor approximations.

The above results naturally lead to several questions. What is the approximability of Cap-SNDP? Should we expect a poly-logarithmic approximation or even a constant factor approximation? If not, what are interesting and useful special cases to consider? And do the knapsack cover inequalities help in the general case? What is the approximability of Cap-SNDP if one allows multiple copies? Does this relaxed version of the problem allow a constant factor approximation?

In this paper we obtain several new positive and negative results for Cap-SNDP that provide new insights into the questions above.

### 1.1 Our Results

We first discuss results for Cap-SNDP where multiple copies are not allowed. We initiate our study by considering the global connectivity version of Cap-SNDP where we want a min-cost subgraph with global min-cut at least $R$; in other words, there is a "uniform" requirement $R_{i j}=R$ for all pairs $(i, j)$. We refer to this as the Cap-R-Connected Subgraph problem; the special case when all capacities are unit corresponds to the classical minimum cost $\lambda$-edge-connected (spanning) subgraph problem, which is known to be APX-hard [12]. We show the following positive result for arbitrary capacities.

Theorem 1.1. There is a randomized $O(\log n)$-approximation algorithm for the Cap- $R$-Connected Subgraph problem. Moreover, for any $\gamma \geq 1$, there is a randomized $O(\gamma \log n)$-approximation algorithm with running time $n^{O(\gamma)}$ for "nearly uniform" Cap-SNDP when all pairwise requirements are in $[R, \gamma R]$.

To prove Theorem 1.1, we begin with a natural LP relaxation for the problem. Almost all positive results previously obtained for the unit capacity case are based on this relaxation. As remarked already, this LP has an unbounded integrality gap even for a graph with two nodes (and hence for Cap- $R$ Connected Subgraph). We strengthen the relaxation by adding the valid knapsack cover inequalities. Although we do not know of a polynomial time algorithm to separate over these inequalities, following [6], we find a violated inequality only if the current fractional solution does not satisfy certain useful properties. Our main technical tool both for finding a violated inequality and subsequently rounding the fractional solution is Karger's theorem on the number of small cuts in undirected graphs [16].

We believe the approach outlined above may be useful in other network design applications. As a concrete illustration, we use it to solve an interesting and natural generalization of Cap- $R$-Connected Subgraph, namely, the $k$-Way- $\mathcal{R}$-Connected Subgraph problem. The input consists of $(k-1)$ integer requirements $R_{1}, \ldots R_{k-1}$, such that $R_{1} \leq R_{2} \leq \ldots \leq R_{k-1}$. The goal is to find a minimum-cost subgraph $H$ of $G$ such that for each $1 \leq i \leq k-1$, the capacity of any $(i+1)$-way cut of $G$ is at least $R_{i}{ }^{1}$ It is easy to see that Cap- $R$-Connected Subgraph is precisely the $k$-Way- $\mathcal{R}$-Connected Subgraph, with $k=2$. Note that the $k$-Way- $\mathcal{R}$-Connected Subgraph problem is not a special case of the general Cap-SNDP as the cut requirements for the former problem are not expressible as pairwise connectivity constraints. Interestingly, our techniques for Cap- $R$-Connected Subgraph can be naturally extended to handle the multiway cut requirements, yielding the following generalization of Theorem 1.1,
Theorem 1.2. There is a randomized $O(k \log n)$-approximation algorithm for the $k$-Way- $\mathcal{R}$-Connected Subgraph problem with $n^{O(k)}$ running time.

We remark that even for the unit-capacity case of this problem, it is not clear how to obtain a better ratio than that guaranteed by the above theorem. We discuss more in Section 2.3 ,

Once the pairwise connectivity requirements are allowed to vary arbitrarily, the Cap-SNDP problem seems to become distinctly harder. Surprisingly, the difficulty of the general case starts to manifest even for the simplest representative problem in this setting, where there is only one pair $(s, t)$ with $R_{s t}>0$; we refer to this as the single pair problem. The only known positive result for this seemingly restricted case is a polynomial-factor approximation that follows from the results in [14, 6] for general Cap-SNDP. We give several negative results to suggest that this special case may capture the essential difficulty of Cap-SNDP. In particular, we start by observing that the LP with knapsack cover inequalities has an $\Omega(n)$ integrality gap even for the single-pair problem. 2 Next we show that the single pair problem

[^1]is $\Omega(\log \log n)$-hard to approximate.
Theorem 1.3. The single pair Cap-SNDP problem cannot be approximated to a factor better than $\Omega(\log \log n)$ unless $N P \subseteq$ DTIME $\left(n^{\log \log \log n}\right)$.

The above theorem is a corollary of the results in Chuzhoy et al. 's work on the hardness of related network design problems [9]. We state it as a theorem to highlight the status of the problem. (See Appendix $\AA$ for a brief proof sketch.) We further discuss this connection at the end of this section. We prove a much stronger negative result for the single pair problem in directed graphs. Since in the unitcapacity case, polynomial-time minimum-cost flow algorithms solve the single-pair problem exactly even in directed graphs, the hardness result below shows a stark contrast between the unit-capacity and the non-unit capacity cases.

Theorem 1.4. In directed graphs, the single pair Cap-SNDP cannot be approximated to a factor better than $2^{\log ^{(1-\delta)} n}$ for any $0<\delta<1$, unless $N P \subseteq D T I M E\left(n^{\operatorname{poly} \log (n)}\right)$. Moreover, this hardness holds for instances in which there are only two distinct edge capacities.

Allowing Multiple Copies: Given the negative results above for even the special case of the singlepair Cap-SNDP, it is natural to consider the relaxed version of the problem where multiple copies of an edge can be chosen. Specifically, for any integer $\alpha \geq 0, \alpha$ copies of $e$ can be bought at a cost of $\alpha \cdot c(e)$ to obtain a capacity $\alpha \cdot u(e)$. In some applications, such as in telecommunication networks, this is a reasonable model. As we discussed, this model was considered by Goemans et al. [14 who gave an $O\left(\log R_{\max }\right)$ approximation for Cap-SNDP. This follows from a simple $O(1)$ approximation for the case when all requirements are in $\{0, R\}$. The advantage of allowing multiple copies is that one can group request pairs into classes and separately solve the problem for each class while losing only the number of classes in the approximation ratio. For instance, one easily obtains a 2 -approximation for the single pair problem even in directed graphs, in contrast to the difficulty of the problem when multiple copies are not allowed. Note that this also implies an easy $2 k$ approximation where $k$ is the number of pairs with $R_{i j}>0$. We address the approximability of Cap-SNDP with multiple copies of edges allowed. When $R_{\max }$ is large, we improve the $\min \left\{2 k, O\left(\log R_{\max }\right)\right\}$-approximation discussed above via the following.

Theorem 1.5. In undirected graphs, there is an $O(\log k)$-approximation algorithm for Cap-SNDP with multiple copies, where $k$ is the number of pairs with $R_{i j}>0$.

Both our algorithm and analysis are inspired by the $O(\log k)$-competitive online algorithm for the Steiner forest problem by Berman and Coulston [5], and the subsequent adaptation of these ideas for the priority Steiner forest problem by Charikar et al. [7]. However, we believe the analysis of our algorithm is more transparent (although it gets weaker constants) than the original analysis of [5].

We complement our algorithmic result by showing that the multiple copy version is $\Omega(\log \log n)$ hard to approximate. This hardness holds even for the single-source Cap-SNDP where we are given a source node $s \in V$, and a set of terminals $T \subseteq V$, such that $R_{i j}>0$ iff $i=s$ and $j \in T$. Observe that single-source Cap-SNDP is a simultaneous generalization of the classical Steiner tree problem ( $R_{i j} \in\{0,1\}$ ) as well as both Cap- $R$-Connected Subgraph and single-pair Cap-SNDP.

Theorem 1.6. In undirected graphs, single source Cap-SNDP with multiple copies cannot be approximated to a factor better than $\Omega(\log \log n)$ unless $N P \subseteq D T I M E\left(n^{\log \log \log n}\right)$.

The above theorem, like Theorem 1.3, also follows easily from the results of 9]. For completeness, we provide a proof of Theorem [1.6 in Appendix A. We note that the hardness reduction above creates
instances with super-polynomially large capacities. For such instances, our $O(\log k)$-approximation strongly improves on the previously known approximation guarantees.

Related Work: Network design has a large literature in a variety of areas including computer science and operations research. Practical and theoretical considerations have resulted in numerous models and results. Due to space considerations it is infeasible even to give a good overview of closely related work. We briefly mention some work that allows the reader to compare the model we consider here to related models. As we mentioned earlier, our version of Cap-SNDP is a direct generalization of SNDP and hence is concerned with (capacitated) connectivity between request node pairs. We refer the reader to the survey [17] and some recent and previous papers [14, 15, 13, 10, 11, 19 , for pointers to literature on network design for connectivity. A different model arises if one wishes to find a min-cost subgraph that supports multicommodity flow for the request pairs; in this model each node pair $(i, j)$ needs to routes a flow of $R_{i j}$ in the chosen graph and these flows simultaneously share the capacity of the graph. We observe that if multiple copies of an edge are allowed then this problem is essentially equivalent to the non-uniform buy-at-bulk network design problem. Buy-at-bulk problems have received substantial attention; we refer the reader to 8 for several pointers to this work. If multiple copies are not allowed, the approximability of this flow version is not well-understood; for example if the flow for each pair is only allowed to be routed on a single path, then even checking feasibility of a given subgraph is NP-Hard since the problem captures the well-known edge-disjoint paths and unsplittable flow problems. Andrews and Zhang [2] have recently considered special cases of this problem with uniform capacities while allowing some congestion (that is, a few copies) on the chosen edges.

The $k$-Way- $\mathcal{R}$-Connected Subgraph problem that we consider does not appear to have been considered previously even in the unit-capacity case.

## 2 The Cap-R-Connected Subgraph problem

In this section, we prove Theorem 1.1, giving an $O(\log n)$-approximation for the Cap- $R$-Connected Subgraph problem. Here, we assume each $R_{i j}=R$; the extension to the case when requirements are "nearly uniform" is deferred to Appendix B.1. We start by writing a natural linear program relaxation for the problem; the integrality gap of this LP can be arbitrarily large. To deal with this, we introduce additional valid inequalities, called the knapsack cover inequalities, that must be satisfied by any integral solution. We show how to round this strengthened LP, obtaining an $O(\log n)$-approximation.

### 2.1 The Standard LP Relaxation and Knapsack-Cover Inequalities

We assume without any loss of generality that the capacity of any edge is at most $R$. For each subset $S \subseteq 2^{V}$, we use $\delta(S)$ to denote the set of edges with exactly one endpoint in $S$. For a set of edges $A$, we use $u(A)$ to denote $\sum_{e \in A} u(e)$. We say that a set of edges $A$ satisfies (the cut induced by) $S$ if $u(A \cap \delta(S)) \geq R$. Note that we wish to find the cheapest set of edges which satisfies every subset $\emptyset \neq S \subset V$. The following is the LP relaxation of the standard integer program capturing the problem.

$$
\begin{array}{lc} 
& \min \sum_{e \in E} c(e) x_{e}  \tag{StdLP}\\
\forall S \subseteq V, & \sum_{e \in \delta(S)} u(e) x_{e} \geq R \\
\forall e \in E, & 0 \leq x_{e} \leq 1
\end{array}
$$

The following example shows that (Std LP) can have integrality gap as bad as $R$.
Example 1: Consider a graph $G$ on three vertices $p, q, r$. Edge $p q$ has cost 0 and capacity $R$; edge $q r$ has cost 0 and capacity $R-1$; and edge $p r$ has $\operatorname{cost} C$ and capacity $R$. To achieve a global min-cut of size at least $R$, any integral solution must include edge $p r$, and hence must have cost $C$. In contrast, in (Std LP) one can set $x_{p r}=1 / R$, and obtain a total cost of $C / R$.

In the previous example, any integral solution in which the mincut separating $r$ from $\{p, q\}$ has size at least $R$ must include edge $p r$, even if $q r$ is selected. The following valid inequalities are introduced precisely to enforce this condition. More generally, let $S$ be a set of vertices, and $A$ be an arbitrary set of edges. Define $R(S, A)=\max \{0, R-u(A \cap \delta(S))\}$ be the residual requirement of $S$ that must be satisfied by edges in $\delta(S) \backslash A$. That is, any feasible solution has $\sum_{e \in \delta(S) \backslash A} u(e) x_{e} \geq R(S, A)$. However, any integral solution also satisfies the following stronger requirement

$$
\sum_{e \in \delta(S) \backslash A} \min \{R(S, A), u(e)\} x_{e} \geq R(S, A)
$$

and thus these inequalities can be added to the LP to strengthen it. These additional inequalities are referred to as Knapsack-Cover inequalities, or simply KC inequalities, and were first used by [6] in design of approximation algorithms for Cap-SNDP.

Below, we write a LP relaxation, (KC LP), strengthened with the knapsack cover inequalities. Note that the original constraints correspond to KC inequalities with $A=\emptyset$; we simply write them explicitly for clarity.

$$
\begin{array}{rrr} 
& \min \sum_{e \in E} c(e) x_{e} & \text { (KC LP) }  \tag{KCLP}\\
\forall S \subseteq V, & \sum_{e \in \delta(S)} u(e) x_{e} \geq R & \text { (Original Constraints) } \\
\forall A \subseteq E, \forall S \subseteq V, & \sum_{e \in \delta(S) \backslash A} \min (u(e), R(S, A)) x_{e} \geq R(S, A) & \text { (KC-inequalities) } \\
\forall e \in E, & 0 \leq x_{e} \leq 1 &
\end{array}
$$

The Linear Program ( $\overline{K C L P}$ ), like the original (Std LP), has exponential size. However, unlike the (Std LP), we do not know of the existence of an efficient separation oracle for this. Nevertheless, as we show below, we do not need to solve (KC LP) ; it suffices to get to what we call a good fractional solution.

Definition 2.1. Given a fractional solution $x$, we say an edge $e$ is nearly integral if $x_{e} \geq \frac{1}{40 \log n}$, and we say e is highly fractional otherwise.

Definition 2.2. For any $\alpha \geq 1$, a cut in a graph $G$ with capacities on edges, is an $\alpha$-mincut if its capacity is within a factor $\alpha$ of the minimum cut of $G$.

Theorem 2.3. [Theorems 4.7.6 and 4.7.7 of [16]] The number of $\alpha$-mincuts in an n-vertex graph is at most $n^{2 \alpha}$. Moreover, the set of all $\alpha$-mincuts can be found in $O\left(n^{2 \alpha} \log ^{2} n\right)$ time with high probability.

Given a fractional solution $x$ to the edges, we let $A_{x}$ denote the set of nearly integral edges, that is, $A_{x}:=\left\{e \in E: x_{e} \geq \frac{1}{40 \log n}\right\}$. Define $\hat{u}(e)=u(e) x_{e}$ to be the fractional capacity on the edges. Let $\mathcal{S}:=\{S \subseteq V: \hat{u}(\delta(S)) \leq 2 R\}$. A solution $x$ is called good if it satisfies the following three conditions:
(a) The global mincut in $G$ with capacity $\hat{u}$ is at least $R$, i.e. $x$ satisfies the original constraints.
(b) The KC inequalities are satisfied for the set $A_{x}$ and the sets in $\mathcal{S}$. Note that if (a) is satisfied, then by Theorem [2.3, $|\mathcal{S}| \leq n^{4}$.
(c) $\sum_{e \in E} c(e) x_{e}$ is at most the value of the optimum solution to (KC LP).

Note that a good solution need not be feasible for ( (KC LP) as it is required to satisfy only a subset of KC-inequalities. We use the ellipsoid method to get such a solution. Such a method was also used in [6].

Lemma 2.4. There is a randomized algorithm that computes a good fractional solution with high probability.
Proof: We start by guessing the optimum value $M$ of (KC LP) and add the constraint $\sum_{e \in E} c(e) x_{e} \leq$ $M$ to the constraints of (KC LP). If the guessed value is too small, a good solution may not exist; however, a simple binary search suffices to identify the smallest feasible value of $M$. With this constraint in place, we will use the ellipsoid method to compute a solution that satisfies (a), (b), and (c) with high probability. Since we do not know of a polynomial-time separation oracle for KC inequalities, we will simulate a separation oracle that verifies condition (b), a subset of KC inequalities, in polynomial time. Specifically, we give a randomized polynomial time algorithm such that given a solution $x$ that violates condition (b), the algorithm detects the violation with high probability and outputs a violated KC inequality. We now describe the entire process.

Given a solution $x$ we first check if condition (a) is satisfied. This can be done in polynomial time by $O(n)$ max-flow computations. If (a) is not satisfied, we have found a violated constraint. Once we have a solution that satisfies (a), we know that $|\mathcal{S}| \leq n^{4}$. By Theorem [2.3, the set $\mathcal{S}$ can be computed in polynomial-time with high probability. Thus we can check condition (b) in polynomial-time, and with high-probability find a violating constraint for (b) if one exists. Once we have a solution that satisfies both (a) and (b), we check if $\sum_{e \in E} c(e) x_{e} \leq M$. If not, we have once again found a violated constraint for input to the ellipsoid algorithm. Thus in polynomially many rounds, where each round runs in polynomial-time, the ellipsoid algorithm combined with the simulated separation oracle, either returns a solution $x$ that satisfies (a), (b), and $\sum_{e \in E} c(e) x_{e} \leq M$, with high probability, or proves that the system is infeasible. Using binary search, we find the smallest $M$ for which a solution $x$ is returned satisfying conditions (a), (b) and $\sum_{e \in E} c(e) x_{e} \leq M$. Since $M$ is less than the optimum value of (KC LP), we get that the returned $x$ is a good fractional solution with high probability.

### 2.2 The Rounding and Analysis

Given a good fractional solution $x$, we now round it to get a $O(\log n)$ approximation to the Cap- $R$ Connected Subgraph problem. A useful tool for our analysis is the following Chernoff bound (see [18], for instance):

Lemma 2.5. Let $X_{1}, X_{2}, \ldots X_{k}$ be a collection of independent random variables in $[0,1]$, let $X=$ $\sum_{i=1}^{k} X_{i}$, and let $\mu=\mathbb{E}[X]$. The probability that $X \leq(1-\delta) \mu$ is at most $e^{-\mu \delta^{2} / 2}$.

We start by selecting $A_{x}$, the set of all nearly integral edges. Henceforth, we lose the subscript and denote the set as simply $A$. Let $F=E \backslash A$ denote the set of all highly fractional edges; for each edge $e \in F$, select it with probability $\left(40 \log n \cdot x_{e}\right)$. Let $F^{*} \subseteq F$ denote the set of selected highly fractional edges. The algorithm returns the set of edges $E_{A}:=A \cup F^{*}$.

It is easy to see that the expected cost of this solution $E_{A}$ is $O(\log n) \sum_{e \in E} c(e) x_{e}$, and hence by condition (c) above, within $O(\log n)$ times that of the optimal integral solution. Thus, to prove Theorem 1.1, it suffices to prove that with high probability, $E_{A}$ satisfies every cut in the graph $G$;
we devote the rest of the section to this proof. We do this by separately considering cuts of different capacities, where the capacities are w.r.t $\hat{u}$ (recall that $\left.\hat{u}(e)=u(e) x_{e}\right)$. Let $\mathcal{L}$ be the set of cuts of capacity at least $2 R$, that is, $\mathcal{L}:=\{S \subseteq V: \hat{u}(\delta(S))>2 R\}$.

Lemma 2.6. $\operatorname{Pr}\left[\forall S \in \mathcal{L}: u\left(E_{A} \cap \delta(S)\right) \geq R\right] \geq 1-\frac{1}{2 n^{10}}$.
Proof: We partition $\mathcal{L}$ into sets $\mathcal{L}_{2}, \mathcal{L}_{3}, \cdots$ where $\mathcal{L}_{j}:=\{S \subseteq V: j R<\hat{u}(\delta(S)) \leq(j+1) R\}$. Note that Theorem 2.3 implies $\left|\mathcal{L}_{j}\right| \leq n^{2(j+1)}$ by condition (a) above. Fix $j$, and consider an arbitrary cut $S \in \mathcal{L}_{j}$. If $u(A \cap \delta(S)) \geq R$, then $S$ is clearly satisfied by $E_{A}$. Otherwise, since the total $\hat{u}$-capacity of $S$ is at least $j R$, we have $\hat{u}(F \cap \delta(S)) \geq \hat{u}(\delta(S))-u(A \cap \delta(S)) \geq(j-1) R$. Thus

$$
\sum_{e \in F \cap \delta(S)} \frac{u(e)}{R} x_{e} \geq(j-1)
$$

Recall that an edge $e \in F$ is selected in $F^{*}$ with probability $\left(40 \log n \cdot x_{e}\right)$. Thus, for the cut $S$, the expected value of $\sum_{e \in F^{*} \cap \delta(S)} \frac{u(e)}{R} \geq 40(j-1) \log n$. Since $u(e) / R \leq 1$, we can apply Lemma 2.5) to get that the probability that $S$ is not satisfied is at most $e^{-16 \log n(j-1)}=1 / n^{16(j-1)}$. Applying the union bound, the probability that there exists a cut in $\mathcal{L}_{j}$ not satisfied by $E_{A}$ is at most $n^{2(j+1)} / n^{16(j-1)}=$ $n^{18-14 j}$. Thus probability that some cut in $\mathcal{L}$ is not satisfied is bounded by $\sum_{j \geq 2} n^{18-14 j} \leq 2 n^{-10}$ if $n \geq 2$. Hence with probability at least $1-1 / 2 n^{10}, A \cup F^{*}$ satisfies all cuts in $\mathcal{L}$.

One might naturally attempt the same approach for the cuts in $\mathcal{S}$ (recall that $\mathcal{S}=\{S \subseteq V$ : $\hat{u}(\delta(S)) \leq 2 R\}$ ) modified as follows. Consider any cut $S$, which is partly satisfied by the nearly integral edges $A$. The fractional edges contribute to the residual requirement of $S$, and since $x_{e}$ is scaled up for fractional edges by a factor of $40 \log n$, one might expect that $F^{*}$ satisfies the residual requirement, with the $\log n$ factor providing a high-probability guarantee. This intuition is correct, but the KC inequalities are crucial. Consider Example 1; edge pr is unlikely to be selected, even after scaling. In the statement of Lemma [2.5, it is important that each random variable takes values in $[0,1]$; thus, to use this lemma, we need the expected capacity from fractional edges to be large compared to the maximum capacity of an individual edge. But the KC inequalities, in which edge capacities are "reduced", enforce precisely this condition. Thus we get the following lemma using a similar analysis as above.

Lemma 2.7. $\operatorname{Pr}\left[\forall S \in \mathcal{S}: u\left(\delta\left(E_{A} \cup \delta(S)\right)\right) \geq R\right] \geq 1-\frac{1}{n^{12}}$.
The $O(\log n)$-approximation guarantee for the Cap- $R$-Connected Subgraph problem stated in Theorem 1.1 follows from the previous two lemmas.

### 2.3 The $k$-Way- $\mathcal{R}$-Connected Subgraph Problem

The $k$-Way- $\mathcal{R}$-Connected Subgraph problem that we define is a natural generalization of the wellstudied min-cost $\lambda$-edge-connected subgraph problem. The latter problem is motivated by applications to fault-tolerant network design where any $\lambda-1$ edge failures should not disconnect the graph. However, there may be situations in which global $\lambda$-connectivity may be too expensive or infeasible. For example the underlying graph $G$ may have a single cut-edge but we still wish a subgraph that is as close to 2 -edge-connected as possible. We could model the requirement by $k$-Way- $\mathcal{R}$-Connected Subgraph (in the unit-capacity case) by setting $R_{1}=1$ and $R_{2}=3$; that is, at least 3 edges have to be removed to partition the graph into 3 disconnected pieces.

We briefly sketch the proof of Theorem 1.2, We work with a generalization of (KC LP) to $i$-way cuts, with an original constraint for each $i+1$-way cut, $1 \leq i \leq k-1$, and with KC inequalities added.

The algorithm is to select all nearly integral edges $e$ (those with $x_{e} \geq \frac{1}{40 k \log n}$ ), and select each of the remaining (highly fractional) edges $e$ with probability $40 k \log n \cdot x_{e}$. The analysis is very similar to that of Theorem 1.1 and hence moved to the appendix; we use the following lemma on counting $k$-way cuts in place of Theorem 2.3.

Lemma 2.8 (Lemma 11.2.1 of [16]). In an $n$-vertex graph, the number of $k$-way cuts with capacity at most $\alpha$ times that of a minimum $k$-way cut is at most $n^{2 \alpha(k-1)}$.

It would be interesting to explore algorithms and techniques for other more general variants of the $k$-Way- $\mathcal{R}$-Connected Subgraph problem that we consider here.

## 3 Single-Pair Cap-SNDP

In this section we show that the integrality gap with KC inequalities is $\Omega(n)$ even for single-pair Cap-SNDP in undirected graphs. Moreover, when the underlying graph is directed, we show that the single-pair problem is hard to approximate to within a factor of $2^{\log ^{(1-\delta)} n}$ for any $\delta>0$.

### 3.1 Integrality Gap with KC Inequalities

We show that for any positive integer $R$, there exists a single-pair Cap-SNDP instance $G$ with $(R+2)$ vertices such that the integrality gap of the natural LP relaxation strengthened with KC inequalities is $\Omega(R)$. The instance $G$ consists of a source vertex $s$, a sink vertex $t$, and $R$ other vertices $v_{1}, v_{2}, \ldots, v_{R}$.
There is an edge of capacity 2 and cost 1 (call these small edges) between $s$ and each $v_{i}$, and an edge of capacity $R$ and cost $R$ between each $v_{i}$ and $t$ (large edges). We have $R_{s t}=R$. Clearly, an optimal integral solution must select at least $R / 2$ of the large edges (in addition to small edges), and hence has cost greater than $R^{2} / 2$. The instance is depicted in the accompanying figure: Label $(u, c)$ on an edge denotes capacity $u$ and cost $c$.

We now describe a feasible LP solution: set $x_{e}=1$ on each small edge $e$, and $x_{e^{\prime}}=2 / R$ on each large edge $e^{\prime}$. The cost of this solution is $R$ from the small edges, and $2 R$ from the large edges, for a total of $3 R$. This is a factor of $R / 6$ smaller than the optimal integral solution,
 proving the desired integrality gap.

It remains only to verify that this is indeed a feasible solution to (KC LP). Consider the constraint corresponding to sets $S, A$. As edges in $A \backslash \delta(S)$ play no role, we may assume $A \subseteq \delta(S)$. If $A$ includes a large edge, or at least $R / 2$ small edges, the residual requirement $R(S, A)$ that must be satisfied by the remaining edges of $\delta(S)$ is 0 , and so the constraint is trivially satisfied. Let $A$ consist of $a<R / 2$ small edges; the residual requirement is thus $R-2 a$. Let $\delta(S)$ contain $i$ large edges and thus $R-i$ small edges. Now, the contribution to the left side of the constraint from small edges in $\delta(S) \backslash A$ is $2(R-i-a)=(R-2 a)+(R-2 i)$. Therefore, the residual requirement is satisfied by small edges alone unless $i>R / 2$. But the contribution of large edges is $i \cdot \frac{2}{R} \cdot(R-2 a)$ which is greater than $R-2 a$ whenever $i>R / 2$. Thus, we satisfy each of the added KC inequalities.

### 3.2 Hardness of Approximation in Directed Graphs

We now prove Theorem 1.4 via a reduction from the label cover problem [4].

Definition 3.1 (Label Cover Problem). The input consists of a bipartite graph $G(A \cup B, E)$ such that the degree of every vertex in $A$ is $d_{A}$ and degree of every vertex in $B$ is $d_{B}$, a set of labels $L_{A}$ and a set of labels $L_{B}$, and a relation $\pi_{(a, b)} \subseteq L_{A} \times L_{B}$ for each edge $(a, b) \in E$. Given a labeling $\phi: A \cup B \rightarrow L_{A} \cup L_{B}$, an edge $e=(a, b) \in E$ is said to be consistent iff $(\phi(a), \phi(b)) \in \pi_{(a, b)}$. The goal is to find a labeling that maximizes the fraction of consistent edges.

The following hardness result for the label-cover problem is a well-known consequence of the PCP theorem [3] and Raz's Parallel Repetition theorem [20].

Theorem 3.2 ([3, [20). For any $\epsilon>0$, there does not exist a poly-time algorithm to decide if a given instance of label cover problem has a labeling where all edges are consistent (Yes-Instance), or if no labeling can make at least $\frac{1}{\gamma}$ fraction of edges to be consistent for $\gamma=2^{\log ^{1-\epsilon} n}$ (NO-Instance), unless $N P \subseteq D T I M E\left(n^{\mathrm{polylog}(n)}\right)$.

We now give a reduction from label cover to the single-pair Cap-SNDP in directed graphs. In our reduction, the only non-zero capacity values will be $1, d_{A}$, and $d_{B}$. We note that Theorem 3.2 holds even when we restrict to instances with $d_{A}=d_{B}$. Thus our hardness result will hold on single-pair Cap-SNDP instances where there are only two distinct non-zero capacity values.

Given an instance $I$ of the label cover problem with $m$ edges, we create in polynomial-time a directed instance $I^{\prime}$ of single-pair Cap-SNDP such that if $I$ is a Yes-Instance then $I^{\prime}$ has a solution of cost at most $2 m$, and otherwise, every solution to $I^{\prime}$ has cost $\Omega\left(m \gamma^{\frac{1}{4}}\right)$. This establishes Theorem 1.4 when we choose $\epsilon=\delta / 2$.

The underlying graph $G^{\prime}\left(V^{\prime}, E^{\prime}\right)$ for the single-pair Cap-SNDP instance is constructed as follows. The set $V^{\prime}$ contains a vertex $v$ for every $v \in A \cup B$. We slightly abuse notation and refer to these sets of vertices in $V^{\prime}$ as $A$ and $B$ as well. Furthermore, for every vertex $a \in A$, and for every label $\ell \in L_{A}$, the set $V^{\prime}$ contains a vertex $a(\ell)$. Similarly, for every vertex $b \in B$, and for every label $\ell \in L_{B}$, the set $V^{\prime}$ contains a vertex $b(\ell)$. Finally, $V^{\prime}$ contains a source vertex $s$ and a sink vertex $t$. The set $E^{\prime}$ contains the following directed edges:

- For each vertex $a$ in $A$, there is an edge from $s$ to the vertex $a$ of cost 0 and capacity $d_{A}$. For each vertex $b \in B$, there is an edge from $b$ to $t$ of cost 0 and capacity $d_{B}$.
- For each vertex $a \in A$, and for all labels $\ell$ in $L_{A}$, there is an edge from $a$ to $a(\ell)$ of cost $d_{A}$ and capacity $d_{A}$. For each vertex $b \in B$, and for all labels $\ell$ in $L_{B}$, there is an edge from $b(\ell)$ to $b$ of cost $d_{B}$ and capacity $d_{B}$. These two types of edges are the only edges with non-zero cost.
- For every edge $(a, b) \in E$, and for every pair of labels $\left(\ell_{a}, \ell_{b}\right) \in \pi_{(a, b)}$, there is an edge from $a\left(\ell_{a}\right)$ to $b\left(\ell_{b}\right)$ of cost 0 and capacity 1 .

This completes the description of the network $G^{\prime}$. The requirement $R_{s t}$ between $s$ and $t$ is $m$, the number of edges in the label cover instance. It is easy to verify that the size of the graph $G^{\prime}$ is at most quadratic in the size of the label cover instance, and that $G^{\prime}$ can be constructed in polynomial-time. The lemmas below analyze the cost of Yes-Instance and No-Instance instances.

Lemma 3.3. If the label cover instance is a Yes-Instance, then $G^{\prime}$ contains a subgraph of cost $2 m$ which can realize a flow of value $m$ from $s$ to $t$.
Proof:Let $\phi$ be any labeling that consistently labels all edges in $G(A \cup B, E)$. Also, let $E_{1} \subseteq E^{\prime}$ be the set of all edges of cost 0 in $E^{\prime}$, and let $E_{2} \subseteq E^{\prime}$ be the set of edges $\{(a, a(\phi(a))) \mid a \in A\} \cup\{(b(\phi(b)), b)$ : $b \in B\}$. We claim that $E_{1} \cup E_{2}$ is a feasible solution for the single-pair Cap-SNDP instance. Note that the total cost of edges in $E_{1} \cup E_{2}$ is $|A| d_{A}+|B| d_{B}=2 m$. We now exhibit a flow of value $m$ from $s$ to
$t$ in $G^{\prime \prime}\left(V^{\prime}, E_{1} \cup E_{2}\right)$. A flow of value $d_{A}$ is sent along the path $s \rightarrow a \rightarrow a(\phi(a))$ for all $a \in A$. From $a(\phi(a))$, a unit of flow is sent to the $d_{A}$ vertices of the form $\{b(\phi(b)) \mid b \in B$ and $(a, b) \in E\}$; this is feasible because $\phi$ consistently labels all edges in $E$. Thus each vertex of the form $b(\phi(b))$ where $b \in B$ receives $d_{B}$ units of flow, since the degree of $b$ is $d_{B}$ in $G$. A flow of value $d_{B}$ is sent to $t$ along the path $b(\phi(b)) \rightarrow b \rightarrow t$. Thus $s$ sends out a flow of value $|A| d_{A}=m$, or equivalently, $t$ receives a flow of value $|B| d_{B}=m$.

Lemma 3.4. If the label cover instance is a No-Instance, then any subgraph of $G^{\prime \prime}$ that realizes a flow of $m$ units from $s$ to $t$ has cost $\Omega\left(m \gamma^{\frac{1}{4}}\right)$.
Proof: Let $\rho=\gamma^{1 / 4} / 2$, and $M=32 / 15$. Assume by way of contradiction, that there exists a subgraph $G^{\prime \prime}\left(V^{\prime}, E^{\prime \prime}\right)$ of $G^{\prime}$ of cost strictly less than $\frac{\rho m}{M}$ that realizes $m$ units of flow from $s$ to $t$. We say a vertex $a \in A$ is light if the number of edges of the form $\left\{(a, a(\ell)) \mid \ell \in L_{A}\right\}$ in $G^{\prime \prime}$ is less than $\rho$. Similarly, we say a vertex $b \in B$ is light if the number of edges of the from $\left\{(b(\ell), b) \mid \ell \in L_{B}\right\}$ in $G^{\prime \prime}$ is less than $\rho$. All other vertices in $A \cup B$ are referred to as heavy vertices. Note that at most $1 / M$ fraction of vertices in $A$ could be heavy, for otherwise the total cost of the edges in $E^{\prime \prime}$ would exceed $\frac{|A|}{M} \cdot \rho \cdot d_{A}=\frac{\rho m}{M}$. Similarly, at most $1 / M$ fraction of vertices in $B$ could be heavy.

Now fix any integral $s$ - $t$ flow $f$ of value $m$ in $G^{\prime \prime}$; an integral flow exists since all capacities are integers. We start by deleting from $G^{\prime \prime}$ all heavy vertices. Since at most $1 / M$ fraction of either $A$ or $B$ are deleted, the total residual flow in this network is at least $\left(1-\frac{2}{M}\right) m=\frac{m}{16}$ (recall that $M=32 / 15$ ) since at most $d_{A}$ units of flow can transit through a vertex in $A$, and at most $d_{B}$ units of flow can transit through a vertex in $B$.

Let $F$ be a decomposition of the residual flow into unit flow paths. Note that $|F|=m / 16$. By construction of $G^{\prime}$, every flow path $f \in F$ is of the form $s \rightarrow a \rightarrow \ell_{a} \rightarrow \ell_{b} \rightarrow b \rightarrow t$ where the pair $\left(\ell_{a}, \ell_{b}\right) \in \pi_{(a, b)}$. We say that an edge $(a, b) \in E$ is a good edge if there is a flow path $f$ of the above form, and we say $f$ is a certificate for edge $(a, b)$ being good. Note that every flow path $f$ is a certificate of exactly one edge $(a, b)$. We claim that there are at least $\frac{m}{16 \rho^{2}}$ good edges in $G$. It suffices to show that for any edge $(a, b) \in E$, at most $\rho^{2}$ flow paths in $F$ can certify that $(a, b)$ as a good edge. Since $a$ and $b$ are both light vertices, we know that $\left|\left\{\left(a, \ell_{a}\right) \mid \ell_{a} \in L_{A}\right\} \cap E^{\prime \prime}\right| \leq \rho$ and $\left|\left\{\left(\ell_{b}, b\right) \mid \ell_{b} \in L_{B}\right\} \cap E^{\prime \prime}\right| \leq \rho$. Now using the fact that each edge $\left(\ell_{a}, \ell_{b}\right)$ has unit capacity, it follows that at most $\rho^{2}$ paths in $F$ can certify $(a, b)$ as a good edge. Hence number of good edges in $E$ is at least $\frac{m}{16 \rho^{2}}$.

We now show existence of a labeling $\phi$ that makes at least $\frac{1}{\gamma}$ fraction of the edges to be consistent, contradicting that we were given a No-Instance of label cover. For a vertex $a \in A$, let $\Gamma(a):=\left\{\ell_{a} \in\right.$ $\left.L_{A} \mid\left(a, \ell_{a}\right) \in E^{\prime \prime}\right\}$. Similarly, we define $\Gamma(b)$ for each vertex $b \in B$. Consider the following random label assignment: each vertex $a \in A$ is assigned uniformly at random a label from $\Gamma(a)$, and each vertex in $B$ is assigned uniformly at random a label in $\Gamma(b)$. For any good edge $(a, b)$, the probability that the random labeling makes it consistent is at least $\frac{1}{\rho^{2}}$ since $|\Gamma(a)|$ and $|\Gamma(b)|$ are both less than $\rho$ (as $a$ and $b$ are light), and there exists an $\ell_{a} \in \Gamma_{A}$ and $\ell_{b} \in \Gamma_{B}$ such that $\left(\ell_{a}, \ell_{b}\right) \in \pi_{(a, b)}$. Thus, in expectation, at least $\frac{1}{\rho^{2}}$ fraction of good edges are made consistent by the random assignment. Hence there exists a labeling $\phi$ that $\frac{m}{16 \rho^{4}}=\frac{m}{\gamma}$ edges in $G$ consistent.

Since the graph $G^{\prime}$ can be constructed from $G$ in poly-time, it follows that a poly-time ( $\gamma^{1 / 4} / 5$ )approximation algorithm for single-pair Cap-SNDP would give a poly-time algorithm to decide whether a given instance of label cover is a Yes-Instance or a No-Instance.

## 4 Cap-SNDP with Multiple Copies Allowed

We now consider the version of Cap-SNDP when multiple copies of any edge $e$ can be chosen; that is, for any integer $\alpha \geq 0, \alpha$ copies of $e$ can be bought at a cost $\alpha \cdot c(e)$ to obtain a capacity of $\alpha \cdot u(e)$. Allowing multiple copies makes the problem easier, and Goemans et al. [14] give a $O\left(\log R_{\max }\right)$ factor approximation algorithm for the problem. In this section, we complement this result with a $O(\log k)$ factor approximation algorithm, where $k$ is the number of $(i, j)$ pairs with $R_{i j}>0.3$ Our algorithm is inspired by the work of Berman and Coulston [5] on online Steiner Forest. For notational convenience, we rename the pairs $\left(s_{1}, t_{1}\right), \cdots,\left(s_{k}, t_{k}\right)$, and denote the requirement $R_{s_{i}, t_{i}}$ as $R_{i}$; the vertices $s_{i}, t_{i}$ are referred to as terminals. We also assume that the pairs are so ordered that $R_{1} \geq R_{2} \geq \cdots \geq R_{k}$.

We first give an intuitive overview of the algorithm. The algorithm considers the pairs in decreasing order of requirements, and maintains a forest solution connecting the pairs that have been already been processed; that is, if we retain a single copy of each edge in the partial solution constructed so far, we obtain a forest $F$. For any edge $e$ on the path in $F$ between $s_{j}$ and $t_{j}$, the total capacity of copies of $e$ will be at least $R_{j}$. When considering $s_{i}, t_{i}$, we connect them as cheaply as possible, assuming that edges previously selected for $F$ have 0 cost. (Note that this can be done since we are processing the pairs in decreasing order of requirements and for each edge already present in $F$, the capacity of its copies is at least $R_{i}$.) The key step of the algorithm is that in addition to connecting $s_{i}$ and $t_{i}$, we also connect the pair to certain other components of $F$ that are "nearby". The cost of these additional connections can be bounded by the cost of the direct connection costs between the pairs. These additional connections are useful in allowing subsequent pairs of terminals to be connected cheaply. In particular, they allow us to prove a $O(\log k)$ upper bound on the approximation factor.

We now describe the algorithm in more detail. The algorithm maintains a forest $F$ of edges that have already been bought; $F$ satisfies the invariant that, after iteration $i-1$, for each $j \leq i-1, F$ contains a unique path between $s_{j}$ and $t_{j}$. In iteration $i$, we consider the pair $s_{i}, t_{i}$. We define the cost function $c_{i}(e)$ as $c_{i}(e):=0$ for edges $e$ already in $F$, and $c_{i}(e):=c(e)+\frac{R_{i}}{u(e)} c(e)$, for edges $e \notin F$. Note that for an edge $e \notin F$, the $\operatorname{cost} c_{i}(e)$ is sufficient to buy enough copies of $e$ to achieve a total capacity of $R_{i}$. Thus it suffices to connect $s_{i}$ and $t_{i}$ and pay cost $c_{i}(e)$ for each edge; in the Cap-SNDP solution we would pay at most this cost and get a feasible solution. However, recall that our algorithm also connects $s_{i}$ and $t_{i}$ to other "close" components; to describe this process, we introduce some notation:

For any vertices $p$ and $q$, we use $d_{i}(p, q)$ to denote the distance between $p$ and $q$ according to the metric given by edge costs $c_{i}(e)$. We let $\ell_{i}:=d_{i}\left(s_{i}, t_{i}\right)$ be the cost required to connect $s_{i}$ and $t_{i}$, given the current solution $F$. We also define the class of a pair $\left(s_{j}, t_{j}\right)$, and of a component:

- For each $j \leq i$, we say that pair $\left(s_{j}, t_{j}\right)$ is in class $h$ if $2^{h} \leq \ell_{j}<2^{h+1}$.

Equivalently, $\operatorname{class}(j)=\left\lfloor\log \ell_{j}\right\rfloor$.

- For each connected component $X$ of $F, \operatorname{class}(X)=\max _{\left(s_{j}, t_{j}\right) \in X} \operatorname{class}(j)$.

Now, the algorithm connects $s_{i}$ (respectively $t_{i}$ ) to component $X$ if $d_{i}\left(s_{i}, X\right)\left(\right.$ resp. $\left.d_{i}\left(t_{i}, X\right)\right) \leq$ $2^{\min \{c l a s s(i), c l a s s(X)\}}$. That is, if $X$ is close to the pair $\left(s_{i}, t_{i}\right)$ compared to the classes they are in, we connect $X$ to the pair. As we show in the analysis, this extra connection cost can be charged to some pair $\left(s_{j}, t_{j}\right)$ in the component $X$. The complete algorithm description is given below.

[^2]```
CAP-SNDP-MC:
\(F \leftarrow \emptyset \quad\langle\langle F\) is the forest solution returned \(\rangle\rangle\)
For \(i \leftarrow 1\) to \(k\)
    For each edge \(e \in F, c_{i}(e) \leftarrow 0\)
    For each edge \(e \notin F, c_{i}(e) \leftarrow c(e)+\left(R_{i} / u(e)\right) c(e)\)
    \(\ell_{i} \leftarrow d_{i}\left(s_{i}, t_{i}\right)\)
    Add to \(F\) a shortest path (of length \(\ell_{i}\) ) from \(s_{i}\) to \(t_{i}\) under distances \(c_{i}(e)\)
    \(\operatorname{class}(i) \leftarrow\left\lfloor\log \ell_{i}\right\rfloor\)
    For each connected component \(X\) of \(F\)
        If \(d_{i}\left(s_{i}, X\right) \leq 2^{\min \{c l a s s}(i)\), class \(\left.(X)\right\}\)
            Add to \(F\) a shortest path connecting \(s_{i}\) and \(X\)
    For each connected component \(X\) of \(F\)
        If \(d_{i}\left(t_{i}, X\right) \leq 2^{\min \{c l a s s(i), \text { class }(X)\}}\)
            Add to \(F\) a shortest path connecting \(t_{i}\) and \(X\)
    Buy \(\left\lceil R_{i} / u_{e}\right\rceil\) copies of each edge \(e\) added during this iteration.
```

We prove that this algorithm CAP-SNDP-MC gives an $O(\log k)$ approximation.

The structure of our proof is as follows: Recall that $\ell_{i}$ was the direct connection cost between $s_{i}$ and $t_{i}$; in addition to paying $\ell_{i}$ to connect these vertices, the algorithm also buys additional edges connecting $s_{i}$ and $t_{i}$ to existing components. We first show (in Lemma 4.1) that the total cost of extra edges bought can be charged to the direct connection costs; thus, it suffices to show that $\sum_{i} \ell_{i} \leq O(\log k) \mathrm{OPT}$, where OPT is the cost of an optimal solution. To prove this (Lemma 4.2), we bucket the pairs ( $s_{i}, t_{i}$ ) into $O(\log k)$ groups based on class $(i)$, and show that in each bucket $h, \sum_{i: c l a s s}(i)=h{ }_{i} \leq O(\mathrm{OPT})$.

Lemma 4.1. The total cost of all edges bought by CAP-SNDP-MC is at most $9 \sum_{i=1}^{k} \ell_{i}$.
Proof: Let $F_{i}$ denote the set of edges added to $F$ during iteration $i$. First, note the total cost paid for copies of edge $e \in F_{i}$ is $\left\lceil\frac{R_{i}}{u(e)}\right\rceil c(e)<c(e)+\frac{R_{i}}{u_{e}} c(e)=c_{i}(e)$. Thus, it suffices to show:

$$
\sum_{i=1}^{k} \sum_{e \in F_{i}} c_{i}(e) \leq 9 \sum_{i=1}^{k} \ell_{i}
$$

We prove that the total cost of the additional edges bought is at most $8 \sum_{i=1}^{k} \ell_{i}$; this clearly implies the desired inequality. It is not true that for each $i$, the total cost of additional edges bought during iteration $i$ is at most $8 \ell i$. Nonetheless, a careful charging scheme proves the needed bound on total cost. In iteration $i$, suppose we connect the pair $\left(s_{i}, t_{i}\right)$ to the components $X_{1}, \ldots, X_{r}$. We charge the cost of connecting ( $s_{i}, t_{i}$ ) and component $X_{j}$ to the connection cost $\ell_{j}$ of a pair $\left(s_{j}, t_{j}\right)$ in $X_{j}$. This is possible since we know the additional connection cost is at most $2^{\text {class }\left(X_{j}\right)}$. Care is required to ensure no pair is overcharged. To do so, we introduce some notation.

At any point during the execution of the algorithm, for any current component $X$ of $F$, we let Leader $(X)$ be a pair $\left(s_{i}, t_{i}\right) \in X$ such that class $(i)=\operatorname{class}(X)$. For integers $h \leq \operatorname{class}(X)$, $h$-Leader $(X)$ will denote a pair $\left(s_{j}, t_{j}\right)$ in $X$; we explain how this pair is chosen later. (Initially, $h$-Leader $(X)$ is undefined for each component $X$.)

Now, we have to account for additional edges bought during iteration $i$; these are edges on a shortest path connecting $s_{i}$ (or $t_{i}$ ) to some other component $X$; we assume w.l.o.g. that the path is from $s_{i}$ to $X$. Consider any such path $P$ connecting $s_{i}$ to a component $X$; we have $\sum_{e \in P} c_{i}(e)=d_{i}\left(s_{i}, X\right) \leq$ $2^{\min \{c l a s s(i), \text { class }(X)\}}$. Let $h=\left\lfloor\log d_{i}\left(s_{i}, X\right)\right\rfloor$ : Charge all edges on this path to $h$-Leader $(X)$ if it is
defined; otherwise, charge all edges on the path to Leader $(X)$. In either case, the pair $\left(s_{i}, t_{i}\right)$ becomes the $h$-Leader of the new component just formed. Note that a pair $\left(s_{i}, t_{i}\right)$ could simultaneously be the $h_{1}$-Leader, $h_{2}$-Leader, etc. for a component $X$ if $\left(s_{i}, t_{i}\right)$ connected to many components during iteration $i$. However, it can never be the $h$-Leader of a component for $h>\operatorname{class}(i)$, and once it has been charged as $h$-Leader, it is never charged again as $h$-Leader. Also observe that if a pair is in a component $X$ whose $h$-Leader is defined, subsequently, it always stays in a component in which the $h$-Leader is defined.

For any $i$, we claim that the total charge to pair $\left(s_{i}, t_{i}\right)$ is at most $8 \ell_{i}$, which completes the proof. Consider any such pair: any charges to the pair occur when it is either Leader or $h$-Leader of its current component. First, consider charges to $\left(s_{i}, t_{i}\right)$ as Leader of a component. Such a charge can only occur when connecting some $s_{j}$ (or $t_{j}$ ) to $X$. Furthermore, if $h=\left\lfloor\log d_{j}\left(s_{j}, X\right)\right\rfloor \leq \operatorname{class}(X)=$ class $(i)$, the $h$-Leader ( $X$ ) must be currently undefined, for otherwise the $h$-Leader $(X)$ would have been charged. Subsequently, the $h$-Leader of the component containing $\left(s_{i}, t_{i}\right)$ is always defined, and so ( $s_{i}, t_{i}$ ) will never again be charged as a Leader $(X)$ by a path of length in $\left[2^{h}, 2^{h+1}\right)$. Therefore, the total charge to $\left(s_{i}, t_{i}\right)$ as Leader of a component is at most $\sum_{h=1}^{\text {class }(i)} 2^{h+1}<2^{\text {class(i)+2 }} \leq 4 \ell_{i}$.

Finally, consider charges to $\left(s_{i}, t_{i}\right)$ as $h$-Leader of a component. As observed above, $h \leq \operatorname{class}(i)$. Also for a fixed $h$, a pair is charged at most once as $h$-Leader. Since the total cost charged to $\left(s_{i}, t_{i}\right)$ as $h$-Leader is at most $2^{h+1}$; summing over all $h \leq \operatorname{class}(i)$, the total charge is less than $2^{\text {class }(i)+2}=4 \ell_{i}$.

Thus, the total charge to $\left(s_{i}, t_{i}\right)$ is at most $4 \ell_{i}+4 \ell_{i}=8 \ell_{i}$, completing the proof.
Lemma 4.2. If OPT denotes the cost of an optimal solution to the instance of Cap-SNDP with multiple copies, then $\sum_{i=1}^{k} \ell_{i} \leq 64(\lceil\log k\rceil+1)$ OPT.
Proof: Let $C_{h}$ denote $\sum_{i: c l a s s ~}(i)=h \quad \ell_{i}$. Clearly, $\sum_{i=1}^{k} \ell_{i}=\sum_{h} C_{h}$. The lemma follows from the two sub-claims below:
Sub-Claim 1: $\quad \sum_{h} C_{h} \leq(2(\lceil\log k\rceil+1)) \cdot \max _{h} C_{h}$
Sub-Claim 2: For each $h, C_{h} \leq 32 \mathrm{OPT}$.
Proof of Sub-Claim 1: Let $h^{\prime}=\max _{i} \operatorname{class}(i)$. We have $C_{h^{\prime}} \geq 2^{h^{\prime}}$, and for any terminal $i$ such that class $(i) \leq h^{\prime}-(\lceil\log k\rceil+1)$, we have $\ell_{i} \leq \frac{2^{h^{\prime}+1}}{2 k}$. Thus, the total contribution from such classes is at most $\frac{2^{h^{\prime}}}{k} \cdot k=2^{h^{\prime}}$, and hence:

$$
\begin{aligned}
\sum_{h=h^{\prime}-\lceil\log k\rceil}^{h^{\prime}} C_{h} & \geq \frac{\sum_{h} C_{h}}{2}, \text { which implies } \\
\max _{h^{\prime}-\lceil\log k\rceil \leq h \leq h^{\prime}} C_{h} & \geq \frac{\sum_{h} C_{h}}{2(\lceil\log k\rceil+1)} .
\end{aligned}
$$

It remains to show Sub-Claim 2, that for each $h, C_{h} \leq 32 \mathrm{OPT}$. Fix $h$. Let $\mathcal{S}_{h}$ denote the set of pairs $s_{i}, t_{i}$ such that class $(i)=h$. Our proof will go via the natural primal and dual relaxations for the Cap-SNDP problem. In particular, we will exhibit a solution to the dual relaxation of cost $C_{h} / 32$. To do so we will require the following claim. Define ball $\left(s_{i}, r\right)$, a ball of radius $r$ around $s_{i}$ as containing the set of vertices $v$ such that $d_{i}\left(s_{i}, v\right) \leq r$ and the set of edges $e=u v$ such that $d_{i}\left(s_{i},\{u, v\}\right)+c_{i}(e) \leq r$. An edge $e$ is partially within the ball if $d_{i}\left(s_{i},\{u, v\}\right)<r<d_{i}\left(s_{i},\{u, v\}\right)+c_{i}(e)$. Subsequently, we assume for ease of exposition that no edges are partially contained within the balls we consider; this can be achieved by subdividing the edges as necessary. Similarly, we define ball $\left(t_{i}, r\right)$, the ball of radius $r$ around $t_{i}$. Two balls are said to be disjoint if they contain no common vertices.

Claim 4.3. There exists a subset of pairs, $\mathcal{S}_{h}^{\prime} \subseteq \mathcal{S}_{h},\left|\mathcal{S}_{h}^{\prime}\right| \geq\left|\mathcal{S}_{h}\right| / 2$, and a collection of $\left|\mathcal{S}_{h}^{\prime}\right|$ disjoint balls of radius $2^{h} / 4$ centred around either $s_{i}$ or $t_{i}$, for every pair $\left(s_{i}, t_{i}\right) \in \mathcal{S}_{h}^{\prime}$.

We prove this claim later; we now use it to complete the proof of Sub-Claim 2. First we describe the LP. Let the variable $x_{e}$ denote whether or not edge $e$ is in the Cap-SNDP solution. Let $\mathcal{P}_{i}$ be the set of paths from $s_{i}$ to $t_{i}$. For each $P \in \mathcal{P}_{i}$, variable $f_{P}$ denotes how much flow $t$ sends to the root along path $P$. We use $u_{i}(e)$ to refer to $\min \left\{R_{i}, u(e)\right\}$, the effective capacity of edge $e$ for pair $\left(s_{i}, t_{i}\right)$.

$$
\begin{aligned}
& \text { Primal min } \sum_{e \in E} c_{e} x_{e} \\
& \sum_{P \in \mathcal{P}_{i}} f_{P} \geq R_{i} \quad(\forall i \in[k]) \\
& \sum_{P \in \mathcal{P}_{t} \mid e \in P} f_{P} \leq u_{i}(e) x_{e} \quad(\forall i \in[k], e \in E) \\
& x_{e}, f_{P} \geq 0
\end{aligned}
$$

$$
\begin{aligned}
& \text { Dual } \max \sum_{t \in T} R_{i} \alpha_{i} \\
& \sum_{i} u_{i}(e) \beta_{i, e} \leq c_{e} \quad(\forall e \in E) \\
& \alpha_{i} \leq \sum_{e \in P} \beta_{i, e} \quad\left(\forall i \in[k], P \in \mathcal{P}_{i}\right) \\
& \alpha_{i}, \beta_{i, e} \geq 0
\end{aligned}
$$

We now describe a feasible dual solution of value at least $C_{h} / 32$ using Claim4.3, For $\left(s_{i}, t_{i}\right) \in \mathcal{S}_{h}^{\prime}$, if there is a ball $B$ around $s_{i}$ (or equivalently $t_{i}$ ), we define $\beta_{i, e}=c(e) / u_{i}(e)$ for each edge in the ball. Since the balls are disjoint, the first inequality of the dual is clearly satisfied. Set $\alpha_{i}=2^{h} / 8 R_{i}$. For any path $P \in \mathcal{P}_{i}$, we have

$$
\sum_{e \in P} \beta_{i, e}=\frac{1}{R_{i}} \sum_{e \in P \cap B} \frac{R_{i} c(e)}{u_{i}(e)} \geq \frac{1}{2 R_{i}} \sum_{e \in P \cap B} \frac{R_{i} c(e)}{u(e)}+c(e) \geq \frac{1}{2 R_{i}} \sum_{e \in P \cap B} c_{i}(e) \geq \frac{1}{2 R_{i}} \frac{2^{h}}{4}=\alpha_{i}
$$

where the first inequality used $u_{i}(e) \leq R_{i}$, the second follows from the definition of $c_{i}(e)$, and the last inequality follows from the definition of ball $\left(s_{i}, 2^{h} / 4\right)$. Thus, $\alpha_{i}=2^{h} / 8 R_{i}$ is feasible along with these $\beta_{i, e}$ 's. This gives a total dual value of

$$
\frac{2^{h}}{8} \cdot\left|\mathcal{S}_{h}^{\prime}\right| \geq \frac{2^{h}}{16} \cdot\left|\mathcal{S}_{h}\right| \geq \frac{1}{32} \sum_{i \in \mathcal{S}_{h}} \ell_{i}=\frac{C_{h}}{32}
$$

where the last inequality follows from the fact that $\operatorname{class}(i)=h$. This proves the lemma modulo Claim 4.3, which we now prove.
Proof of Claim 4.3: We process the pairs in $\mathcal{S}_{h}$ in the order they are processed by the original algorithm and grow the balls. We abuse notation and suppose these pairs are $\left(s_{1}, t_{1}\right), \ldots,\left(s_{p}, t_{p}\right)$. We maintain a collection of disjoint balls of radius $r=2^{h} / 4$, initially empty.

At stage $i$, we try to grow a ball of radius $r$ around either $s_{i}$ or $t_{i}$. If this is not possible, the ball around $s_{i}$ intersects that around some previous terminal in $\mathcal{S}_{h}^{\prime}$, say $s_{j}$; similarly, the ball around $t_{i}$ intersects that of a previous terminal, say $t_{\ell}$. Let $v$ be a vertex in ball $\left(s_{i}, r\right)$ and ball $\left(s_{j}, r\right)$. We have $d_{i}\left(s_{i}, s_{j}\right) \leq d_{i}\left(s_{i}, v\right)+d_{i}\left(v, s_{j}\right) \leq d_{i}\left(s_{i}, v\right)+d_{j}\left(v, s_{j}\right)<2^{h} / 2$. (The second inequality follows because for any $j<i$ and any edge $e, c_{i}(e) \leq c_{j}(e)$.) Similarly, we have $d_{i}\left(t_{i}, t_{\ell}\right)<2^{h} / 2$.

Now, we observe that $s_{j}$ and $t_{\ell}$ could not have been in the same component of $F$ at the beginning of iteration $i$ of CAP-SNDP-MC; otherwise $d_{i}\left(s_{i}, t_{i}\right) \leq d_{i}\left(s_{i}, s_{j}\right)+d_{i}\left(t_{i}, t_{\ell}\right)<2^{h}$, contradicting that $\operatorname{class}(i)=h$. But since $d_{i}\left(s_{i}, s_{j}\right) \leq 2^{h} / 2$ and $\operatorname{class}(i)=\operatorname{class}(j)=h$, we connect $s_{i}$ to the component of $s_{j}$ during iteration $i$; likewise, we connect $t_{i}$ to the component of $t_{\ell}$ during this iteration. Hence, at the end of the iteration, $s_{i}, t_{i}, s_{j}, t_{\ell}$ are all in the same component. As a result, the number of components of $F$ containing pairs of $\mathcal{S}_{h}$ decreases by at least one during the iteration.

It is now easy to complete the proof: During any iteration of $F$ corresponding to a pair $\left(s_{i}, t_{i}\right) \in \mathcal{S}_{h}$, the number of components of $F$ containing pairs of $\mathcal{S}_{h}$ can go up by at most one. Say that an iteration succeeds if we can grow a ball of radius $r$ around either $s_{i}$ or $t_{i}$, and fails otherwise. During any iteration that fails, the number of components decreases by at least one; as the number of components is always non-negative, the number of iterations which fail is no more than the number which succeed. That is, $\left|\mathcal{S}_{h}^{\prime}\right| \geq\left|\mathcal{S}_{h}-\mathcal{S}_{h}^{\prime}\right|$.

Theorem 1.5 is now a straightforward consequence of Lemmas 4.1 and 4.2:
Proof of Theorem 1.5: The total cost of edges bought by the algorithm is at most $\sum_{i=1}^{k} \sum_{e \in F_{i}} c_{i}(e) \leq$ $9 \sum_{i=1}^{k} \ell_{i}$, by Lemma 4.1. But $\sum_{i=1}^{k} \ell_{i} \leq 64(\lceil\log k\rceil+1)$ OPT, by Lemma 4.2, and hence the total cost paid by CAP-SNDP-MC is at most $O(\log k) \mathrm{OPT}$.

## 5 Conclusions

In this paper we made progress on addressing the approximability of Cap-SNDP. We gave an $O(\log n)$ approximation for the Cap- $R$-Connected Subgraph problem, which is a capacitated generalization of the well-studied min-cost $\lambda$-edge-connected subgraph problem. Can we improve this to obtain an $O(1)$ approximation or prove super-constant factor hardness of approximation? We also highlighted the difficulty of Cap-SNDP by focusing on the single pair problem, and showing both super-constant hardness and an $\Omega(n)$ integrality gap example, even for the LP with KC inequalities. We believe that understanding the single pair problem is the key to understanding the general case. In particular, we do not have a non-trivial algorithm even for instances in which the edge capacities are either 1 or $U$; this appears to capture much of the difficulty of the general problem. As we noted, allowing multiple copies of edges makes the problem easier; in practice, however, it may be desirable to not allow too many copies of an edge to be used. It is therefore of interest to examine the approximability of CapSNDP if we allow only a small number of copies of an edge. Does the problem admit a non-trivial approximation if we allow $O(1)$ copies or, say, $O(\log n)$ copies? This investigation may further serve to delineate the easy versus difficult cases of Cap-SNDP.

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## A Hardness of Approximation for Cap-SNDP in Undirected Graphs

In this section, we prove Theorem 1.6 via a reduction from the Priority Steiner Tree problem. In the Priority Steiner Tree problem, the input is an undirected graph $G(V, E)$ with a cost $c(e)$ and a priority $P(e) \in\{1,2, \ldots, k\}$ for each edge $e$. (We assume $k$ is the highest and 1 the lowest priority.) We are also given a root $r$ and a set of terminals $T \subseteq V-\{r\}$; each terminal $t \in T$ has a desired priority $P(t)$. The goal is to find a minimum-cost Steiner Tree in which the unique path from each terminal $t$ to the root consists only of edges of priority $P(t)$ or higher $4^{4}$

Chuzhoy et al. 9] showed that one cannot approximate the Priority Steiner Tree problem within a factor better than $\Omega(\log \log n)$ unless $N P \subseteq D T I M E\left(n^{\log \log \log n}\right)$, even when all edge costs are 0 or 1. Here, we show an approximation-preserving reduction from this problem to Cap-SNDP with multiple copies; this also applies to the basic Cap-SNDP problem, as the copies of edges do not play a significant role in the reduction.

Given an instance $\mathcal{I}_{p s t}$ of Priority Steiner Tree on graph $G(V, E)$ with edge costs in $\{0,1\}$, we construct an instance $\mathcal{I}_{\text {cap }}$ of Cap-SNDP defined on the graph $G$ as the underlying graph. Fix $R$ to be any integer greater than $2 m^{3}$ where $m$ is the number of edges in the graph $G$. We now assign a capacity of $u(e)=R^{i}$ to each edge $e$ with priority $P(e)=i$ in $\mathcal{I}_{p s t}$. Each edge $e$ of cost 0 in $\mathcal{I}_{\text {pst }}$ has cost $c(e)=1$ in $\mathcal{I}_{\text {cap }}$, and each edge $e$ of cost 1 in $\mathcal{I}_{\text {pst }}$ has cost $c(e)=m^{2}$ in $\mathcal{I}_{\text {cap }}$. Finally, for each terminal $t$, set $R_{t r}=R^{i}$ if $P(t)=i$; for every other pair of vertices $(p, q), R_{p q}=0$.

Let $C$ denotes the cost of an optimal solution to $\mathcal{I}_{p s t}$; note that $C \leq m$; we now argue that $\mathcal{I}_{p s t}$ has an optimal solution of cost $C$ iff $\mathcal{I}_{\text {cap }}$ has an optimal solution of of cost between $C m^{2}$ and $C m^{2}+m<(C+1) m^{2}$. Given a solution $E^{*}$ to $\mathcal{I}_{p s t}$ of cost $C$, simply select the same edges for $\mathcal{I}_{\text {cap }} ;$ the cost in $\mathcal{I}_{\text {cap }}$ is at most $C m^{2}+m$ since in $\mathcal{I}_{\text {cap }}$, we pay 1 for each edge in $E^{*}$ that has cost 0 in $\mathcal{I}_{\text {pst }}$. This is clearly a feasible solution to $\mathcal{I}_{\text {cap }}$ as each terminal $t$ has a path to $r$ in $E^{*}$ containing only edges with priority at least $P(t)$, which is equivalent to having capacity at least $R_{t r}$. Conversely, given a solution $E^{\prime}$ to $\mathcal{I}_{\text {cap }}$ with cost in $\left[C m^{2},(C+1) m^{2}\right)$, select a single copy of each edge in $E^{\prime}$ as a solution to $\mathcal{I}_{p s t}$; clearly the total cost is at most $C$. To see that this is a feasible solution, suppose that $E^{\prime}$ did not contain a path from some terminal $t$ to the root $r$ using edges of priority $P(t)$ or more. Then there must be a cut separating $t$ from $r$ in which all edges of $E^{\prime}$ have capacity at most $R^{P(t)-1}$. But since $E^{\prime}$ supports a flow of $R^{P(t)}$ from $t$ to $r$, it must use at least $R$ edges (counting with multiplicity); this implies that the cost of $E^{\prime}$ is at least $R \geq(C+1) m^{2}$, a contradiction.

We remark that a similar reduction also proves $\Omega(\log \log n)$ hardness for the single-pair Cap-SNDP problem without multiple copies: One can effectively encode an instance of the single-source FixedCharge Network Flow (FCNF, [9]), very similar to single-source Cap-SNDP with multiple copies, as an instance of single-pair Cap-SNDP without multiple copies: Create a new sink $t^{*}$, and connect $t^{*}$ to each original terminal $t$ with a single edge of cost 0 and capacity $R_{t r}$. The only way to send flow $\sum_{t \in T} R_{t r}$ flow from $t^{*}$ to the source $s$ is for each terminal $t$ to send $R_{t r}$ to $s$. Thus, Theorem 1.3 is a simple consequence of the $\Omega(\log \log n)$ hardness for single-source FCNF 9].

[^3]
## B Omitted Proofs

## B. 1 Proof of Theorem 1.1: Near Uniform Cap-SNDP

The algorithm described in Section 2 can be extended to the case where requirements are nearly uniform, that is, if $R_{p q} \in[R, \gamma R]$ for all pairs $(p, q) \in V \times V$. We obtain an $O(\gamma \log n)$-approximation, while increasing the running time by a factor of $O\left(n^{4 \gamma}\right)$. We work with a similar LP relaxation; for each set $S \subseteq 2^{V}$, we use $R(S)=\max _{p \in S, q \notin S}\left\{R_{p q}\right\}$ to denote the requirement of $S$. Now, the original constraints are of the form

$$
\sum_{e \in \delta(S)} u(e) x_{e} \geq R(S)
$$

for each set $S$, and we define the residual requirement for a set as $R(S, A)=\min \{0, R(S)-u(A \cap \delta(S))\}$. The KC inequalities use this new definition of $R(S, A)$.

Given a fractional solution $x$ to the KC LP, we modify the definitions of highly fractional and nearly integral edges: An edge $e$ is said to be nearly integral if $x_{e} \geq \frac{1}{40 \gamma \log n}$, and highly fractional otherwise. Again, for a fractional solution $x$, we let $A_{x}$ denote the set of nearly integral edges; the set $\mathcal{S}$ of small cuts is now $\{S \subseteq V: \hat{u}(\delta(S)) \leq 2 \gamma R\}$. From the cut-counting theorem, $|\mathcal{S}| \leq n^{4 \gamma}$. We use $\mathcal{L}$ to denote the set of large cuts, the sets $\{S \subseteq V: \hat{u}(\delta(S))>2 \gamma R\}$.

As before, a fractional solution $x$ is good if the original constraints are satisfied, and the KC Inequalities are satisfied for the set of edges $A_{x}$ and the sets in $\mathcal{S}$. These constraints can be checked in time $O\left(n^{4 \gamma+2} \log ^{2} n\right)$, so following the proof of Lemma 2.4, for constant $\gamma$, we can find a good fractional solution in polynomial time.

The rounding and analysis proceed precisely as before: For each highly fractional edge $e$, we select it for the final solution with probability $40 \gamma \log n \cdot x_{e}$. The expected cost of this solution is at most $O(\gamma \log n)$ times that of the optimal integral solution, and analogously to the proofs of Lemmas 2.6 and 2.7, one can show that the solution satisfies all cuts with high probability. This completes the proof of Theorem 1.1,

## B. 2 Proof of Theorem 1.2

To prove Theorem 1.2, we work with the generalization of (KC LP ) given below. For any $i$-way cut $\mathcal{C}$ and for any set of edges $A$, we use $R(\mathcal{C}, A)$ to be $\max \left\{0, R_{i}-u(A \cap \delta(C)\}\right.$

$$
\begin{array}{lrl} 
& \min \sum_{e \in E} c(e) x_{e} & \text { ( } k \text {-way KC LP) } \\
\forall A \subseteq E, \forall i, \forall i \text {-way cuts } \mathcal{C}, & \forall i, \forall i \text {-way cuts } \mathcal{C}, & \sum_{e \in \delta(\mathcal{C})} u(e) x_{e} \geq R_{i} \\
\text { (Original Constraints) } \\
\sum_{e \in \delta(\mathcal{C}) \backslash A} \min \{u(e), R(\mathcal{C}, A)\} x_{e} \geq R(\mathcal{C}, A) & \text { (KC-inequalities) } \\
\forall e \in E, & 0 \leq x_{e} \leq 1 &
\end{array}
$$

As before, given a fractional solution $x$ to this LP, we define $A_{x}$ (the set of nearly integral edges) to be $\left\{e \in E: x_{e} \geq \frac{1}{40 k \log n}\right\}$. Define $\hat{u}(e)=u(e) x_{e}$ to be the fractional capacity on the edges. Let

[^4]$\mathcal{S}_{i}:=\left\{\mathcal{C}: \mathcal{C}\right.$ is an $i+1$-way cut and $\left.\hat{u}(\delta(C)) \leq 2 R_{i}\right\}$. The solution $x$ is said to be good if it satisfies the following three conditions:
(a) If the capacity of $e$ is $\hat{u}(e)$, the capacity of any $i+1$-way cut in $G$ is at least $R_{i}$; equivalently $x$ satisfies the original constraints.
(b) The KC inequalities are satisfied for the set $A_{x}$ and the sets in $\mathcal{S}_{i}$, for each $1 \leq i \leq k-1$. Note that if (a) is satisfied, then by Lemma 2.8, $\left|\mathcal{S}_{i}\right| \leq n^{4 i}$.
(c) $\sum_{e \in E} c(e) x_{e}$ is at most the value of the optimum solution to the linear program ( $k$-way KC LP).

Following the proof of Lemma 2.4, it is straightforward to verify that there is a randomized algorithm that computes a good fractional solution with high probability in $n^{O(k)}$ time.

Once we have a good fractional solution, our algorithm is to select $A_{x}$, the set of nearly integral edges, and to select each highly fractional edge $e \in E \backslash A_{x}$ with probability $40 k \log n \cdot x_{e}$. If $F^{*}$ denotes the highly fractional edges that were selected, we return the solution $A_{x} \cup F^{*}$. As before, it is trivial to see that the expected cost of this solution is $O(k \log n)$ times that of the optimal integral solution.

We show below that for any $i \leq k-1$, we satisfy all $i+1$-way cuts with high probability; taking the union bound over the $k-1$ choices of $i$ yields the theorem.

As in Lemmas 2.6 and 2.7, we separately consider the "large" and "small" $i+1$-way cuts. First, consider any small cut $\mathcal{C}$ in $\mathcal{S}_{i}$. From the Chernoff bound (Lemma 2.5) and the KC inequality for $\mathcal{C}$ and $A_{x}$, it follows that the probability we fail to satisfy $\mathcal{C}$ is at most $1 / n^{19 k}$. From the cut-counting Lemma 2.8, there are at most $n^{4 i}<n^{4 k}$ such small cuts, so we satisfy all the small $i+1$ way cuts with probability at least $1-\frac{1}{n^{15 k}}$.

For the large $i+1$-way cuts $\mathcal{L}$, we separately consider cuts of differing capacities. For each $j \geq 2$, let $\mathcal{L}(j)$ denote the $i+1$-way cuts $\mathcal{C}$ such that $j R_{i} \leq \hat{u}(\mathcal{C}) \leq(j+1) R_{i}$. Consider any cut $\mathcal{C} \in \mathcal{L}_{j}$; if $u\left(A_{x} \cap \delta(C)\right) \geq R_{i}$, then the cut $\mathcal{C}$ is clearly satisfied. Otherwise, $\hat{u}\left(\delta(\mathcal{C}) \backslash A_{x}\right) \geq(j-1) R_{i}$. But since we selected each edge $e$ in $\delta(\mathcal{C}) \backslash A_{x}$ for $F^{*}$ with probability $40 k \log n \cdot x_{e}$, the Chernoff bound implies that we do not satisfy $\mathcal{C}$ with probability at most $\frac{1}{n^{19 k(j-1)}}$. The cut-counting Lemma 2.8 implies there are most $n^{2 i(j+1)}<n^{2 k(j+1)}$ such cuts, so we fail to satisfy any cut in $\mathcal{L}(j)$ with probability at most $n^{21-17 j}$. Taking the union bound over all $j$, the failure probability is at most $2 n^{-13}$.

This figure "GapExample.png" is available in "png" format from: http://arxiv.org/ps/1009.5734v1


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[^1]:    ${ }^{1} \mathrm{An} i$-way cut $\mathcal{C}$ of a graph $G(V, E)$ is a partition of its vertices into $i$ non-empty sets $V_{1}, \ldots, V_{i}$; we use $\delta(\mathcal{C})$ to denote the set of edges with endpoints in different sets of the partition $\mathcal{C}$. The capacity of an $i$-way cut $\mathcal{C}$ is the total capacity of edges in $\delta(\mathcal{C})$.
    ${ }^{2}$ In 6] it is mentioned that there is a series-parallel graph instance of Cap-SNDP such that the LP with knapsack-cover inequalities has an integrality gap of at least $\lfloor\beta(G) / 2\rfloor+1$. However, no example is given; it is not clear if the gap applied to a single pair instance or if $\beta(G)$ could be as large as $n$ in the construction.

[^2]:    ${ }^{3}$ Note that we overload the letter ' $k$ ', previously used in the definition of the $k$-Way- $\mathcal{R}$-Connected Subgraph problem; this should cause no ambiguity as we discuss only pairwise connectivity requirements in this section.

[^3]:    ${ }^{4}$ It is easy to see that a minimum-cost subgraph containing such a path for each terminal is a tree; given any cycle, one can remove the edge of lowest priority.

[^4]:    ${ }^{5}$ For ease of notation, we assume that for any edge $e, u(e) \leq R_{1}$. This is not without loss of generality, but the proof can be trivially generalized: In the constraint for each $i+1$-way cut $\mathcal{C}$ such that $e \in \delta(\mathcal{C})$, simply use the minimum of $u(e)$ and $R_{i}$.

