Matroid and Knapsack Center Problems*

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Abstract

In the classic k-center problem, we are given a metric graph, and the objective is to select k nodes as centers such that the maximum distance from any vertex to its closest center is minimized. In this paper, we consider two important generalizations of k-center, the matroid center problem and the knapsack center problem. Both problems are motivated by recent content distribution network applications. Our contributions can be summarized as follows:

- 1. We consider the matroid center problem in which the centers are required to form an independent set of a given matroid. We show this problem is NP-hard even on a line. We present a 3-approximation algorithm for the problem on general metrics. We also consider the outlier version of the problem where a given number of vertices can be excluded as outliers from the solution. We present a 7-approximation for the outlier version.
- 2. We consider the (multi-)knapsack center problem in which the centers are required to satisfy one (or more) knapsack constraint(s). It is known that the knapsack center problem with a single knapsack constraint admits a 3-approximation. However, when there are at least two knapsack constraints, we show this problem is not approximable at all. To complement the hardness result, we present a polynomial time algorithm that gives a 3-approximate solution such that one knapsack constraint is satisfied and the others may be violated by at most a factor of $1 + \epsilon$. We also obtain a 3-approximation for the outlier version that may violate the knapsack constraint by $1 + \epsilon$.

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1 Introduction

The k-center problem is a fundamental facility location problem. In the basic version, we are given a metric space (V, d) and are asked to locate a set $S \subseteq V$ of at most k vertices as centers and to assign the other vertices to the centers, so as to minimize the maximum distance from any vertex to its assigned center, or more formally, to minimize $\max_{v \in V} \min_{u \in S} d(v, u)$. In the demand version of the k-center problem, each vertex v has a positive demand r(v), and our goal is to minimize the maximum weighted distance from any vertex to the centers, i.e., $\max_{v \in V} \min_{u \in S} r(v)d(v, u)$. It is well known that the k-center problem is NP-hard and admits a polynomial time 2-approximation even for the demand version [14, 17], and that no polynomial time $(2 - \epsilon)$ -approximation algorithm exists unless P = NP [14].

In this paper, we conduct a systematic study on two generalizations of the k-center problem and their variants. The first one is the matroid center problem, denoted by MatCenter, which is almost the same as the k-center problem except that, instead of the cardinality constraint on the set of centers, now the centers are required to form an independent set of a given matroid. A finite matroid \mathcal{M} is a pair (V, \mathcal{I}) , where V is a finite set (called the ground set) and \mathcal{I} is a collection of subsets of V. Each element in \mathcal{I} is called an independent set. Moreover, $\mathcal{M} = (V, \mathcal{I})$ satisfies the following three properties: (1) $\emptyset \in \mathcal{I}$; (2) if $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$; (3) for all $A, B \in \mathcal{I}$ with |A| > |B|, there exists an element $e \in A \setminus B$ such that $B \cup \{e\} \in \mathcal{I}$. Following the conventions in the literature, we assume the matroid \mathcal{M} is given by an independence oracle which, given a subset $S \subseteq V$, decides whether $S \in \mathcal{I}$. For more information about the theory of matroids, see, e.g., [29].

The second problem we study is the knapsack center problem (denoted as KnapCenter), another generalization of k-center in which the chosen centers are subject to (one or more) knapsack constraints. More formally, in KnapCenter, there are m nonnegative weight functions w_1, \ldots, w_m on V, and m weight budgets $\mathcal{B}_1, \ldots, \mathcal{B}_m$. Let $w_i(V') := \sum_{v \in V'} w_i(v)$ for all $V' \subseteq V$. A solution takes a set of vertices $S \subseteq V$ as centers such that $w_i(S) \leq \mathcal{B}_i$ for all $1 \leq i \leq m$. The objective is still to minimize the maximum service cost of any vertex in V (the service cost of v equals $\min_{c \in S} d(v, c)$, or $\min_{c \in S} r(v)d(v, c)$ in the demand version). In this paper, we are interested only in the case where the number m of knapsack constraints is a constant. We note that the special case with only one knapsack constraint was studied in [18] under the name of weighted k-center, which already generalizes the basic k-center problem.

Both MatCenter and KnapCenter are motivated by important applications in content distribution networks [16, 22]. In a content distribution network, there are several types of servers and a set of clients to be connected to the servers. Often there is a budget constraint on the number of deployed servers of each type [16]. We would like to deploy a set of servers subject to these budget constraints in order to minimize the maximum service cost of any client. The budget constraints correspond to finding an independent set in a partition matroid.* We can also use a set of knapsack constraints to capture the budget constraints for all types (we need one knapsack constraint for each type). Motivated by such applications, Hajiaghayi et al. [16] first studied the red-blue median problem in which there are two types (red and blue) of facilities, and the goal is to deploy at most k_r red facilities and k_b blue facilities so as to minimize the sum of service costs. Subsequently, Krishnaswamy et al. [22] introduced a more general matroid median problem which seeks to select

^{*} Let B_1, B_2, \ldots, B_b be a collection of disjoint subsets of V and d_i be integers such that $1 \leq d_i \leq |B_i|$ for all $1 \leq i \leq b$. We say a set $I \subseteq V$ is independent if $|I \cap B_i| \leq d_i$ for $1 \leq i \leq b$. All such independent sets form a partition matroid.

a set of facilities that is an independent set in a given matroid and the *knapsack median* problem in which the set of facilities must satisfy a knapsack constraint. The work mentioned above uses the sum of service costs as the objective (the *k*-median objective), while our work aims to minimize the maximum services cost (the *k*-center objective), which is another popular objective in the clustering and network design literature.

1.1 Our Results

For MatCenter, we show the problem is NP-hard to approximate within a factor of $2 - \epsilon$ for any constant $\epsilon > 0$, even on a line. Note that the k-center problem on a line can be solved exactly in polynomial time [5]. We present a 3-approximation algorithm for MatCenter on general metrics. This improves the constant factors implied by the approximation algorithms for matroid median [22, 3] (see Section 2.2 for details).

Next, we consider the outlier version of MatCenter, denoted as Robust-MatCenter, where one can exclude at most n-p nodes as outliers. We obtain a 7-approximation for Robust-MatCenter. Our algorithm is a nontrivial generalization of the greedy algorithm of Charikar et al. [2], which only works for the outlier version of the basic k-center. However, their algorithm and analysis do not extend to our problem. In their analysis, if at least p nodes are covered by k disks (with radius 3 times OPT), they have found a set of k centers and obtained a 3-approximation. However, in our case, we may not be able to open enough centers in the covered region, due to the matroid constraint. Therefore, we need to search for centers globally. To this end, we carefully construct two matroids and argue that their intersection provides a desirable answer (the construction is similar to that for the non-outlier version, but more involved).

We next deal with the KnapCenter problem. We show that for any f > 0, the existence of an f-approximation algorithm for KnapCenter with more than one knapsack constraint implies P = NP. This is a sharp contrast with the case with only one knapsack constraint, for which a 3-approximation exists [18] and is known to be optimal [7]. Given this strong inapproximability result, it is then natural to ask whether efficient approximation algorithms exist if we are allowed to slightly violate the constraints. We answer this question affirmatively. We provide a polynomial time algorithm that, given an instance of KnapCenter with a constant number of knapsack constraints, finds a 3-approximate solution that is guaranteed to satisfy one constraint and violate each of the others by at most a factor of $1 + \epsilon$ for any fixed $\epsilon > 0$. This generalizes the result of [18] to the multi-constraint case. Our algorithm also works for the demand version of the problem.

We then consider the outlier version of the knapsack center problem, which we denote by Robust-KnapCenter. We present a 3-approximation algorithm for Robust-KnapCenter that violates the knapsack constraint by a factor of $1 + \epsilon$ for any fixed $\epsilon > 0$. Our algorithm can be regarded as a "weighted" version of the greedy algorithm of Charikar et al. [2] which only works for the unit-weight case. However, their charging argument does not apply to the weighted case. We instead adopt a more involved algebraic approach to prove the performance guarantee. We translate our algorithm into inequalities involving point sets, and then directly manipulate the inequalities to establish our desired approximation ratio. The total weight of our chosen centers may exceed the budget by the maximum weight of any client, which can be turned into a $1 + \epsilon$ multiplicative factor by the partial enumeration technique. We leave open the question whether there is a constant factor approximation for Robust-KnapCenter that satisfies the knapsack constraint.

1.2 Related Work

For the basic k-center problem, Hochbaum and Shmoys [17, 18] and Gonzalez [14] developed 2-approximation algorithms, which are the best possible if $P \neq NP$ [14]. The former algorithms are based on the idea of the threshold method, which originates from [10]. On some special metrics like the shortest path metrics on trees, k-center (with or without demands) can typically be solved in polynomial time by dynamic programming. By exploring additional structures of the metrics, even linear or quasi-linear time algorithms can be obtained; see e.g. [5, 8, 11] and the references therein. Several generalizations and variations of k-center have also been studied in a variety of application contexts; see, e.g. [1, 25, 20, 4, 9, 21].

A problem closely related to k-center is the well-known k-median problem, whose objective is to minimize the sum of service costs of all nodes instead of the maximum one. Hajiaghayi et al. [16] introduced the red-blue median problem that generalizes k-median, and presented a constant factor approximation based on local search. Krishnaswamy et al. [22] introduced the more general matroid median problem and presented a 16-approximation algorithm based on LP rounding, whose ratio was improved to 9 by Charikar and Li [3] using a more careful rounding scheme. Another generalization of k-median is the knapsack median problem studied by Kumar [23], which requires to open a set of centers with a total weight no larger than a specified value. Kumar gave a (large) constant factor approximation for knapsack median, which was improved by Charikar and Li [3] to a 34-approximation. Several other classical problems have also been investigated recently under matroid or knapsack constraints, such as minimum spanning tree [32], maximum matching [15], and submodular maximization [24, 30].

For the k-center formulation, it is well known that a few distant vertices (outliers) can disproportionately affect the final solution. Such outliers may significantly increase the cost of the solution, without improving the level of service to the majority of clients. To deal with outliers, Charikar et al. [2] initiated the study of the robust versions of k-center and other related problems, in which a certain number of points can be excluded as outliers. They gave a 3-approximation for robust k-center, and showed that the problem with forbidden centers (i.e., some points cannot be centers) is inapproximable within $3 - \epsilon$ unless P = NP. For robust k-median, they presented a bicriteria approximation algorithm that returns a $4(1 + 1/\epsilon)$ -approximate solution in which the number of excluded outliers may violate the upper bound by a factor of $1 + \epsilon$. Later, Chen [6] gave a truly constant factor approximation (with a very large constant) for the robust k-median problem. McCutchen and Khuller [26] and Zarrabi-Zadeh and Mukhopadhyay [31] considered the robust k-center problem in a streaming context.

2 The Matroid Center Problem

In this section, we consider the matroid center problem and its outlier version. A useful ingredient of our algorithms is the (weighted) matroid intersection problem defined as follows. We are given two matroids $\mathcal{M}_1(V, \mathcal{I}_1)$ and $\mathcal{M}_2(V, \mathcal{I}_2)$ defined on the same ground set V. Each element $v \in V$ has a weight $w(v) \geq 0$. The goal is to find a common independent set S in the two matroids, i.e., $S \in \mathcal{I}_1 \cap \mathcal{I}_2$, such that the total weight $w(S) = \sum_{v \in S} w(v)$ is maximized. It is well known that this problem can be solved in polynomial time (e.g., see [29]).

2.1 NP-hardness of Matroid Centers on a Line

In contrast to the basic k-center problem on a line which can be solved in near-linear time [5], we show that MatCenter is NP-hard even on a line. We actually prove the following stronger theorem.

Theorem 1. It is NP-hard to approximate MatCenter on a line within a factor strictly better than 2, even when the given matroid is a partition matroid.

Proof. In a partition matroid, each element in the ground set is colored using one of the h colors and we are given h integers b_1, b_2, \ldots, b_h . The collection of all independent sets is defined to be all subsets that contain at most b_1 elements of color 1, at most b_2 elements of color 2, and so on.

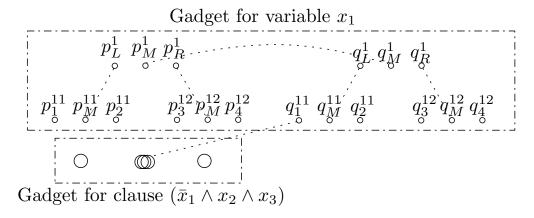


Figure 1: A variable gadget and a clause gadget.

We use the 3SAT problem for the reduction. Without loss of generality, we assume that each literal (including all variables x_i and their negation \bar{x}_i) appears exactly four times in the 3DNF. Given a 3DNF, we create a MatCenter instance as follows. The points appear in groups. Each group consists of r ($r \geq 3$) points with r-2 points in the middle, one to the left and one to the right. The left and right points are 1 unit distance away from the midpoints. Different groups are very far away from each other. Therefore, in order to make the maximum radius at most one, we need to either select one of the midpoints in each group or select at least the two points not in the middle. For each variable x_i , we create a variable gadget as follows. The gadget consists of 6 groups, each having 3 points:

$$(p_L^i,p_M^i,p_R^i),(q_L^i,q_M^i,q_R^i),(p_1^{i1},p_M^{i1},p_2^{i1}),(p_3^{i2},p_M^{i2},p_4^{i2}),(q_1^{i1},q_M^{i1},q_2^{i1}),(q_3^{i2},q_M^{i2},q_4^{i2}).$$

For two points p and q, we use [p,q] to indicate that we assign a new color to p and q. The color assignment for the gadget is defined by the following pairs:

$$[p_M^i,q_M^i],[p_L^i,p_M^{i1}],[p_R^i,p_M^{i2}],[q_L^i,q_M^{i1}],[q_R^i,q_M^{i2}].$$

We are allowed to choose at most one point as a center from each color class. Points $p_1^{i1}, p_2^{i1}, p_3^{i2}, p_4^{i2}$ are called *positive portals* of x_i and points $q_1^{i1}, q_2^{i1}, q_3^{i2}, q_4^{i2}$ are called *negative portals* of x_i . See Figure 1 for an example. For each clause, we create a clause gadget, which is a group of 5 points. We have 3 points in the middle (co-located at the same place), each corresponding to a literal in the clause. If the point corresponds to a positive (negative) literal, say x_i (or \bar{x}_i), the point is paired

with one of the positive (negative) portals of x_i and we assign the pair a new color. We also require that at most one point can be chosen as a center in this pair. Each portal can be paired at most once. Since each literal appears exactly 4 times, we have enough portals for the clause gadgets. All the left and right points of all clause gadgets have the same color but we are allowed to choose none of them as centers.

We can show that the optimal radius for the MatCenter instance is 1 if and only if the 3DNF formula is satisfiable. First, suppose the 3DNF is satisfiable. If x_i is TRUE in a truth assignment, then we pick $p_M^i, p_M^{i1}, p_M^{i2}$ and $p_1^{i1}, p_2^{i1}, p_3^{i2}, p_4^{i2}$ as centers. Otherwise, we pick $q_M^i, q_M^{i1}, q_M^{i1}$ and $q_1^{i1}, q_2^{i1}, q_3^{i2}, q_4^{i2}$ as centers. It is straightforward to verify the independence property. For each group, at least one of the midpoints is selected. Thus, the optimal solution is 1. Given the correspondence, the reverse direction can be proved similarly and we omit it.

2.2 A 3-Approximation for MatCenter

In fact, we can obtain a constant approximation for MatCenter by using the constant approximation for the matroid median problem [22, 3], which roughly gives a 9-approximation for MatCenter. The idea is given below.

We say a space V with a distance function d satisfies the (λ, c) -relaxed triangle inequality (TI) for some λ and c, if $d(a_0, a_c) \leq \lambda \sum_{i=1}^c d(a_{i-1}, a_i)$ for all $a_0, a_1, \ldots, a_c \in V$. (Thus a metric space satisfies the (1, c)-relaxed TI for all $c \geq 1$.) By examining the algorithms in [22, 3] for the matroid median problem, we notice that they can actually give a $(\mu\lambda)$ -approximation for matroid median where μ is some universal constant, if the underlying space satisfies the (λ, c_0) -relaxed TI for some algorithm-dependent c_0 .[†] (Roughly speaking, c_0 is the maximum number of times that the triangle inequality is used for bounding the distance between a client and a facility.) Now, given an instance of MatCenter with metric space (V, d), we define a new distance function d' as $d'(a, b) = (d(a, b))^p$ for all $a, b \in V$, where p > 2 is a parameter whose value will be specified later. By the convexity of the function $f(x) = x^p$ when $p \geq 2$, for all $c \geq 1$ and $a_0, a_1, \ldots, a_c \in V$, we have $(\sum_{i=1}^c d(a_{i-1}, a_i)/c)^p \leq \sum_{i=1}^c d(a_{i-1}, a_i)^p/c$, and thus

$$d'(a_0, a_c) = d(a_0, a_c)^p \le \left(\sum_{i=1}^c d(a_{i-1}, a_i)\right)^p$$

$$\le c^{p-1} \sum_{i=1}^c d(a_{i-1}, a_i)^p = c^{p-1} \sum_{i=1}^c d'(a_{i-1}, a_i).$$

Therefore (V, d') satisfies the (c^{p-1}, c) -relaxed TI for all $c \geq 1$. In particular, it satisfies the (c_0^{p-1}, c_0) -relaxed TI where c_0 is the algorithm-dependent parameter mentioned before. We now solve the matroid median problem on the instance with the new distance function d'. Let OPT denote the optimal objective value of MatCenter on the original instance. Then it is clear that the optimal cost of matroid median on the new instance is at most $|V| \cdot \mathsf{OPT}^p$. By our previous observation, the algorithms of [22, 3] give a solution of cost at most $\mu c_0^{p-1} |V| \mathsf{OPT}^p$. Transforming the distance function back to d, the maximum service cost of any client is at most

[†]We note that Golovin et al. [13] claimed (without a proof) that, in our notations, most existing approximation algorithms for k-median achieve an $O(\lambda)$ -approximation on spaces satisfying $(\lambda, 2)$ -relaxed TI. By a scrutiny of the existing k-median algorithms, we are not able to reproduce the same result and the correct approximation ratio should be roughly $O(\lambda^{c_0})$. However, the results of [13] are not affected in any essential way since this only changes the constant hidden in the big-oh notation.

Algorithm 1: Algorithm for MatCenter on G_i

- 1 Initially, $C \leftarrow \emptyset$, and mark all vertices in V as uncovered.
- $\mathbf{2}$ while V contains uncovered vertices \mathbf{do}
- **3** Pick an uncovered vertex v. Set $\mathsf{B}(v) \leftarrow \mathsf{B}(v, d(e_i))$ and $C \leftarrow C \cup \{v\}$.
- 4 Mark all vertices in $B(v, 2d(e_i))$ as covered.
- 5 end
- 6 Define a partition matroid $\mathcal{M}_{\mathsf{B}} = (V, \mathcal{I})$ with partition $\{\{\mathsf{B}(v)\}_{v \in C}, V \setminus \cup_{v \in C} \mathsf{B}(v)\}$ (note that $\{\mathsf{B}(v)\}_{v \in C}$ are disjoint sets by Lemma 1), where \mathcal{I} is the set of subsets of V that contains at most 1 element from every $\mathsf{B}(v)$ and 0 element from $V \setminus \cup_{v \in C} \mathsf{B}(v)$.
- 7 Solve the unweighted (or, unit-weight) matroid intersection problem between \mathcal{M}_{B} and \mathcal{M} to get an optimal intersection \mathcal{S} . If $|\mathcal{S}| < |C|$, then we declare a failure and try the next G_i . Otherwise, we succeed and return \mathcal{S} as the set of centers.

 $(\mu c_0^{p-1}|V|\mathsf{OPT}^p)^{1/p} = c_0^{1-1/p}(\mu|V|)^{1/p}\mathsf{OPT}$. By choosing $p = \Omega(|V|)$, this can produce a $(c_0 + \epsilon)$ -approximation for MatCenter for any fixed $\epsilon > 0$. Using the algorithm of [3] this roughly gives a 9-approximation.

We next present a 3-approximation for MatCenter, thus improving the ratio derived from the matroid median algorithms [22, 3]. Also, compared to their LP-based algorithms, ours is simpler, purely combinatorial, and easy to implement. We begin with the description of our algorithm. Regard the metric space as a (complete) graph G = (V, E) where each edge $\{u, v\}$ has length d(u, v). Let B(v, r) be the set of vertices that are at most r unit distance away from v (it depends on the underlying graph). Let $e_1, e_2, \ldots, e_{|E|}$ be the edges in a non-decreasing order of their lengths. We consider each spanning subgraph G_i of G that contains only the first i edges, i.e., $G_i = (V, E_i)$ where $E_i = \{e_1, \ldots, e_i\}$. We run Algorithm 1 on each G_i and take the best solution.

Lemma 1. For any two distinct $u, v \in C$, B(u) and B(v) are disjoint sets.

Proof. Suppose we are working on G_i and there is a node w that is in both $\mathsf{B}(u)$ and $\mathsf{B}(v)$. Then we know $d(w,u) \leq d(e_i)$ and $d(w,v) \leq d(e_i)$. Thus, $d(u,v) \leq 2d(e_i)$. But this contradicts with the fact that the distance between every two nodes in C must be larger than $2d(e_i)$.

Theorem 2. Algorithm 1 produces a 3-approximation for MatCenter.

Proof. Suppose the maximum radius of any cluster in an optimal solution is r^* and a set of optimal centers is C^* . Consider the algorithm on G_i with $d(e_i) = r^*$ (r^* must be the length of some edge). First we claim that there exists an intersection of \mathcal{M} and \mathcal{M}_B of size |C|. In fact, we show there is a subset of C^* that is such an intersection. For each node u, let a(u) be an optimal center in C^* that is at most $d(e_i)$ away from u. Consider the set $\mathcal{S}^* = \{a(u)\}_{u \in C}$. Since \mathcal{S}^* is a subset of C^* , it is an independent set of \mathcal{M} by the definition of matroid. It is also easy to see that $a(u) \in B(u)$ for each $u \in C$. Therefore, \mathcal{S}^* is also independent in \mathcal{M}_B , which proves our claim. Thus, the algorithm returns a set \mathcal{S} that contains exactly 1 element from each B(v) with $v \in C$. According to the algorithm, for each $v \in V$ there exists $u \in C$ that is at most $2d(e_i)$ away, and this u is within distance $d(e_i)$ from the (unique) element in $B(u) \cap \mathcal{S}$. Thus every node of V is within a distance $3d(e_i) = 3r^*$ from some center in \mathcal{S} .

2.3 Dealing with Outliers: Robust-MatCenter

We now consider the outlier version of MatCenter, denoted as Robust-MatCenter, in which an additional parameter p is given and the goal is to place centers (which must form an independent set) such that after excluding at most |V| - p nodes as outliers, the maximum service cost of any node is minimized. For p = |V|, we have the standard MatCenter. In this section, we present a 7-approximation for Robust-MatCenter.

Our algorithm bears some similarity to the 3-approximation algorithm for robust k-center by Charikar et al. [2], who also showed that robust k-center with forbidden centers cannot be approximated within $3-\epsilon$ unless P=NP. However, their algorithm for robust k-center does not directly yield any approximation ratio for the forbidden center version. In fact, robust k-center with forbidden centers is a special case of Robust-MatCenter since forbidden centers can be easily captured by a partition matroid. We briefly describe the algorithm in [2]. Assume we have guessed the right optimal radius r. For each $v \in V$, call B(v,r) the disk of v and B(v,3r) the $expanded\ disk$ of v. Repeat the following step k times: Pick an uncovered vertex as a center such that its disk covers the most number of uncovered nodes, then mark all nodes in the corresponding expanded disk as covered. Using a clever charging argument they showed that at least p nodes can be covered, which gives a 3-approximation. However, their algorithm and analysis do not extend to our problem in a straightforward manner. The reason is that even if at least p nodes are covered, we may not be able to find enough centers in the covered region due to the matroid constraint. In order to remedy this issue, we need to search for centers in the entire graph, which also necessitates a more careful charging argument to show that we can cover at least p nodes.

Now we describe our algorithm and prove its performance guarantee. For each $1 \le i \le {|V| \choose 2}$, we run Algorithm 2 on the graph G_i defined as before. We need the following simple lemma.

Lemma 2. \mathcal{M}_1 is a matroid.

Proof. It is straightforward to verify that the first and second matroid properties hold. We only need to verify the third property. Suppose A and B are two independent sets of \mathcal{M}_1 and |A| > |B|. We know the set V(A) (resp., V(B)) of vertices that appear in A (resp., B) is an independent set of \mathcal{M} . Since |V(A)| = |A| and |V(B)| = |B|, |V(A)| > |V(B)|. Hence, there is a vertex $v \in V(A) \setminus V(B)$ such that $V(B) \cup \{v\}$ is independent. We add to B the pair in A that involves v and it is easy to see the resulting set is also independent in \mathcal{M}_1 .

Theorem 3. Algorithm 2 produces a 7-approximation for Robust-MatCenter.

Proof. Assume the maximum radius of any cluster in an optimal solution is r^* and the set of optimal centers is C^* . For each $v \in C^*$, let O(v) denote the optimal disk $B(v, r^*)$. As before, we claim that our algorithm succeeds if $d(e_i) = r^*$. It suffices to show the existence of an intersection of \mathcal{M}_1 and \mathcal{M}_2 with a weight at least p. We next construct such an intersection \mathcal{S}' from the optimal center set C^* . The high level idea is as follows. Let the disk centers in C be v_1, v_2, \ldots, v_k (according to the order that our algorithm chooses them). Note that v_1, v_2, \ldots, v_k are the centers chosen by the greedy procedure in the first part of the algorithm, but not the centers returned at last. We process these centers one by one. Initially, \mathcal{S}' is empty. As we process a new center v_j , we may add $(v, \mathsf{E}(v_j))$ for some $v \in C^*$ to \mathcal{S}' . Moreover, we charge each newly covered node in any optimal disk to some nearby node in the expanded disk $\mathsf{E}(v_j)$. (Note that this is the key difference between our charging argument and that of [2]; in [2], a node may be charged to some node far away.) We maintain that all nodes in $\cup_{v \in C^*} \mathsf{O}(v)$ covered by $\cup_{i'=1}^j \mathsf{E}(v_{i'})$ are charged after processing v_j . Thus,

Algorithm 2: Algorithm for Robust-MatCenter on G_i

- 1 Initially, set $C \leftarrow \emptyset$ and mark all vertices in V as uncovered.
- 2 while V contains uncovered vertices do
- Pick an uncovered vertex v such that $B(v, d(e_i))$ covers the most number of uncovered elements.
- 4 $B(v) \leftarrow B(v, d(e_i))$. (B(v) is called the disk of v.)
- 5 $\mathsf{E}(v) \leftarrow \mathsf{B}(v, 3d(e_i)) \setminus \bigcup_{u \in C} \mathsf{E}(u)$. ($\mathsf{E}(v)$ is called the expanded disk of v. This definition ensures that all expanded disks in $\{\mathsf{E}(u)\}_{u \in C}$ are pairwise disjoint.)
- 6 $C \leftarrow C \cup \{v\}$. Mark all vertices in E(v) as covered.

7 end

- 8 Create a set U of (vertex, expanded disk) pairs, as follows: For each $v \in V$ and $u \in C$, if $\mathsf{B}(v,d(e_i)) \cap \mathsf{B}(u,3d(e_i)) \neq \emptyset$, we add $(v,\mathsf{E}(u))$ to U. The weight $w((v,\mathsf{E}(u)))$ of the pair $(v,\mathsf{E}(u))$ is $|\mathsf{E}(u)|$.
- **9** Define two matroids \mathcal{M}_1 and \mathcal{M}_2 over U as follows:
 - A subset $\{(v_i, \mathsf{E}(u_i))\}$ is independent in \mathcal{M}_1 if all v_i 's in the subset are distinct and form an independent set in \mathcal{M} .
 - A subset $\{(v_i, \mathsf{E}(u_i))\}$ is independent in \mathcal{M}_2 if all $\mathsf{E}(u_i)$'s in the subset are distinct. (It is easy to see \mathcal{M}_2 is a partition matroid.)
- 10 Solve the matroid intersection problem between \mathcal{M}_1 and \mathcal{M}_2 optimally (note that the independence oracles for \mathcal{M}_1 and \mathcal{M}_2 can be easily simulated in polynomial time). Let \mathcal{S} be an optimal intersection. If $w(\mathcal{S}) < p$, then we declare a failure and try the next G_i . Otherwise, we succeed and return $V(\mathcal{S})$ as the set of centers, where $V(\mathcal{S}) = \{v \mid (v, \mathsf{E}(u)) \in \mathcal{S} \text{ for some } u \in C\}$.

eventually, all nodes covered by the optimal solution (i.e., $\cup_{v \in C^*} O(v)$) are charged to the expanded disks selected by our algorithm. We also make sure that each node in any expanded disk in S' is being charged to at most once. Therefore, the weight of S' is at least $|\cup_{v \in C^*} O(v)| \geq p$.

Now, we present the details of the construction of S'. If every node in O(v) for some $v \in C^*$ is charged, we say O(v) is *entirely charged*. Consider the step when we process $v_j \in C$. We distinguish the following cases.

- 1. Suppose there is a node $v \in C^*$ such that O(v) is not entirely charged and O(v) intersects $B(v_j)$. Then add $(v, E(v_j))$ to S' (if there are multiple such v's, we only add one of them). We charge the newly covered nodes in $\bigcup_{v \in C^*} O(v)$ (i.e., the nodes in $(\bigcup_{v \in C^*} O(v)) \cap E(v_j)$) to themselves (we call this charging rule I). Note that O(v) is entirely charged after this step since $O(v) \subseteq B(v_j, 3r^*)$.
- 2. Suppose $B(v_j)$ does not intersect O(v) for any $v \in C^*$, but there is some node $v \in C^*$ such that O(v) is not entirely charged and $O(v) \cap E(v_j) \neq \emptyset$. Then we add $(v, E(v_j))$ to S' and charge all newly covered nodes in O(v) (i.e., the node in $O(v) \cap E(v_j)$) to $B(v_j)$ (we call this charging rule II). Since $B(v_j)$ covers the most number of uncovered elements when v_j is added, there are enough vertices in $B(v_j)$ to charge. Obviously, O(v) is entirely charged after this step. If there is some other node $u \in C^*$ such that O(u) is not entirely charged and $O(u) \cap E(v_j) \neq \emptyset$,

then we charge each newly covered node (i.e., nodes in $O(u) \cap E(v_j)$) in O(u) to itself using rule I.

3. If $E(v_j)$ does not intersect with any optimal disk O(v) that is not entirely charged, then we simply skip this iteration and continue to the next v_j .

It is easy to see that all covered nodes in $\cup_{v \in C^*} O(v)$ are charged in the process and each node is being charged to at most once. Indeed, consider a node u in $B(v_j)$. If $B(v_j)$ intersects some O(v), then u may be charged by rule I and, in this case, no further node can be charged to u again. If $B(v_j)$ does not intersect any O(v), then u may be charged by rule II. This also happens at most once. It is obvious that in this case, no node can be charged to u using rule I. For a node $u \in E(v_j) \setminus B(v_j)$, it can be charged at most once using rule I. Moreover, by the charging process, all nodes in $\cup_{v \in C^*} O(v)$ are charged to the nodes in some expanded disks that appear in S'. Therefore, the total weight of S is at least p. We can see that each vertex in V(S') is also in C^* and appears at most one. Therefore, S' is independent in \mathcal{M}_1 . Clearly, each E(u) appears in S' at most once. Hence, S' is also independent in \mathcal{M}_2 , which proves our claim.

Since S is an optimal intersection, we know the expanded disks in S contain at least p nodes. By the requirement of \mathcal{M}_1 , we can guarantee that the set of centers forms an independent set in \mathcal{M} . For each $(v, \mathsf{E}(u))$ in S, we can see that every node v' in $\mathsf{E}(u)$ is within a distance $7d(e_i)$ from v, as follows. Suppose $u' \in \mathsf{B}(v, d(e_i)) \cup \mathsf{B}(u, 3d(e_i))$ (because $\mathsf{B}(v, d(e_i)) \cup \mathsf{B}(u, 3d(e_i)) \neq \emptyset$ for any pair $(v, \mathsf{E}(u)) \in U$). By the triangle inequality, $d(v', v) \leq d(v', u) + d(u, u') + d(u', v) \leq 3d(e_i) + 3d(e_i) + d(e_i) = 7d(e_i)$. This completes the proof of the theorem.

3 The Knapsack Center Problem

In this section, we study the KnapCenter problem and its outlier version. Recall that an input of KnapCenter consists of a metric space (V,d), m nonnegative weight functions w_1,\ldots,w_m on V, and m budgets $\mathcal{B}_1,\ldots,\mathcal{B}_m$. The goal is to select a set of centers $\mathcal{S}\subseteq V$ with $w_i(\mathcal{S})\leq \mathcal{B}_i$ for all $1\leq i\leq m$, so as to minimize the maximum service cost of any vertex in V. In the outlier version of KnapCenter, we are given an additional parameter $p\leq |V|$, and the objective is to minimize $cost_p(\mathcal{S}):=\min_{V'\subseteq V:|V'|\geq p}\max_{v\in V'}\min_{i\in\mathcal{S}}d(v,i)$, i.e., the maximum service cost of any non-outlier node after excluding at most |V|-p nodes as outliers.

3.1 Approximability of KnapCenter

When there is only one knapsack constraint (i.e., m=1), the problem degenerates to the weighted k-center problem for which a 3-approximation algorithm exists [18]. However, as we show in Theorem 4, the situation changes dramatically even if there are only two knapsack constraints.

Theorem 4. For any f > 0, if there is an f-approximation algorithm for KnapCenter with two knapsack constraints, then P = NP.

Proof. To prove the theorem, we present a reduction from the partition problem, which is well-known to be NP-hard [12], to the KnapCenter problem with two knapsack constraints. In the partition problem, we are given a multiset of positive integers $S = \{s_1, s_2, \ldots, s_n\}$, and the goal is to decide whether S can be partitioned into two subsets such that the sum of numbers in one subset equals the sum of numbers in the other subset.

Given an instance $S = \{s_1, s_2, \dots, s_n\}$ of the partition problem, we construct an instance \mathcal{I} of the KnapCenter problem as follows. The set of clients is $V = \{a_i, b_i \mid 1 \leq i \leq n\}$. The distance metric d is defined as $d(a_i, b_i) = 0$ for all $1 \leq i \leq n$, and $d(a_i, a_j) = d(a_i, b_j) = d(b_i, b_j) = 1$ for all $i \neq j$. It is easy to verify that d is indeed a metric. Every client in V has a unit demand. There are two weight functions w_1 and w_2 specified as follows: for each $1 \leq i \leq n$, $w_1(a_i) = s_i$, $w_1(b_i) = 0$, $w_2(a_i) = 0$, and $w_2(b_i) = s_i$. The two corresponding weight budgets are $\mathcal{B}_1 = \mathcal{B}_2 = T/2$, where $T = \sum_{j=1}^n s_j$. This finishes the construction of \mathcal{I} .

We show that S can be partitioned into two subsets of equal sum if and only if \mathcal{I} has a solution of cost 0. First consider the "if" direction. Assume that \mathcal{I} admits a solution of cost 0. Clearly, for each $1 \leq i \leq n$, the solution must take at least one of $\{a_i, b_i\}$ as a center, and we assume w.l.o.g. that it takes exactly one of a_i and b_i (just choosing an arbitrary one if both are taken). Let I_1 be the set of indices i for which a_i is taken as a center in the solution. Then $I_2 = \{1, 2, \ldots, n\} \setminus I_1$ consists of all indices i for which b_i is taken by the solution. Considering the first weight constraint, we have $T/2 = \mathcal{B}_1 \geq \sum_{i \in I_1} w_1(a_i) + \sum_{i \in I_2} w_1(b_i) = \sum_{i \in I_1} s_i$. Similarly, by the second weight constraint, we get $T/2 \geq \sum_{i \in I_2} s_i$. Since $\sum_{i \in I_1} s_i + \sum_{i \in I_2} s_i = \sum_{i=1}^n s_i = T$, it holds that $\sum_{i \in I_1} s_i = \sum_{i \in I_2} s_i = T/2$. Therefore, S can be partitioned into two subsets of equal sum.

We next prove the "only if" part. Suppose there exists $I_1 \subseteq \{1, 2, ..., n\}$ such that $\sum_{i \in I_1} s_i = T/2$. In the instance \mathcal{I} , we take $\mathcal{T} := \{a_i \mid i \in I_1\} \cup \{b_j \mid j \in \{1, 2, ..., n\} \setminus I_1\}$ as the set of centers. It only remains to show that \mathcal{T} satisfies both the weight constraints, which is easy to verify: $\sum_{v \in \mathcal{T}} w_1(v) = \sum_{i \in I_1} s_i = T/2 \leq \mathcal{B}_1$, and $\sum_{v \in \mathcal{T}} w_2(v) = \sum_{j \in \{1, 2, ..., n\} \setminus I_1} s_j = T - \sum_{j \in I_1} s_j = T/2 \leq \mathcal{B}_2$. This proves the "only if" direction.

Since the optimal objective value of \mathcal{I} is 0, any f-approximate solution is in fact an optimal one. Hence, if KnapCenter with two constraints and unit demands allows an f-approximation algorithm for any f > 0, then the partition problem can be solved in polynomial time, which implies P = NP. The proof of Theorem 4 is thus complete.

It is then natural to ask whether a constant factor approximation can be obtained if the constraints can be relaxed slightly. We show in Theorem 5 that this is achievable (even for the demand version). Before proving the theorem we first present some high-level ideas of our algorithm, shown as Algorithm 3. The algorithm first guesses the optimal cost OPT, and then chooses a collection of disjoint disks of radius OPT according to some rules. It can be shown that there exists a set of centers consisting of exactly one point from each disk that gives a 3-approximate solution and satisfies all the knapsack constraints. We then reduce the remaining task to another problem called the group multi-knapsack problem, which will formally be defined in the following proof.

Theorem 5. For any fixed $\epsilon > 0$, there is a 3-approximation algorithm for KnapCenter with a constant number of knapsack constraints, which is guaranteed to satisfy one constraint and violate each of the others by at most a factor of $1 + \epsilon$.

In what follows we prove Theorem 5. We first present our algorithm for KnapCenter in Algorithm 3 that we use to prove Theorem 5. The algorithm works for the more general version where each vertex v has a demand r(v) and the service cost of v is $\min_{i \in \mathcal{S}} r(v)d(v,i)$ when taking \mathcal{S} as the set of centers.

Given an instance of the KnapCenter problem, suppose Algorithm 3 correctly guesses the optimal objective value OPT. (This can be equivalently realized by running the algorithm for all $\binom{|V|}{2}$ possibilities and taking the best solution among all the candidates.) The algorithm greedily finds

Algorithm 3: Algorithm for KnapCenter with multiple constraints

- 1 Guess the optimal objective value OPT.
- **2** For each client $v \in V$, let $B(v) \leftarrow B(v, \mathsf{OPT})$ be the disk of v. Let $\mathcal{T} \leftarrow \emptyset$.
- **3 while** there exists $i \in V$ such that $B(i) \cap B(j) = \emptyset$ for all $j \in \mathcal{T}$ do
- 4 Choose such an i with maximum demand, and let $\mathcal{T} \leftarrow \mathcal{T} \cup \{i\}$.
- 5 end
- 6 Create an instance \mathcal{I} of the group multi-knapsack problem as $\mathcal{I} = (\{B(i)\}_{i \in \mathcal{T}}, \{w_j, \mathcal{B}_j\}_{1 \leq j \leq m})$ (recall that m = O(1)), and get a solution \mathcal{S} by applying the algorithm indicated by Lemma 3.
- 7 return S

a collection of mutually disjoint disks $\{B(i)\}_{i\in\mathcal{T}}$, and then constructs a set of centers by selecting exactly one point from each disk using some algorithm for the group multi-knapsack problem, which we will define later.

Call a set $S \subseteq V$ standard if S consists of exactly one point from each of the disks $\{B(i)\}_{i \in \mathcal{T}}$. We first show that there exists a standard set S such that $w_j(S) \leq \mathcal{B}_j$ for all $1 \leq j \leq m$, i.e., S fulfills all the knapsack constraints. Suppose $\mathcal{O} \subseteq V$ is the set of centers opened in some optimal solution. Then, for each $i \in \mathcal{T}$, there exists $j \in \mathcal{O}$ such that $r(i)d(i,j) \leq \mathsf{OPT}$, and thus $j \in \mathsf{B}(i)$. Hence, we can choose from each $\mathsf{B}(i)$ exactly one point that belongs to \mathcal{O} , and these points are distinct because the disks are pairwise disjoint. Let S denote the set of these points. Clearly, S is a standard and is a subset of \mathcal{O} , and thus $w_j(S) \leq w_j(\mathcal{O}) \leq \mathcal{B}_j$ for all $1 \leq j \leq m$. This proves the existence of a standard set that satisfies all the knapsack constraints.

We will reduce the remaining task to another problem called the group multi-knapsack problem, which we define as follows. Suppose we are given a collection of pairwise disjoint sets $\{S_i\}_{1 \leq i \leq n}$. Let $S = \bigcup_{i=1}^n S_i$. For some fixed integer $m \geq 1$, there are m nonnegative weight functions defined on the items of S, which we denote by w_1, \ldots, w_m , and m weight limits B_1, \ldots, B_m . A solution is a subset $S' \subseteq S$ that consists of exactly one element from each of the n sets S_1, \ldots, S_n . The goal is find a solution S' such that $w_j(S') \leq B_j$ for all $1 \leq j \leq m$, provided that such solution exists. For our purpose, we require the number of constraints to be a constant. This problem is new to our knowledge, and may be useful in other applications. By Lemma 3 (which will be presented and proved later), we can find in polynomial time a solution that satisfies one constraint and violates each of the others by a small factor.

Now come back to the KnapCenter problem. By Lemma 3, line 6 of Algorithm 3 produces in polynomial time a standard set S that satisfies one constraint and violates each of the others by a factor of at most $1 + \epsilon$. (We notice that, when running Algorithm 3 with an incorrect value of OPT, there may not exist any standard set, in which case the algorithm may return an empty set. We shall simply ignore such solutions.)

It now only remains to show that, by designating S as the set of centers, the maximum service cost of any client is at most $3 \cdot \mathsf{OPT}$. Suppose $S \cap \mathsf{B}(i) = \{t_i\}$ for each $i \in \mathcal{T}$. It suffices to prove that, for each $j \in V$, there exists $i \in \mathcal{T}$ such that $r(j)d(j,t_i) \leq 3 \cdot \mathsf{OPT}$. We consider two cases.

- 1. $j \in \mathcal{T}$. Since $t_i \in \mathsf{B}(j)$, we have $r(j)d(j,t_i) \leq \mathsf{OPT} \leq 3\mathsf{OPT}$ by the definition of $\mathsf{B}(j)$.
- 2. $j \notin \mathcal{T}$. Then $\mathsf{B}(j) \cap \mathsf{B}(i) \neq \emptyset$ for some $i \in \mathcal{T}$, otherwise j should be added to \mathcal{T} by the algorithm. Let $\mathcal{Q} = \{i \in \mathcal{T} \mid \mathsf{B}(i) \cap \mathsf{B}(j) \neq \emptyset\}$. If r(i) < r(j) for all $i \in \mathcal{Q}$, then the algorithm

Algorithm 4: Algorithm for Robust-KnapCenter

```
1 Guess the optimal objective value OPT.
2 For each v \in V, let B(v) \leftarrow B(v, \mathsf{OPT}) and E(v) \leftarrow B(v, \mathsf{3OPT}).
3 \mathcal{S} \leftarrow \emptyset; \mathcal{C} \leftarrow \emptyset (the points in \mathcal{C} are covered and those in V \setminus \mathcal{C} are uncovered).
4 while w(\mathcal{S}) < \mathcal{B} and V \setminus \mathcal{C} \neq \emptyset do
```

- Choose $i \in V \setminus \mathcal{S}$ that maximizes $\frac{|\mathsf{B}(i) \setminus \mathcal{C}|}{w(i)}$. $\mathcal{S} \leftarrow \mathcal{S} \cup \{i\}; \mathcal{C} \leftarrow \mathcal{C} \cup \mathsf{E}(i) \text{ (i.e., mark all uncovered points in } \mathsf{E}(i) \text{ as covered)}$.
- 7 end
- $_8$ return $\mathcal S$

will choose j before choosing all $i \in \mathcal{Q}$, which contradicts with the assumption that $j \notin \mathcal{T}$. Thus, there exists $i \in \mathcal{Q}$ for which $r(i) \geq r(j)$. Consider this particular i, and choose an arbitrary $i' \in B(i) \cap B(j)$. We have

```
r(j)d(j,t_i) \leq r(j)(d(j,i') + d(i,i') + d(i,t_i)) (triangle inequality)
              \leq r(j)d(j,i') + r(i)d(i,i') + r(i)d(i,t_i) (because r(i) \geq r(j))
              \leq OPT + OPT + OPT (due to the definition of disks)
              = 3 \cdot \mathsf{OPT}.
```

Combining the two cases, we have shown that the service cost with centers in S is at most three times the optimal cost, which completes the proof.

Finally, we need the following Lemma 3, which is used in the above argument. The group multi-knapsack problem is similar to the multiple knapsack problem (i.e., the knapsack problem with multiple resource constraints), and the (standard) technique for the latter can be easily adapted to solve the group multi-knapsack problem (see, e.g., [28, 19]). Another way to deduce Lemma 3 is by applying the ϵ -approximate Pareto curve method introduced by Papadimitriou and Yannakakis [27]. For sake of completeness, we give a proof of Lemma 3 in Appendix A.

Lemma 3. For any fixed $\epsilon > 0$, there is a polynomial time algorithm that, given an instance of group multi-knapsack for which a solution satisfying all weight constraints exists, constructs in polynomial time a solution that satisfies one constraint and violates each of the others by at most a factor of $1 + \epsilon$.

Dealing with Outliers: Robust-KnapCenter 3.2

We now study Robust-KnapCenter, the outlier version of KnapCenter. Here we consider the case with one knapsack constraint (with weight function w and budget \mathcal{B}) and unit demand. Our main theorem is as follows.

Theorem 6. There is a 3-approximation algorithm for Robust-KnapCenter that violates the knapsack constraint by at most a factor of $1 + \epsilon$ for any fixed $\epsilon > 0$.

We present our algorithm for Robust-KnapCenter as Algorithm 4. We assume that $\mathcal{B} < w(V)$, since otherwise the problem is trivial. We also set $A/0 := \infty$ for A > 0 and 0/0 := 0, which makes line 5 work even if w(i) = 0. Our algorithm can be regarded as a "weighted" version of that of Charikar et al. [2], but the analysis is much more involved. We next prove the following theorem, which can be used together with the partial enumeration technique to yield Theorem 6. Note that, if all clients have unit weight, Theorem 7 will guarantee a 3-approximate solution S with w(S) < B + 1, which implies $w(S) \le B$. So it actually gives a 3-approximation without violating the constraint. Thus, our result generalizes that of Charikar et al. [2].

Theorem 7. Given an input of the Robust-KnapCenter problem, Algorithm 4 returns a set S with $w(S) < B + \max_{v \in V} w(v)$ such that $cost_p(S) \leq 3\mathsf{OPT}$.

Proof. We call B(v) the disk of v and E(v) the expanded disk of v. Assume w.l.o.g. that the algorithm returns $S = \{1, 2, ..., q\}$ where q = |S|, and that the centers are chosen in the order 1, 2, ..., q. We first observe that B(1), ..., B(q) are pairwise disjoint, which can be seen as follows. By standard use of the triangle inequality, we have $B(i) \subseteq E(j)$ and $B(j) \subseteq E(i)$ for any $i, j \in V$ such that $B(i) \cap B(j) \neq \emptyset$. Therefore, if there exists $1 \leq i < j \leq q$ such that $B(j) \cap B(i) \neq \emptyset$, then all points in B(j) are marked "covered" when choosing i, and hence choosing j cannot cover any more point, contradicting with the way in which the centers are chosen (note that the algorithm terminates when all points have been covered). So the q disks B(1), ..., B(q) are pairwise disjoint.

For ease of notation, let $\mathsf{B}(V') := \bigcup_{v \in V'} \mathsf{B}(v)$ and $\mathsf{E}(V') := \bigcup_{v \in V'} \mathsf{E}(v)$ for $V' \subseteq V$. By the condition of the WHILE loop, $w(\{1,\ldots,q-1\}) < \mathcal{B}$, and thus $w(\mathcal{S}) < \mathcal{B} + w(q) \leq \mathcal{B} + \max_{v \in V} w(v)$. It remains to prove $cost_p(\mathcal{S}) \leq 3\mathsf{OPT}$. Note that this clearly holds if the expanded disks $\mathsf{E}(1),\ldots,\mathsf{E}(q)$ together cover at least p points. Thus, it suffices to show that $|\mathsf{E}(\mathcal{S})| \geq p$. If $w(\mathcal{S}) < \mathcal{B}$, then all points in V are covered by $\mathsf{E}(\mathcal{S})$ due to the termination condition of the WHILE loop, and thus $|\mathsf{E}(\mathcal{S})| = |V| \geq p$. In the rest of the proof, we deal with the case $w(\mathcal{S}) \geq \mathcal{B}$.

For each $v \in V$, let f(v) be the minimum $i \in \mathcal{S}$ such that $\mathsf{B}(v) \cap \mathsf{B}(i) \neq \emptyset$; let $f(v) = \infty$ if no such i exists (i.e., if disk $\mathsf{B}(v)$ is disjoint from all disks centered in \mathcal{S}). Suppose $\mathcal{O} = \{o_1, o_2, \dots, o_m\}$ is an optimal solution, in which the centers are ordered such that $f(o_1) \leq \cdots \leq f(o_m)$. Since the optimal solution is also feasible, we have $|\mathsf{B}(\mathcal{O})| \geq p$. Hence, to prove $|\mathsf{E}(\mathcal{S})| \geq p$, we only need to show $|\mathsf{E}(\mathcal{S})| \geq |\mathsf{B}(\mathcal{O})|$. For any sets A and B, we have $|A| = |A \setminus B| + |A \cap B|$. Therefore,

$$\begin{split} &|E(\mathcal{S})| - |B(\mathcal{O})| \\ &= (|E(\mathcal{S}) \setminus B(\mathcal{O})| + |E(\mathcal{S}) \cap B(\mathcal{O})|) - (|B(\mathcal{O}) \setminus E(\mathcal{S})| + |E(\mathcal{S}) \cap B(\mathcal{O})|) \\ &= |E(\mathcal{S}) \setminus B(\mathcal{O})| - |B(\mathcal{O}) \setminus E(\mathcal{S})| \\ &\geq |B(\mathcal{S}) \setminus B(\mathcal{O})| - |B(\mathcal{O}) \setminus E(\mathcal{S})| \quad \text{(because } B(\mathcal{S}) \subseteq E(\mathcal{S})). \end{split} \tag{1}$$

As $B(1), \ldots, B(q)$ are pairwise disjoint,

$$|\mathsf{B}(\mathcal{S}) \setminus \mathsf{B}(\mathcal{O})| = |\cup_{i \in \mathcal{S}} (\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O}))| = \sum_{i \in \mathcal{S}} |\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O})|,$$

and

$$|\mathsf{B}(\mathcal{O})\setminus\mathsf{E}(\mathcal{S})|=|\cup_{j=1}^m\left(\mathsf{B}(o_j)\setminus\mathsf{E}(\mathcal{S})\right)|\leq \sum_{j=1}^m|\mathsf{B}(o_j)\setminus\mathsf{E}(\mathcal{S})|.$$

Thus,

$$|\mathsf{E}(\mathcal{S})| - |\mathsf{B}(\mathcal{O})| \ge \sum_{i \in \mathcal{S}} |\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O})| - \sum_{j=1}^{m} |\mathsf{B}(o_j) \setminus \mathsf{E}(\mathcal{S})|. \tag{2}$$

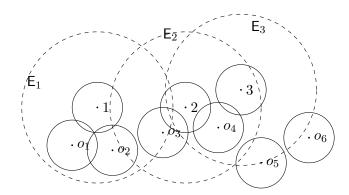


Figure 2: An example of the algorithm for Robust-KnapCenter. The algorithm returns $S = \{1, 2, 3\}$, and the optimal solution opens $\{o_1, o_2, \ldots, o_6\}$. Disks and extended disks are represented by (small) circles and (large) dashed circles, respectively. In this case, we have $f(o_1) = f(o_2) = 1, f(o_3) = f(o_4) = 2, f(o_5) = f(o_6) = \infty$, and thus t = 5. Then, $R(1) = \{1, 2\}, R(2) = \{3, 4\}, \text{ and } R(3) = \emptyset$.

Let t be the unique integer in $\{1, \ldots, m+1\}$ such that $f(o_j) \leq |\mathcal{S}|$ for all $1 \leq j \leq t-1$ and $f(o_j) = \infty$ for all $t \leq j \leq m$. (That is, each disk $\mathsf{B}(o_j)$ $(1 \leq j \leq t-1)$ intersects with $\mathsf{B}(i)$ for some $i \in \mathcal{S}$, while the remaining $\mathsf{B}(o_t), \ldots, \mathsf{B}(o_m)$ are disjoint from all the disks of points in \mathcal{S} . Such t exists because $f(o_1) \leq \cdots \leq f(o_m)$. See Figure 2 for an example.) Then, for all $1 \leq j \leq t-1$, we have $\mathsf{B}(o_j) \cap \mathsf{B}(f(o_j)) \neq \emptyset$, and thus $\mathsf{B}(o_j) \subseteq \mathsf{E}(f(o_j)) \subseteq \mathsf{E}(\mathcal{S})$, implying that $|\mathsf{B}(o_j) \setminus \mathsf{E}(\mathcal{S})| = 0$ for all $1 \leq j \leq t-1$. Combining with the inequality (2), we have

$$|\mathsf{E}(\mathcal{S})| - |\mathsf{B}(\mathcal{O})| \ge \sum_{i \in \mathcal{S}} |\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O})| - \sum_{j=t}^{m} |\mathsf{B}(o_j) \setminus \mathsf{E}(\mathcal{S})|. \tag{3}$$

Hence, it suffices to prove that

$$\sum_{i \in \mathcal{S}} |\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O})| - \sum_{i=t}^{m} |\mathsf{B}(o_i) \setminus \mathsf{E}(\mathcal{S})| \ge 0. \tag{4}$$

The inequality is trivial when t=m+1. Thus, we assume in what follows that $t \leq m$, i.e., $\mathsf{B}(o_m)$ is disjoint from $\mathsf{B}(1), \mathsf{B}(2), \ldots, \mathsf{B}(q)$. Before proving (4), we introduce some notations. For each $i \in \mathcal{S}$, define $\mathsf{R}(i) := \{j \mid 1 \leq j \leq m; f(o_j) = i\}$, and let $l(i) := \min\{j \mid j \in \mathsf{R}(i)\}$ and $q(i) := \max\{j \mid j \in \mathsf{R}(i)\}$ be the minimum index and maximum index in $\mathsf{R}(i)$, respectively (let $l(i) = q(i) = \infty$ if $\mathsf{R}(i) = \emptyset$). By the definitions of $f(\cdot)$ and t, each $\mathsf{R}(i)$ is a set of consecutive integers (or empty), and $\{\mathsf{R}(i)\}_{i\in\mathcal{S}}$ forms a partition of $\{1,2,\ldots,t-1\}$. Also, q(i) = l(i+1)-1 if $l(i+1) \neq \infty$. See Figure 2 for an illustration of the notations.

Consider an arbitrary $i \in \mathcal{S}$. For each j such that $l(i+1) \leq j \leq t-1$, we know that $j \in \mathsf{R}(i')$ for some i' > i, i.e., $f(o_j) = i' > i$, and thus $\mathsf{B}(o_j) \cap \mathsf{B}(i) = \emptyset$. By the definition of t, we also have $\mathsf{B}(o_j) \cap \mathsf{B}(i) = \emptyset$ for all $t \leq j \leq m$. Therefore,

$$\mathsf{B}(o_j)\cap\mathsf{B}(i)=\emptyset \text{ for all } j \text{ s.t. } \min\{t,l(i+1)\}\leq j\leq m. \tag{5}$$

(Here we take the minimum of l(i+1) and t because l(i+1) may be ∞ .)

We next try to lower-bound $|B(i) \setminus B(\mathcal{O})|$ in order to establish (4). Equality (5) tells us that $B(o_i) \cap B(i) \neq \emptyset$ implies $j \in R(1) \cup \cdots \cup R(i)$. In consequence,

$$\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O}) = \mathsf{B}(i) \setminus \cup_{j=1}^{m} \mathsf{B}(o_j) = \mathsf{B}(i) \setminus \cup_{j \in \mathsf{R}(1) \cup \dots \cup \mathsf{R}(i)} \mathsf{B}(o_j). \tag{6}$$

For each $j \in \mathsf{R}(i')$ with $1 \le i' \le i-1$, $\mathsf{B}(o_j) \cap \mathsf{B}(i') \ne \emptyset$, and thus $\mathsf{B}(o_j) \subseteq \mathsf{E}(i') \subseteq \mathsf{E}(\{1,2,\ldots,i-1\})$. For convenience, define $\mathsf{E}_{< i} := \mathsf{E}(\{1,2,\ldots,i-1\})$. Then, from (6) we get $\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O}) \supseteq \mathsf{B}(i) \setminus (\mathsf{E}_{< i} \cup \bigcup_{j \in \mathsf{R}(i)} \mathsf{B}(o_j))$, and hence

$$|\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O})| \ge |\mathsf{B}(i) \setminus (\mathsf{E}_{< i} \cup \bigcup_{j \in \mathsf{R}(i)} \mathsf{B}(o_{j}))|$$

$$= |\mathsf{B}(i) \setminus (\mathsf{E}_{< i} \cup \bigcup_{j \in \mathsf{R}(i)} (\mathsf{B}(o_{j}) \setminus \mathsf{E}_{< i}))| \text{ (because } A \cup \bigcup_{i} B_{i} = A \cup \bigcup_{i} (B_{i} \setminus A))$$

$$= |(\mathsf{B}(i) \setminus \mathsf{E}_{< i}) \setminus \bigcup_{j \in \mathsf{R}(i)} (\mathsf{B}(o_{j}) \setminus \mathsf{E}_{< i})| \ge |\mathsf{B}(i) \setminus \mathsf{E}_{< i}| - \sum_{j \in \mathsf{R}(i)} |\mathsf{B}(o_{j}) \setminus \mathsf{E}_{< i}|. \tag{7}$$

Now consider the particular execution of line 5 in which i is chosen and added to \mathcal{S} . Note that (5) holds for all $i \in \mathcal{S}$. Thus, for all $1 \leq i' \leq i-1$ and $\min\{t, l(i'+1)\} \leq j \leq m$, $\mathsf{B}(o_j)$ is disjoint from $\mathsf{B}(i')$, which in particular implies $o_j \notin \mathsf{B}(i')$. By considering all $i' \in \{1, \ldots, i-1\}$ and noting that $l(i) \geq l(i'+1)$, we have $o_j \notin \mathsf{B}(\{1,2,\ldots,i-1\})$ for all $\min\{t,l(i)\} \leq j \leq m$. This further indicates that $\{1,2,\ldots,i-1\} \cap \{o_j \mid \min\{t,l(i)\} \leq j \leq m\} = \emptyset$. Recall that $1,2,\ldots,i-1$ are all the points added to \mathcal{S} before i. Therefore, no point in $\{o_j \mid \min\{t,l(i)\} \leq j \leq m\}$ was chosen before i. By our way of choosing centers (see line 5), we have

$$\frac{|\mathsf{B}(i) \setminus \mathsf{E}_{< i}|}{w(i)} \ge \frac{|\mathsf{B}(o_j) \setminus \mathsf{E}_{< i}|}{w(o_j)} \text{ for all } j \text{ s.t. } \min\{t, l(i)\} \le j \le m. \tag{8}$$

Hence, for all $j \in R(i)$,

$$|\mathsf{B}(o_j) \setminus \mathsf{E}_{< i}| \le \frac{w(o_j)}{w(i)} |\mathsf{B}(i) \setminus \mathsf{E}_{< i}|.$$

Substituting the above inequality into (7) gives

$$|\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O})| \geq |\mathsf{B}(i) \setminus \mathsf{E}_{< i}| - \sum_{j \in \mathsf{R}(i)} \frac{w(o_j)}{w(i)} |\mathsf{B}(i) \setminus \mathsf{E}_{< i}|$$

$$= \left(1 - \frac{\sum_{j \in \mathsf{R}(i)} w(o_j)}{w(i)}\right) |\mathsf{B}(i) \setminus \mathsf{E}_{< i}|. \tag{9}$$

By (8) we also have

$$|\mathsf{B}(i) \setminus \mathsf{E}_{< i}| \ge w(i) \cdot \max_{t \le j \le m} \frac{|\mathsf{B}(o_j) \setminus \mathsf{E}_{< i}|}{w(o_j)} \ge w(i) \cdot \frac{\sum_{j=t}^m |\mathsf{B}(o_j) \setminus \mathsf{E}_{< i}|}{\sum_{j=t}^m w(o_j)},$$

where we use the inequality $\max_j \frac{A_j}{B_j} \ge \frac{\sum_j A_j}{\sum_j B_j}$ when $B_j \ge 0$ for all j. Plugging this inequality into (9) and noting that $\mathsf{E}_{< i} \subseteq \mathsf{E}(\mathcal{S})$, we obtain:

$$|\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O})| \geq \left(1 - \frac{\sum_{j \in \mathsf{R}(i)} w(o_j)}{w(i)}\right) w(i) \cdot \frac{\sum_{j=t}^m |\mathsf{B}(o_j) \setminus \mathsf{E}_{

$$= \frac{w(i) - \sum_{j \in \mathsf{R}(i)} w(o_j)}{\sum_{j=t}^m w(o_j)} \cdot \sum_{j=t}^m |\mathsf{B}(o_j) \setminus \mathsf{E}_{

$$\geq \frac{w(i) - \sum_{j \in \mathsf{R}(i)} w(o_j)}{\sum_{j=t}^m w(o_j)} \cdot \sum_{j=t}^m |\mathsf{B}(o_j) \setminus \mathsf{E}(\mathcal{S})|. \tag{10}$$$$$$

Applying (10) for all $i \in \mathcal{S}$ and summing the resulting inequalities up, we get

$$\sum_{i \in \mathcal{S}} |\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O})| \ge \frac{\sum_{i \in \mathcal{S}} (w(i) - \sum_{j \in \mathsf{R}(i)} w(o_j))}{\sum_{j=t}^m w(o_j)} \cdot \sum_{j=t}^m |\mathsf{B}(o_j) \setminus \mathsf{E}(\mathcal{S})|$$

$$= \frac{\sum_{i \in \mathcal{S}} w(i) - \sum_{i \in \mathcal{S}} \sum_{j \in \mathsf{R}(i)} w(o_j)}{\sum_{j=t}^m w(o_j)} \cdot \sum_{j=t}^m |\mathsf{B}(o_j) \setminus \mathsf{E}(\mathcal{S})|$$

$$= \frac{w(\mathcal{S}) - \sum_{j=1}^{t-1} w(o_j)}{\sum_{j=t}^m w(o_j)} \cdot \sum_{j=t}^m |\mathsf{B}(o_j) \setminus \mathsf{E}(\mathcal{S})|, \tag{11}$$

where the last equality holds because $\{R(i)\}_{i\in\mathcal{S}}$ is a partition of $\{1, 2, \dots, t-1\}$.

Recall that we are dealing with the case of $w(S) \geq \mathcal{B}$. Since \mathcal{O} is an optimal solution meeting the weight constraint, $w(\mathcal{O}) = \sum_{j=1}^{m} w(o_j) \leq \mathcal{B} \leq w(S)$. Therefore, by (11) we have

$$\sum_{i \in \mathcal{S}} |\mathsf{B}(i) \setminus \mathsf{B}(\mathcal{O})| \ge \frac{\sum_{j=1}^m w(o_j) - \sum_{j=1}^{t-1} w(o_j)}{\sum_{j=t}^m w(o_j)} \cdot \sum_{j=t}^m |\mathsf{B}(o_j) \setminus \mathsf{E}(\mathcal{S})| = \sum_{j=t}^m |\mathsf{B}(o_j) \setminus \mathsf{E}(\mathcal{S})|,$$

which immediately gives (4). This completes the proof of Theorem 7.

At the end of this section, we prove Theorem 6 using Theorem 7 and the partial enumeration technique. Fix a parameter $\epsilon > 0$. Given an instance \mathcal{I} of Robust-KnapCenter, call a point $v \in V$ heavy if $w(v) \geq \epsilon \cdot \mathcal{B}$. Let $\mathcal{O} \subseteq V$ be the set of centers taken by the optimal solution of \mathcal{I} (without violating the knapsack constraint), and \mathcal{H} be the set of heavy centers in \mathcal{O} . Let OPT denote the optimum objective value. Clearly, $|\mathcal{H}| \leq \mathcal{B}/(\epsilon \cdot \mathcal{B}) = 1/\epsilon$. We guess the elements of \mathcal{H} by trying all possible cases (at most $|V|^{1/\epsilon} = |V|^{O(1)}$ possibilities) and using the best solution. We then construct a new instance \mathcal{I}' of Robust-KnapCenter as follows: the metric space is the same as that of \mathcal{I} , the weight function w' is defined as w'(v) = 0 for $v \in \mathcal{H}$ and w'(v) = w(v) for $v \in V \setminus \mathcal{H}$, and the weight budget is $\mathcal{B}' = \mathcal{B} - w(\mathcal{H})$. It is easy to see that opening \mathcal{O} in \mathcal{I}' gives a feasible solution of cost OPT. Note that the maximum weight of any point in \mathcal{I}' is at most $\epsilon \cdot \mathcal{B}$. Hence, by Theorem 7, we can find in polynomial time a solution \mathcal{S} such that $cost(\mathcal{S}) \leq 3$ OPT and $w'(\mathcal{S}) < \mathcal{B} - w(\mathcal{H}) + \epsilon \cdot \mathcal{B}$. We use \mathcal{S} as our solution to the original instance \mathcal{I} . Then, $cost(\mathcal{S}) \leq 3$ OPT and $w(\mathcal{S}) \leq w'(\mathcal{S}) + w(\mathcal{H}) < (1 + \epsilon)\mathcal{B}$. The proof is complete.

4 Concluding Remarks and Open Problems

We gave a 3-approximation algorithm for MatCenter and the best known inapproximability bound is $2-\epsilon$. For Robust-MatCenter, we give a 7-approximation while the current best known lower bound is $3-\epsilon$ due to the hardness of robust k-center with forbidden centers [2]. It would be interesting to close these gaps. (Note that MatCenter includes as a special case the k-center problem with forbidden centers, i.e., some points are not allowed to be chosen as centers. It is known that another generalization of the latter, namely the k-supplier problem, is NP-hard to approximate within $3-\epsilon$ [18].) For Robust-KnapCenter, it is interesting to explore whether constant factor approximation exists while not violating the knapsack constraint. It is also open whether there is a constant factor approximation for the demand version (even for the unit-weight case). Finally, extending our results for Robust-KnapCenter to the multi-constraint case seems intriguing and may require essentially different ideas.

References

- [1] G. Aggarwal, T. Feder, K. Kenthapadi, S. Khuller, R. Panigrahy, D. Thomas, and A. Zhu. Achieving anonymity via clustering. In *PODS*, pages 153–162, 2006.
- [2] M. Charikar, S. Khuller, D. Mount, and G. Narasimhan. Algorithms for facility location problems with outliers. In *SODA*, pages 642–651, 2001.
- [3] M. Charikar and S. Li. A dependent LP-rounding approach for the k-median problem. In *ICALP*, pages 194–205, 2012.
- [4] S. Chechik and D. Peleg. The fault tolerant capacitated k-center problem. In SIROCCO, pages 13–24, 2012.
- [5] D. Chen and H. Wang. Efficient algorithms for the weighted k-center problem on a real line. In ISAAC, pages 584-593, 2011.
- [6] K. Chen. A constant factor approximation algorithm for k-median clustering with outliers. In SODA, pages 826–835, 2008.
- [7] J. Chuzhoy, S. Guha, E. Halperin, S. Khanna, G. Kortsarz, R. Krauthgamer, and J. Naor. Asymmetric k-center is log* n-hard to approximate. J. ACM, 52(4):538–551, 2005.
- [8] R. Cole. Slowing down sorting networks to obtain faster sorting algorithms. *J. ACM*, 34(1):200–208, 1987.
- [9] M. Cygan, M. Hajiaghayi, and S. Khuller. LP rounding for k-centers with non-uniform hard capacities. In FOCS, pages 273–282, 2012.
- [10] J. Edmonds and D. Fulkerson. Bottleneck extrema. J. Combin. Theory, 8(3):299–306, 1970.
- [11] G. Frederickson. Parametric search and locating supply centers in trees. In WADS, pages 299–319, 1991.
- [12] M. Garey and D. Johnson. Computers and Intractability: A Guide to the Theory of NP-Completeness. W. H. Freeman, 1979.
- [13] D. Golovin, A. Gupta, A. Kumar, and K. Tangwongsan. All-norms and all- L_p -norms approximation algorithms. In FSTTCS, pages 199–210, 2008.
- [14] T. Gonzalez. Clustering to minimize the maximum intercluster distance. *Theor. Comput. Sci.*, 38:293–306, 1985.
- [15] F. Grandoni and R. Zenklusen. Approximation schemes for multi-budgeted independence systems. In ESA, pages 536–548, 2010.
- [16] M. Hajiaghayi, R. Khandekar, and G. Kortsarz. Budgeted red-blue median and its generalizations. In ESA, pages 314–325, 2011.
- [17] D. Hochbaum and D. Shmoys. A best possible heuristic for the k-center problem. *Math. Oper. Res.*, pages 180–184, 1985.

- [18] D. Hochbaum and D. Shmoys. A unified approach to approximation algorithms for bottleneck problems. *J. ACM*, 33(3):533–550, 1986.
- [19] H. Kellerer, U. Pferschy, and D. Pisinger. Knapsack Problems. Springer Verlag, 2004.
- [20] S. Khuller, R. Pless, and Y. Sussmann. Fault tolerant k-center problems. Theor. Comput. Sci., 242(1-2):237-245, 2000.
- [21] S. Khuller, B. Saha, and K. K. Sarpatwar. New approximation results for resource replication problems. In *APPROX-RANDOM*, volume 7408 of *LNCS*, pages 218–230. Springer, 2012.
- [22] R. Krishnaswamy, A. Kumar, V. Nagarajan, Y. Sabharwal, and B. Saha. The matroid median problem. In *SODA*, 2011.
- [23] A. Kumar. Constant factor approximation algorithm for the knapsack median problem. In *SODA*, pages 824–832, 2012.
- [24] J. Lee, V. S. Mirrokni, V. Nagarajan, and M. Sviridenko. Non-monotone submodular maximization under matroid and knapsack constraints. In *STOC*, pages 323–332, 2009.
- [25] J. Li, K. Yi, and Q. Zhang. Clustering with diversity. In ICALP, pages 188–200, 2010.
- [26] R. McCutchen and S. Khuller. Streaming algorithms for k-center clustering with outliers and with anonymity. In RANDOM, pages 165–178, 2008.
- [27] C. H. Papadimitriou and M. Yannakakis. On the approximability of trade-offs and optimal access of Web sources. In *FOCS*, pages 86–92, 2000.
- [28] D. Pisinger. Algorithms for Knapsack Problems. PhD thesis, Department of Computer Science, University of Copenhagen, 1995.
- [29] A. Schrijver. Combinatorial Optimization: Polyhedra and Efficiency. Springer-Verlag, Berlin, 2003.
- [30] J. Vondrák, C. Chekuri, and R. Zenklusen. Submodular function maximization via the multilinear relaxation and contention resolution schemes. In *STOC*, pages 783–792, 2011.
- [31] H. Zarrabi-Zadeh and A. Mukhopadhyay. Streaming 1-center with outliers in high dimensions. In *CCCG*, pages 83–86, 2009.
- [32] R. Zenklusen. Matroidal degree-bounded minimum spanning trees. In *SODA*, pages 1512–1521, 2012.

A Proof of Lemma 3

Let $\mathcal{I} = (\{\mathcal{S}_i\}_{1 \leq i \leq n}, \{w_j, \mathcal{B}_j\}_{1 \leq j \leq m})$ be an instance of the group multi-knapsack problem, for which there exists a solution satisfying all the weight constraints; we will call such a solution good. Let $\mathcal{S} = \bigcup_{i=1}^n \mathcal{S}_i$ and $w_{\max} = \max_{v \in \mathcal{S}; 1 \leq j \leq m} w_j(v)$. When m = 1, we can simply choose from each \mathcal{S}_i the element $v \in \mathcal{S}_i$ with the smallest $w_1(v)$. In what follows, we assume $m \geq 2$. If there exists $1 \leq j \leq m$ such that $w_{\max} > \mathcal{B}_j$, then the element having the weight w_{\max} cannot appear in any

good solution, and we will modify the instance by removing it from S. Hence, we also assume that $w_{\text{max}} \leq \mathcal{B}_j$ for all $1 \leq j \leq m$.

We apply the scaling technique that has been widely used in the design of PTASs for knapsacklike problems. Fix $\epsilon > 0$, and define $A := \epsilon \cdot w_{\text{max}}/n$. For each $v \in \mathcal{S}$, define

$$w'_j(v) = \lfloor w_j(v)/A \rfloor$$
 for all $1 \le j \le m-1$, and $w'_m(v) = w_m(v)$.

Also define

$$\mathcal{B}'_j = \min\{\lfloor \mathcal{B}_j/A \rfloor, \lfloor n^2/\epsilon \rfloor\} \text{ for all } 1 \leq j \leq m-1, \text{ and } \mathcal{B}'_m = \mathcal{B}_m.$$

(The choice of the "special" index m can be arbitrary; it indicates the constraint that we wish to satisfy.) We have $w'_j(v) \in \{0, 1, \dots, K\}$ for all $1 \leq j \leq m-1$ and $v \in \mathcal{S}$, where $K = \lfloor w_{\max}/A \rfloor = \lfloor n/\epsilon \rfloor$. Create a new instance $\mathcal{I}' = (\{\mathcal{S}_i\}_{1 \leq i \leq n}, \{w'_j, \mathcal{B}'_j\}_{1 \leq j \leq m})$). For the original instance \mathcal{I} , we know that there exists a good solution $\mathcal{T} \subseteq \mathcal{S}$. Using the inequality $\lfloor a \rfloor + \lfloor b \rfloor \leq \lfloor a+b \rfloor$, we obtain that for each $1 \leq j \leq m-1$,

$$w'_{j}(\mathcal{T}) = \sum_{v \in \mathcal{T}} \lfloor w_{j}(v)/A \rfloor$$

$$\leq \min\{\lfloor \sum_{v \in \mathcal{T}} w_{j}(v)/A \rfloor, n \cdot \lfloor n/\epsilon \rfloor\}$$

$$\leq \min\{\lfloor \mathcal{B}_{j}/A \rfloor, \lfloor n^{2}/\epsilon \rfloor\}$$

$$= \mathcal{B}'_{j}.$$

Also, $w'_m(\mathcal{T}) = w_m(\mathcal{T}) \leq \mathcal{B}_m = \mathcal{B}'_m$. Therefore, \mathcal{T} is also a good solution of \mathcal{I}' . For $i \in \{1, 2, ..., n\}$, a subset $\mathcal{T} \subseteq \mathcal{S}$ is called *i-standard* if \mathcal{T} consists of exactly one element from each of the i sets $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_i$. Thus a solution of \mathcal{I}' is just an n-standard subset, and vice versa. For each tuple $(i, p_1, p_2, \ldots, p_{m-1})$ where $i \in \{1, \ldots, n\}$ and $(\forall 1 \leq j \leq m-1)p_j \in \{0, 1, \ldots, \mathcal{B}'_j\}$, let $F(i, p_1, p_2, \ldots, p_{m-1})$ denote the minimum possible value of p_m for which there exists an i-standard subset \mathcal{T} such that $w_j(\mathcal{T}) \leq p_j$ for all $1 \leq j \leq m$, and let $\mathcal{T}(i, p_1, p_2, \ldots, p_{m-1})$ be an (arbitrary) such i-standard subset. If such p_m does not exist, then we let $F(i, p_1, p_2, \ldots, p_{m-1}) = \infty$ and $\mathcal{T}(i, p_1, p_2, \ldots, p_{m-1}) = \emptyset$. Since \mathcal{I}' admits a good solution, it is easy to see that

$$F(n, \mathcal{B}'_1, \mathcal{B}'_2, \dots, \mathcal{B}'_{m-1}) \le \mathcal{B}'_m.$$

Our goal is thus to find $\mathcal{T}(n, \mathcal{B}'_1, \dots, \mathcal{B}'_{m-1})$. Note that the number of tuples (i, p_1, \dots, p_{m-1}) is at most $n \cdot \prod_{j=1}^{m-1} \mathcal{B}'_j \leq n(n^2/\epsilon)^{m-1} = n^{O(1)}$, since m and ϵ are both constants.

We now compute all $F(i, p_1, p_2, ..., p_{m-1})$ and find the corresponding i-standard subsets by dynamic programming. The base case is i = 1. For each tuple $(1, p_1, p_2, ..., p_{m-1})$, let $\mathcal{R} = \{v \in \mathcal{S}_1 \mid (\forall 1 \leq j \leq m-1)w_j(v) \leq p_j\}$. If $\mathcal{R} \neq \emptyset$, then clearly $F(1, p_1, ..., p_{m-1}) = \min_{v \in \mathcal{R}} w_m(v)$, and we set $\mathcal{T}(1, p_1, ..., p_{m-1})$ to be the vertex $v \in \mathcal{R}$ that achieves the minimum $w_m(v)$. If $\mathcal{R} = \emptyset$, then $F(1, p_1, ..., p_{m-1}) = \infty$ and $\mathcal{T}(1, p_1, ..., p_{m-1}) = \emptyset$.

Next we derive the transition function for computing $F(i, p_1, p_2, ..., p_{m-1})$ for $i \geq 2$. We enumerate all possible $v \in S_i$ that may belong to $\mathcal{T}(i, p_1, ..., p_{m-1})$. Then, it is easy to see that

$$F(i, p_1, \dots, p_{m-1}) = \min_{v \in S_i} \{ w_m(v) + F(i-1, p_1 - w_1(v), p_2 - w_2(v), \dots, p_{m-1} - w_{m-1}(v)) \}.$$

(We assume $F(i', p'_1, \dots, p'_{m-1}) = \infty$ if $p'_j < 0$ for some j.)

If $F(i, p_1, \ldots, p_{m-1}) = \infty$, then we let $\mathcal{T}(i, p_1, \ldots, p_{m-1}) = \emptyset$; otherwise, assuming the minimum value is attained at $v \in \mathcal{S}_i$, we set

$$\mathcal{T}(i, p_1, \dots, p_{m-1}) = \{v\} \cup \mathcal{T}(i-1, p_1 - w_1(v), \dots, p_{m-1} - w_{m-1}(v)).$$

In this way, we can correctly compute the values of every $F(i, p_1, \ldots, p_{m-1})$ and find the set $\mathcal{T}(i, p_1, \ldots, p_{m-1})$ witnessing the value. Since there are only $n^{O(1)}$ tuples and the time spent on each tuple is polynomial in the number of elements, the computation can be done in polynomial time.

As argued before, $\mathcal{T} = \mathcal{T}(n, \mathcal{B}'_1, \mathcal{B}'_2, \dots, \mathcal{B}'_{m-1})$ is a good solution to \mathcal{I}' , provided that the original instance \mathcal{I} has a good solution. Now we take \mathcal{T} as our solution to \mathcal{I} . (We note that, if the original instance \mathcal{I} is not guaranteed to have a good solution, then we may have $F(n, \mathcal{B}'_1, \dots, \mathcal{B}'_{m-1}) > \mathcal{B}'_m$, in which case we will simply return an empty set. This can happen when Algorithm 3 is executed with an incorrect value of OPT.) We have $w_m(\mathcal{T}) = w'_m(\mathcal{T}) \leq \mathcal{B}'_m = \mathcal{B}_m$. For each $1 \leq j \leq m-1$, $w'_j(v) = \lfloor w_j(v)/A \rfloor > w_j(v)/A - 1$, and thus we have

$$\sum_{v \in \mathcal{T}} w_j(v) \leq \sum_{v \in \mathcal{T}} (A \cdot w'_j(v) + A) = A \cdot \sum_{v \in \mathcal{T}} w'_j(v) + nA$$

$$\leq A \cdot \mathcal{B}'_j + n \cdot \epsilon \cdot w_{\max}/n$$

$$\leq \mathcal{B}_j + \epsilon \cdot w_{\max}$$

$$\leq (1 + \epsilon)\mathcal{B}_j \text{ (since } w_{\max} \leq \mathcal{B}_j).$$

Therefore, \mathcal{T} is a solution of \mathcal{I} that satisfies one of the constraints and violates the others by at most a factor of $1+\epsilon$. (It is easy to see that, by modifying the definitions of $\{w'_j\}$ and $\{\mathcal{B}'_j\}$, we can make any one of the constraints to be the satisfied one.) The proof of Lemma 3 is thus complete.