

Matrix sparsification and the sparse null space problem

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Abstract

We revisit the matrix problems **sparse null space** and **matrix sparsification**, and show that they are equivalent. We then proceed to seek algorithms for these problems: We prove the hardness of approximation of these problems, and also give a powerful tool to extend algorithms and heuristics for sparse approximation theory to these problems.

1 Introduction

In this paper, we revisit the matrix problems **sparse null space** and **matrix sparsification**.

The **sparse null space** problem was first considered by Pothén in 1984 [27]. The problem asks, given a matrix A , to find a matrix N that is a full null matrix for A – that is, N is full rank and the columns of N span the null space of A . Further, N should be sparse, i.e. contain as few nonzero values as possible. The **sparse null space** problem is motivated by its use to solve Linear Equality Problems (LEPs) [9]. LEPs arise in the solution of constrained optimization problems via generalized gradient descent, segmented Lagrangian, and projected Lagrangian methods. Berry *et al.* [4] consider the **sparse null space** problem in the context of the dual variable method for the Navier-Stokes equations, or more generally in the context of null space methods for quadratic programming. Gilbert and Heath [16] noted that among the numerous applications of the **sparse null space** problem arising in solutions of underdetermined system of linear equations, is the efficient solution to the force method (or flexibility method) for structural analysis, which uses the null space to create multiple linear systems. Finding a sparse null space will decrease the run time and memory required for solving these systems. More recently, it was shown [36, 26] that the **sparse null space** problem can be used to find correlations between small numbers of time series, such as financial stocks. The decision version of the **sparse null space** problem is known to be NP-Complete [9], and only heuristic solutions have been suggested for the minimization problem [9, 16, 4].

The **matrix sparsification** problem is of the same flavor as **sparse null space**. One is given a full rank matrix A , and the task is to find another matrix B that is equivalent to A under elementary column operations, and contains as few nonzero values as possible. Many fundamental matrix operations are greatly simplified by first sparsifying a matrix (see [12]) and the problem has applications in areas such as machine learning [30] and in discovering cycle bases of graphs [20]. But there seem to be only a small number of heuristics for **matrix sparsification** ([7] for example), or algorithms under limiting assumptions ([17] considers matrices that satisfy the Haar condition), but no general approximation algorithms. McCormick [22] established that the decision version of this problem is NP-Complete.

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For these two classic problems, we wish to investigate potentials and limits of approximation algorithms both for the general problems and for some variants under simplifying assumptions. To this end, we will need to consider the well-known vector problems `min unsatisfy` and `exact dictionary representation` (elsewhere called the `sparse approximation` or `highly nonlinear approximation problem` [32]).

The `min unsatisfy` problem is an intuitive problem on linear equations. Given a system $Ax = b$ of linear equations (where A is an integer $m \times n$ matrix and b is an integer m -vector), the problem is to provide a rational n -vector x ; the measure to be minimized is the number of equations not satisfied by $Ax = b$. The term “`min unsatisfy`” was first coined by Arora *et al.* [2] in a seminal paper on the hardness of approximation, but they claim that the the NP-Completeness of the decision version of this problem is implicit in a 1978 paper of Johnson and Preparata [18]. Arora *et al.* demonstrated that it is hard to approximate `min unsatisfy` to within a factor $2^{\log^{5-o(1)} n}$ of optimal (under the assumption that NP does not admit a quasi-polynomial time deterministic algorithm). This hardness result holds over \mathbb{Q} , and stronger results are known for finite fields [10]. For this problem, Berman and Karpinski [3] gave a randomized $\frac{m}{c \log m}$ -approximation algorithm (where c is a constant). We know of no heuristics studied for this problem.

The `exact dictionary representation` problem is the fundamental problem in sparse approximation theory (see [23]). In this problem, we are given a matrix of dictionary vectors D and a target vector s , and the task is to find the smallest set $D' \subset D$ such that a linear combination of the vectors of D' is equal to s . This problem and its variants have been well studied. According to Temlyakov [31], a variant of this problem may be found as early as 1907, in a paper of Schmidt [28]. The decision version of this problem was shown to be NP-Complete by Natarajan [24]. (See [21] for further discussion.)

The field of sparse approximation theory has become exceedingly popular: For example, SPAR05 was largely devoted to it, as was the SparseLand 2006 workshop at Princeton, and a mini-symposium at NYU’s Courant Institute in 2007. The applications of sparse approximation theory include signal representation and recovery [8, 25], amplitude optimization [29] and function approximation [24]. When the dictionary vectors are Fourier coefficients, this problem is a classic problem in Fourier analysis, with applications in data compression, feature extraction, locating approximate periods and similar data mining problems [37, 14, 15, 6]. There is a host of results for this problem, though all are heuristics or approximations under some qualifying assumptions. In fact, Amaldi and Kann [1] showed that this problem (they called it RVLS – ‘relevant variables in the linear system’) is as hard to approximate as `min unsatisfy`, though their result seems to have escaped the notice of the sparse approximation theory community.

Our contribution. As a first step, we note that the matrix problems `sparse null space` and `matrix sparsification` are equivalent, and that the vector problems `min unsatisfy` and `exact dictionary representation` are equivalent as well. Note that although these equivalences are straightforward, they seem to have escaped researchers in this field. For example, [5] claimed that the `sparse null space` problem is computationally more difficult than `matrix sparsification`.)

We then proceed to show that `matrix sparsification` is hard to approximate, via a reduction from `min unsatisfy`. We will thereby show that the two matrix problems are hard to approximate within a factor $2^{\log^{5-o(1)} n}$ of optimal (assuming NP does not admit quasi-polynomial time deterministic algorithms).

This hardness result for `matrix sparsification` is important in its own right, but it further leads us to ask what *can* be done for this problem. Specifically, what restrictions or simplifying assumptions may be made upon the input matrix to make `matrix sparsification` problem tractable? In addressing this question, we provide the major contribution of this paper and show how to adapt the vast number of heuristics and algorithms for `exact dictionary representation` to solve `matrix sparsification` (and hence `sparse null space` as well). This allows us to conclude, for example, that `matrix sparsification` admits a randomized $\frac{m}{c \log m}$ -approximation algorithm, and also to give limiting conditions under which a known ℓ_1 relaxation scheme for `exact dictionary matching` solves `matrix sparsification` exactly. Our results also carry over to relaxed version of these problems, where the input is extended by an error term δ which relaxes a constraint.

These versions are defined in the appendix (§B), although we omit the proof of their equivalence. All of our results assume that the vector variables are over \mathbb{Q} .

An outline of our paper follows: In Section 2 we review some linear algebra and introduce notation. In closing the preliminary section (2.3), we prove equivalences between the two matrix problems and the two vector problems. In Section 3 we prove that **matrix sparsification** is hard to approximate, and in Section 4 we show how to adapt algorithms for exact dictionary representation to solve **matrix sparsification**.

2 Preliminaries

In this section we review some linear algebra, introduce notation and definitions, and formally state our four problems.

2.1 Linear algebra and notation.

Matrix and vector properties. Given a set V of n m -dimensional column vectors, an m -vector $v \notin V$ is *independent* of the vectors of V if there is no linear combination of vectors in V that equals v . A set of vectors is independent if each vector in the set is independent of the rest.

Now let the vectors of V be arranged as columns of an $m \times n$ matrix A ; we refer to a column of A as a_i , and to a position in A as a_{ij} . We define $\#\text{col}(A)$ to be the number of columns of A . The column *span* of A ($\text{col}(A)$) is the (infinite) set of column vectors that can be produced by a linear combination of the columns of A . The *column rank* of A is the dimension of the column space of A ($\text{rank}(A) = \dim(\text{col}(A))$); it is the size of the maximal independent subset in the columns of A . If the column rank of A is equal to n , then the columns of A are independent, and A is said to be *full rank*.

Other matrices may be produced from A using *elementary column operations*. These include multiplying columns by a nonzero factor, interchanging columns, and adding a multiple of one column to another. These operations produce a matrix A' which has the same column span as A ; we say A and A' are *column equivalent*. It can be shown that A, A' are column equivalent iff $A' = AX$ for some invertible matrix X .

Let R be a set of rows of A , and C be a set of columns. $A(R, C)$ is the submatrix of A restricted to R and C . Let $A(:, C)$ ($A(R, :)$) be the submatrix of A restricted to all rows of A and to columns in C (restricted to the rows of R and all columns in A). A *square matrix* is an $m \times m$ matrix. A square matrix is *nonsingular* if it is invertible.

Null space. The *null space* (or *kernel*) of A ($\text{null}(A)$) is the set of all nonzero n -length vectors b for which $Ab = 0$. The rank of A 's null space is called the *corank* of A . The rank-nullity theorem states that for any matrix A , $\text{rank}(A) + \text{corank}(A) = n$. Let N be a matrix consisting of column vectors in the null space of A ; we have that $AN = 0$. If the rank of N is equal to the corank of A then N is a *full null matrix* for A .

Given matrix A , a full null matrix for A can be constructed in polynomial time. Similarly, given a full rank matrix N , polynomial time is required to construct a matrix A for which N is a full null matrix [26].

Notation. Throughout this paper, we will be interested in the number of zero and nonzero entries in a matrix A . Let $\text{nnz}(A)$ denote the number of nonzero entries in A . For a vector x , let $\|x\|_0$ denote the number of nonzero entries in x . This notation refers to the quasi-norm ℓ_0 , which is not a true norm since $\lambda\|x\|_0 \neq \|\lambda x\|_0$, although it does honor the triangle inequality.

For vector x , let x_i be the value of the i^{th} position in x . The *support* of x ($\text{supp}(x)$) is the set of indices in x which correspond to nonzero values, $i \in \text{supp}(x) \Leftrightarrow x_i \neq 0$.

The notation $A|B$ indicates that the rows of matrix B are concatenated to the rows of matrix A . The notation $\begin{pmatrix} A \\ B \end{pmatrix}$ indicates that the columns of B are appended to the columns of A . $M = A \otimes B$ denotes the Kronecker product of two matrices, where M is formed by multiplying each individual entry in A by the entire matrix B . (If A is $m \times n$, B is $p \times q$, then M is $mp \times nq$.)

By *equivalent* problems, we mean that reductions between them preserve approximation factors. A formal definition of approximation equivalence is found in Appendix A.

2.2 Minimization problems

In this section, we formally state the four major minimization problems discussed in this paper. The first two problems have vector solutions, and the second two problems have matrix solutions. Our results hold when the variables are over \mathbb{Q} , although these problems can be defined over \mathbb{R} . \mathcal{I}_F is the set of input instances, $S_F(x)$ is the solution space for $x \in \mathcal{I}_F$, $M_F(x, y)$ is the objective metric for $x \in \mathcal{I}_F$ and $y \in S_F(x)$.

exact dictionary representation (EDR)

$\mathcal{I}_{\text{EDR}} = \langle D, s \rangle$, $m \times n$ matrix D , vector s with $s \in \text{col}(D)$

$S_{\text{EDR}}(D, s) = \{v \in \mathbb{Q}^n : Dv = s\}$

$m_{\text{EDR}}(\langle D, s \rangle, v) = \|v\|_0$

min unsatisfy (MU)

$\mathcal{I}_{\text{MU}} = \langle A, y \rangle$, $m \times n$ matrix A , vector $y \in \mathbb{Q}^m$

$S_{\text{MU}}(A, y) = \{x : x \in \mathbb{Q}^n\}$

$m_{\text{MU}}(\langle A, y \rangle, x) = \|y - Ax\|_0$

sparse null space (SNS)

$\mathcal{I}_{\text{SNS}} = \text{matrix } A$

$S_{\text{SNS}}(A) = \{N : N \text{ is a full null matrix for } A\}$

$m_{\text{SNS}}(A, N) = \text{nnz}(N)$

matrix sparsification (MS)

$\mathcal{I}_{\text{MS}} = \text{full rank } m \times n \text{ matrix } B$

$S_{\text{MS}}(B) = \{\text{matrix } N : N = BX \text{ for some invertible matrix } X\}$

$m_{\text{MS}}(B, N) = \text{nnz}(N)$

2.3 Equivalences

In closing the preliminary section, we show that **min unsatisfy** and **exact dictionary representation** are equivalent. We then show that **matrix sparsification** and **matrix sparsification** are equivalent. The type of equivalence is formally stated in definition 14, and guarantees exact equality of approximation factors among polynomial-time algorithms (Corollary 16).

Equivalence of vector problems. Here we show that **EDR** and **min unsatisfy** are equivalent.

We reduce **EDR** to **min unsatisfy**. Given input $\langle D, s \rangle$ to **EDR**, we seek a vector v with minimum $\|v\|_0$ that satisfies $Dv = s$. Let y be any vector that satisfies $Dy = s$, and A be a full null matrix for D . (These can be derived in polynomial time.) Let $x = \text{MU}(A, y)$ and $v = y - Ax$. We claim that v is a solution to **EDR**. First note that v satisfies $Dv = s$: $Dv = D(y - Ax) = Dy - DAx = s - 0 = s$. Now, the call to $\text{MU}(A, y)$ returned a vector x for which $\|y - Ax\|_0 = \|v\|_0$ is the minimization measure; and, as x ranges over \mathbb{R}^n ,

the vector $v = y - Ax$ ranges over all vectors with $Dv = s$. Hence, the oracle for `min unsatisfy` directly minimizes $\|v\|_0$, and so v is a solution to `EDR`.

We now reduce `min unsatisfy` to `EDR`. Given input $\langle A, y \rangle$ to `min unsatisfy`, we seek a vector x which minimizes $\|y - Ax\|_0$. We may assume that A is full rank. (Otherwise, we can simply take any matrix \tilde{A} whose columns form a basis of $\text{col}(A)$, and it follows easily that $\|MU(A, y)\| = \|MU(\tilde{A}, y)\|$.) Find (in polynomial time) a matrix D such that A is a full null matrix for D (this can be achieved by finding D^T as a null matrix of A^T). Let $s = Dy$, and $v = \text{EDR}(D, s)$. Since $Dv = s$ we have that $D(y - v) = Dy - Dv = 0$, from which we conclude that $y - v$ is in the null space of D , and therefore in the column space of A . It follows that we can find an x such that $Ax = y - v$. We claim that x solves the instance of `min unsatisfy`: It suffices to note that the call to `EDR}(D, s)` minimizes $\|v\|_0 = \|y - Ax\|_0$, and that as v ranges over $\{v : Dv = s\}$, the vector $Ax = y - v$ ranges over all of $\text{col}(A)$. In conclusion,

Lemma 1 *The problems exact dictionary representation and min unsatisfy are equivalent.*

Equivalence of matrix problems. Here we demonstrate that sparse null space and matrix sparsification are equivalent. Recall that in the description of `matrix sparsification` on input matrix B , we required that B be full rank, $\#\text{col}(B) = \text{rank}(B)$. (We could in fact allow $\#\text{col}(B) > \text{rank}(B)$, but this would trivially result in $\#\text{col}(B) - \text{rank}(B)$ zero columns in the solution, and these columns are not interesting.) We will need the following lemma:

Lemma 2 *Let B be a full null matrix for $m \times n$ matrix A . The following statements are equivalent: (1) $N = BX$ for some invertible matrix X . (2) N is a full null matrix for A .*

Proof. In both cases, N and B must have the same number of columns, the same rank, and the same span. This is all that is required to demonstrate either direction. \square

We can now prove that `sparse null space` and `matrix sparsification` are equivalent. The problem `sparse null space` may be solved utilizing an oracle for `matrix sparsification`. Given input A to `sparse null space`, create (in polynomial time) a matrix B which is a full null matrix for A , and let $N = \text{MS}(B)$. We claim that N is a solution to `SNS}(A)`. Since $N = BX$ for some invertible matrix X , by Lemma 2 N is a full null matrix for A . Therefore the call to `MS}(B)` is equivalent to a call to `MS}(N)`, which solves `sparse null space` on A .

We show that `matrix sparsification` can be solved using an oracle for `sparse null space`. Given input B to `matrix sparsification`, create (in polynomial time) matrix A such that B is a full null matrix for A . Let $N = \text{SNS}(A)$. We claim that N is a solution to `MS}(B)`. By the lemma, $N = BX$ for some invertible matrix X , so N can be derived from B via elementary row reductions. The call to `SNS}(A)` finds an optimally sparse N , which is equivalent to solving `min unsatisfy` on B . In conclusion,

Lemma 3 *The problems matrix sparsification and sparse null space are equivalent.*

3 Hardness of approximation for matrix problems

In this section, we prove the hardness of approximation of `matrix sparsification` (and therefore `sparse null space`). This motivates the search for heuristics or algorithms under simplifying assumptions for `matrix sparsification`, which we undertake in the next section. For the reduction, we will need a relatively dense matrix which we know cannot be further sparsified. We will prove the existence of such a matrix in the first subsection.

3.1 Unsparsifiable matrices

Any $m \times n$ matrix A may be column reduced to contain at most $(m - r + 1)r$ nonzeros, where $r = \text{rank}(A)$. For example, Gaussian elimination on the columns of the matrix will accomplish this sparsification. We will say that a rank r , $m \times n$ matrix A is *completely unsparsifiable* if and only if, for any invertible matrix X , $\text{nnz}(AX) \geq (m - r + 1)r$. A matrix A is *optimally sparse* if, for any invertible X , $\text{nnz}(AX) \geq \text{nnz}(A)$. The main result of this section follows.

Theorem 4 *Let A be an $m \times n$ matrix with $m \geq n$. If every square submatrix of A is nonsingular, then A has rank n and is completely unsparsifiable. Moreover, in such case the matrix $\begin{pmatrix} I \\ A \end{pmatrix}$ is optimally sparse, where I is the $n \times n$ identity matrix.*

Before attempting a proof of the theorem, we need a few intermediate results.

Lemma 5 *Matrix A is optimally sparse if and only if, for any vector $x \neq 0$, $\|Ax\|_0 \geq \max_{i \in \text{supp}(x)} \|a_i\|_0$.*

Proof. Suppose that there exists an x that, for some $i \in \text{supp}(x)$, $\|Ax\|_0 < \|a_i\|_0$. Then we may replace the matrix column a_i by Ax , and create a matrix with the same rank as A which is sparser than A ; a contradiction. Similarly, suppose that A is not optimally sparse, so that there exists $B = AX$ with $\text{nnz}(B) < \text{nnz}(A)$, for some invertible X . Assume without loss of generality that the diagonal of X is full, $x_{ii} \neq 0$ (otherwise just permute the columns of X to make it so). Then there must exist an index $j \in [n]$ with $\|b_j\|_0 < \|a_j\|_0$, and we have $\|Ax_j\|_0 = \|b_j\|_0 < \|a_j\|_0 \leq \max_{i \in \text{supp}(x_j)} \|a_i\|_0$, since $x_{jj} \neq 0$. \square

A submatrix $A(R, C)$ is *row-inclusive* iff $r \notin R$ implies that $A(r, C)$ is not in the row span of $A(R, C)$. In other words, $A(R, C)$ includes all the rows of $A(:, C)$ which are in the row span of this submatrix. A submatrix $A(R, C)$ is a *candidate submatrix* of A (written $A(R, C) \triangleleft A$) if and only if $A(R, C)$ is both row-inclusive and $\text{rank}(A(R, C)) = |C| - 1$. This last property is equivalent to stating that the columns of $A(R, C)$ form a *circuit* – they are minimally linearly dependent. We can potentially zero out $|R|$ entries of A by using the column dependency of $A(R, C)$; being row-inclusive means there would be exactly $|R|$ zeros in the modified column of A .

The next lemma demonstrates the close relationship between candidate submatrices and vectors x which may sparsify A as in Lemma 5.

Lemma 6 *For any $m \times n$ matrix A : (1) For any $x \neq 0$ and $i \in \text{supp}(x)$, there exists $A(R, C) \triangleleft A$ for which $|R| \geq m - \|Ax\|_0$, and $i \in C \subset \text{supp}(x)$. (2) For any $A(R, C) \triangleleft A$ there exists a vector x for which $\text{supp}(x) = C$ and $\|Ax\|_0 = m - |R|$.*

Proof. Part 1: Let $R' = [m] - \text{supp}(Ax)$ (where $[m] = \{1, 2, \dots, m\}$), and choose C so that $i \in C \subset \text{supp}(x)$, and the columns of $A(R', C)$ form a circuit. (Note that the columns $A(R', \text{supp}(x))$ are dependent since $A(R', :)x = 0$). Now expand R' to R so that $A(R, C)$ is row-inclusive. Then $\text{rank}(A(R, C)) = \text{rank}(A(R', C)) = |C| - 1$, so that $A(R, C) \triangleleft A$.

Part 2: Since the columns of $A(R, C)$ form a circuit, there is an \tilde{x} with $\tilde{x}_i \neq 0 \forall i$ and $A(R, C)\tilde{x} = 0$. Then $\dim(\text{col}(A(R, C)^T)) = |C| - 1 = \dim(\text{null}(\tilde{x}^T))$ and also $\text{col}(A(R, C)^T) \subset \text{null}(\tilde{x}^T)$, which together imply $\text{col}(A(R, C)^T) = \text{null}(\tilde{x}^T)$. So $A(r, C)\tilde{x} = 0$ is true iff $r \in R$ (using the fact that $A(R, C)$ is row-inclusive). Now choose x so that $x(C) = \tilde{x}$ and all other coordinates are zero; then $\text{supp}(Ax) = [m] - R$. \square

The following is an immediate consequence of the lemma, and is crucial to our proof of Theorem 4.

Corollary 7 *The $m \times n$ matrix A is optimally sparse if and only if there is no candidate submatrix $A(R, C) \triangleleft A$ with $m - |R| < \|a_i\|_0$ for some $i \in C$.*

We are now ready to prove the theorem.

Proof of Theorem 4. Let $B = \begin{pmatrix} I \\ A \end{pmatrix}$. We prove that B is optimally sparse. Suppose $B(R, C) \triangleleft B$. Let $R_I = R \cap [n]$ and $R_A = R - [n]$. Now $B(R_I, C)$ is a submatrix of I with dependent columns, so $B(R_I, C) = 0$. By row-inclusiveness, R_I must include all zero rows in $B([n], C)$, so $|R_I| = n - |C|$. Since $B(R_I, C) = 0$, it follows that $\text{rank}(B(R_A, C)) = \text{rank}(B(R, C)) = |C| - 1$, and $|R_A| \geq |C| - 1$. Any $|C| \times |C|$ subsquare of $B(R_A, C)$ would make the rank at least $|C|$, so we must have $|R_A| < |C|$; thus $|R_A| = |C| - 1$. Combined with $|R_I| = n - |C|$, this implies that $|R| = n - 1$. Then $m + n - |R| = m + 1 = \|b_i\|_0$ for any column b_i of B , proving that B is optimally sparse by corollary 7.

Recall that Gaussian elimination on matrix $A \rightarrow G$ yields $\text{nnz}(G) = (m - n + 1)n$. Now suppose there is an invertible matrix X with $\text{nnz}(AX) < (m - n + 1)n$. Then $\text{nnz}(BX) = \text{nnz}\left(\begin{pmatrix} X \\ AX \end{pmatrix}\right) < n^2 + (m - n + 1)n = (m + 1)n$, contradicting the optimal sparsity of B . Hence no such X exists and A is completely unsparsifiable. \square

3.2 Efficiently building an unsparsifiable matrix

The next lemma establishes that we can easily construct an unsparsifiable matrix with a given column, a useful fact for the reductions to follow.

Lemma 8 *If $n \times n$ matrix $M = (M_{ij})$ has entries $m_{ij} = i^{p_j}$ for distinct positive reals p_1, p_2, \dots, p_n , then every subsquare of M is nonsingular.*

Proof. Let f be a *signomial* (a polynomial allowed to have nonintegral exponents). We define $\text{positive_zeros}(f) := \{x : x > 0 \ \& \ f(x) = 0\}$ and $\text{\#sign_changes}(f) := \#\{i : \mu_i \mu_{i+1} < 0\}$, where $f = \sum_i \mu_i x^{p_i}$, and no $\mu_i = 0$. A slight generalization of Descartes' rule of signs [35] states that

$$\text{\#positive_zeros}(f) \leq \text{\#sign_changes}(f) \tag{1}$$

Consider any $k \times k$ subsquare $M(R, C)$ given by $R = \{r_1, \dots, r_k\}, C = \{c_1, \dots, c_k\} \subset [n]$, and any nonzero vector $\mu \in \mathbb{R}^k$. Then $M(R, C) \cdot \mu$ matches the signomial $f(x) = \sum \mu_i x^{p_{c_i}}$ evaluated at $x = r_1, \dots, r_k$. Using (1), $\#\{i : f(r_i) = 0\} \leq \text{\#sign_changes}(f) < k$, so that some $f(r_i) \neq 0$, and $M(R, C)\mu \neq 0$. Hence the subsquare has a trivial kernel, and is nonsingular. \square

To avoid problems of precision, we will choose powers of p_j to be consecutive integers beginning at 0. This yields the Vandermonde matrix over \mathbb{Q} . It can also be shown, by an elementary cardinality argument, that a random matrix (using a non-atomic distribution) is unsparsifiable with probability 1 over infinite fields. The above lemma avoids any probability and allows us to construct such a matrix as quickly as we can iterate over the entries.

3.3 Reduction for matrix problems

After proving the existence of an unsparsifiable matrix in the last section, we can now prove the hardness of approximation of matrix sparsification. We reduce min unsatisfy to matrix sparsification. Given an instance $\langle A, y \rangle$ of min unsatisfy , we create a matrix M such that matrix sparsification on M solves the instance of min unsatisfy .

Before describing the reduction, we outline the intuition behind it. We wish to create a matrix M with many copies of y and some copies of A . The number of copies of y should greatly outnumber the number of copies of A . The desired approximation bounds will be achieved by guaranteeing that M is composed mostly of zero entries and of copies of y . It follows that minimizing the number of nonzero entries in the matrix (solving **matrix sparsification**) will reduce to minimizing the number of nonzero entries in the copies of y by finding a sparse linear combination of y with some other dictionary vectors (solving **min unsatisfy**).

The construction is as follows: Given an instance $\langle A, y \rangle$ of **min unsatisfy** (where A is an $m \times n$ matrix, $y \in R^m$, and $q \geq p$ are free parameters), take an optimally sparse $(p+q) \times p$ matrix $\begin{pmatrix} I_p \\ X \end{pmatrix}$ as given by Lemma 8 and Theorem 4 (where I_p is a $p \times p$ identity matrix), and create matrix $M_l = \begin{pmatrix} I_p \\ X \end{pmatrix} \otimes y = \begin{pmatrix} I_p \otimes y \\ X \otimes y \end{pmatrix}$ (of size $(p+q)m \times p$). Further create matrix $I_q \otimes A$ (of size $qm \times qn$), and take matrix 0 (of size $pm \times qn$) and form matrix $M_r = \begin{pmatrix} 0 \\ I_q \otimes A \end{pmatrix}$ (of size $(p+q)m \times qn$). Append M_r to the right of M_l to create matrix $M = M_l | M_r$ of size $(p+q)m \times (p+qn)$. We can summarize this construction as $M = \begin{pmatrix} I_p \otimes y & 0 \\ X \otimes y & I_q \otimes A \end{pmatrix}$.

M_l is composed of $p+pq$ m -length vectors, all corresponding to copies of y . M_r is composed of qn m -length vectors, all corresponding copies of vectors in A . By choosing $p = q = n^2$, we ensure that the term pq is larger than qn by a factor of n . Note that M now contains $O(n^3)$ columns.

It follows that the number of zeros in M depends mostly on the number of zeros induced by a linear combination of dictionary vectors that include y . Because M_l is unsparsifiable, vectors in the rows of M_l will not contribute to sparsifying other vectors in these rows; only vectors in M_r (which are copies of the vectors of A) may sparsify vectors in M_l (which are copies of the vectors in y). It follows that an approximation to **matrix sparsification** will yield a similar approximation – within a factor of $1+n^{-\frac{1}{3}}$ – to **min unsatisfy**, and that **matrix sparsification** is hard to approximate within a factor $2^{\log^{.5-o(1)} n^{1/3}} = 2^{\log^{.5-o(1)} n}$ of optimal (assuming NP does not admit quasi-polynomial time deterministic algorithms).

4 Solving matrix sparsification through min unsatisfy

In the previous section we showed that **matrix sparsification** is hard to approximate. This motivates the search for heuristics and algorithms under simplifying assumptions for **matrix sparsification**. In this section we show how to extend algorithms and heuristics for **min unsatisfy** to apply to **matrix sparsification** – and hence **sparse null space** – while preserving approximation guarantees. (Note that this result is distinct from the hardness result; neither one implies the other.)

We first present an algorithm for **matrix sparsification** which is in essence identical to the one given by Coleman and Pothen [9] for **sparse null space**. The algorithm assumes the existence of an oracle for a problem we will call the **sparsest independent vector problem**. The algorithm makes a polynomial number of queries to this oracle, and yields an optimal solution to **matrix sparsification**.

The **sparsest independent vector problem** takes full-rank input matrices A and B , where the columns of B are a contiguous set of right-most columns from A (informally, one could say that B is a suffix of A , in terms of columns). The output is the sparsest vector in the span of A but not in the span of B . For convenience, we add an extra output parameter — a column of $A \setminus B$ which can be replaced by the sparsest independent vector while preserving the span of A . More formally, **sparsest independent vector** is defined as follows. (See §A for the definition of a problem instance.)

sparsest independent vector (SIV)

$\mathcal{I}_{\text{SIV}} = \langle A, B \rangle$; A is an $m \times n$ full rank matrix with $A = (C|B)$ for some non-empty matrix C .

$S_{\text{SIV}}(A, B) = \{a : a \in \text{col}(A), a \notin \text{col}(B)\}$

$m_{\text{SIV}}(\langle A, B \rangle, a) = \text{nnz}(a)$

The following algorithm reduces matrix sparsification on an $m \times n$ input matrix A to making a polynomial number of queries to an oracle for sparsest independent vector:

```

Algorithm Matrix_Sparsification( $A$ )
 $B \leftarrow$  null
for  $i = n$  to 1:
     $\langle b_i, a_j \rangle = \text{SIV}(A, B)$ 
     $A \leftarrow (A \setminus \{a_j\} | b_i)$ 
     $B \leftarrow (b_i | B)$ 
return  $B$ 

```

This greedy algorithm sparsifies the matrix A by generating a new matrix B one column at a time. The first-added column (b_n) is the sparsest possible, and each subsequent column is the next sparsest. It is decidedly non-obvious why such a greedy algorithm would actually succeed; we refer the reader to [9] where it is proven that greedy algorithms yield an optimal result on matroids such as the set of vectors in $\text{col}(A)$. Our first contribution is in expanding the result of [9] as follows.

Lemma 9 *Let subroutine SIV in algorithm Matrix_Sparsification be a λ -approximation oracle for sparse independent vector. Then the algorithm yields a λ -approximation to matrix sparsification.*

Proof. Given $m \times n$ matrix A , suppose \tilde{C} exactly solves $\text{MS}(A)$, and that the columns $\tilde{c}_1, \dots, \tilde{c}_n$ of \tilde{C} are sorted in decreasing order by number of nonzeros. Let $s_i = \|\tilde{c}_i\|_0$; then $s_1 \geq s_2 \geq \dots \geq s_n$. As already mentioned, given a true oracle to sparsest independent vector, algorithm Matrix_Sparsification would first discover a column with s_n nonzeros, then a column with s_{n-1} nonzeros, etc.

Now suppose algorithm Matrix_Sparsification made calls to a λ -approximation oracle for sparse independent vector. The first column generated by the algorithm, call it b_n , will have at most λs_n nonzeros, since the optimal solution has s_n nonzeros. The second column generated will have at most λs_{n-1} nonzeros, since the optimal solution to the call to SIV has no more than s_{n-1} nonzeros: even if b_n is suboptimal, it is true that at least one of \tilde{c}_n or \tilde{c}_{n-1} is an optimal solution to $\text{SIV}(A, b_n)$.

More generally, the i^{th} column found by the algorithm has no more than λs_i nonzeros, since at least one of $\{\tilde{c}_n, \dots, \tilde{c}_i\}$ is an optimal solution to the i^{th} query to SIV. Thus we have $\text{nnz}(B) = \sum_i \|b_i\|_0 \leq \sum \lambda \|\tilde{c}_i\|_0 = \lambda \text{nnz}(\tilde{C})$, and may conclude that the algorithm yields a λ -approximation to matrix sparsification. \square

It follows that in order to utilize the aforementioned algorithm for matrix sparsification, we need some algorithm for sparsest independent vector. This is in itself problematic, as the sparsest independent vector problem is hard to approximate – in fact, we will demonstrate later that sparsest independent vector is as hard to approximate as min unsatisfy. Hence, although we have extended the algorithm of [9] to make use of an approximation oracle for sparsest independent vector, the benefit of this algorithm remains unclear.

To this end, we will show how to solve sparsest independent vector while making queries to an approximate oracle for min unsatisfy. This algorithm preserves the approximation ratio of the oracle. This implies that *all* algorithms for min unsatisfy immediately carry over to sparsest independent vector, and further that they carry over to matrix sparsification as well. This also implies a useful tool for applying heuristics for min satisfy to the other problems.

The problem sparsest independent vector on input $\langle A, B \rangle$ asks to find the sparsest vector in the span of A but not in the span of B . It is not difficult to see that min unsatisfy solves a similar problem: Given a matrix A and target vector y not in the span of A , find the sparsest vector in the span of $(A|y)$ but not in the span of A . Hence, if we query the oracle for min unsatisfy once for each vector $a_j \notin \text{col}(B)$, one of

these queries must return the solution for the sparsest independent vector problem. This discussion implies the following algorithm:

```

Algorithm Sparse_Independent_Vector( $A, B$ )
 $s \leftarrow m + 1$ 
for  $j = 1$  to  $n$ :
  if  $a_j \notin \text{col}(B)$  :
     $A_j \leftarrow A \setminus \{a_j\}$ 
     $x \leftarrow \text{MU}(A_j, a_j)$ 
     $c' \leftarrow A_j x - a_j$ 
    if  $\|c'\|_0 < s$ 
       $c \leftarrow c'$ ;  $s \leftarrow \|c\|_0$ ;  $\alpha \leftarrow a_j$ 
return  $\langle c, \alpha \rangle$ 

```

Note that when this algorithm is given a λ -approximate oracle for `min unsatisfy`, it yields a λ -approximate algorithm for `sparsest independent vector`. (In this case, the approximation algorithm is valid over the field for which the oracle is valid.)

We conclude this section by giving hardness results for `sparsest independent vector` by reduction from `min unsatisfy`; we show that any instance $\langle A, b \rangle$ of `min unsatisfy` may be modeled as an instance $\langle A', B' \rangle$ of `sparsest independent vector`: Let $A' = A|y$, and $B' = A$. This suffices to force the linear combination to include y . It follows that `sparsest independent vector` is as hard to approximate as `min unsatisfy`, and in fact that the two problems are approximation equivalent.

4.1 Approximation algorithms

We have presented a tool for extending algorithms and heuristics for exact dictionary representation to `min unsatisfy` and then directly to the matrix problems. When these algorithms make assumptions on the dictionary of EDR, it is necessary to investigate how these assumptions carry over to the other problems.

To this end, we consider here one of the most popular heuristic for EDR – ℓ_1 -minimization – and the case where it is guaranteed to provide the optimal result. The heuristic is to find a vector v that satisfies $Dv = s$, while minimizing $\|v\|_1$ instead of $\|v\|_0$. (See [34, 33, 11] for more details.) In [13], Fuchs shows that under the following relatively simple condition ℓ_1 -minimization provides the optimal answer to EDR.

In the following, we write $\text{sgn}(x)$ to indicate $\frac{x}{|x|}$, or zero if $x = 0$. Given a matrix D whose columns are divided into two submatrices D_0 and D_1 , we may write $D = (D_0 \ D_1)$, even though D_0 and D_1 may not be contiguous portions of the full matrix. (The reader may view this as permuting the columns of D before splitting into D_0 and D_1 .)

Theorem 10 (Fuchs) *Suppose that $s = Dv$, and that $\|v\|_0$ is minimal (so that this v solves $\text{EDR}(D, s)$). Split $D = (D_0 \ D_1)$ so that D_0 contains all the columns in the support of v . Accordingly, we split the vector $v = \begin{pmatrix} v_0 \\ 0 \end{pmatrix}$, in which all coordinates of v_0 are nonzero.*

If there exists a vector h so that $D_0^T h = \text{sgn}(v_0)$, and $\|D_1^T h\|_\infty < 1$, then $\|v\|_1 < \|w\|_1$ for all vectors $w \neq v$ with $Dw = s$.

We extend this result to each of our major problems.

Theorem 11 *min unsatisfy.* Suppose, for a given A, y pair, that x minimizes $\|y - Ax\|_0$. Split $y = \begin{pmatrix} y_0 \\ y_1 \end{pmatrix}$ and $A = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}$ so that A_1 is maximal such that $y_1 = A_1 x$, and let $v = y_0 - A_0 x$. If there is a matrix u with $\|u\|_\infty < 1$ and $A_1^T u = -A_0^T \text{sgn}(v)$, then our reduction of $\text{MU}(A, y)$ to an ℓ_1 approximation of $\text{EDR}(D, s)$ gives the truly optimal answer.

matrix sparsification. For a given $m \times n$ matrix B , suppose C minimizes $\text{nnz}(C)$ such that $C = BX$ for invertible X . For any $i \in [n]$, split column $c_i = \begin{pmatrix} c_{i,0} \\ \bar{0} \end{pmatrix}$ so that $c_{i,0}$ is completely nonzero, and, respectively, $B = \begin{pmatrix} B_{i,0} \\ B_{i,1} \end{pmatrix}$, so that $c_{i,0} = B_{i,0} x_i$. If, for all $i \in [n]$, there exists vector u_i with $\|u_i\|_\infty < 1$ and $B_{i,1}^T u_i = -B_{i,0}^T \text{sgn}(c_{i,0})$, then our reduction algorithm to an ℓ_1 approximation of EDR via *min unsatisfy* will give a truly optimal answer to this *MS* instance.

sparse null space For a given matrix A with corank c , suppose matrix V solves $\text{SNS}(A)$. For each $i \in [c]$, split column $v_i = \begin{pmatrix} v_{i,0} \\ \bar{0} \end{pmatrix}$ so that $v_{i,0}$ is completely nonzero and, respectively, $A = \begin{pmatrix} A_{i,0} & A_{i,1} \end{pmatrix}$ so that $A_{i,0} v_{i,0} = 0$. If, for all $i \in [c]$, there exists vector h_i with $\|A_{i,1}^T h_i\|_\infty < 1$ and $A_{i,0}^T h_i = \text{sgn}(v_{i,0})$, then our reduction to an ℓ_1 approximation of EDR via *matrix sparsification* and *min unsatisfy* gives a truly optimal answer to this *SNS* instance.

Proof. *min unsatisfy.* As in our reduction from *MU* to *EDR*, we find matrix D with $DA = 0$ and vector $s = Dy$. Then

$$\begin{pmatrix} \text{sgn}(v_0) \\ u \end{pmatrix} \in \text{null}(A^T) = \text{col}(D^T) \implies \exists h : D^T h = \begin{pmatrix} \text{sgn}(v) \\ u \end{pmatrix}.$$

Splitting $D = \begin{pmatrix} D_0 & D_1 \end{pmatrix}$, we see that $D_0^T h = \text{sgn}(v)$ and $\|D_1^T h\|_\infty < 1$, exactly what is required for theorem 10, showing that ℓ_1 minimization gives the answer $D_0 v_0$. Since $D_0 v_0 = \begin{pmatrix} D_0 & D_1 \end{pmatrix} \begin{pmatrix} v \\ \bar{0} \end{pmatrix} = D(y - Ax) = s$, this completes the proof.

matrix sparsification. We write $A \setminus i$ to denote matrix A with the i^{th} column removed. In our reduction of *MS* to *MU*, we need to solve instances of *MU* over equations of the form $(B \setminus i)x = b_i$. According to the *MU* portion of this theorem, it suffices to show that $(B_{i,1} \setminus i)^T u_i = -(B_{i,0} \setminus i)^T \text{sgn}(c_{i,0})$. The condition for this portion of the theorem implies this, since removing any corresponding rows from a matrix equation of the form $Ax = By$ still preserves the equality.

sparse null space. As in our reduction from *SNS* to *MS*, we find a matrix B such that A is a full null matrix for B . For any i , let $u_i = A_{i,1}^T h_i$ so that $A^T h_i = \begin{pmatrix} \text{sgn}(v_{i,0}) \\ u_i \end{pmatrix}$. Then $\begin{pmatrix} \text{sgn}(v_{i,0}) \\ u_i \end{pmatrix} \in \text{col}(A^T) = \text{null}(B^T)$, and $B_{i,1}^T u_i = -B_{i,0}^T \text{sgn}(v_{i,0})$, which is exactly what is necessary for *matrix sparsification* to function through ℓ_1 approximation. \square

The following intuitive conditions give insight into which matrices are amenable to ℓ_1 approximations. A^+ denotes $(A^T A)^{-1} A^T$, the pseudoinverse of A .

Corollary 12 *min unsatisfy.* Suppose matrix $A = \begin{pmatrix} A_0 \\ A_1 \end{pmatrix}$ is split by an optimal answer as in theorem 11. If $\text{row}(A_0) \subset \text{row}(A_1)$ and $\|(A_1^T)^+ A_0^T\|_{1,1} < 1$, our ℓ_1 approximation scheme will give a truly optimal answer.

matrix sparsification. Suppose matrix $B = \begin{pmatrix} B_{i,0} \\ B_{i,1} \end{pmatrix}$ is split by the columns of an optimal answer $C = BX$ as in theorem 11. If, for any i , $\text{row}(B_{i,0}) \subset \text{row}(B_{i,1})$ and $\|(B_{i,1}^T)^+ B_{i,0}^T\|_{1,1} < 1$, then our ℓ_1 approximation will give the optimal answer.

sparse null space. Suppose matrix $A = \begin{pmatrix} A_{i,0} & A_{i,1} \end{pmatrix}$ is split by the columns of an optimal answer V with $AV = 0$ as in theorem 11. If $\text{col}(A_{i,0}) \subset \text{col}(A_{i,1})$ and $\|A_{i,0}^+ A_{i,1}\|_{1,1} < 1 \forall i$, then our ℓ_1 approximation will give an optimal answer.

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A Approximation equivalence

Here we define approximation equivalence. Some of our notation and definitions are inspired by [1], which itself built upon [19].

Definition 13 *An optimization problem is a four-tuple $F = \{\mathcal{I}_F, S_F, M_F, \text{opt}_F\}$, where \mathcal{I}_F is the set of input instances, $S_F(x)$ is the solution space for $x \in \mathcal{I}_F$, $M_F(x, y)$ is the objective metric for $x \in \mathcal{I}_F$ and $y \in S_F(x)$, and $\text{opt}_F \in \{\min, \max\}$.*

We will assume throughout the paper that $\text{opt}_F = \min$.

For any optimization problem F and $x \in \mathcal{I}_F$, we define $F(x) = \text{argmin}_{y \in S_F(x)} M_F(x, y)$ and $\|F(x)\| = M_F(x, F(x))$. An approximation \tilde{F} to F is any map on \mathcal{I}_F with $\tilde{F}(x) \in S_F(x)$. We write $\|\tilde{F}(x)\|$ for $M_F(x, \tilde{F}(x))$. \tilde{F} is a λ -approximation for F when, for all $x \in \mathcal{I}_F$, $\frac{\|\tilde{F}(x)\|}{\|F(x)\|} \leq \lambda(|x|)$.

Definition 14 *Given optimization problems F and G , an exact reduction from F to G is a pair $\langle t_1, t_2 \rangle$ that satisfies the following: (1) $t_1, t_2 \in P$. (2) $t_1 : \mathcal{I}_F \rightarrow \mathcal{I}_G$ and for all $x \in \mathcal{I}_F$, $y \in S_G(t_1(x))$, we have $t_2(x, y) \in S_F(x)$. (3) For all $x \in \mathcal{I}_F$, $y \in S_G(t_1(x))$, we have $M_F(x, t_2(x, y)) = M_G(t_1(x), y)$. (4) For all $x \in \mathcal{I}_F$, $\|F(x)\| \geq \|G(t_1(x))\|$.*

We write $F \preceq G$. We write $F \sim G$ to denote that $F \preceq G$ and $G \preceq F$, and call these problems equivalent.

Theorem 15 *If $F \preceq G$ and G admits a λ -approximation, then so does F .*

Proof. We are given that for G , there exists \tilde{G} with $\frac{\|\tilde{G}(x)\|}{\|G(x)\|} \leq \lambda$ for all $x \in \mathcal{I}_G$. Let $\tilde{F}(x) = t_2(x, \tilde{G}(t_1(x)))$, where $\langle t_1, t_2 \rangle$ is the $F \preceq G$ exact reduction. It suffices to show that $\frac{\|\tilde{F}(x)\|}{\|F(x)\|} \leq \frac{\|\tilde{G}(x')\|}{\|G(x')\|}$, where $x' = t_1(x)$. By the fourth item of the definition, it suffices to demonstrate that $\|\tilde{F}(x)\| \leq \|\tilde{G}(x')\|$. By the third item, $\|\tilde{F}(x)\| = M_F(x, t_2(x, \tilde{G}(x'))) = M_G(x', \tilde{G}(x')) = \|\tilde{G}(x')\|$. \square

In fact, it can be shown that $\frac{\|\tilde{F}(x)\|}{\|F(x)\|} = \frac{\|\tilde{G}(x')\|}{\|G(x')\|}$.

Corollary 16 *If $F \sim G$, then F admits a λ -approximation if and only if G admits a λ -approximation.*

B Relaxed Versions

The following problems are variations which work with more approximate solution spaces. This can be considered as allowing some noise in either the inputs or outputs. It is not difficult to extend the above proofs to see that these four problems are equivalent as well.

Relaxed Dictionary Representation (RDR)

$\mathcal{I}_{\text{RDR}} = \langle D, s, \delta \rangle$, $m \times n$ matrix D , vector s with $s \in \text{col}(D)$, $\delta \geq 0$

$S_{\text{RDR}}(D, s, \delta) = \{ \langle v, w \rangle \text{ each in } \mathbb{R}^n : D(v - w) = s, \|w\| \leq \delta \}$

$m_{\text{RDR}}(\langle D, s \rangle, \langle v, w \rangle) = \|v\|_0$

Relaxed MinUnsatisfy (RMU)

$\mathcal{I}_{\text{RMU}} = \langle A, y, \delta \rangle$, $m \times n$ matrix A , vector $y \in \mathbb{R}^m$, $\delta \geq 0$

$S_{\text{RMU}}(A, y, \delta) = \{ \langle x \in \mathbb{R}^n, w \in \mathbb{R}^m \rangle : \|w\| \leq \delta \}$

$m_{\text{RMU}}(\langle A, y \rangle, \langle x, w \rangle) = \|y - Ax + w\|_0$

Relaxed Sparse Null Space (RSNS)

$\mathcal{I}_{\text{RSNS}} = \langle A, \delta \rangle$, matrix A , and $\delta \geq 0$

$S_{\text{RSNS}}(A, \delta) = \{ \langle M, N \rangle : N \text{ is a full null matrix for } A, \|M\| \leq \delta \}$

$m_{\text{RSNS}}(\langle A, \delta \rangle, \langle M, N \rangle) = \text{nnz}(M + N)$

Relaxed Matrix Sparsification (RMS)

$\mathcal{I}_{\text{RMS}} = \langle B, \delta \rangle$, matrix B , and $\delta \geq 0$

$S_{\text{RMS}}(B, \delta) = \{ \langle M, N \rangle : N = BX, X \text{ invertible}, \|M\| \leq \delta \}$

$m_{\text{RMS}}(\langle B, \delta \rangle, \langle M, N \rangle) = \text{nnz}(M + N)$