# On the Number of Edges of Fan-Crossing Free Graphs* 

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#### Abstract

A graph drawn in the plane with $n$ vertices is $k$-fan-crossing free for $k \geqslant 2$ if there are no $k+1$ edges $g, e_{1}, \ldots, e_{k}$, such that $e_{1}, e_{2}, \ldots, e_{k}$ have a common endpoint and $g$ crosses all $e_{i}$. We prove a tight bound of $4 n-8$ on the maximum number of edges of a 2 -fan-crossing free graph, and a tight $4 n-9$ bound for a straight-edge drawing. For $k \geqslant 3$, we prove an upper bound of $3(k-1)(n-2)$ edges. We also discuss generalizations to monotone graph properties.


## 1 Introduction

A topological graph $G$ is a graph drawn in the plane: vertices are points, and the edges of the graph are drawn as Jordan curves connecting the vertices. Edges are not allowed to pass through vertices other than their endpoints. We will assume the topological graph to be simple, that is, any pair of its edges have at most one point in common (so edges with a common endpoint do not cross, and edges cross at most once). Figure 1 (a-b) shows configurations that are not allowed.

If there are no crossings between edges, then the graph is planar, and Euler's formula implies that it has at most $3 n-6$ edges, where $n$ is the number of vertices. What can be said if we relax this restriction-that is, we permit some edge crossings?

For instance, a topological graph is called $k$-planar if each edge is crossed at most $k$ times. Pach and Tóth [PT97] proved that a $k$-planar graph on $n$ vertices has at most $(k+3)(n-2)$ edges for $0 \leqslant k \leqslant 4$, and at most $4.108 \sqrt{ } k n$ edges for general $k$. The special case of 1-planar graphs has recently received some attention, especially in the graph drawing community. Pach and Tóth's bound is $4 n-8$, and this is tight: starting with a planar graph $H$ where every face is a quadrilateral, and adding both diagonals results in a 1-planar graph with $4 n-8$ edges. However, Didimo [Did13] showed that straight-line 1-planar graphs have at most $4 n-9$ edges, showing that Fáry's theorem does not generalize to 1-planar graphs. This bound is tight, as he constructed an infinite family of straight-line 1-planar graphs with $4 n-9$ edges. Hong et al. [HELP12] characterize the 1-planar graphs that can be drawn as straight-line 1-planar graphs. Grigoriev and Bodlaender [GB07] showed that testing if a given graph is 1-planar is NP-hard.

A topological graph is called $k$-quasi planar if it does not contain $k$ pairwise crossing edges. It is conjectured that for any fixed $k$ the number of edges of a $k$-quasi planar graph is linear in the number of vertices $n$. Agarwal et al. $\left[\mathrm{AAP}^{+} 97\right]$ proved this for straight-line 3-planar graphs, Pach et al. [PRT02] for general 3-planar graphs, Ackerman [Ack09] for 4-planar graphs, and Fox et al. [FPS13] prove a bound of the form $O\left(n \log ^{1+o(1)} n\right)$ for $k$-planar graphs.

A different restriction on crossings arises in graph drawing: Humans have difficulty reading graph drawings where edges cross at acute angles, but graph drawings where edges cross at right angles are nearly as readable

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Figure 1: (a) and (b) shows illegal embeddings of edges of a graph. (c) is a 4-fan crossing, (d) a fan crossing. as planar ones. A right-angle crossing graph (RAC graph) is a topological graph with straight edges where edges that cross must do so at right angle. Didimo et al. [DEL11] showed that an RAC graph on $n$ vertices has at most $4 n-10$ edges. Testing whether a given graph is an RAC graph is NP-hard [ABS12]. Eades and Liotta [EL13] showed that an extremal RAC graph, that is, an RAC graph with $n$ vertices and $4 n-10$ edges, is 1-planar, and is the union of two maximal planar graphs sharing the same vertex set.

A radial $(p, q)$-grid in a graph $G$ is a set of $p+q$ edges such that the first $p$ edges are all incident to a common vertex, and each of the first $p$ edges crosses each of the remaining $q$ edges. Pach et al. [PPST05] proved that a graph without a radial $(p, q)$-grid, for $p, q \geqslant 1$, has at most $8 \cdot 24^{q} p n$ edges.

We will call a radial $(k, 1)$-grid a $k$-fan crossing. In other words, a fan crossing is formed by an edge $g$ crossing $k$ edges $e_{1}, \ldots, e_{k}$ that are all incident to a common vertex, see Figure 1 (c). A topological graph is $k$-fan-crossing free if it does not contain a $k$-fan crossing. We are particularly interested in the special case $k=2$. For brevity, let us call a 2-fan crossing simply a fan crossing (shown in Figure 1 (d)), and a 2-fan-crossing free graph a fan-crossing free graph.

By Pach et al.'s result, a $k$-fan-crossing free graph on $n$ vertices has at most $192 k n$ edges, and a fan-crossing free graph has therefore at most $384 n$ edges. We improve this bound by proving the following theorem.

Theorem 1. A fan-crossing free graph on $n \geqslant 3$ vertices has at most $4 n-8$ edges. If the graph has straight edges, it has at most $4 n-9$ edges. Both bounds are tight for $n \geqslant 10$.

A 1-planar graph is fan-crossing free, so Theorem 1 generalizes both Pach and Tóth's and Didimo's bound. We also extend their lower bounds by giving tight constructions for every value of $n$.

In an RAC graph all edges crossed by a given edge $g$ are orthogonal to it and therefore parallel to each other, implying that an RAC graph is fan-crossing free. Our theorem, therefore, "nearly" implies Didimo et al.'s bound: a fan-crossing free graph has at most one edge more than an RAC graph.

We can completely characterize extremal fan-crossing free graphs, that is, fan-crossing free graphs on $n$ vertices with $4 n-8$ edges: Any such graph consists of a planar graph $H$ where each face is a quadrilateral, together with both diagonals for each face. This implies the same properties obtained by Eades and Liotta for extremal RAC graphs: An extremal fan-crossing free graph is 1-planar, and is the union of two maximal planar graphs.

For $k$-fan-crossing free graphs with $k \geqslant 3$, we obtain the following result.
Theorem 2. A $k$-fan-crossing free graph on $n \geqslant 3$ vertices has at most $3(k-1)(n-2)$ edges, for $k \geqslant 3$.
This bound is not tight, and the best lower-bound construction we are aware of has only about $k n$ edges.
Most of the graph families discussed above have a common pattern: the subgraphs obtained by taking the edges crossed by a given edge $e$ may not contain some forbidden subgraph. We can formalize this notion as follows: For a topological graph $G$ and an edge $e$ of $G$, let $G_{e}$ denote the subgraph of $G$ containing exactly those edges that cross $e$.

A graph property $\mathcal{P}$ is called monotone if it is preserved under edge-deletions. In other words, if $G$ has $\mathcal{P}$ and $G^{\prime}$ is obtained from $G$ by deleting edges, then $G^{\prime}$ must have $\mathcal{P}$. Given a monotone graph property $\mathcal{P}$, we define a derived graph property $\mathcal{P}^{*}$ as follows: A topological graph $G$ has $\mathcal{P}^{*}$ if for every edge $e$ of $G$ the subgraph $G_{e}$ has $\mathcal{P}$. Some examples are:

- If $\mathcal{P}$ is the property that a graph does not contain a path of length two, then $\mathcal{P}^{*}$ is the property of being fan-crossing free;
- if $\mathcal{P}$ is the property of having at most $k$ edges, then $\mathcal{P}^{*}$ is $k$-planarity;


Figure 2: Left: A 7 -star. Right: A 3 -star has at most one arrow.

- if $\mathcal{P}$ is planarity, then $\mathcal{P}^{*}$ is 3-quasi-planarity.

We can consider $\mathcal{P}^{*}$ for other interesting properties $\mathcal{P}$, such as not containing a path of length $k$, or not containing a $K_{2,2}$.

We prove the following very general theorem:
Theorem 3. Let $\mathcal{P}$ be a monotone graph property such that any graph on $n$ vertices that has $\mathcal{P}$ has at most $O\left(n^{1+\alpha}\right)$ edges, for a constant $0 \leqslant \alpha \leqslant 1$. Let $G$ be a graph on $n$ vertices that has $\mathcal{P}^{*}$. If $\alpha>0$, then $G$ has $O\left(n^{1+\alpha}\right)$ edges. If $\alpha=0$, then $G$ has $O\left(n \log ^{2} n\right)$ edges.

This immediately covers many interesting cases. For instance, a graph where no edge crosses a path of length $k$, for a constant $k$, has at most $O\left(n \log ^{2} n\right)$ edges. Graphs where no edge crosses a $K_{2,2}$ have at most $\Theta\left(n^{3 / 2}\right)$ edges (and this is tight, as there are graphs with $\Theta\left(n^{3 / 2}\right)$ edges that do not contain a $K_{2,2}$, implying that no edge can cross a $K_{2,2}$ ).

Paper organization. Section 2 tackles the problem in the simplest settings involving a single simplyconnected "face" of a fan-crossing free graph. Section 3 extends this argument to the whole graph. Section 4 describes lower bound constructions, and the straight-line case. In Section 5, the argument is extended to the $k$-fan case. Section 6 proves the bound for the case where we consider the forbidden structure to be a hereditary property defined on the intersection graphs induced by the edges. Finally, Section 7 ends the paper with a discussion and some open problems.

## 2 A combinatorial puzzle: Arrows and fans

At the core of our bound lies a combinatorial question that we can express as follows: An $m$-star is a regular $m$-gon $\psi$ with a set of arrows. An arrow is a ray starting at a vertex of $\psi$, pointing into the interior of $\psi$, and exiting through an edge of $\psi$.

We require the set of edges and arrows to be fan-crossing free - that is, no edge or arrow intersects two arrows or an edge and an arrow incident to the same vertex. The left side of Figure 2 shows a 7 -star. The dashed arrows are impossible - each of them forms a fan crossing with the solid edges and arrows.

The question is: How many arrows can an m-star possess?
Observation 1. A 3-star has at most one arrow.
Proof. An arrow from a vertex has to exit the triangle through the opposing edge, so no vertex has two arrows. But two arrows from different vertices will also form a fan crossing, see the right side of Figure 2.

It is not difficult to see that a 4 -star possesses at most 2 arrows. The reader may enjoy constructing $m$-stars with $2 m-6$ arrows, for $m \geqslant 4$. We conjecture that this bound is tight. In the following, we will only prove a weaker bound that is sufficient to obtain tight results for fan-crossing free graphs.


Figure 3: Left: properties (A) to (F), right: property (G) for the proof of Lemma 4.
While we have posed the question in a geometric setting, it is important to realize that it is a purely combinatorial question. We can represent the $m$-star by writing its sequence of vertices and indicating when an arrow exits $\psi$. Whether or not three edges/arrows form a fan crossing can be determined from the ordering of their endpoints along the boundary of $\psi$ alone.

Let $C=v_{1}, \ldots, v_{m}$ be the sequence of vertices of $\psi$ in counter-clockwise order, such that the $i$ th boundary edge of $\psi$ is $e_{i}=v_{i} v_{i+1}$ (all indices are modulo $m$ ). Consider an arrow $e$ starting at $v_{i}$. It exits $\psi$ through some edge $e_{j}$, splitting $\psi$ into two chains $v_{i+1} \ldots v_{j}$ and $v_{j+1} \ldots v_{i-1}$. The length of $e$ is the number of vertices on the shorter chain.

We will call an arrow short if it has length one. A long arrow is an arrow of length larger than one.
Lemma 4. For $m \geqslant 4$, an $m$-star $\psi$ has at most $2 m-8$ long arrows.
Proof. The proof is by induction over $m$.
Any arrow in a 4-star partitions the boundary into chains of length one and length two, and so there are no long arrows, proving the claim for $m=4$.

We suppose now that $m>4$ and that the claim holds for all $4 \leqslant m^{\prime}<m$. We delete all short arrows, and let $L$ denote the remaining set of arrows, all of which are now long arrows. Let $e$ be an arrow of shortest length $\ell$ in $\psi$. Without loss of generality, we assume that $e$ starts in $v_{1}$ and exits through edge $e_{\ell+1}=v_{\ell+1} v_{\ell+2}$. Then, the following properties hold (see Figure 3):
(A) Every arrow starting in $v_{2}, \ldots, v_{\ell+1}$ must cross $e$, as otherwise it would be shorter than $e$.
(B) There is no arrow that starts in $v_{\ell+1}$. By (A), such an arrow must cross $e$, and so it forms a fan crossing with $e$ and $e_{\ell+1}$.
(C) At most one arrow starts in $v_{i}$, for $i=2, \ldots, \ell$. Indeed, two arrows starting in $v_{i}$, for $i=2, \ldots, \ell$, must cross $e$ by (A), and so they form a fan crossing with $e$.
(D) No arrow starting in $v_{\ell+2}$ exits through $e_{2}, \ldots, e_{\ell}$, as then it would be shorter than $e$.
(E) An arrow starting in $v_{\ell+2}$ and exiting through $e_{1}$ cannot exist either, as it forms a fan crossing with $e$ and $e_{1}$.
(F) No arrow starting in $v_{m}$ crosses $e_{1}, \ldots, e_{\ell-1}$, as then it would be shorter than $e$.
(G) The following two arrows cannot both exist: An arrow $e^{\prime}$ starting in $v_{m}$ and exiting through $e_{\ell}$, and an arrow $e^{\prime \prime}$ starting in $v_{\ell}$. Indeed, if both $e^{\prime}$ and $e^{\prime \prime}$ are present, then either $e^{\prime \prime}$ exits through $e_{m}$ and forms a fan crossing with $e$ and $e_{m}$, or $e^{\prime \prime}$ intersects $e^{\prime}$ and so $e^{\prime}, e^{\prime \prime}$, and $e_{\ell}$ form a fan crossing (see the right side of Figure 3).
We now create an $(m-\ell+1)$-star $\varphi$ by removing the vertices $v_{2} \ldots v_{\ell}$ with all their incident arrows from $\psi$, such that $v_{1}$ and $v_{\ell+1}$ are consecutive on the boundary of $\varphi$. An arrow that exits $\psi$ through one of the edges $e_{1} \ldots e_{\ell}$ exits $\varphi$ through the new edge $g=v_{1} v_{\ell+1}$.

Let $L^{\prime} \subset L$ be the set of arrows of $\varphi$, that is, the arrows of $\psi$ that do not start from $v_{2} \ldots v_{\ell}$. Among the arrows in $L^{\prime}$, there are one or two short arrows: the arrow $e$, and the arrow $e^{\prime}$ starting in $v_{m}$ and exiting through $e_{\ell}$ in $\psi$ (and therefore through $g$ in $\varphi$ ) if it exists. We set $q=1$ if $e^{\prime}$ exists, and else $q=0$.

We delete from $\varphi$ those one or two short arrows, and claim that there is now no fan crossing in $\varphi$. Indeed, a fan crossing would have to involve the new edge $g=v_{1} v_{\ell+1}$. But any arrow that crosses $g$ must also cross $e$, and there is no arrow starting in $v_{\ell+1}$ by (B).

Since $\ell \geqslant 2$, we have $m-\ell+1<m$, and so by the inductive assumption $\varphi$ has at most $2(m-\ell+1)-8=$ $2 m-2 \ell-6$ long arrows. Since there are $1+q$ short arrows in $L^{\prime}$, we have $\left|L^{\prime}\right| \leqslant 2 m-2 \ell-5+q$. By (C) and (G), we have $|L|-\left|L^{\prime}\right| \leqslant \ell-1-q$. It follows that

$$
|L| \leqslant\left|L^{\prime}\right|+\ell-1-q \leqslant 2 m-\ell-6 \leqslant 2 m-8 .
$$



It remains to count the short arrows. Let $e$ be a short arrow, say starting in $v_{i}$ and exiting through $e_{i+1}$. Let us call $v_{i+1}$ the witness of $e$. We observe that no arrow $e^{\prime}$ can start in this witness - $e^{\prime}$ would form a fan crossing with $e$ and $e_{i+1}$. The vertex $v_{i+1}$ can serve as the witness of only one short arrow: The only other possible short arrow $e^{\prime \prime}$ with witness $v_{i+1}$ starts in $v_{i+2}$ and exits through $e_{i}$. However, $e, e^{\prime \prime}$, and $e_{i}$ form a fan crossing.

We can now bound the number of arrows of an $m$-star.
Lemma 5. For $m \geqslant 3$, an $m$-star $\psi$ has at most $3 m-8$ arrows. The bound is attained only for $m=3$.
Proof. By Observation 1, the claim is true for $m=3$. We consider $m>3$. By Lemma 4, there are at most $2 m-8$ long arrows. Each short arrow has a unique witness. If all vertices are witnesses then there is no arrow, and so we can assume that at most $m-1$ vertices serve as witnesses, and we have at most $m-1$ short arrows, for a total of $3 m-9$ arrows.

## 3 The upper bound for fan-crossing free graphs



Let $G=(V, E)$ be a fan-crossing free graph. We fix an arbitrary maximal planar subgraph $H=\left(V, E^{\prime}\right)$ of $G$. Let $K=E \backslash E^{\prime}$ be the set of edges of $G$ that is not in $H$. Since $H$ is maximal, every edge in $K$ must cross at least one edge of $H$. We will replace each edge of $K$ by two arrows.
Let $e \in K$ be an edge connecting vertices $v$ and $u$. The initial segment of $e$ must lie inside a face $\psi$ of $H$ incident to $v$, the final segment must lie inside a face $\varphi$ of $H$ incident to $u$. It is possible that $\psi=\varphi$, but in that case the edge $e$ does not entirely lie in the face, as $H$ is maximal. We replace $e$ by two arrows: one arrow starting in $v$ and passing through $\psi$ until it exits $\psi$ through some edge; another arrow starting in $u$ and passing through $\varphi$ until it exits $\varphi$ through some edge.

In this manner, we replace the set of edges $K$ by a set of $2|K|$ arrows. The result is a planar graph whose faces have been adorned with arrows. The collection of edges and arrows is fan-crossing free.

Every edge of $H$ is incident to two faces of $H$, which can happen to be identical. If we distinguish the sides of an edge, the boundary of each face $\psi$ of $H$ consists of simple chains of edges. If $\psi$ is bounded, one chain forms the outer boundary of $\psi$ that makes $\psi$ bounded, while all other chains bound holes inside $\psi$; if $\psi$ is unbounded, then all chains bound holes in $\psi$.


If the graph $H$ is connected, then the boundary of each face consists of a single chain. Let $\psi$ be such a face whose boundary chain consists of $m$ edges (where edges that bound $\psi$ on both sides are counted twice). Then $\psi$ has at most $3 m-8$ arrows. This follows immediately from Lemma 5: Recall that $m$-stars can be defined purely combinatorially. Whether three edges form a fan crossing can be decided solely by the ordering of their endpoints along the boundary chain. The boundary of a simply connected face is a single closed chain, and so Lemma 5 applies to this setting, see the side figure.

Unfortunately, we cannot guarantee that $H$ is connected. The following lemma bounds the number of arrows of a face $\psi$ in terms of its complexity and its number of boundary chains. The complexity of a face is the total number of edges of all its boundary chains, where edges that are incident to the face on both sides are counted twice.

Lemma 6. A face of $H$ of complexity $m$ bounded by $p$ boundary chains possesses at most $3 m+8 p-16$ arrows. The bound can be attained only when $m=3$ and $p=1$.

We will prove the lemma below, but let us first observe how it implies the upper bound on the number of edges of fan-crossing free graphs.

Lemma 7. A fan-crossing free graph $G$ on $n$ vertices has at most $4 n-8$ edges.
Proof. Let $m$ be the number of edges, let $r$ be the number of faces, and let $p$ be the number of connected components of $H$. Let $\mathcal{F}$ be the set of faces of $H$. For a face $\psi \in \mathcal{F}$, let $m(\psi)$ denote the complexity of $\psi$, let $p(\psi)$ denote the number of boundary chains of $\psi$, and let $a(\psi)$ denote the number of arrows of $\psi$.

We have $\sum_{\psi \in \mathcal{F}} m(\psi)=2 m$ and $\sum_{\psi \in \mathcal{F}}(p(\psi)-1)=p-1$ (each component is counted in its unbounded face, except that we miss one hole in the global unbounded face).

The graph $G$ has $|E|=m+|K|$ edges. Using Lemma 6 we have

$$
\begin{aligned}
2|E| & =2 m+2|K|=\sum_{\psi \in \mathcal{F}} m(\psi)+\sum_{\psi \in \mathcal{F}} a(\psi) \\
& \leqslant \sum_{\psi \in \mathcal{F}}(4 m(\psi)+8 p(\psi)-16) \\
& =4 \sum_{\psi \in \mathcal{F}} m(\psi)+8 \sum_{\psi \in \mathcal{F}}(p(\psi)-1)-8 r \\
& =8 m+8 p-8-8 r .
\end{aligned}
$$

By Euler's formula, we have $n-m+r=1+p$, so $m-r=n-1-p$, and we have

$$
2|E| \leqslant 8(m-r)+8 p-8=8 n-8-8 p+8 p-8=8 n-16 .
$$

It remains to fill in the missing proof.
Proof of Lemma 6. Let $\psi$ be a face of $H$, and let $m=m(\psi)$ and $p=p(\psi)$ be its complexity and its number of boundary components. A boundary component is a chain of edges, and could possibly degenerate to a single isolated vertex.

We say that two boundary chains $\xi$ and $\zeta$ are related if an arrow starting in a vertex of $\xi$ ends in an edge of $\zeta$, or vice versa. Consider the undirected graph whose nodes are the boundary chains of $\psi$ and whose arcs connect boundary chains that are related. If this graph has more than one connected component, we can bound the number of arrows separately for each component, and so in the following we can assume that all boundary chains are (directly or indirectly) related.


Figure 4: Building a bridge between $\xi$ and $\zeta$.
Consider two related boundary chains $\xi$ and $\zeta$. By assumption there must be an arrow $e$, starting at a vertex $v \in \xi$, and ending in an edge $u_{1} u_{2}$ of $\zeta$. We create a new vertex $z$ on $\zeta$ at the intersection point of $e$ and $u_{1} u_{2}$, split the boundary edge $u_{1} u_{2}$ into two edges $u_{1} z$ and $z u_{2}$, and insert the two new boundary edges $v z$ and $z v$, see Figure 4. This operation has increased the complexity of $\psi$ by three. Note that some arrows of $\psi$ might be crossing the new boundary edges - these arrows will now be shortened, and end on the new boundary edge.

The two boundary chains $\xi$ and $\zeta$ have now merged into a single boundary chain. In effect, we have turned an arrow into a "bridge" connecting two boundary chains. No fan crossing is created, since all edges and arrows already existed. We do create a new vertex $z$, but no arrow starts in $z$, and so this vertex cannot cause a fan crossing.

We insert $p-1$ bridges in total and connect all $p$ boundary chains. In this manner, we end up with a face $\varphi$ whose boundary is a single chain consisting of $m^{\prime}=m+3(p-1)$ edges.

If $m^{\prime}=3$, then $\varphi$ has at most one arrow, by Observation 1. This case happens only for $m=3$ and $p=1$, and is the only case where the bound is tight.

If $m^{\prime}>3$, then we can apply Lemma 4 to argue that $\varphi$ has at most $2 m^{\prime}-8$ long arrows. To count the short arrows, we observe that the vertex $z$ created in the bridge-building process cannot be the witness of a short arrow: such a short arrow would imply a fan crossing in the original face $\psi$. It is also not the starting point of any arrow.

It follows that building a bridge increases the number of possible witnesses by only one (the vertex $v$ now appears twice on the boundary chain). There are thus at most $m+p-1$ possible witnesses in $\varphi$. However, if all of these vertices are witnesses, then there is no arrow at all, and so there are at most $m+p-2$ short arrows.

Finally, we converted $p-1$ arrows of $\psi$ into bridges to create $\varphi$. The total number of arrows of $\psi$ is therefore at most

$$
\begin{aligned}
2 m^{\prime}-8+(m+p-2)+(p-1) & =2(m+3 p-3)-8+(m+p-2)+(p-1) \\
& =3 m+8 p-17 .
\end{aligned}
$$

## 4 Lower bounds and the straight-line case

Consider a fan-crossing free graph $G$ with $4 n-8$ edges. This means that the inequality in the proof of Lemma 7 must be an equality. In particular, every face $\psi$ of $H$ must have exactly $3 m(\psi)+8 p(\psi)-16$ arrows. By Lemma 6 , this is only possible if $\psi$ is a triangle, and so we have proven

Lemma 8. A fan-crossing free graph $G$ with $4 n-8$ edges contains a planar triangulation $H$ of its vertex set. Each triangle of $H$ possesses exactly one arrow.

Note that the arrow must necessarily connect a vertex of the triangle with the opposite vertex in the triangle adjacent along the opposite edge, as otherwise, it forms a fan crossing, and so we get the following second characterization of extremal fan-crossing free graphs:

Lemma 9. A fan-crossing free graph $G$ with $4 n-8$ edges contains a planar graph $Q$ on its vertex set, where each face of $Q$ is a quadrilateral. $G$ is obtained from $Q$ by adding both diagonals for each face of $Q$.

By Euler's formula, a planar graph $Q$ on $n$ vertices whose faces are quadrilaterals has $2 n-4$ edges and $n-2$ faces. Adding both diagonals to each face of $Q$, we obtain a fan-crossing free graph $G$ with $4 n-8$ edges. However, it turns out that this construction needs to be done carefully: Otherwise, diagonals of two distinct faces of $Q$ can connect the same two vertices, and the result is not a simple graph. And indeed, it turns out that for $n \in\{7,9\}$, no (simple) fan-crossing free graph with $4 n-8$ edges exists! When $n \geqslant 8$ is a multiple of four, Figure 5 shows planar graphs $Q$ where every bounded face is a convex quadrilateral. Since all their diagonals are straight, no multiple edges can arise. Only the two diagonals of the unbounded face are not straight and need to be checked individually. We will return to other values of $n$ below.


Figure 5: Planar graphs with convex quadrilateral faces for $n \in\{8,12,16 \ldots\}$.
Lemma 8 implies immediately that a fan-crossing free graph with $4 n-8$ edges cannot exist if all edges are straight: Since the unbounded face of $H$ is a triangle, it cannot contain a straight arrow, and so any fan-crossing free graph drawn with straight edges has at most $4 n-9$ edges. This bound is tight: for any $n \geqslant 6$, we can construct a planar graph $Q$ such that two faces of $Q$ are triangles, and all other faces are convex quadrilaterals. Euler's formula implies that $Q$ has $2 n-3$ edges and $n-1$ faces, and adding both diagonals to each quadrilateral face results in a graph with $2 n-3+2(n-3)=4 n-9$ edges. The construction of $Q$ is shown in Figure 6 . The upper row shows the construction when $n \geqslant 6$ is a multiple of three. If $n \equiv 1(\bmod 3)$, we replace the two innermost triangles as shown in the lower left of the figure. If $n \equiv 2(\bmod 3)$, the two innermost triangles are replaced as in the lower right figure.

Lemma 10. A fan-crossing free graph drawn with straight edges has at most $4 n-9$ edges. This bound is tight for $n \geqslant 6$.


Figure 6: Adding both diagonals for each quadrilateral face results in a straight-line fan-crossing free graph with $4 n-9$ edges.

We observe that the extremal fan-crossing free graph constructed for $n=6$ has 15 edges and is therefore
the complete graph $K_{6}$. It follows that the complete graph $K_{n}$ is fan-crossing free for $n \leqslant 6$, and it remains to discuss extremal fan-crossing free graphs for $n \geqslant 7$ when the edges are not necessarily straight.

Lemma 11. Extremal fan-crossing free graphs with $4 n-8$ edges exist for $n=8$ and all $n \geqslant 10$. For $n \in\{7,9\}$, extremal fan-crossing free graphs have $4 n-9$ edges.

Proof. Let $n \in\{7,9\}$, and assume $G$ is a fan-crossing free graph on $n$ vertices with $4 n-8$ edges. By Lemma 9 , $G$ contains a planar graph $Q$ whose faces are quadrilaterals. We claim that $Q$ has a vertex $v$ of degree two. This implies that the two quadrilaterals incident to $v$ have diagonals connecting the same two vertices, a contradiction. It follows that a fan-crossing free graph with $4 n-8$ edges does not exist. By Lemma 10 , a fan-crossing free graph with $4 n-9$ edges does exist.

For $n=7$, the claim is immediate: $Q$ has $2 n-4=10$ edges, and so its average degree is $20 / 7<3$, and a vertex of degree two must exist.

For $n=9$, the total degree of $Q$ is $2(2 n-4)=28$. We assume there is no vertex of degree two. Since nine vertices of degree three already contribute 27 to the total degree, it follows that there is one vertex $w$ of degree four, while the other eight vertices have degree three. Since $Q$ contains only even cycles, it is a bipartite graph, and the vertices can be partitioned into two classes $A$ and $B$. Let $w \in A$, and let $k=|A|-1$. The total degree of vertices in $A$ and in $B$ is identical, so we have $4+3 k=3(8-k)$, or $6 k=20$, a contradiction.

The case $n=8$ was already handled in Figure 5, so consider $n \geqslant 10$. We will again start with a planar graph $Q$ with quadrilateral faces. To avoid diagonals connecting identical vertices, we would like to make all faces convex. This is obviously not possible when drawing the graph in the plane, and so we will draw the graph on the sphere, such that all faces are spherically-convex quadrilaterals.

For even $n$, we place a vertex at the North pole and at the South pole each. The remaining $n-2$ vertices form a zig-zag chain near the equator, distributing the points equally on two circles of equal latitude, see lefthand side of Figure 7 (for $n=10$ ). For odd $n$, we place two vertices close to the North pole, one vertex at the South pole, and let the remaining $n-3$ vertices again form a zig-zag chain near the equator, see the right-hand side of Figure 7 (for $n=11$ ). One of the resulting quadrilaterals is long and skinny, but it is still spherically convex.


Figure 7: Spherically-convex quadrilateralization for even $n$ and odd $n$.

## $5 k$-fan-crossing free graphs for $k \geqslant 3$

Our proof of Theorem 2 has the same structure as for the case $k=2$ : Let $G=(V, E)$ be a $k$-fan-crossing free graph for $k \geqslant 3$. We fix an arbitrary maximal planar subgraph $H=\left(V, E^{\prime}\right)$ of $G$, and let $K=E \backslash E^{\prime}$. We replace each edge in $K$ by two arrows, so we end up with a planar graph $H$ whose faces have been adorned with $2|K|$ arrows in total.

Lemma 12. A $k$-fan-crossing free face of $H$ of complexity $m \geqslant 3$ bounded by $p$ boundary chains possesses at most $3(k-1)(m+2 p-4)-2 m+3$ arrows, for $k \geqslant 3$.

As in the case $k=2$, we prove Lemma 12 by converting arrows connecting different boundary components of the face into bridges, until we have obtained a single boundary chain. The technical details are more complicated, and so we first discuss how Lemma 12 implies Theorem 2.

Proof of Theorem 2. Let $m$ be the number of edges, let $r$ be the number of faces, and let $p$ be the number of connected components of $H$. Let $\mathcal{F}$ be the set of faces of $H$. For a face $\psi \in \mathcal{F}$, let $m(\psi)$ denote the complexity of $\psi$, let $p(\psi)$ denote the number of boundary chains of $\psi$, and let $a(\psi)$ denote the number of arrows of $\psi$. As observed before, we have $\sum_{\psi \in \mathcal{F}} m(\psi)=2 m$ and $\sum_{\psi \in \mathcal{F}}(p(\psi)-1)=p-1$.

The graph $G$ has $|E|=m+|K|$ edges. Since $m(\psi) \geqslant 3$, Lemma 12 implies $m(\psi)+a(\psi) \leqslant 3(k-1)(m(\psi)+$ $2 p(\psi)-4)$, and so

$$
\begin{aligned}
2|E| & =2 m+2|K|=\sum_{\psi \in \mathcal{F}} m(\psi)+\sum_{\psi \in \mathcal{F}} a(\psi) \\
& \leqslant \sum_{\psi \in \mathcal{F}}((3 k-3)(m(\psi)+2(p(\psi)-1)-2)) \\
& =3(k-1)\left(\sum_{\psi \in \mathcal{F}} m(\psi)+2 \sum_{\psi \in \mathcal{F}}(p(\psi)-1)-2 r\right) \\
& =6(k-1)(m+(p-1)-r)=6(k-1)(n-2) .
\end{aligned}
$$

In the last line we used Euler's formula $n-m+r=1+p$ for $H$.
To prove Lemma 12, we need a $k$-fan equivalent of Lemma 5 . One additional difficulty is now keeping the contribution of the additional vertices formed by creating bridges low. We need a few new definitions.

Again, an $m$-star is a regular $m$-gon $\psi$ with a set of arrows. We require the set of edges and arrows to be $k$-fan-crossing free. We number the vertices counter-clockwise as $v_{1}, v_{2}, \ldots, v_{m}$, and denote the edge between $v_{i}$ and $v_{i+1}$ as $e_{i}$ (where index arithmetic is again modulo $m$ ). We let $a_{i j}$ denote the number of arrows starting in vertex $v_{i}$ and exiting through edge $e_{j}$. The degree $a_{i}$ of vertex $v_{i}$ is the total number of arrows starting in $v_{i}$. We define the length of an arrow and the notions of short and long arrows as before.

We need to distinguish different kinds of vertices. If vertex $v_{i}$ has degree $a_{i}=0$, we call it void. If, in addition, its left neighbor $v_{i-1}$ has no short arrow passing over $v_{i}$, that is, if $a_{i}=0$ and $a_{i-1, i}=0$, then we call it left-light. Similarly, if $a_{i}=0$ and $a_{i+1, i-1}=0$, then we call $v_{i}$ right-light. A vertex is light when it is left-light or right-light. A vertex is heavy when its degree is non-zero. Finally, we do not allow a left-light vertex to be adjacent to a right-light vertex. What this means is that if a sequence of consecutive vertices, say $v_{1}, v_{2}, \ldots, v_{t}$ is a maximal sequence of zero-degree vertices, then either they are all left-light (if $a_{m, 1}=0$ ), or they are all right-light (if $a_{t+1, t-1}=0$ ), or only $t-1$ of them are light and the last one is counted as void.

We can now formulate our lemma.
Lemma 13. For $m \geqslant 3, k \geqslant 3$, and $h \geqslant 2$, an $m$-star $\psi$ with $h$ heavy vertices, $\lambda$ light vertices, and $\nu$ void (but not light) vertices has at most $(3 k-5) h+k \lambda+(2 k-3) \nu-(6 k-9)$ arrows.

Before we prove Lemma 13, let us first see how it implies Lemma 12.
Proof of Lemma 12. We consider a face $\psi$ of $H$ of complexity $m$ bounded by $p$ boundary components. As in the proof of Lemma 6 , we can assume that all boundary components are related. Thus there must be boundary components $\xi, \zeta$ and an arrow starting at a vertex $v \in \xi$ and ending in an edge $u_{1} u_{2}$ of $\zeta$. More precisely, there could be at most $k-1$ such arrows starting in $v$ and ending in $u_{1} u_{2}$. We pick the two extreme arrows, that is, the one closest to $u_{1}$ and the one closest to $u_{2}$, and convert them to bridges, creating two new vertices $z_{1}, z_{2}$ on $u_{1} u_{2}$, see Figure 8.

Inserting these two bridges increases the complexity of the face by three. However, $z_{1}$ and $z_{2}$ are by construction light vertices (in Figure $8, z_{1}$ is right-light, $z_{2}$ is left-light). There are at most $k-1$ arrows from $v$ to $u_{1} u_{2}$ that are deleted from the face.


Figure 8: Building a bridge between $\xi$ and $\zeta$.
We continue in this manner until the face is bounded by a single boundary chain. Every bridge adds one heavy vertex and two light vertices to the complexity of the face. (Note that later bridges could possibly create vertices on bridges built earlier, but that doesn't stop the vertices from being light.) We finally obtain a face with $m+p-1 \geqslant 2$ heavy vertices and $2(p-1)$ light vertices, and during the process we deleted at most $(k-1)(p-1)$ arrows. Applying Lemma 13, the total number of arrows of $\psi$ is at most

$$
\begin{aligned}
(3 k-5) & (m+p-1)+2 k(p-1)-(6 k-9)+(k-1)(p-1) \\
& =(3 k-5) m+(3 k-5+2 k+k-1)(p-1)-(6 k-9) \\
& =(3 k-3) m-2 m+(6 k-6)(p-1)-(6 k-6)+3 \\
& =3(k-1)(m+2 p-4)-2 m+3 .
\end{aligned}
$$

Finally, it remains to prove the bound on the number of arrows of a $k$-fan-crossing free $m$-star.
Proof of Lemma 13. Let $A(h, \lambda, \nu)$ denote the maximum number of arrows of an $m$-star with $h$ heavy vertices, $\lambda$ light vertices, and $\nu$ void but not light vertices, where $m=h+\lambda+\nu$. We set

$$
B(h, \lambda, \nu)=(3 k-5) h+k \lambda+(2 k-3) \nu-(6 k-9)
$$

and show by induction over $m$ that $A(h, \lambda, \nu) \leqslant B(h, \lambda, \nu)$ under the assumption that $m \geqslant 3, k \geqslant 3, h \geqslant 2$.
We have several base cases. The reader may enjoy verifying that the claim holds for the triangle and the quadrilateral:

$$
\begin{aligned}
& A(3,0,0)=3 k-6=B(3,0,0) \\
& A(2,0,1)=2 k-4=B(2,0,1) \\
& A(2,1,0)=k-1=B(2,1,0) \\
& A(4,0,0)=5 k-9 \leqslant 6 k-11=B(4,0,0), \\
& A(3,0,1)=4 k-6 \leqslant 5 k-9=B(3,0,1), \\
& A(3,1,0)=3 k-5 \leqslant 4 k-6=B(3,1,0), \\
& A(2,0,2)=4 k-8 \leqslant 4 k-7=B(2,0,2), \\
& A(2,1,1)=3 k-5 \leqslant 3 k-4=B(2,1,1), \text { and } \\
& A(2,2,0)=2 k-2 \leqslant 2 k-1=B(2,2,0)
\end{aligned}
$$

The second base case is when $m>4$ and all vertex degrees are at most $k-2$. In this case we have $A(h, \lambda, \nu) \leqslant(k-2) h$. If $h \geqslant 3$ then

$$
(3 k-5) h-(6 k-9)-(k-2) h=(2 k-3)(h-3) \geqslant 0,
$$

and the claim holds. If $h=2$ then $\lambda+\nu>2$ and so

$$
A(2, \lambda, \nu) \leqslant 2 k-4 \leqslant 2 k-1=B(2,2,0) \leqslant B(2, \lambda, \nu)
$$



Figure 9: When all arrows are short.
The third base case is when $m>4$ and $h=2$, and the two heavy vertices are adjacent. Let $v_{1}$ and $v_{2}$ denote the two heavy vertices. No arrows start from the other vertices $v_{3}, \ldots, v_{m}$. If no arrows starting from $v_{1}$ and $v_{2}$ intersect each other, then

$$
A(2, \lambda, \nu) \leqslant(k-1)(m-2)=(k-1) \lambda+(k-1) \nu \leqslant B(2, \lambda, \nu) .
$$

Assume now that there is at least one arrow from $v_{1}$ that intersects an arrow from $v_{2}$. Let $x_{1}$ denote the rightmost arrow from $v_{1}$ and let $x_{i}$ (for $i>1$ ) denote the $i$ th arrow from $x_{1}$ in counter-clockwise order. Similarly, let $y_{1}$ denote the leftmost arrow from $v_{2}$ and let $y_{i}($ for $i>1)$ denote the $i$ th arrow from $v_{2}$ in clockwise order. We claim that only the arrows $x_{1}, \ldots, x_{k-1}$ from $v_{1}$ can intersect the arrows $y_{1}, \ldots, y_{k-1}$ from $v_{2}$. Indeed, $x_{1}$ cannot intersect more than $k-1$ arrows starting from $v_{2}$, since otherwise, it forms a $k$-fan crossing. If $x_{1}$ intersects $k-1$ arrows from $v_{2}$, it has to intersect $y_{1}, \ldots, y_{k-1}$. Since $x_{1}$ is the rightmost arrow from $v_{1}, y_{j}$ for $j \geqslant k$ must lie on the right side of $x_{1}$, which implies that $y_{j}$ cannot intersect any arrow from $v_{1}$. Similarly, since $y_{1}$ is the leftmost arrow from $v_{2}, x_{j}$ for $j \geqslant k$ cannot intersect any arrow from $v_{2}$, proving the claim. This implies that there can exist at most $(k-1)(m-2)+(k-1)=(k-1)(m-1)$ arrows in total, and so

$$
\begin{gathered}
{[(3 k-5)(2+\nu+\lambda)+(k-1) \nu+(2 k-3) \lambda-(6 k-9)]-(k-1)(1+\nu+\lambda)} \\
\quad=(2 k-4)(1+\nu+\lambda)+(k-1) \nu+(2 k-3) \lambda-(3 k-4) \geqslant 0,
\end{gathered}
$$

for $k \geqslant 3$ and $\nu+\lambda \geqslant 3$, and thus $A(2, \nu, \lambda) \leqslant B(2, \nu, \lambda)$.
We now turn to the inductive step. Consider an $m$-star $\psi$ with $A(h, \lambda, \nu)$ arrows, where $m>4, h \geqslant 2$, at least one vertex has degree at least $k-1$, and if $h=2$ then the two heavy vertices are not adjacent.

We consider first the case where there are no long arrows: all arrows are short. By renumbering, we can assume that $v_{2}$ has degree $a_{2} \geqslant k-1$. This implies $a_{13}=0$ and $a_{31}=0$, see Figure 9. We now construct an ( $m-1$ )-star $\varphi$ by deleting $v_{2}$ and its arrows, and inserting the edge $g=v_{1} v_{3}$. Since $\psi$ has only short arrows, the only arrows intersecting $g$ are $a_{m 1} \leqslant k-1$ arrows starting in $v_{m}$ and $a_{42} \leqslant k-1$ arrows starting in $v_{4}$. Since $m>4$ the vertices $v_{m}$ and $v_{4}$ are distinct, and this implies that $\varphi$ has no $k$-fan-crossing. The vertex $v_{2}$ is obviously heavy and has $a_{2} \leqslant 2(k-1)$. If $v_{3}$ is left-light in $\psi$, it is still left-light in $\varphi$ since $a_{13}=0$. Similarly, if $v_{1}$ is right-light in $\psi$, it remains so in $\varphi$. (It cannot happen that $v_{1}$ is right-light and $v_{3}$ is left-light in $\psi$, as then $a_{2}=0<k-1$.) We thus have

$$
A(h, \lambda, \nu) \leqslant 2 k-2+B(h-1, \lambda, \nu) \leqslant B(h, \lambda, \nu) .
$$

It remains to consider the case where $\psi$ has long arrows. Let $e$ denote the shortest long arrow in $\psi$ and let $\ell$ denote its length. Renumbering the vertices we can assume that $e$ starts in $v_{1}$ and exits through edge $e_{\ell+1}=v_{\ell+1} v_{\ell+2}$. See Figure 10. The following property holds:
(A) Every long arrow starting in $v_{2}, \ldots, v_{\ell+1}$ must cross $e$, as otherwise it would be shorter than $e$.

Let $a_{2 e}$ denote the number of arrows starting in $v_{2}$ that cross $e$. These arrows cannot form a $k$-fan, so we have $a_{2 e} \leqslant k-1$. An arrow starting in $v_{2}$ that does not cross $e$ is short by (A) and must exit through $e_{3}$. This implies that $a_{2}=a_{2 e}+a_{23} \leqslant 2(k-1)$.


Figure 10: $e$ is the shortest long arrow.


Figure 11: Two cases where $g$ creates a $k$-fan in $\varphi$.
We now create an $(m-1)$-star $\varphi$ by deleting $v_{2}$ and all its arrows, deleting the $a_{12}$ arrows starting in $v_{1}$ and exiting through $e_{3}$, deleting the $a_{31}$ arrows starting in $v_{3}$ and exiting through $e_{1}$, and inserting the new edge $g=v_{1} v_{3}$.

If $\varphi$ has a $k$-fan crossing, it must involve the new edge $g$. There are three ways in which this could happen:
Case (a): $g$ and $k-1$ arrows starting in $v_{1}$ are intersected by an arrow $x$ that also intersects $e_{2}$. See left side of Figure 11.
Case (b): $g$ and $k-1$ arrows starting in $v_{3}$ are intersected by an arrow $y$ that also intersects $e_{1}$. See right side of Figure 11.
Case (c): $g$ intersects $k$ arrows starting in the same vertex.
We first observe that case (c) cannot happen: If the $k$ arrows intersecting $g$ also intersect $e$, they form a $k$-fan crossing in $\psi$. If there is an arrow that does not intersect $e$, it must start in one of $v_{4}, \ldots, v_{\ell+1}$, and so it is short by (A). To intersect $g$, these arrows must start in $v_{4}$ and exit through $e_{2}$, and there are at most $(k-1)$ such arrows.

If case (a) happens, we delete from $\varphi$ one more arrow, namely, the "lowest" arrow starting in $v_{1}$. If case (b) happens, we take out the "lowest" arrow starting in $v_{3}$. This ensures that $\varphi$ has no $k$-fan crossing. The inductive assumption holds for $\varphi$, since it has $m-1$ vertices and at least two heavy vertices. The latter follows since $v_{1}$ has at least the arrow $e$ and is therefore heavy, so $v_{1}$ and $v_{2}$ cannot be the two only heavy vertices of $\psi$.

We set $t_{x}=1$ if case (a) happens, and $t_{x}=0$ otherwise; similarly we set $t_{y}=1$ if case (b) happens and $t_{y}=0$ otherwise. Or original face $\psi$ has

$$
\Delta:=a_{2 e}+a_{23}+a_{12}+a_{31}+t_{x}+t_{y}
$$

arrows more than the new face $\varphi$.
We collect a few more properties:
(B) If $a_{23}=k-1$ then case (a) cannot happen: Since $x$ exits through $e_{2}$, it intersects the $k-1$ arrows starting in $v_{2}$ and exiting through $e_{3}$, and in addition the edge $e_{2}$ incident to $v_{2}$.
(C) If $a_{23} \geqslant 1$ then case (b) cannot happen: An arrow starting in $v_{2}$ and exiting through $e_{3}$ intersects all arrows starting in $v_{3}$ as well as the edge $e_{3}$, and so $a_{3}<k-1$. But $v_{3}$ needs to have $k-1$ arrows for case (b) to occur.
(D) If $a_{31}=k-1$ then case (a) cannot happen: Since $x$ exits through $e_{2}$, it intersects the $k-1$ arrows starting in $v_{3}$ and exiting through $e_{1}$, and in addition the edge $e_{2}$ incident to $v_{3}$.
(E) If $a_{12} \geqslant k-2$ then case (b) cannot happen: If $y$ intersects $e$, then it intersects $e, e_{1}$, and $k-2$ arrows from $v_{1}$ to $e_{2}$, a $k$-fan crossing. If $y$ does not intersect $e$, then $y$ must start in $v_{2}, \ldots, v_{\ell+1}$, and so by (A), $y$ is short. To cross $e_{1}, y$ must therefore start in $v_{3}$, and so $y$ does not cross $g$.
We now distinguish several cases to bound $\Delta$ :
Case 1: $a_{2}=a_{2 e}+a_{23} \geqslant k-1$. In this case, $a_{12}=a_{31}=0$, and so $\Delta=a_{2}+t_{x}+t_{y}$. Since $a_{2 e} \leqslant k-1$, we have by (B) and (C):

\[

\]

Case 2: $\quad a_{2} \leqslant k-2$ and $\max \left(a_{12}, a_{31}\right)=k-1$. If $a_{12}=k-1$ then $a_{31}=0$, and if $a_{31}=k-1$ then $a_{12}=0$. By (D) and (E), we have

$$
\Delta=a_{2}+\left(a_{12}+a_{31}\right)+\left(t_{x}+t_{y}\right) \leqslant(k-2)+(k-1)+1=2 k-2 .
$$

Case 3: $\max \left\{a_{2}, a_{12}, a_{31}\right\} \leqslant k-2$. By (E), $a_{12}=k-2$ implies $t_{y}=0$, and so

$$
\begin{aligned}
& \text { If } a_{12}=k-2: \quad \Delta \leqslant \begin{array}{c}
a_{2} \\
k-2
\end{array}{ }^{a_{12}} \begin{array}{c}
a_{31} \\
k-2
\end{array} \begin{array}{c}
t_{x} \\
k-2 \\
1
\end{array}+\begin{array}{c}
t_{y} \\
0
\end{array}=3 k-5 \text {. } \\
& \text { If } a_{12}<k-2: \quad \Delta \leqslant k-2+k-3+k-2+1+1=3 k-5 \text {. }
\end{aligned}
$$

We now have all the ingredients to complete the inductive argument. Clearly $v_{1}$ is heavy because of the arrow $e$. We distinguish the types of $v_{2}$ and $v_{3}$ (note that case 1 can occur only when $v_{2}$ is heavy):

- If $v_{2}$ is heavy and $v_{3}$ is not left-light, then we have

$$
A(h, \lambda, \nu) \leqslant \max (2 k-2,3 k-5)+B(h-1, \lambda, \nu)=B(h, \lambda, \nu) .
$$

- If $v_{2}$ is heavy and $v_{3}$ is left-light, then $v_{3}$ might become void in $\varphi$. In this case $a_{31}=a_{23}=0$. The bound in case 3 improves to $\Delta \leqslant 2 k-3 \leqslant 2 k-2$ :

$$
A(h, \lambda, \nu) \leqslant 2 k-2+B(h-1, \lambda-1, \nu+1)=B(h, \lambda, \nu)
$$

- If $v_{2}$ is right-light, then $a_{2}=a_{31}=0$. If $v_{2}$ is left-light and $v_{3}$ is not left-light, then $a_{2}=a_{12}=0$. Either way, in case 2 the bound improves to $\Delta \leqslant k$, and in case 3 it improves to $\Delta \leqslant k-1$. We have

$$
A(h, \lambda, \nu) \leqslant k+B(h, \lambda-1, \nu)=B(h, \lambda, \nu) .
$$

- If $v_{2}$ and $v_{3}$ are both left-light, then $v_{3}$ might become void in $\varphi$. We have $a_{2}=a_{3}=a_{12}=0$. We are thus in case 3 and have the improved bound $\Delta \leqslant 2$. This gives

$$
A(h, \lambda, \nu) \leqslant 2+B(h, \lambda-2, \nu+1)<B(h, \lambda, \nu) .
$$

- If $v_{2}$ is void and $v_{3}$ is not left-light, then $a_{2}=0$. In case 2 this implies $\Delta \leqslant k$, in case 3 it implies $\Delta \leqslant 2 k-3$. Since $k \leqslant 2 k-3$ we have

$$
A(h, \lambda, \nu) \leqslant 2 k-3+B(h, \lambda, \nu-1)=B(h, \lambda, \nu) .
$$

- Finally, if $v_{2}$ is void and $v_{3}$ is left-light, then $v_{3}$ might become void in $\varphi$. We have $a_{2}=a_{3}=0$. In case 2 this implies $\Delta \leqslant k$, in case 3 it implies $\Delta \leqslant k-1$. We have

$$
A(h, \lambda, \nu) \leqslant k+B(h, \lambda-1, \nu)=B(h, \lambda, \nu)
$$

This completes the inductive step.

## 6 The general bound

We now prove Theorem 3. The proof makes use of the following lemma by Pach et al. [PSS94]:
Lemma 14 ([PSS94, Theorem 2.1]). Let $G$ be a graph with $n$ vertices of degree $d_{1}, \ldots, d_{n}$ and crossing number $\chi$. Then there is a subset $E$ of $b$ edges of $G$ such that removing $E$ from $G$ creates components of size at most $2 n / 3$, and

$$
b^{2} \leqslant(1.58)^{2}\left(16 \chi+\sum_{i=1}^{n} d_{i}^{2}\right) .
$$

Proof of Theorem 3. Let $G$ be a graph on $n$ vertices with $m$ edges having property $\mathcal{P}^{*}$. Since each edge $e$ crosses a graph that has property $\mathcal{P}$, the crossing number of $G$ is at most $\chi \leqslant O\left(m n^{1+\alpha}\right)$. The degree of any vertex is bounded by $n-1$, and so we have $d_{i}^{2} \leqslant n \cdot d_{i}$. It follows using Lemma 14 that there exists a set $E$ of $b$ edges in $G$ such that

$$
b^{2} \leqslant O\left(\chi+\sum_{i=1}^{n} d_{i}^{2}\right) \leqslant O\left(m n^{1+\alpha}+n \sum_{i=1}^{n} d_{i}\right) \leqslant O\left(m n^{1+\alpha}+m n\right) \leqslant O\left(m n^{1+\alpha}\right)
$$

and removing $E$ from $G$ results in components of size at most $2 n / 3$.
We recursively subdivide $G$. Level 0 of the subdivision is $G$ itself. We obtain level $i+1$ from level $i$ by decomposing each component of level $i$ using Lemma 14.

Consider a level $i$. It consists of $k$ components $G_{1}, \ldots, G_{k}$. Component $G_{j}$ has $n_{j}$ vertices and $m_{j}$ edges, where $n_{j} \leqslant\left(\frac{2}{3}\right)^{i} n$. The total number of edges at level $i$ is $r=\sum_{j=1}^{k} m_{j}$.

Using the Cauchy-Schwarz inequality for the vectors $\sqrt{m_{j}}, \sqrt{n_{j}}$, we have

$$
\sum_{j=1}^{k} \sqrt{m_{j} n_{j}} \leqslant \sqrt{\sum_{j=1}^{k} m_{j}} \sqrt{\sum_{j=1}^{k} n_{j}}=\sqrt{r n} \leqslant \sqrt{m n}
$$

We first consider the case $\alpha>0$. The number of edges needed to subdivide $G_{j}$ is $O\left(\sqrt{m_{j} n_{j}^{1+\alpha}}\right)$. We bound this using $n_{j} \leqslant\left(\frac{2}{3}\right)^{i} n$ as $O\left(\sqrt{m_{j} n_{j}}\left(\left(\frac{2}{3}\right)^{i} n\right)^{\alpha / 2}\right)$, and we obtain that the total number of edges removed between levels $i$ and $i+1$ is bounded by $O\left(\sqrt{m n}\left(\left(\frac{2}{3}\right)^{i} n\right)^{\alpha / 2}\right)$. Since $\left(\frac{2}{3}\right)^{\alpha / 2}<1$, summing over all levels results in a geometric series, and so the total number of edges removed is $O\left(\sqrt{m n^{1+\alpha}}\right)$. But this implies that the total number of edges in the graph is bounded as

$$
m \leqslant O\left(\sqrt{m n^{1+\alpha}}\right)
$$

and squaring both sides and dividing by $m$ results in

$$
m \leqslant O\left(n^{1+\alpha}\right)
$$

Next, consider the case $\alpha=0$. The number of edges removed between levels $i$ and $i+1$ is bounded by $O(\sqrt{m n})$. Adding over all $O(\log n)$ levels shows that

$$
m \leqslant O(\sqrt{m n} \log n)
$$

Again, squaring and dividing by $m$ leads to

$$
m \leqslant O\left(n \log ^{2} n\right)
$$

## 7 Conclusions

We have proven bounds on the number of edges of $k$-fan-crossing free graphs. For $k=2$ our bound is tight, and we could even characterize the extremal graphs. In comparison, the bound we obtain for $k>2$ is much weaker. For $k=3$, for instance, our bound is $6 n-12$. The best lower bound we are aware of is the construction of Pach and Tóth [PT97] of a 2-planar graph with $5 n-10$ edges. For general $k \geqslant 2$, a lower bound of $n(k-1) / 2$ follows by considering a $(k-1)$-regular graph, that is a graph where every vertex has degree $k-1$. Clearly it is $k$-fan-crossing free and has $n(k-1) / 2$ edges. For $k \ll n$ this can be improved by a factor two: Consider a $\sqrt{n} \times \sqrt{n}$ integer grid of vertices. Connect every vertex to $2(k-1)$ neighbors within a $O(\sqrt{k}) \times O(\sqrt{k})$ grid in a symmetric way - that is, whenever we connect $(x, y)$ with $(x+s, y+t)$, we also connect with $(x-s, y-t)$. Then no line can intersect more than $k-1$ edges incident to a common vertex, and so the graph is $k$-fan-crossing free. Except for vertices near the boundary of the grid, every vertex has degree $2(k-1)$, so the number of edges is $(k-1)(n-O(\sqrt{n k})) \approx k n$. By contrast, our upper bound is $\approx 3 k n$.

The weakness in our technique is that it analyzes arrows separately for every face of $H$. When $H$ is a triangulation, it has $3 n-6$ edges and $2 n-4$ triangles. Each triangle can have $3 k-6$ arrows, so we could have $(2 n-4)(3 k-6)$ arrows, implying $3 n-6+(2 n-4)(3 k-6) / 2=3(k-1)(n-2)$ edges. Our bound is thus the best bound that can be obtained with this technique. For $k=3$, it is possible to improve it slightly by observing that two adjacent triangles of $H$ can only have four arrows in total.

A $(k-1)$-planar graph is automatically $k$-fan-crossing free. Lemma 9 implies that an extremal 2 -fan-crossing free graph is 1 -planar. Is the same statement true for $k=3$ ? It certainly doesn't hold for large $k$, as Pach and Tóth's bound on the number of edges of $k$-planar graphs is only $O(\sqrt{k} n)$ [PT97]. Already for $k=4$, 3-planarity is a stronger condition than absence of a 4 -fan crossing: Pach et al. [PRTT06] showed that 3-planar graphs have at most $5.5(n-2)$ edges. A 4 -fan-crossing free graph with $6 n-12$ edges can be constructed by starting with a triangulation, and adding the "dual" of every edge: for every pair of adjacent triangles, connect the two vertices not shared between the triangles.

The crossing number of a 1-planar graph on $n$ vertices is at most $n-2$ (this implies the bound $4 n-8$ on the number of edges). In contrast, the crossing number of a fan-crossing free graph can be quadratic. For instance, start with the complete graph $K_{q}$ on $q$ vertices. It has $\binom{q}{2}$ edges and crossing number $\Omega\left(q^{4}\right)$. Now subdivide every edge into three edges by inserting two vertices very close to the original vertices. The resulting graph is fan-crossing free and has $n=q+2\binom{q}{2}$ vertices. Since any drawing of this graph can be converted into a drawing of $K_{q}$ with the same number of crossings, it has crossing number $\Omega\left(q^{4}\right)=\Omega\left(n^{2}\right)$. (The same construction could be done with any graph whose crossing number is quadratic in the number of edges, such as an expander graph or even a random graph.)

A natural next question to ask is if our techniques can be used for graphs that do not contain a radial ( $p, q$ )-grid for $q>1$, and if we can find tighter bounds than Pach et al. [PPST05].

In Theorem 3, we have given a rather general bound on the number of edges of graphs that exclude certain crossing patterns. The theorem shows that for graph properties $\mathcal{P}$ that imply that the number of edges grows as $\Theta\left(n^{1+\alpha}\right)$, for $\alpha>0$, the size of the entire graph is bounded by the same function. For the interesting case $\alpha=0$, which arises for instance for fan-crossing free graphs, our bound includes an extra $\log ^{2} n$-term. Is this term an artifact of our proof technique, or are the examples of graph properties where $\mathcal{P}$ implies a linear number of edges, but graphs with $\mathcal{P}^{*}$ and a superlinear number of edges exist?

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