# ASYMPTOTIC LATTICE PATH ENUMERATION USING DIAGONALS 

STEPHEN MELCZER AND MARNI MISHNA


#### Abstract

This work presents new asymptotic formulas for family of walks in Weyl chambers. The models studied here are defined by step sets which exhibit many symmetries and are restricted to the first orthant. The resulting formulas are very straightforward: the exponential growth of each model is given by the number of steps, while the sub-exponential growth depends only on the dimension of the underlying lattice and the number of steps moving forward in each coordinate. These expressions are derived by analyzing the singular variety of a multivariate rational function whose diagonal counts the lattice paths in question. Additionally, we show how to compute subdominant growth for these models, and how to determine first order asymptotics for excursions.


## 1. Introduction

The reflection principle and its various incarnations have been indispensable in the study of the lattice path models, particularly in the discovery of explicit enumerative formulas. Two examples include the formulas for the family of reflectable walks in Weyl chambers of Gessel and Zeilberger [15], and various approaches using the widely applied kernel method [8, 10, 19, 9]. In these guises, the reflection principle is often a key element in the solution when the resulting generating function is shown to be D-finite ${ }^{1}$. This is no coincidence: the connection is an expression for the generating function as a diagonal of a rational function. More precisely, in works such as [15, 9], the analysis results in generating functions expressed as rational sub-series extractions, which can be easily converted to diagonal expressions. Unfortunately, the resulting explicit representations of generating functions can be cumbersome to manipulate. For example, much recent work on walks in Weyl chambers has led to expressions which are determinants of large matrices with Bessel function entries [17, 16, 30]. Here, we aim to determine asymptotics for a family of lattice path models arising naturally among those restricted to positive orthants - which correspond to walks in certain Weyl chambers - while avoiding such unwieldly representations. This is acheived by working directly with the diagonal expressions obtained through the recently developed machinery on analytic combinatorics in several variables [25].

Coupling these techniques - diagonal representation and analytic combinatorics in several variables - yields explicit, yet simple, asymptotic formulas for families of lattice paths. The focus of this article is $d$-dimensional models whose set of allowable steps is symmetric with respect to any axis; we call these models highly symmetric walks. The techniques of analytic combinatorics in several variables apply in a rather straightforward way to derive dominant asymptotics for the number of walks ending anywhere and give an effective procedure to calculate descending terms in the asymptotic expansions. Furthermore, we also consider the subfamily of walks that return to the origin (known as excursions). Once our equations are established, they are suitable input to existing implementations such as that of Raichev [26] (however, in practice one can calculate only the first few terms in these expansions).

The highly symmetric walks we present are amenable to a kernel method treatment. In particular, they fit well into the ongoing study of lattice path classes restricted to an orthant and taking only

[^0]"small" steps $[9,7]$. This collection of models forms a little universe exhibiting many interesting phenomena, and recent work in two and three dimensions has used novel applications of algebra and analysis, along with new computational techniques, to determine exact and asymptotic enumeration formulas. One key predictor of the nature of a model's generating function (whether it is rational, algebraic, or transcendental D-finite, or none of these) is the order of a group that is associated to each model. This group has its origins in the probabilistic study of random walks, namely [12], and when the group is finite it can sometimes be used to write generating functions as the positive part of an explicit multivariate rational Laurent series. The intimate relation between the generating function of the walks and the nature of the generating function is explored in [7, 23].

For highly symmetric models in two and three dimensions, this group coincides with that of a Weyl group for walks in the Weyl chambers $A_{1}^{2}$ and $A_{1}^{3}$, respectively. Indeed, one can use either viewpoint to generalize the study of highly symmetric models to models in arbitrary dimension. As these viewpoints are largely isomorphic, and the kernel method viewpoint is more self-contained, we begin this article by working through a straightforward generalization of the kernel method in order to write the generating function for higher dimensional highly symmetric walks as diagonals of rational functions. We then perform an asymptotic analysis of the coefficients of counting generating functions using techniques from the study of analytic combinatorics in several variables, and consequently link some of the combinatorial symmetries in a walk model to both analytic properties of the generating function and geometric properties of an associated variety. After this is complete, we examine how this connects to the notion of walks in Weyl chambers, use results from their study to determine asymptotic results about excursions, and discuss how the Weyl chamber viewpoint can be used in future work to examine larger classes of lattice path models through diagonals. Next we specify the walks we study in order to precisely state our main results.
1.1. Highly Symmetric Walks. Concretely, the lattice path models we consider are restricted as follows. For a fixed dimension $d$, we define a model by its step set $\mathcal{S} \subseteq\{ \pm 1,0\}^{d} \backslash\{\mathbf{0}\}$ and say that $\mathcal{S}$ is symmetric about the $x_{k}$ axis if $\left(i_{1}, \ldots, i_{k}, \ldots, i_{d}\right) \in \mathcal{S}$ implies $\left(i_{1}, \ldots,-i_{k}, \ldots, i_{d}\right) \in \mathcal{S}$. We further impose a non-triviality condition: for each coordinate there is at least one step in $\mathcal{S}$ which moves in the positive direction of that coordinate (this implies that for each coordinate there is a walk in the model which moves in that coordinate).

The number of walks taking steps in $\mathcal{S}$ which are restricted to the positive orthant $\mathbb{N}^{d}=\mathbb{Z}_{\geq 0}^{d}$ are studied by expressing the counting generating functions of such models as positive parts of multivariate rational Laurent series, which are then converted to diagonals of rational functions in $d+1$ variables. A first consequence is that all of these models have D-finite generating functions (since D-finite functions are closed under the diagonal operation).

After the above manipulations, these models are very well suited to the asymptotic enumeration methods for diagonals of rational functions outlined in [25], in particular the cases which were developed by Pemantle, Raichev and Wilson in [24] and [27]. Following these methods, we study the singular variety of the denominator of this rational function to determine related asymptotics. The condition of having a symmetry across each axis ensures that the variety is smooth and allows us to calculate the leading asymptotic term explicitly. This is not generally the case, in our experience, and hence we focus on this particular kind of restriction.
1.2. Main results. We present two main results in this work. The first appears as Theorem 3.4.

Theorem. Let $\mathcal{S} \subseteq\{-1,0,1\}^{d} \backslash\{\mathbf{0}\}$ be a set of unit steps in dimension d. If $\mathcal{S}$ is symmetric with respect to each axis, and $\mathcal{S}$ takes a positive step in each direction, then the number of walks of length $n$ taking steps in $\mathcal{S}$, beginning at the origin, and never leaving the positive orthant has

| $\mathcal{S}$ | Asymptotics | $\mathcal{S}$ | Asymptotics |
| :---: | :---: | :---: | :---: |
| $\nexists$ | $\frac{4}{\pi \sqrt{1 \cdot 1}} \cdot n^{-1} \cdot 4^{n}=\frac{4}{\pi} \cdot \frac{4^{n}}{n}$ | $X$ | $\frac{4}{\pi \sqrt{2 \cdot 2}} \cdot n^{-1} \cdot 4^{n}=\frac{2}{\pi} \cdot \frac{4^{n}}{n}$ |
| $\not X$ | $\frac{6}{\pi \sqrt{3 \cdot 2}} \cdot n^{-1} \cdot 6^{n}=\frac{\sqrt{6}}{\pi} \cdot \frac{6^{n}}{n}$ | $\boxed{X}$ | $\frac{8}{\pi \sqrt{3 \cdot 3}} \cdot n^{-1} \cdot 8^{n}=\frac{8}{3 \pi} \cdot \frac{8^{n}}{n}$ |

Table 1. The four highly symmetric models with unit steps in the quarter plane.
asymptotic expansion

$$
s_{n}=\left[\left(s^{(1)} \cdots s^{(d)}\right)^{-1 / 2} \pi^{-d / 2}|\mathcal{S}|^{d / 2}\right] \cdot n^{-d / 2} \cdot|\mathcal{S}|^{n}+O\left(n^{-(d+1) / 2} \cdot|\mathcal{S}|^{n}\right),
$$

where $s^{(k)}$ denotes the number of steps in $\mathcal{S}$ which have $k^{\text {th }}$ coordinate 1 .
This formula is easy to apply to any given model, and for certain infinite families as well.
Example 1. When $d=2$ there are four non-isomorphic highly symmetric walks in the quarter plane, listed in Table 1. Applying Theorem 3.4 verifies the asymptotic results guessed previously by [5].
Example 2. Let $\mathcal{S}=\{-1,0,1\}^{d} \backslash\{\mathbf{0}\}$, the full set of possible steps. This is symmetric across each axis. We compute that $|\mathcal{S}|=3^{d}-1$, and $s^{(j)}=3^{d-1}$ for all $j$ and so

$$
s_{n} \sim\left(\frac{\left(3^{d}-1\right)^{d / 2}}{3^{d(d-1) / 2} \cdot \pi^{d / 2}}\right) \cdot n^{-d / 2} \cdot\left(3^{d}-1\right)^{n}
$$

Example 3. Let $e_{k}=(0, \ldots, 0,1,0, \ldots, 0)$ be the $k^{\text {th }}$ standard basis vector in $\mathbb{R}^{d}$, and consider the set of steps $\mathcal{S}=\left\{e_{1},-e_{1}, \ldots, e_{d},-e_{d}\right\}$. Then the number of walks of length $n$ taking steps from $\mathcal{S}$ and never leaving the positive orthant has asymptotic expansion

$$
s_{n} \sim\left(\frac{2 d}{\pi}\right)^{d / 2} n^{-d / 2}(2 d)^{n} .
$$

The second main result is a comparable statement for excursions, Theorem 7.2.
Theorem. Let $\mathcal{S} \subseteq\{-1,0,1\}^{d} \backslash\{\mathbf{0}\}$ be a set of unit steps in dimension d. If $\mathcal{S}$ is symmetric with respect to each axis, and $\mathcal{S}$ takes a positive step in each direction, then the number of walks $e_{n}$ of length $n$ taking steps in $\mathcal{S}$, beginning and ending at the origin, and never leaving the positive orthant satisfies

$$
e_{n}=O\left(\frac{|\mathcal{S}|^{n}}{n^{3 d / 2}}\right)
$$

1.3. Organization of the paper. The article is organized as follows. Section 2 describes how to express the generating function using an orbit sum by applying the kernel method, following the strategy described in [9]. We then derive Equation (9), which describes the generating function as the diagonal of a rational power series in multiple variables. Section 3 justifies why the work of Pemantle and Wilson [25] is applicable, with the asymptotic results computed in Section 3.3. We discuss the sub-dominant growth, and compute an example in Section 5. Section 6 discusses the


Figure 1. The strategem of determining asymptotics via the generalized kernel method for symmetric walks.
differential equations satisfied by these generating functions, and how to use creative telescoping techniques to find them. We tabulate some small examples. We conclude with a discussion of how these walks fit into the context of walks in Weyl chambers, which allows us to obtain results on the asymptotics of walk excursions, and also to consider other families of walks.

## 2. Deriving a diagonal expression for the generating function

Fix a dimension $d$ and a highly symmetric set of steps $\mathcal{S} \subseteq\{ \pm 1,0\}^{d} \backslash\{\mathbf{0}\}$. Recall this means that $\left(i_{1}, \ldots, i_{k}, \ldots, i_{d}\right) \in \mathcal{S}$ implies $\left(i_{1}, \ldots,-i_{k}, \ldots, i_{d}\right) \in \mathcal{S}$. In this section we derive a functional equation for a multivariate generating function, apply the orbit sum method to derive a closed expression related to this generating function, and conclude by writing the univariate counting generating function for the number of walks as the complete diagonal of a rational function.

The following notation is used throughout:

$$
\bar{z}_{i}=z_{i}^{-1} ; \quad \mathbf{z}=\left(z_{1}, \ldots, z_{d}\right) ; \quad \mathbf{i}=\left(i_{1}, i_{2}, \ldots, i_{d}\right) \in \mathbb{Z}^{d} ; \quad \mathbf{z}^{\mathbf{i}}=z_{1}^{i_{1}} \cdots z_{d}^{i_{d}}
$$

and we write $\mathbb{Q}\left[z_{k}, \overline{z_{k}}\right]$ to refer to the ring of Laurent polynomials in the variable $z_{k}$.
2.1. A functional equation. To begin, we define the generating function:

$$
\begin{equation*}
F(\mathbf{z}, t)=\sum_{\substack{n \geq 0 \\ \mathbf{i} \in \mathbb{Z}^{d}}} s_{\mathbf{i}}(n) \mathbf{z}^{\mathbf{i}} t^{n}=\sum_{n \geq 0}\left(\sum_{\mathbf{i} \in \mathbb{Z}^{d}} s_{\mathbf{i}}(n) z_{1}^{i_{1}} \cdots z_{d}^{i_{d}}\right) t^{n} \in \mathbb{Q}\left[z_{1}, \bar{z}_{1}, \ldots, z_{d}, \bar{z}_{d}\right] \llbracket t \rrbracket, \tag{1}
\end{equation*}
$$

where $s_{\mathbf{i}}(n)$ counts the number of walks of length $n$ taking steps from $\mathcal{S}$ which stay in the positive orthant and end at lattice point $\mathbf{i} \in \mathbb{Z}^{d}$. Note that the series $F(\mathbf{1}, t)$ is the generating function for the total number of walks in the orthant, and we can recover the series for walks ending on the hyperplane $z_{k}=0$ by setting $z_{k}=0$ in the series $F(\mathbf{z}, t)$ (the variables $z_{1}, \ldots, z_{d}$ are referred to as catalytic variables in the literature, as they are present during the analysis and removed at the end of the 'reaction' via specialization to 1 ). We also define the function (known as either the
characteristic polynomial or the inventory of $\mathcal{S}$ ) by

$$
\begin{equation*}
S(\mathbf{z})=\sum_{\mathbf{i} \in \mathcal{S}} \mathbf{z}^{\mathbf{i}}=\left[t^{1}\right] F(\mathbf{z}, t) \in \mathbb{Q}\left[z_{1}, \bar{z}_{1}, \ldots, z_{d}, \bar{z}_{d}\right] \tag{2}
\end{equation*}
$$

In many recent analyses of lattice walks, functional equations are derived by translating the following description of a walk into a generating function equation: a walk is either an empty walk, or a shorter walk followed by a single step. To ensure the condition that the walks remain in the positive orthant, we must not count walks that add a step with a negative $k$-th component to a walk ending on the hyperplane $z_{k}=0$. To account for this, it is sufficient to subtract an appropriate multiple of $F$ from the functional equation: $t \bar{z}_{k} F\left(z_{1}, \ldots, z_{k-1}, 0, z_{k+1}, \ldots, z_{d}, t\right)$, however if a given step has several negative components we must use inclusion and exclusion to prevent over compensation.

This can be made explicit. Let $\mathcal{S} \subseteq\{1,0,-1\}^{d}$ define a $d$-dimensional lattice model restricted to the first orthant, and let $F(\mathbf{z}, t)$ be the generating function for this model, counting the number of walks of length $n$ with marked endpoint. Let $V=\{1, \ldots, d\}$, so that it is the set of coordinates $j$ for which there is at least one step in $\mathcal{S}$ with -1 in the $j$-th coordinate (this is the full set of indices by our assumptions). Then, by translating the combinatorial recurrence described above, we see that $F(\mathbf{z}, t)$ satisfies the functional equation

$$
\begin{align*}
\left(z_{1} \cdots z_{d}\right) F(\mathbf{z}, t)=\left(z_{1} \cdots z_{d}\right) & +t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z}) F(\mathbf{z}, t) \\
& -t \sum_{V^{\prime} \subseteq V}(-1)^{\left|V^{\prime}\right|}\left[\left(z_{1} \cdots z_{d}\right) S(\mathbf{z}) F(\mathbf{z}, t)\right]_{\left\{z_{j}=0: j \in V^{\prime}\right\}} \tag{3}
\end{align*}
$$

Basic manipulations then give the following result.
Lemma 2.1. Let $F(\mathbf{z}, t)$ be the multivariate generating function described above. Then

$$
\begin{equation*}
\left(z_{1} \cdots z_{d}\right)(1-t S(\mathbf{z})) F(\mathbf{z}, t)=\left(z_{1} \cdots z_{d}\right)+\sum_{k=1}^{d} A_{k}\left(z_{1}, \ldots, z_{k-1}, z_{k+1} \ldots, z_{d}, t\right) \tag{4}
\end{equation*}
$$

for some $A_{k} \in \mathbb{Q}\left[z_{1}, \ldots, z_{k-1}, z_{k+1}, \ldots, z_{d}\right] \llbracket t \rrbracket$.
Example 4. Set $\mathcal{S}=\left\{e_{1},-e_{1}, \ldots, e_{d},-e_{d}\right\}$. In this case $S(\mathbf{z})=\sum_{j=1}^{d}\left(z_{j}+\bar{z}_{j}\right)$, so $\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})$ vanishes when at least two of the $z_{j}$ are zero, and the generating function satisfies
$\left(z_{1} \cdots z_{d}\right)(1-t S(\mathbf{z})) F(\mathbf{z}, t)=\left(z_{1} \cdots z_{d}\right)+\sum_{k=1}^{d} t\left(z_{1} \ldots z_{k-1} z_{k+1} \ldots z_{d}\right) F\left(z_{1}, \ldots, z_{j-1}, 0, z_{j+1} \ldots, z_{d}\right)$.
2.2. The Orbit Sum Method. The orbit sum method, when it applies, has three main steps: find a suitable group $\mathcal{G}$ of rational maps; apply the elements of the group to the functional equation and form a telescoping sum; and (ultimately) represent the generating function of a model as the positive series extraction of an explicit rational function. Bousquet-Mélou and Mishna [9] illustrate the applicability in the case of lattice walks, and it has been adapted to several dimensions [4].
2.2.1. The group $\mathcal{G}$. For any $d$-dimensional model, we define the group $\mathcal{G}$ of $2^{d}$ rational maps by

$$
\begin{equation*}
\mathcal{G}:=\left\{\left(z_{1}, \ldots, z_{d}\right) \mapsto\left(z_{1}^{i_{1}}, \ldots, z_{d}^{i_{d}}\right):\left(i_{1}, \ldots, i_{d}\right) \in\{-1,1\}^{d}\right\} \tag{5}
\end{equation*}
$$

Given $\sigma \in \mathcal{G}$, we can consider $\sigma$ as a map on $\mathbb{Q}\left[z_{1}, \bar{z}_{1}, \ldots, z_{d}, \bar{z}_{d}\right] \llbracket t \rrbracket$ through the group action defined by $\sigma(A(\mathbf{z}, t)):=A(\sigma(\mathbf{z}), t)$. Due to the symmetry of the step set across each axis, one can verify that $\sigma(S(\mathbf{z}))=S(\sigma(\mathbf{z}))=S(\mathbf{z})$ always holds. The fact that this group does not depend on the step set of the model - only on the dimension $d$ - is crucial to obtaining the general results here. When $d$ equals two, the group $\mathcal{G}$ matches the group used by [12] and [9]. As we will see in Section $7, \mathcal{G}$
corresponds to the Weyl group of the Weyl chamber $A_{1}^{d}$, where the step set $\mathcal{S}$ can be studied in the context of Gessel and Zeilberger [15].
2.2.2. A telescoping sum. Next we apply each of the $2^{d}$ elements of $\mathcal{G}$ to Equation (4), and take a weighted sum. Define $\operatorname{sgn}(\sigma)=(-1)^{r}$, where $r=\#\left\{k: \sigma\left(z_{k}\right)=\bar{z}_{k}\right\}$, and let $\sigma_{k}$ be the map which sends $z_{k}$ to $\bar{z}_{k}$ and fixes all other components of $\left(z_{1}, \ldots, z_{d}\right)$.

Lemma 2.2. Let $F(\mathbf{z}, t)$ be the generating function counting the number of walks of length $n$ with marked endpoint. Then, as elements of the ring $\mathbb{Q}\left[z_{1}, \bar{z}_{1}, \ldots, z_{d}, \bar{z}_{d}\right] \llbracket t \rrbracket$,

$$
\begin{equation*}
\sum_{\sigma \in \mathcal{G}} \operatorname{sgn}(\sigma) \cdot \sigma\left(z_{1} \cdots z_{d}\right) \sigma(F(\mathbf{z}, t))=\frac{\sum_{\sigma \in \mathcal{G}} \operatorname{sgn}(\sigma) \cdot \sigma\left(z_{1} \cdots z_{d}\right)}{1-t S(\mathbf{z})} \tag{6}
\end{equation*}
$$

Proof. For each $\sigma \in \mathcal{G}$ we have $\operatorname{sgn}(\sigma)=-\operatorname{sgn}\left(\sigma_{k} \sigma\right)$ and, for the $A_{k}$ in Equation (4),

$$
\sigma\left(A_{k}\left(z_{1}, \ldots, z_{k-1}, z_{k+1} \ldots, z_{d}, t\right)\right)=\left(\sigma_{k} \sigma\right)\left(A_{k}\left(z_{1}, \ldots, z_{k-1}, z_{k+1} \ldots, z_{d}, t\right)\right)
$$

Thus, we can apply each $\sigma \in \mathcal{G}$ to Equation (4) and sum the results, weighted by $\operatorname{sgn}(\sigma)$, to cancel each $A_{k}$ term on the right hand side. Minor algebraic manipulations, along with the fact that the group elements fix $S\left(z_{1}, \ldots, z_{d}\right)$, then give Equation (6).
2.2.3. Positive series extraction. Next, we note that each term in the expansion of

$$
\sigma_{1}\left(z_{1}, \ldots, z_{d}\right) \sigma_{1}(F(\mathbf{z}, t))=-\left(\bar{z}_{1} z_{2} \cdots z_{d}\right) F\left(\bar{z}_{1}, z_{2}, \ldots, z_{d}, t\right) \in \mathbb{Q}\left[z_{1}, \bar{z}_{1}, \ldots, z_{d}, \bar{z}_{d}\right] \llbracket t \rrbracket
$$

has a negative power of $z_{1}$. In fact, except for when $\sigma$ is the identity any summand $\sigma\left(z_{1} \cdots z_{d}\right) \sigma(F(\mathbf{z}, t))$ on the left hand side of Equation (6) contains a negative power of at least one variable in any term of its expansion.

With this in mind, for an element $A(\mathbf{z}, t) \in \mathbb{Q}\left[z_{1}, \bar{z}_{1}, \ldots, z_{d}, \bar{z}_{d}\right] \llbracket t \rrbracket$ we let $\left[z_{k}^{Z}\right] A(\mathbf{z}, t)$ denote the sum of all terms of $A(\mathbf{z}, t)$ which contain only non-negative powers of $z_{k}$. Lemma 2.3 then follows from the identity

$$
\sum_{\sigma \in \mathcal{G}} \operatorname{sgn}(\sigma) \cdot \sigma\left(z_{1} \cdots z_{d}\right)=\left(z_{1}-\bar{z}_{1}\right) \cdots\left(z_{d}-\bar{z}_{d}\right),
$$

which can be proven by induction.
Lemma 2.3. Let $F(\mathbf{z}, t)$ be the generating function counting the number of walks of length $n$ with marked endpoint. Then

$$
\begin{equation*}
F(\mathbf{z}, t)=\left[z_{1}^{Z}\right] \cdots\left[z_{d}^{\geq}\right] R(\mathbf{z}, t), \tag{7}
\end{equation*}
$$

where

$$
R(\mathbf{z}, t)=\frac{\left(z_{1}-\bar{z}_{1}\right) \cdots\left(z_{d}-\bar{z}_{d}\right)}{\left(z_{1} \cdots z_{d}\right)(1-t S(\mathbf{z}))} .
$$

Since the class of D-finite functions is closed under positive series extraction - as shown in [22] - an immediate consequence is the following.

Corollary 2.4. Under the above conditions on $\mathcal{S}$, the generating functions $F(\mathbf{z}, t)$ and (thus) $F(\mathbf{1}, t)$ are $D$-finite functions.
2.3. The generating function as a diagonal. Given an element

$$
B(\mathbf{z}, t)=\sum_{n \geq 0}\left(\sum_{\mathbf{i} \in \mathbb{Z}^{d}} b_{\mathbf{i}}(n) z_{1}^{i_{1}} \cdots z_{d}^{i_{d}}\right) t^{n} \in \mathbb{Q}\left[z_{1}, \bar{z}_{1}, \ldots, z_{d}, \bar{z}_{d}\right] \llbracket t \rrbracket,
$$

we let $\Delta$ denote the (complete) diagonal operator

$$
\Delta B(\mathbf{z}, t):=\sum_{n \geq 0} b_{n, \ldots, n}(n) t^{n}
$$

There is a natural correspondence between the diagonal operator and extracting the positive part of a multivariate power series, as in Equation (7).

Proposition 2.5. Let $B(\mathbf{z}, t)$ be an element of $\mathbb{Q}\left[z_{1}, \bar{z}_{1}, \ldots, z_{d}, \bar{z}_{d}\right] \llbracket t \rrbracket$. Then

$$
\begin{equation*}
\left.\left[z_{1}^{\geq}\right] \cdots\left[z_{d}^{\geq}\right] B(\mathbf{z}, t)\right|_{z_{1}=1, \ldots, z_{d}=1}=\Delta\left(\frac{B\left(\bar{z}_{1}, \ldots, \bar{z}_{d}, z_{1} \cdots z_{d} \cdot t\right)}{\left(1-z_{1}\right) \cdots\left(1-z_{d}\right)}\right) . \tag{8}
\end{equation*}
$$

Proof. Suppose that $B$ has the expansion

$$
B(\mathbf{z}, t)=\sum_{n \geq 0}\left(\sum_{\mathbf{i} \in \mathbb{Z}^{d}} b_{\mathbf{i}}(n) z_{1}^{i_{1}} \cdots z_{d}^{i_{d}}\right) t^{n} .
$$

Then the right hand side of Equation (8) is given by

$$
\Delta\left(\sum_{k \geq 0} z_{1}^{k}\right) \cdots\left(\sum_{k \geq 0} z_{d}^{k}\right)\left(\sum_{n \geq 0}\left(\sum_{\mathbf{i} \in \mathbb{Z}^{d}} b_{\mathbf{i}}(n) z_{1}^{n-i_{1}} \cdots z_{d}^{n-i_{d}}\right) t^{n}\right)
$$

so that the coefficient of $t^{n}$ in the diagonal is the sum of all terms $b_{\mathbf{i}}(n)$ with $i_{1}, \ldots, i_{d} \geq 0$ (by assumption there are only finitely many which are non-zero). But this is exactly the coefficient of $t^{n}$ on the left hand side.

We note also that in the context of lattice path models with step set $\mathcal{S} \subseteq\{ \pm 1,0\}^{d} \backslash\{\mathbf{0}\}$, the modified generating function $F\left(\bar{z}_{1}, \ldots, \bar{z}_{d}, z_{1} \cdots z_{d} \cdot t\right)$ is actually a power series in the variables $z_{1}, \ldots, z_{d}, t$ (as a walk cannot move farther on the integer lattice than its number of steps). Combining Lemma 2.3 and Proposition 2.5 implies that the generating function for the number of walks can be represented as $F(\mathbf{1}, t)=\Delta\left(\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)}\right)$, where

$$
\begin{align*}
\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)} & =\frac{\left(1-z_{1}^{2}\right) \cdots\left(1-z_{d}^{2}\right)}{1-t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})} \cdot \frac{1}{\left(1-z_{1}\right) \cdots\left(1-z_{d}\right)} \\
& =\frac{\left(1+z_{1}\right) \cdots\left(1+z_{d}\right)}{1-t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})} \tag{9}
\end{align*}
$$

To be precise, $G(\mathbf{z}, t)$ and $H(\mathbf{z}, t)$ are defined as the numerator and denominator of Equation (9).
Example 5. For the walks defined by $\mathcal{S}=\left\{e_{1},-e_{1}, \ldots, e_{d},-e_{d}\right\}$, we have

$$
\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)}=\frac{\left(1+z_{1}\right) \cdots\left(1+z_{d}\right)}{1-t \sum_{k=1}^{n}\left(1+z_{k}^{2}\right)\left(z_{1} \cdots z_{k-1} z_{k+1} \cdots z_{d}\right)} .
$$

Note that this rational function is not unique, in the sense that there are other rational functions whose diagonals yield the same counting sequence.
2.4. The singular variety associated to the kernel. Here, we pause to note that the combinatorial symmetries of the step sets that we consider affect the geometry of the variety of $H(\mathbf{z}, t)$ - called the singular variety. This has a direct impact on both the asymptotics of the counting sequence under consideration and the ease with which its asymptotics are computed. In particular, any factors of the form $\left(1-z_{k}\right)$ present in the denominator of this rational function before simplification could have given rise to non-simple poles and thus made the singular variety non-smooth. Although non-smooth varieties can be handled in many cases - see [25] - having a smooth singular variety is the easiest situation in which one can work in the multivariate setting. Understanding the interplay between the step set symmetry and the singular variety geometry, and in the process dealing with the non-smooth cases, is promising future work.

## 3. Analytic combinatorics in several variables

Following the work of Pemantle and Wilson [24] and Raichev and Wilson [27], we can determine the dominant asymptotics for the diagonal of the multivariate power series $\frac{G(\mathbf{z}, t)}{H(\mathbf{z}, t)}$ by studying the variety (complex set of zeroes) $\mathcal{V} \subseteq \mathbb{C}^{d+1}$ of the denominator

$$
H(\mathbf{z}, t)=1-t \cdot\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})
$$

To begin, a particular set of singular points - called the critical points - containing all singular points which could affect the asymptotics of $\Delta(G / H)$ are computed in Section 3.1. The set of critical points is then refined to those which determine the dominant asymptotics up to an exponential decay in Section 3.2; this refined set is called the set of minimal points as they are the critical points which are 'closest' to the origin in a sense made precise below. The enumerative results come from calculating a Cauchy residue type integral, and after determining the minimal points we determine asymptotics in Section 3.3 using pre-computed formulas for such integrals which can be found in [25]. In fact, up to polynomial decay there is only one singular point which determines dominant asymptotics for each model - the point $\rho=(\mathbf{1}, 1 /|\mathcal{S}|)$ - and this uniformity aids greatly in computing the quantities required in the analysis of a general step set, in order to obtain Theorem 3.4.

We first verify our claim in the previous section that the variety is smooth (that is, at every point on $\mathcal{V}$ one of the partial derivatives $H_{z_{k}}$ or $H_{t}$ does not vanish). Indeed, any non-smooth point on $\mathcal{V}$ would have to satisfy both

$$
\begin{aligned}
1-t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z}) & =H=0 \\
\text { and } \quad-\left(z_{1} \cdots z_{d}\right) S(\mathbf{z}) & =H_{t}=0
\end{aligned}
$$

which can never occur. Equivalently, this shows that at each point in $\mathcal{V}$ there exists a neighbourhood $N \subseteq \mathbb{C}^{d+1}$ such that $\mathcal{V} \cap N$ is a complex submanifold of $N$.
3.1. Critical points. The next step is to find the critical points. Determined through an appeal to stratified Morse theory, for a smooth variety the critical points are precisely those which satisfy the following critical point equations:

$$
H=0, \quad t H_{t}=z_{1} H_{z_{1}}, \quad t H_{t}=z_{2} H_{z_{2}}, \quad \ldots \quad t H_{t}=z_{d} H_{z_{d}}
$$

which we now solve. Given $\mathbf{z} \in \mathbb{C}^{d}$, define

$$
\mathbf{z}_{\hat{k}}:=\left(z_{1}, \ldots, z_{k-1}, z_{k+1}, \ldots, z_{d}\right) \in \mathbb{C}^{d-1}
$$

As each step in $\mathcal{S}$ has coordinates taking values in $\{-1,0,1\}$, we may collect the coefficients of the $k^{\text {th }}$ variable, and use the symmetries present to write

$$
\begin{equation*}
S(\mathbf{z})=\left(\bar{z}_{k}+z_{k}\right) S_{8}^{(k)}\left(\mathbf{z}_{\hat{k}}\right)+S_{0}^{(k)}\left(\mathbf{z}_{\hat{k}}\right) \tag{10}
\end{equation*}
$$

which uniquely defines the Laurent polynomials $S_{1}^{(k)}\left(\mathbf{z}_{\hat{k}}\right)$ and $S_{0}^{(k)}\left(\mathbf{z}_{\hat{k}}\right)$. With this notation the equation $t H_{t}=z_{k} H_{z_{k}}$ becomes

$$
t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})=t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})+t\left(z_{1} \cdots z_{d}\right)\left(z_{k} S_{z_{k}}(\mathbf{z})\right)
$$

which implies

$$
\begin{equation*}
0=t\left(z_{1} \cdots z_{d}\right) \cdot z_{k} S_{z_{k}}(\mathbf{z})=t\left(z_{k}^{2}-1\right)\left(z_{1} \cdots z_{k-1} z_{k+1} \cdots z_{d}\right) S_{1}^{(k)}\left(\mathbf{z}_{\hat{k}}\right) . \tag{11}
\end{equation*}
$$

Note that while $\left(z_{1} \cdots z_{k-1} z_{k+1} \cdots z_{d}\right) S_{1}^{(k)}\left(\mathbf{z}_{\hat{k}}\right)$ is a polynomial, $S_{1}^{(k)}\left(\mathbf{z}_{\hat{k}}\right)$ itself is a Laurent polynomial, so one must be careful when specializing variables to 0 in the expression. This calculation characterizes the critical points of $\mathcal{V}$.

Proposition 3.1. The point $(\mathbf{z}, t)=\left(z_{1}, \ldots, z_{d}, t\right) \in \mathcal{V}$ is a critical point of $\mathcal{V}$ if and only if for each $1 \leq k \leq d$ either:
(1) $z_{k}= \pm 1$ or,
(2) the polynomial $\left(y_{1} \cdots y_{k-1} y_{k+1} \cdots y_{d}\right) S_{1}^{(k)}\left(\mathbf{y}_{\hat{k}}\right)$ has a root at $\mathbf{z}$.

Proof. We have shown above that the critical point equations reduce to Equation (11). Furthermore, if $t$ were zero at a point on $\mathcal{V}$ then $0=H\left(z_{1}, \ldots, z_{n}, 0\right)=1$, a contradiction.

It is interesting to note that the polynomial $\left(y_{1} \cdots y_{k-1} y_{k+1} \cdots y_{d}\right) S_{1}^{(k)}\left(\mathbf{y}_{\hat{k}}\right)$ has combinatorial signifigance, as the subset of $S(\mathbf{z})$ which encodes only the steps which move forwards in their $k^{\text {th }}$ coordinate.
3.2. Minimal points. Among the critical points, only those which are 'closest' to the origin will contribute to the asymptotics, up to an exponentially decaying error. This is analogous to the single variable case, where the singularities of minimum modulus are those which contribute to the dominant asymptotic term. To be precise, for any point $(\mathbf{z}, t) \in \mathbb{C}^{d+1}$ we define the closed polydisk

$$
D(\mathbf{z}, t):=\left\{\left(\mathbf{w}, t^{\prime}\right) \in \mathbb{C}^{d+1}:\left|t^{\prime}\right| \leq|t| \text { and }\left|w_{j}\right| \leq\left|z_{j}\right| \text { for } j=1, \ldots, d\right\} .
$$

The critical point $(\mathbf{z}, t)$ is called strictly minimal if $D(\mathbf{z}, t) \cap \mathcal{V}=\{(\mathbf{z}, t)\}$, and finitely minimal if the intersection contains only a finite number of points, all of which are on the boundary of $D(\mathbf{z}, t)$. Finally, we call a critical point isolated if there exists a neighbourhood of $\mathbb{C}^{d+1}$ where it is the only critical point. In our case, we need only be concerned with isolated finitely minimal points.

Proposition 3.2. The point $\boldsymbol{\rho}=(\mathbf{1}, 1 /|\mathcal{S}|)$ is a finitely minimal point of the variety $\mathcal{V}$. Furthermore, any point in $D(\boldsymbol{\rho}) \cap \mathcal{V}$ is an isolated critical point.

Proof. The point $\rho$ is critical as it lies on $\mathcal{V}$ and its first $d$ coordinates are all one. Suppose ( $\mathbf{w}, t_{\mathbf{w}}$ ) lies in $\mathcal{V} \cap D(\boldsymbol{\rho})$, where we note that any choice of $\mathbf{w}$ uniquely determines $t_{\mathbf{w}}$ on $\mathcal{V}$. Then, as $t_{\mathbf{w}} \neq 0$,

$$
\left|\sum_{\left(i_{1}, \ldots, i_{d}\right) \in \mathcal{S}} w_{1}^{i_{1}+1} \cdots w_{d}^{i_{d}+1}\right|=\left|\left(w_{1} \cdots w_{d}\right) S(\mathbf{w})\right|=\left|\frac{1}{t_{\mathbf{w}}}\right| \geq|\mathcal{S}| .
$$

But $\left(\mathbf{w}, t_{\mathbf{w}}\right) \in D(\boldsymbol{\rho})$ implies $\left|w_{j}\right| \leq 1$ for each $1 \leq j \leq d$. Thus, the above inequality states that the sum of $|\mathcal{S}|$ complex numbers of modulus at most one has modulus $|\mathcal{S}|$. The only way this can occur is if each term in the sum has modulus one, and all terms point in the same direction in the complex plane. By symmetry, and the assumption that we take a positive step in each direction, there are two terms of the form $w_{2}^{i_{2}+1} \cdots w_{d}^{i_{d}+1}$ and $w_{1}^{2} w_{2}^{i_{2}+1} \cdots w_{d}^{i_{d}+1}$ in the sum, so that $w_{1}^{2}$ must be 1 in order for them to point in the same direction. This shows $w_{1}= \pm 1$, and the same argument applies to each $w_{k}$, so there are at most $2^{d}$ points in $\mathcal{V} \cap D(\boldsymbol{\rho})$.

By Proposition 3.1 every such point $\left(\mathbf{w}, t_{\mathbf{w}}\right) \in \mathcal{V} \cap D(\boldsymbol{\rho})$ is critical, and to show it is isolated it is sufficient to prove $S_{1}^{(k)}\left(\mathbf{w}_{\hat{k}}\right) \neq 0$ for all $1 \leq k \leq d$. Indeed, if $S_{1}^{(k)}\left(\mathbf{w}_{\hat{k}}\right)=0$ then $\mathbf{w} \in \mathcal{V}$ implies

$$
\left|t_{\mathbf{w}}\right|=\frac{1}{\left|w_{1} \cdots w_{d} S_{0}^{(k)}\left(\mathbf{w}_{\hat{k}}\right)\right|} \geq \frac{1}{\left|S_{0}^{(k)}\left(\mathbf{w}_{\hat{k}}\right)\right|} \geq \frac{1}{S_{0}^{(k)}(\mathbf{1})}>\frac{1}{|\mathcal{S}|}
$$

by our assumption that $\mathcal{S}$ contains a step which moves forward in the $k^{\text {th }}$ coordinate. This contra$\operatorname{dicts}\left(\mathbf{w}, t_{\mathbf{w}}\right) \in D(\boldsymbol{\rho})$.
3.3. Asymptotics Results. To apply the formulas of [24] we need to define a few quantities. To start, we note that on all of $\mathcal{V}$ we may parametrize the coordinate $t$ as

$$
t(\mathbf{z})=\frac{1}{z_{1} \cdots z_{d} S(\mathbf{z})}
$$

For each point $\left(\mathbf{w}, t_{\mathbf{w}}\right) \in \mathcal{V} \cap D(\boldsymbol{\rho})$, the analysis of [24] shows that the asymptotics of the integral in question which determines asymptotics for a given model depends on the function

$$
\begin{align*}
\tilde{f}^{(\mathbf{w})}(\boldsymbol{\theta}) & =\log \left(\frac{t\left(w_{1} e^{i \theta_{1}}, \ldots, w_{d} e^{i \theta_{d}}\right)}{t_{\mathbf{w}}}\right)+i \sum_{k=1}^{d} \theta_{k} \\
& =\log \left(\frac{S(\mathbf{w})}{e^{i\left(\theta_{1}+\cdots+\theta_{d}\right)} S\left(w_{1} e^{i \theta_{1}}, \ldots, w_{d} e^{i \theta_{d}}\right)}\right)+i\left(\theta_{1}+\cdots+\theta_{d}\right) \\
& =\log S(\mathbf{w})-\log S\left(w_{1} e^{i \theta_{1}}, \ldots, w_{d} e^{i \theta_{d}}\right) \tag{12}
\end{align*}
$$

Let $\mathcal{H}_{\mathbf{w}}$ denote the determinant of the Hessian of $\tilde{f}^{(\mathbf{w})}(\boldsymbol{\theta})$ at $\mathbf{0}$ :

$$
\mathcal{H}_{\mathbf{w}}:=\operatorname{det} \tilde{f}^{\prime \prime}(\mathbf{w})(\mathbf{0})=\left|\begin{array}{cccc}
\tilde{f}_{\theta_{1} \theta_{1}}^{(\mathbf{w})}(\mathbf{0}) & \tilde{f}_{\theta_{1} \theta_{2}}^{(\mathbf{w})}(\mathbf{0}) & \cdots & \tilde{f}_{\theta_{1} \theta_{d}}^{(\mathbf{w})}(\mathbf{0}) \\
\tilde{f}_{\theta_{2} \theta_{1}}^{(\mathbf{w})}(\mathbf{0}) & \tilde{f}_{\theta_{2} \theta_{2}}^{(\mathbf{w})}(\mathbf{0}) & \cdots & \tilde{f}_{\theta_{2} \theta_{d}}^{(\mathbf{w})}(\mathbf{0}) \\
\vdots & \vdots & \ddots & \vdots \\
\tilde{f}_{\theta_{d} \theta_{1}}^{(\mathbf{w})}(\mathbf{0}) & \tilde{f}_{\theta_{d} \theta_{2}}^{(\mathbf{w})}(\mathbf{0}) & \cdots & \tilde{f}_{\theta_{d} \theta_{d}}^{(\mathbf{w})}(\mathbf{0})
\end{array}\right|,
$$

if $\mathcal{H}_{\mathbf{w}} \neq 0$ then we say $\left(\mathbf{w}, t_{\mathbf{w}}\right)$ is non-degenerate. The main asymptotic result of smooth multivaritate analytic combinatorics, in this restricted context, is the following (the original result allows for asymptotic expansions of coefficient sequences more generally defined from multivariate functions than the diagonal sequence).

Theorem 3.3 (Adapted from Theorem 3.5 of [24]). Suppose that the meromorphic function $F(\mathbf{z}, t)=$ $G(\mathbf{z}, t) / H(\mathbf{z}, t)$ has an isolated strictly minimal simple pole at $\left(\mathbf{w}, t_{\mathbf{w}}\right)$. If $t H_{t}$ does not vanish at $\left(\mathbf{w}, t_{\mathbf{w}}\right)$ then there is an asymptotic expansion

$$
\begin{equation*}
c_{n} \sim\left(w_{1} \cdots w_{d} \cdot t\right)^{-n} \sum_{l \geq l_{0}} C_{l} n^{-(d+l) / 2} \tag{13}
\end{equation*}
$$

for constants $C_{l}$, where $l_{0}$ is the degree to which $G$ vanishes near $\left(\mathbf{w}, t_{\mathbf{w}}\right)$. When $G$ does not vanish at $\left(\mathbf{w}, t_{\mathbf{w}}\right)$ then $l_{0}=0$ and the leading term of this expansion is

$$
\begin{equation*}
C_{0}=(2 \pi)^{-d / 2} \mathcal{H}_{\mathbf{w}}^{-1 / 2} \cdot \frac{G\left(\mathbf{w}, t_{\mathbf{w}}\right)}{t H_{t}\left(\mathbf{w}, t_{\mathbf{w}}\right)} \tag{14}
\end{equation*}
$$

In fact, Corollary 3.7 of [24] shows that in the case of a finitely minimal point one can simply sum the contributions of each point. Combining this with the above calculations gives our main result.

Theorem 3.4. Let $\mathcal{S} \subseteq\{-1,0,1\}^{d} \backslash\{\mathbf{0}\}$ be a set of unit steps in dimension d. If $\mathcal{S}$ is symmetric with respect to each axis, and $\mathcal{S}$ takes a positive step in each direction, then the number of walks of length $n$ taking steps in $\mathcal{S}$, beginning at the origin, and never leaving the positive orthant has asymptotic expansion

$$
\begin{equation*}
s_{n}=\left[\left(s^{(1)} \cdots s^{(d)}\right)^{-1 / 2} \pi^{-d / 2}|\mathcal{S}|^{d / 2}\right] \cdot n^{-d / 2} \cdot|\mathcal{S}|^{n}+O\left(n^{-(d+1) / 2} \cdot|\mathcal{S}|^{n}\right), \tag{15}
\end{equation*}
$$

where $s^{(k)}$ denotes the number of steps in $\mathcal{S}$ which have $k^{\text {th }}$ coordinate 1 .
Proof. We begin by verifying that each point $\left(\mathbf{w}, t_{\mathbf{w}}\right) \in \mathcal{V} \cap D(\boldsymbol{\rho})$ satisfies the conditions of Theorem 3.3:

1. $\left(\mathbf{w}, t_{\mathbf{w}}\right)$ is a simple pole: As $\mathcal{V}$ is smooth, the point $\left(\mathbf{w}, t_{\mathbf{w}}\right)$ is a simple pole.
2. $\left(\mathbf{w}, t_{\mathbf{w}}\right)$ is isolated: This is proven in Proposition 3.2.
3. $t H_{t}$ does not vanish at $\left(\mathbf{w}, t_{\mathbf{w}}\right)$ : This follows from $t_{\mathbf{w}} H_{t}\left(\mathbf{w}, t_{\mathbf{w}}\right)=1 /\left(w_{1} \cdots w_{d}\right) \neq 0$.
4. ( $\mathbf{w}, t_{\mathbf{w}}$ ) is non-degenerate: Directly taking partial derivatives in Equation (12) implies

$$
\tilde{f}_{\theta_{j} \theta_{k}}^{(\mathbf{w})}(\mathbf{0})=\left\{\begin{array}{ll}
w_{j} w_{k} \frac{S_{y_{j} y_{k}}(\mathbf{w}) S(\mathbf{w})-S_{y_{j}}(\mathbf{w}) S_{y_{k}}(\mathbf{w})}{S(\mathbf{w})^{2}} & : j \neq k \\
\frac{S_{y_{j} y_{j}}(\mathbf{w}) S(\mathbf{w})+w_{j} S_{y_{j}}(\mathbf{w}) S(\mathbf{w})-S_{y_{j}}(\mathbf{w})^{2}}{S(\mathbf{w})^{2}} & : j=k
\end{array} .\right.
$$

Since $S_{y_{j}}(\mathbf{y})=\left(1-y_{j}^{-2}\right) S_{1}^{(j)}\left(\mathbf{y}_{\hat{j}}\right)$ we see that $S_{y_{j}}(\mathbf{w})=0$. Similarly, one can calculate that $S_{y_{j} y_{j}}(\mathbf{w})=2 S_{1}^{(j)}(\mathbf{w})$ and $S_{y_{j} y_{k}}(\mathbf{w})=0$ for $j \neq k$, so that the Hessian of $\tilde{f}^{(\mathbf{w})}(\boldsymbol{\theta})$ at $\mathbf{0}$ is a diagonal matrix and

$$
\begin{equation*}
\mathcal{H}_{\mathbf{w}}=\frac{2^{d}}{S(\mathbf{w})^{d}} S_{1}^{(1)}(\mathbf{w}) \cdots S_{1}^{(d)}(\mathbf{w}) \tag{16}
\end{equation*}
$$

The proof of Proposition 3.2 implies that $S_{1}^{(k)}(\mathbf{w}) \neq 0$ for any $1 \leq k \leq d$, so each ( $\left.\mathbf{w}, t_{\mathbf{w}}\right)$ is non-degenerate.
Thus, we can apply Corollary 3.7 of [24] and sum the expansions (13) at each point in $\mathcal{V} \cap D(\boldsymbol{\rho})$ to obtain the asymptotic expansion

$$
\begin{equation*}
s_{n} \sim|\mathcal{S}|^{n} \sum_{\mathbf{w} \in \mathcal{V} \cap D(\boldsymbol{\rho})}\left(\sum_{l \geq l_{\mathbf{w}}} C_{l}^{\mathbf{w}} n^{-(d+l) / 2}\right) \tag{17}
\end{equation*}
$$

for constants $C_{l}^{\mathbf{w}}$, where $l_{\mathbf{w}}$ is the degree to which $G(\mathbf{y}, t)$ vanishes near ( $\mathbf{w}, t_{\mathbf{w}}$ ). Since the numerator $G(\mathbf{y}, t)=\left(1+y_{1}\right) \cdots\left(1+y_{d}\right)$ vanishes at all points of $\mathbf{w} \in \mathcal{V} \cap D(\boldsymbol{\rho})$ except for $\boldsymbol{\rho}=(\mathbf{1}, 1 /|\mathcal{S}|)$, the dominant term of (17) is determined only by the contribution of $\mathbf{w}=\boldsymbol{\rho}$. Substituting the value for $\mathcal{H} \boldsymbol{\rho}$ given by Equation (16) into Equation (14) gives the desired asymptotic result.

## 4. Examples

We now give two examples, both of which calculate critical points by directly solving the critical point equations. The first example has only a finite number of critical points, all of which are minimal points. In contrast, the second example contains a curve of critical points (however, as guaranteed by Proposition 3.2, no points on this curve are minimal points).

Example 6. Consider the model in three dimensions restricted to the positive octant taking the eight steps

$$
\mathcal{S}=\{(-1,0, \pm 1),(1,0, \pm 1),(0,1, \pm 1),(0,-1, \pm 1)\}
$$

The kernel equation here is

$$
\begin{aligned}
x y z(1-t S(x, y, z)) F(x, y, z, t) & =x y z-t y\left(z^{2}+1\right) F(0, y, z)-t x\left(z^{2}+1\right) F(x, 0, z) \\
& -t\left(x^{2} y+y^{2} x+y+x\right) F(x, y, 0) \\
& +t x F(x, 0,0)+t y F(0, y, 0)
\end{aligned}
$$

with characteristic polynomial

$$
S(x, y, z)=(x+y+\bar{x}+\bar{y})(z+\bar{z})
$$

The generalized orbit sum method implies $F(1,1,1, t)=\Delta B(x, y, z, t)$ where

$$
\begin{aligned}
B(x, y, z, t) & =\frac{(\bar{x}-x)(\bar{y}-y)(\bar{z}-z)}{\bar{x} \bar{y} \bar{z}(1-t x y z P(\bar{x}, \bar{y}, \bar{z}))} \cdot \frac{1}{(1-x)(1-y)(1-z)} \\
& =\frac{(1+x)(1+y)(1+z)}{1-t\left(z^{2}+1\right)(x+y)(x y+1)}
\end{aligned}
$$

Next, we verify that the denominator $H(x, z, y, t)$ of $B(x, y, z, t)$ is smooth - i.e., that $H$ and its partial derivatives don't vanish together at any point. This can be checked automatically by computing a Gröbner Basis of the ideal generated by $H$ and its partial derivatives.

In pseudo-code: ${ }^{2}$

$$
\begin{aligned}
& >\quad H:=1-t\left(z^{2}+1\right)(x+y)(x y+1): \\
& >\quad \operatorname{GroebnerBasis}\left(\left[H, H_{x}, H_{y}, H_{z}, H_{t}\right], \operatorname{plex}(t, x, y, z)\right)
\end{aligned}
$$

[1]
The critical points can be computed:

$$
\begin{gathered}
>\quad \text { GroebnerBasis }\left(\left[H, t H_{t}-x H_{x}, t H_{t}-y H_{y}, t H_{t}-z H_{z}\right], \operatorname{plex}(t, x, y, z)\right) \\
{\left[z^{2}-1, y^{2}-1, x-y, 8 t-y\right]}
\end{gathered}
$$

This implies that there is a finitely minimal critical point $\boldsymbol{\rho}=(1,1,1,1 / 8)$, where

$$
T(\boldsymbol{\rho}) \cap \mathcal{V}=\{(1,1,1,1 / 8),(1,1,-1,1 / 8),(-1,-1,1,-1 / 8),(-1,-1,-1,-1 / 8)\}
$$

The value of $\mathcal{H}_{\mathbf{w}}$ can be calculated at each point to be $1 / 4$. For instance:

$$
\begin{aligned}
& >\quad f:=\log S(\mathbf{1})-\log S\left(e^{i \theta_{1}}, e^{i \theta_{2}}, e^{i \theta_{3}}\right): \\
& >\quad \operatorname{subs}\left(\theta_{1}=0, \theta_{2}=0, \theta_{3}=0, \operatorname{det}\left(\operatorname{Hessian}\left(f,\left[\theta_{1}, \theta_{2}, \theta_{3}\right]\right)\right)\right)
\end{aligned}
$$

$1 / 4$
Equation (14) then gives the asymptotic result

$$
c_{n} \sim 4 \sqrt{2} \cdot \pi^{-3 / 2} \cdot n^{-3 / 2} \cdot 8^{n}
$$

Example 7. Consider the model in three dimensions restricted to the positive octant taking the twelve steps

$$
\mathcal{S}=\{(-1,0, \pm 1),(1,0, \pm 1),(0,1, \pm 1),(0,-1, \pm 1),( \pm 1,1,0),(1, \pm 1,0)\}
$$

Now, by our previous analysis, $F(1,1,1, t)=\Delta B(x, y, z, t)$ where

$$
\begin{equation*}
B(x, y, z, t)=\frac{(1+x)(1+y)(1+z)}{1-t\left(z^{2}+1\right)(x+y)(x y+1)-t z\left(y^{2}+1\right)\left(x^{2}+1\right)} \tag{18}
\end{equation*}
$$

${ }^{2}$ The input is formatted for Maple version 18.

The denominator $H(x, z, y, t)$ of $B(x, y, z, t)$ can again be verified to be smooth, but the ideal encoding the critical point equations is no longer zero dimensional; i.e., there are an infinite number of solutions of the critical point equations. For instance, the following calculation shows that any point $(1,-1, z, 1 / 4 z)$ with $z \neq 0$ is a non-isolated critical point:

$$
\begin{aligned}
& >\quad H:=1-t\left(z^{2}+1\right)(x+y)(x y+1)-t z\left(y^{2}+1\right)\left(x^{2}+1\right): \\
& >\quad I:=\operatorname{subs}\left(x=1, y=-1,\left[H, t H_{t}-x H_{x}, t H_{t}-y H_{y}, t H_{t}-z H_{z}\right]\right): \\
& >\quad \text { GroebnerBasis }(I, \operatorname{plex}(t, x, y, z))
\end{aligned}
$$

$$
[1-4 t z]
$$

Note that none of these points are minimal - so Proposition 3.2 is not contradicted - since

$$
|(1) \cdot(-1) \cdot(z) \cdot(1 / 4 z)|=1 / 4>\frac{1}{|\mathcal{S}|}
$$

## 5. LOWER ORDER TERMS

Building upon the work of Pemantle and Wilson, Raichev and Wilson [27] refined the asymptotics of Equation (13) and found expressions for the lower order constants $C_{1}, C_{2}, \ldots$, theoretically allowing one to calculate the contribution of each minimal point $\mathbf{w} \in \mathcal{V} \cap D(\boldsymbol{\rho})$. To be explicit, Theorem 3.8 of [27] gives the asymptotic contribution of the minimal point $\mathbf{w}$ as

$$
\begin{align*}
c_{n}^{(\mathbf{w})}=|\mathcal{S}|^{n} \cdot\left[2^{-d} \pi^{-d / 2} S(\mathbf{w})^{d / 2} \cdot\left(S_{1}^{(1)}(\mathbf{w}) \cdots S_{1}^{(d)}(\mathbf{w})\right)^{-1 / 2}\right] \cdot n^{-d / 2} \cdot & \sum_{k=0}^{N-1} n^{-k} L_{k}\left(\tilde{u}^{(\mathbf{w})}, \tilde{f}^{(\mathbf{w})}\right)  \tag{19}\\
& +O\left(|\mathcal{S}|^{n} \cdot n^{-(d-1) / 2-N}\right)
\end{align*}
$$

where, for $\star$ denoting the Hadamard product

$$
\left(a_{1}, \ldots, a_{d}\right) \star\left(b_{1}, \ldots, b_{d}\right)=\left(a_{1} b_{1}, \ldots, a_{d} b_{d}\right)
$$

we have

$$
\begin{aligned}
\tilde{u}^{(\mathbf{w})}(\boldsymbol{\theta}) & :=-\frac{1}{t_{\mathbf{w}}} \cdot \frac{G\left(\mathbf{w} \star e^{i \boldsymbol{\theta}}, t_{\mathbf{w}}\right)}{H_{t}\left(\mathbf{w} \star e^{i \boldsymbol{\theta}}, t_{\mathbf{w}}\right)} \\
g_{\mathbf{w}}(\boldsymbol{\theta}) & :=\log S(\mathbf{w})-\log S\left(\mathbf{w} \star e^{i \boldsymbol{\theta}}\right)-\frac{1}{2} \boldsymbol{\theta} \cdot \tilde{f}^{\prime \prime}(\mathbf{w})(\boldsymbol{\theta}) \cdot \boldsymbol{\theta}^{T} \\
L_{k}\left(\tilde{u}^{(\mathbf{w})}, \tilde{f}^{(\mathbf{w})}\right) & :=\sum_{r=0}^{2 k} \frac{\mathcal{D}^{r+k}\left(\tilde{u}^{(\mathbf{w})} \cdot g_{\mathbf{w}}^{r}\right)(\mathbf{0})}{(-1)^{k} 2^{r+k} r!(r+k)!}
\end{aligned}
$$

and $\mathcal{D}$ is the differential operator

$$
\mathcal{D}=-\sum_{0 \leq r, s \leq d}\left(\operatorname{Inv} \tilde{f}^{\prime \prime(\mathbf{w})}\right)_{r, s} \partial_{\theta_{r}} \partial_{\theta_{s}}=-\frac{S(\mathbf{w})}{2} \sum_{r=0}^{d} \frac{1}{S_{1}^{(1)}(\mathbf{w})} \partial_{\theta_{r}}^{2}
$$

This expression is quite involved - making it hard to derive a general asymptotic theorem with lower order terms - but completely effective for a given step set. The principle difficulty determining enumerative results for explicit models in the smooth case is the identification of points which actually contribute to the asymptotic growth. In the case of highly symmetric walks this is accomplished through the characterization of minimal points given in Proposition 3.2.

Example 8. Consider the two dimensional model with step set $\{N, S, N E, S E, N W, S W\}=\mathbb{X}$, previously computed to have dominant asymptotics

$$
c_{n} \sim \frac{\sqrt{6}}{\pi} \cdot \frac{6^{n}}{n} .
$$

By Proposition 3.2, to find the minimal points we simply need to solve the equation

$$
H(x, y, t)=1-t\left(1+y^{2}+x+x y^{2}+x^{2}+x^{2} y^{2}\right)=0,
$$

in $t$ for all $(x, y) \in\{ \pm 1\}^{2}$, and check whether the corresponding solution $t_{x, y}$ satisfies $\left|t_{x, y}\right|=1 /|\mathcal{S}|=1 / 6$. Of the four possible points, we get only two minimal points: the expected point $\boldsymbol{\rho}=(1,1,1 / 6)$ along with the point $\boldsymbol{\sigma}=(1,-1,1 / 6)$.

Computing the terms in expansion (19) at these two minimal points - aided by the Sage implementation of [26] - gives the asymptotic contributions:

$$
\begin{aligned}
& c_{n}^{(\boldsymbol{\rho})}=6^{n}\left(\frac{\sqrt{6}}{\pi n}-\frac{17 \sqrt{6}}{16 \pi n^{2}}+\frac{605 \sqrt{6}}{512 \pi n^{3}}+O\left(1 / n^{4}\right)\right) \\
& c_{n}^{(\boldsymbol{\sigma})}=(-6)^{n}\left(\frac{\sqrt{6}}{4 \pi n^{2}}-\frac{33 \sqrt{6}}{64 \pi n^{3}}+O\left(1 / n^{4}\right)\right) .
\end{aligned}
$$

Thus, the counting sequence for the number of walks of length $n$ has the asymptotic expansion

$$
c_{n}=6^{n}\left(\frac{\sqrt{6}}{\pi n}-\frac{\sqrt{6}\left(17-4(-1)^{n}\right)}{16 \pi n^{2}}+\frac{\sqrt{6}\left(38720-16896(-1)^{n}\right)}{32768 \pi n^{3}}+O\left(1 / n^{4}\right)\right) .
$$

## 6. FROM DIAGONALS TO DIFFERENTIAL EQUATIONS

As seen in Corollary 2.4, the generating function $F(\mathbf{1}, t)$ will be D-finite for any highly symmetric model $\mathcal{S}$. Indeed, from the expression $F(\mathbf{1}, t)=\Delta G(\mathbf{z}, t) / H(\mathbf{z}, t)$ it is possible in principle to compute an annihilating linear differential equation of $F(\mathbf{1}, t)$ through the use of algorithms for creative telescoping. These algorithms, which are typically grouped into those that perform elimination in an Ore algebra - including the famous algorithm of Zeilberger [31] - and those which use an ansatz of undetermined coefficients, compute differential operators annihilating multivariate integrals and connect to diagonals of rational functions through the relations

$$
\begin{align*}
\frac{1}{2 \pi i} \int_{\Omega} \frac{B\left(z_{1}, z_{2} / z_{1}, z_{3}, \ldots, z_{d}, t\right)}{z_{2}} d z_{2} & =\Delta_{1,2} B(\mathbf{z}, t)  \tag{20}\\
\left(\frac{1}{2 \pi i}\right)^{d} \int_{T} \frac{B\left(z_{1}, z_{2} / z_{1}, z_{3} / z_{2}, \ldots, z_{d} / z_{d-1}, t / z_{d}\right)}{z_{1} z_{2} \cdots z_{d}} d \mathbf{z} & =\Delta B(\mathbf{z}, t), \tag{21}
\end{align*}
$$

where $B(\mathbf{z}, t)$ is analytic in a neighbourhood of the origin, $\Omega$ is an appropriate contour in $\mathbb{C}$ containing the origin, and $T$ is an appropriate torus in $\mathbb{C}^{d}$ containing the origin. The reader is directed to [21] and [6] for details on how these methods work and are implemented in modern computer algebra systems. In Table 2 we have computed annihilators for the four highly symmetric models in two dimensions using an ansatz method developed and implemented in Mathematica by Koutschan [21].

Given an annihilating linear differential operator of the univariate generating function $F(\mathbf{1}, t)$, one can easily compute a linear recurrence relation that the counting sequence $\left(c_{n}\right)$ must satisfy. The

| S | Annihilating DE |
| :---: | :---: |
| $\square$ | $\begin{array}{r} t^{2}(4 t-1)(4 t+1) D_{t}^{3}+2 t(4 t+1)(16 t-3) D_{t}^{2} \\ +\left(-6+28 t+224 t^{2}\right) D_{t}+(12+64 t) \end{array}$ |
| X | $\begin{array}{r} t^{2}(4 t+1)(4 t-1)^{2} D_{t}^{3}+t(4 t-1)\left(112 t^{2}-5\right) D_{t}^{2} \\ +4(8 t-1)\left(20 t^{2}-3 t-1\right) D_{t}+\left(-4-48 t+128 t^{2}\right) \end{array}$ |
| $\nVdash$ | $\begin{array}{r} t^{2}(6 t-1)(6 t+1)(2 t+1)(2 t-1)\left(12 t^{2}-1\right) D_{t}^{3} \\ +t(2 t-1)\left(6048 t^{5}+2736 t^{4}-672 t^{3}-336 t^{2}+6 t+5\right) D_{t}^{2} \\ +\left(-4+16 t+516 t^{2}+96 t^{3}-5520 t^{4}-2304 t^{5}+17280 t^{6}\right) D_{t} \\ +\left(8+132 t+96 t^{2}-1104 t^{3}-1152 t^{4}+3456 t^{5}\right) \end{array}$ |
| * | $\begin{array}{r} -t^{2}(4 t+1)(8 t-1)(2 t-1)(t+1) D_{t}^{3} \\ +t\left(-5+33 t+252 t^{2}-200 t^{3}-576 t^{4}\right) D_{t}^{2} \\ +\left(-4+48 t+468 t^{2}-88 t^{3}-1152 t^{4}\right) D_{t} \\ +\left(12+144 t+72 t^{2}-384 t^{3}\right) \end{array}$ |

Table 2. Annihilating differential equations for the highly symmetric quarter plane models.

| Dimension $d$ | Annihilating DE |
| :---: | ---: |
| 3 | $-t^{3}(2 t-1)(2 t+1)(6 t-1)(6 t+1) D_{t}^{4}$ |
|  | $-4 t^{2}\left(576 t^{4}+36 t^{3}-140 t^{2}-5 t+3\right) D_{t}^{3}$ |
|  | $-4 t\left(2592 t^{4}+324 t^{3}-531 t^{2}-40 t+9\right) D_{t}^{2}$ |
|  | $-8\left(1728 t^{4}+324 t^{3}-282 t^{2}-34 t+3\right) D_{t}$ |
|  | $-24\left(144 t^{3}+36 t^{2}-17 t-3\right)$ |
| 4 | $-t^{4}(4 t-1)(4 t+1)(8 t-1)(8 t+1) D_{t}^{5}$ |
|  | $-4 t^{3}(4 t+1)\left(1536 t^{3}-320 t^{2}-30 t+5\right) D_{t}^{4}$ |
|  | $-4 t^{2}\left(47104 t^{4}+3968 t^{3}-2976 t^{2}-145 t+30\right) D_{t}^{3}$ |
|  | $-12 t\left(45056 t^{4}+5760 t^{3}-2368 t^{2}-191 t+20\right) D_{t}^{2}$ |
|  | $-24\left(21504 t^{4}+3712 t^{3}-848 t^{2}-106 t+5\right) D_{t}$ |
|  | $-96\left(1024 t^{3}+224 t^{2}-24 t-5\right)$ |

TABLE 3. Annihilating differential equations for the models $\left\{e_{1},-e_{1}, \ldots, e_{d},-e_{d}\right\}$.

Birkhoff-Trjitzinsky method (see [29] and [13]) can then be used to determine a basis of solutions to this recurrence. Each element of the basis has dominant asymptotic growth of the form

$$
c_{n}^{(k)} \sim C_{k} \rho^{n} n^{\beta_{k}}(\log n)^{l_{k}},
$$

for computable constants $C_{k}, \rho, \beta_{k}, l_{k}$. Using this technique to approach an asymptotic analysis for lattice walks in restricted regions has been used previously - for instance in the work of Bostan and Kauers [5] on two dimensional lattice walks confined to the positive quadrant - however it is not apparent how the number of walks in a model, $c_{n}$, is represented as a linear combination of the basis elements $c_{n}^{(k)}$. Determining this linear combination is known in the literature as the connection problem, as it describes how the generating function is connected to a local basis of singular solutions. This highlights a severe drawback to using the differential equation for asymptotics, when compared to the methods of this section: there is no known effective procedure to solve the
connection problem in general, even when the coefficients of the differential equation are known to be rational functions (the connection problem is believed by some to be uncomputable [13]). In essence, this implies that the multiplicative growth constant of the dominant asymptotic term cannot be determined rigorously in general (Bostan and Kauers used numerical approximations to non-rigorously solve the connection problem for their work on two dimensional models).

Example 9. As seen in Table 2, the step set univariate generating function $\sum c_{n} t^{n}$ of the quarterplane model $\{N, S, N E, S E, N W, S W\}=\mathbb{Z}$ is annihilated by the differential operator

$$
\begin{aligned}
\mathcal{L} & =\left(-t^{2}+52 t^{4}-624 t^{6}+1728 t^{8}\right) D_{t}^{3}+\left(-5 t+4 t^{2}+348 t^{3}-4080 t^{5}-576 t^{6}+12096 t^{7}\right) D_{t}^{2} \\
& +\left(-4+16 t+516 t^{2}+96 t^{3}-5520 t^{4}-2304 t^{5}+17280 t^{6}\right) D_{t} \\
& +\left(8+132 t+96 t^{2}-1104 t^{3}-1152 t^{4}+3456 t^{5}\right)
\end{aligned}
$$

which implies that the sequence $\left(c_{n}\right)$ satisfies the following linear recurrence relation with polynomial coefficients

$$
\begin{aligned}
0 & =\left(-n^{3}-20 n^{2}-133 n-294\right) c_{n+6}+\left(4 n^{2}+52 n+168\right) c_{n+5}+\left(52 n^{3}+816 n^{2}+4304 n+7620\right) c_{n+4} \\
& +(96 n+384) c_{n+3}+\left(-624 n^{3}-5952 n^{2}-19008 n-20304\right) c_{n+2}+\left(-576 n^{2}-2880 n-3456\right) c_{n+1} \\
& +\left(1728 n^{3}+6912 n^{2}+8640 n+3456\right) c_{n} .
\end{aligned}
$$

Using the Birkhoff-Trjitzinsky method one computes a basis of local solutions at infinity to this degree six linear recurrence relation (the basis given here was computed using the Sage package of [20]):
$c_{n}^{(1)}=\frac{6^{n}}{n}\left(1-\frac{17}{16} n^{-1}+\frac{605}{512} n^{-2}+O\left(n^{-3}\right)\right) \quad c_{n}^{(2)}=\frac{6^{n}}{n^{2}}\left(1-\frac{33}{16} n^{-1}+\frac{1565}{512} n^{-2}+O\left(n^{-3}\right)\right)$
$c_{n}^{(3)}=\frac{(2 \sqrt{3})^{n}}{n^{4}}\left(1-\frac{14+3 \sqrt{3}}{2} n^{-1}+O\left(n^{-2}\right)\right) \quad c_{n}^{(4)}=\frac{(-2 \sqrt{3})^{n}}{n^{4}}\left(1-\frac{14-3 \sqrt{3}}{2} n^{-1}+O\left(n^{-2}\right)\right)$
$c_{n}^{(5)}=\frac{2^{n}}{n^{3}}\left(1-\frac{51}{16} n^{-1}+\frac{3341}{512} n^{-2}+O\left(n^{-3}\right)\right) \quad c_{n}^{(6)}=\frac{(-2)^{n}}{n^{2}}\left(1-\frac{35}{16} n^{-1}+\frac{1805}{512} n^{-2}+O\left(n^{-3}\right)\right)$,
so that $c_{n}=O\left(6^{n} / n\right)$. Note that the results of Example 8 imply

$$
c_{n}=\frac{\sqrt{6}}{\pi} c_{n}^{(1)}+\frac{\sqrt{6}}{4 \pi} c_{n}^{(2)}+O\left((2 \sqrt{3})^{n}\right),
$$

and we can partially resolve the connection problem, however this is only possible because leading term asymptotics for $c_{n}$ were already calculated through the techniques of Pemantle, Raichev, and Wilson.

Although differential operators are very useful data structures for the D-finite functions which they annihilate, the work above illustrates that the representation of $F(\mathbf{1}, t)$ as a rational diagonal can yield easier access to its asymptotic information when coupled with the results of analytic combinatorics in several variables (at least in the smooth case). Furthermore, the combinatorial properties of lattice path models often naturally give representations of their generating functions as rational diagonals, and determining annihilating differential operators for these diagonals can be difficult. Creative telescoping methods - although always improving (see, for example, [6]) - do not scale well with degree and must be calculated on a model by model basis.

## 7. Walks in a Weyl Chamber

In 1992, Gessel and Zeilberger [15] outlined an extension of the reflection principle - originally used by André [2] in the nineteenth century to solve the two candidate ballot problem - to lattice walks on regions preserved under the actions of Coxeter-Weyl finite reflection groups. In this section we show how the highly symmetric walks can be viewed in this context. In addition to giving an alternative view of the calculations presented through the kernel method in Section 2, this view also allows us to determine diagonal expressions for the excursion generating function and permits a seguë to a discussion of how other, non-highly symmetric models, fit into this template.
7.1. Weyl Chambers and Reflectable Walks. The following definitions are taken from Gessel and Zeilberger [15], Grabiner and Magyar [17], and Humphreys [18], and the reader is directed to these manuscripts for more details.

A (reduced) root system is a finite set of vectors $\Phi \subset \mathbb{R}^{n}$ such that

- for any $x, y \in \Phi$, the set $\Phi$ contains the reflection of $y$ through the hyperplane with normal $x$

$$
\sigma_{x}(y)=y-2 \frac{(x, y)}{(x, x)} x
$$

- for any $x, y \in \Phi, x-\sigma_{y}(x)$ is an integer multiple of $y$;
- the only scalar multiples of $x \in \Phi$ to be in $\Phi$ are $x$ and $-x$.

The set of linear transformations generated by the reflections $\sigma_{x}$ is always a finite Coxeter group and is called the Weyl group $W$ of the root system. The complement of the union of the hyperplanes whose normals are the root system is an open set, and a connected component of this open set is called a Weyl chamber. For the root system $\Phi$, a set of positive roots $\Phi^{+}$is a subset of $\Phi$ such that
(1) for each $x \in \Phi$ exactly one of $x$ and $-x$ is in $\Phi^{+}$;
(2) for any two distinct $\alpha, \beta \in \Phi$ such that $\alpha+\beta$ is a root, $\alpha+\beta \in \Phi^{+}$.

An element of $\Phi^{+}$is called a simple root if it cannot be written as a sum of two elements of $\Phi^{+}$, and a maximal set $\Delta$ of simple roots is called a basis for the root system. It can be shown that for a basis $\Delta$ any $x \in \Phi$ is a linear combination of members of $\Delta$ with all non-negative or non-positive coefficients, and that the set $\left\{\sigma_{x}: x \in \Delta\right\}$ generates the Weyl group $W$.

Fix a root system $\Phi$ and a basis $\Delta$, and let

- $\mathcal{S} \subset \mathbb{Z}^{n}$ be a set of steps such that $W \cdot \mathcal{S}=\mathcal{S}$ - i.e., $\mathcal{S}$ is preserved under each element of the Weyl group;
- $L$ be a lattice, restricted to the linear span of elements of $\mathcal{S}$, such that $W \cdot L=L$;
- $C$ be the Weyl chamber

$$
C=\left\{\mathbf{z} \in \mathbb{R}^{n}:(\alpha, \mathbf{z})>0 \text { for all } \alpha \in \Delta\right\} .
$$

The lattice path model in the Weyl chamber $C$ using the steps $\mathcal{S}$ beginning at a point a $\in C$ is the combinatorial class of all sequences of steps in $\mathcal{S}$ beginning at a and never leaving $C$ (when viewed as a walk on $L$ in the typical manner). If, in addition to the requirements above, the two conditions
(1) For all $\alpha \in \Delta$ and $s \in \mathcal{S},(\alpha, s)= \pm k(\alpha)$ or 0 , where $k(\alpha)$ is a constant depending only on $\alpha$;
(2) For all $\alpha \in \Delta$ and $\lambda \in L,(\alpha, \lambda)$ is an integer multiple of $k(\alpha)$ depending only on $\alpha$; are met, we say that the lattice path model is reflectable, and any step $s \in S$ taken from any lattice point inside $C$ will not leave $C$ except possibly to land on its boundary (one of the hyperplanes whose normals are the elements of $\Phi$ ).

The main result of Gessel and Zeilberger [15], after a conversion from constant term extraction to diagonal extraction, is the following.

Theorem 7.1 (Gessel and Zeilberger [15]). Given a reflectable walk as defined above such that $(a, \alpha)$ is an integer multiple of $k(\alpha)$ for each $\alpha \in \Delta$, and an element $b \in C$ such that $(b, \alpha)$ is also an integer multiple of $k(\alpha)$ for each $\alpha \in \Delta$, the generating function for the number of walks which begin at $a$, end at $b$, and stay in $C$ is

$$
\begin{equation*}
F_{a \rightarrow b}(t)=\Delta\left[\frac{1}{1-t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})} \cdot \mathbf{z}^{-\mathbf{b}} \cdot \sum_{w \in W}(-1)^{l(w)} \mathbf{z}^{w(\mathbf{a})}\right], \tag{22}
\end{equation*}
$$

where $l(w)$ is the minimal length of $w$ represented as a product of elements in $\left\{\sigma_{x}: x \in \Delta\right\}$.
If $(b, \alpha)$ is an integer multiple of $k(\alpha)$ for each $\alpha \in \Delta$ and $b \in C$, and the formal power series $\sum_{\mathbf{b} \in C} \mathbf{z}^{-\mathbf{b}}$ exists (see [3] for a discussion on the existence of multivariate Laurent series) then summing Equation (22) over all possible endpoints implies that the generating function for the number of walks beginning at $a$ and staying in $C$ which are allowed to end anywhere is

$$
\begin{equation*}
F_{a}(t)=\Delta\left[\frac{1}{1-t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})} \cdot \sum_{\mathbf{b} \in C} \mathbf{z}^{-\mathbf{b}} \cdot \sum_{w \in W}(-1)^{l(w)} \mathbf{z}^{w(\mathbf{a})}\right] . \tag{23}
\end{equation*}
$$

7.2. Classification of Weyl chambers and reflectable walks. Given two root systems $\Phi_{1} \subset \mathbb{R}^{n}$ and $\Phi_{2} \subset \mathbb{R}^{m}$, one can create a new root system $\Phi_{1} \times \Phi_{2}$ by treating the two vector spaces spanned by the elements of $\Phi_{1}$ and $\Phi_{2}$ as mutually orthogonal subspaces of $\mathbb{R}^{n+m}$. To this end, a root system $\Phi$ is called reducible if it can be decomposed as $\Phi=\Phi_{1} \cup \Phi_{2}$, where $\Phi_{1}$ and $\Phi_{2}$ are root systems whose elements are pairwise orthogonal, and irreducible otherwise.

One of the main results in the study of root systems - which arises in relation to Lie algebras and representation theory - is a complete classification of the irreducible root systems, consisting of four infinite families $\left(A_{n}\right.$ for $n \geq 1, B_{n}$ for $n \geq 2, C_{n}$ for $n \geq 3$, and $D_{n}$ for $\left.n \geq 4\right)$ and five exceptional cases $\left(E_{6}, E_{7}, E_{8}, F_{4}\right.$, and $\left.G_{2}\right)$. The interested reader is directed to Section 11.4 of Humphreys [18] for details and a proof of the classification.

Example 10. There is, up to scaling by a constant, one root system in $\mathbb{R}$ : the system $\Phi_{1}=\{ \pm 1\}$ with basis $\Delta_{1}=\{1\}$, which is called $A_{1}$. From this, the root system $A_{1} \times A_{1}=A_{1}^{2} \subset \mathbb{R}^{2}$ is defined as the direct sum of two copies of $A_{1}$, giving elements $\Phi_{2}=\left\{ \pm e_{1}, \pm e_{2}\right\}$ and basis $\Delta_{2}=\left\{e_{1}, e_{2}\right\}$. In general, for any $d \in \mathbb{N}$ the root system $A_{1}^{d}$ will be the system with elements $\Phi=\left\{ \pm e_{1}, \ldots, \pm e_{d}\right\}$, which admits the basis $\Delta=\left\{e_{1}, \ldots, e_{d}\right\}$.
7.3. Highly symmetric walks are walks in Weyl chambers. The root system $A_{1}^{d}$, described in Example 10, has corresponding Weyl chamber

$$
C=\left\{\mathbf{z}: z_{1}>0 \text { and } z_{2}>0 \text { and } \cdots \text { and } z_{d}>0\right\}=\left(\mathbb{Z}_{>0}\right)^{d}
$$

and it follows directly from the definitions above that a step set $\mathcal{S} \subset \mathbb{Z}^{d}$ is a reflectable walk with respect to $\Delta$ if and only if it is highly symmetric. As $C$ does not include the the hyper-planes $\left\{z_{1}=0\right\}, \cdots,\left\{z_{d}=0\right\}$, we shift the origin of the walks under consideration by starting them at the point $\mathbf{a}=1$. The Weyl group $W$ corresponding to this set of roots is isomorphic to $\mathbb{Z}_{2}^{d}$ (in fact, it is equal to the group $\mathcal{G}$ as defined in Section 2.2) and

$$
\begin{aligned}
\sum_{\mathbf{b} \in C} \mathbf{z}^{-\mathbf{b}} & =\frac{1}{z_{1}-1} \cdots \frac{1}{z_{d}-1} \\
\sum_{w \in W}(-1)^{l(w)} \mathbf{z}^{w(\mathbf{a})} & =\left(z_{1}-\bar{z}_{1}\right) \cdots\left(z_{d}-\bar{z}_{d}\right) .
\end{aligned}
$$

Substitution into Equation 23 recovers Equation (9), shifted by a factor of $\left(z_{1} \cdots z_{d}\right)$ to account for the shifted walk origin a:

$$
F_{\mathbf{1}}(t)=\Delta\left[\frac{\left(1+z_{1}\right) \cdots\left(1+z_{d}\right)}{1-t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})} \cdot\left(z_{1} \cdots z_{d}\right)\right] .
$$

We note that the argument presented in Section 2 - which is a standard generalization of the kernel method - mirrors the proof of Theorem 7.1 given by Gessel and Zeilberger.
7.4. Excursions. Not only can we recover previous results, but we can now give asymptotics for the number of walks which return to the origin. Taking $\mathbf{a}=\mathbf{b}=\mathbf{1}$ in Equation 22, we see that the number of excursions $e_{n}$ is given by

$$
\begin{aligned}
e_{n} & =\left[t^{n}\right] \Delta\left(\frac{\left(z_{1}-\bar{z}_{1}\right) \cdots\left(z_{d}-\bar{z}_{d}\right)}{1-t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})} \cdot\left(z_{1} \cdots z_{d}\right)^{-1}\right) \\
& =\left[t^{n}\right] \Delta\left(\frac{t^{2}\left(z_{1}^{2}-1\right) \cdots\left(z_{d}^{2}-1\right)}{1-t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})} \cdot\left(t z_{1} \cdots z_{d}\right)^{-2}\right) \\
& =\left[t^{n+2}\right] \Delta\left(\frac{t^{2}\left(z_{1}^{2}-1\right) \cdots\left(z_{d}^{2}-1\right)}{1-t\left(z_{1} \cdots z_{d}\right) S(\mathbf{z})}\right) .
\end{aligned}
$$

Note that the form of the final rational function on the right hand side implies that the same minimal points will appear in the analysis of excursion asympotics - however, due to the factors of $\left(z_{1}-1\right) \cdots\left(z_{d}-1\right)$ now present in the numerator the finitely minimal point $\rho=(1, \ldots, 1,1 /|S|)$ will vanish, bringing down the polynomial growth factor of excursions compared to the asymptotics of walks ending anywhere. Furthermore, as more than one minimal point can now determine the dominant asymptotics closed form results are not easily obtainable. Despite that, as the minimal points are still classified by Proposition 3.2, one can use the machinary available to calculate lower terms in asymptotic expansions (as in Section 5) to determine the asymptotics of specific models.

Example 11. Consider the highly symmetric 2D step set $\{N, S, N E, S E, N W, S W\}=\mathbb{X}$. Here we have

$$
e_{n}=\left[t^{n+2}\right] \Delta\left(\frac{t^{2}\left(x^{2}-1\right)\left(y^{2}-1\right)}{1-\left(t x^{2} y^{2}+t y^{2}+t x^{2}+t+t x y^{2}+t x\right)}\right),
$$

and as discussed in Example 8 this rational function has the expected minimal point $\boldsymbol{\rho}=(1,1,1 / 6)$ along with the point $\boldsymbol{\sigma}=(1,-1,1 / 6)$. Computing the terms in expansion (19) at these two minimal points - again aided by the Sage implementation of [26] - gives the asymptotic contributions (after properly shifting index):

$$
e_{n}^{(\boldsymbol{\rho})}=6^{n}\left(\frac{3 \sqrt{6}}{2 \pi n^{3}}+O\left(1 / n^{4}\right)\right) \quad e_{n}^{(\boldsymbol{\sigma})}=(-6)^{n}\left(\frac{3 \sqrt{6}}{2 \pi n^{3}}+O\left(1 / n^{4}\right)\right)
$$

Thus, the counting sequence for the number of excursions of length $n$ has the asymptotic expansion

$$
e_{n}=6^{n}\left(\frac{3 \sqrt{6}}{2 \pi n^{3}}\left(1+(-1)^{n}\right)+O\left(1 / n^{4}\right)\right)
$$

where we note that there are no excursions of odd length.
As the denominator of the rational function under consideration is smooth, and the numerator $t^{2}\left(z_{1}^{2}-1\right) \cdots\left(z_{d}^{2}-1\right)$ vanishes at any minimal point to order $d$, the asymptotic expansion given in Equation (19) implies the following.

Theorem 7.2. Let $\mathcal{S} \subseteq\{-1,0,1\}^{d} \backslash\{\mathbf{0}\}$ be a set of unit steps in dimension d. If $\mathcal{S}$ is symmetric with respect to each axis, and $\mathcal{S}$ takes a positive step in each direction, then the number of walks $e_{n}$ of length $n$ taking steps in $\mathcal{S}$, beginning and ending at the origin, and never leaving the positive orthant satisfies

$$
e_{n}=O\left(\frac{|\mathcal{S}|^{n}}{n^{3 d / 2}}\right) .
$$

## 8. Conclusion

The purpose of this article, aside from the specific combinatorial results it contains, is to reinforce the notion that there are many possibilities for studying lattice walks in restricted regions through the use of diagonals and analytic combinatorics in several variables: in this context the diagonal data structure often permits analysis in general dimension. Furthermore, walks with symmetry across each axis all have a smooth singular variety, making them the perfect entry point to this confluence of the kernel method, the reflection principle and analytic combinatorics of several variables.
8.1. Generalizations: Other Weyl Chambers. A major goal moving forward is to deal with more general step set models. As a first attempt, we have also considered reflectable walks in $A_{2}$, and some other related models, and this gives nice diagonal expressions for the generating functions for the models with group order 6 in the classification of [9]. However, the expressions are far more difficult to analyze with these asymptotic techniques, since the expressions no longer fall in the simplest, smooth case.

More generally, Grabiner and Magyar [17] have classified, for each irreducible root system $\Phi$, the step sets which give rise to a reflectible lattice path model in the corresponding Weyl chamber. This combinatorial classification gives a large collection of future objects to study through the means of analytic combinatorics in several variables. Assuming one can get the generating function for the number of walks in a more general setting as a rational diagonal, results on asymptotics can be reduced to an analysis of this rational function. Both [25] and [28] give results for singular varieties which are non-smooth, but whose critical points are multiple points. Due to the constraints on the rational functions arising from the combinatorial nature of lattice paths in restricted regions, there is hope for a completely systematic treatment which allows for some non symmetries.

This leads to the natural question, can the infamous Gessel walks be expressed as walk in a Weyl Chamber? A positive answer could result in a far simpler path to a generating function expression than those presently known, even the methods explicitly derived by humans [1], and a negative answer might help explain why it has resisted simpler approaches.

Furthermore, the asymptotic enumeration of excursions has received much attention lately, due to the recent work of Denisov and Wachtel. It could be interesting to link their work to expressions using diagonals in the case of D-finite models. The results of [7] suggest very compelling evidence that the boundary between D-finite models and non-D-finite models leaves strong traces in the asymptotic enumeration.
8.2. Are all D-finite models diagonals? Across the study of lattice path models to date, it has been true that every model with a D-finite generating function is accompanied by an expression of the generating function as a diagonal of a rational function (or equivalent). A conjecture of Christol [11] posits that any globally bounded D-finite function (which includes power series convergent at the origin with integer coefficients) can be written as the diagonal of a multivariate rational function. Could one prove a lattice path version of this conjecture? More practically, could such a result be made effective with an automatic method of writing known D-finite functions as diagonals?

Finally, it would be interesting to understand if there is a direct combinatorial interpretation for the diagonal operator acting of rational functions. Recent work of Garrabrant and Pak [14] gives a
tiling interpretation of diagonals of $\mathbb{N}$-rational functions. Our rationals here are very combinatorial, although they have some negative coefficients. Very possibly a signed version of their construction might capture the diagonals that we build.

## 9. Acknowledgments

The authors would like to thank Manuel Kauers for the construction in Proposition 2.5, and illuminating discussions on diagonals of generating functions, and the anonymous referees of an extended abstract of this work for their comments and suggestions. We are also grateful to Mireille Bousquet-Mélou for pointing out some key references.

## References

[1] A.Bostan, I. Kurkova, and K. Raschel. A human proof of Gessel's lattice path conjecture. http://arxiv.org/abs/1309.1023.
[2] D. André. Solution directe du problème résolu par M. Bertrand. C. R. Acad. Sci., Paris(105):436-437, 1887.
[3] A. Aparicio-Monforte and M. Kauers. Formal Laurent series in several variables. Expo. Math., 31(4):350-367, 2013.
[4] A. Bostan, M. Bousquet-Mélou, M. Kauers, and S. Melczer. On lattice walks confined to the positive octant. 2014. Submitted, http://arxiv.org/abs/1409.3669.
[5] A. Bostan and M. Kauers. Automatic classification of restricted lattice walks. In Proceedings of FPSAC 2009, Discrete Math. Theor. Comput. Sci. Proc., AK, pages 201-215, 2009.
[6] A. Bostan, P. Lairez, and B. Salvy. Creative telescoping for rational functions using the griffiths-dwork method. In Proceedings of the International Symposium on Symbolic and Algebraic Computation (ISSAC), New York, NY, USA. ACM., pages 93-100, 2013.
[7] A. Bostan, K. Raschel, and B. Salvy. Non-d-finite excursions in the quarter plane. J. Comb. Theory, Ser. A, 121(0):45-63, 2014.
[8] M. Bousquet-Mélou. Walks in the quarter plane: Kreweras' algebraic model. Ann. Appl. Probab., 15(2):14511491, 2005.
[9] M. Bousquet-Mélou and M. Mishna. Walks with small steps in the quarter plane. In Algorithmic Probability and Combinatorics, volume 520 of Contemp. Math., pages 1-40. Amer. Math. Soc., 2010.
[10] M. Bousquet-Mélou and M. Petkovšek. Walks confined in a quadrant are not always D-finite. Theoret. Comput. Sci., 307(2):257-276, 2003. Random generation of combinatorial objects and bijective combinatorics.
[11] G. Christol. Globally bounded solutions of differential equations. In Analytic number theory (Tokyo, 1988), volume 1434 of Lecture Notes in Math., pages 45-64. Springer Berlin Heidelberg, 1990.
[12] G. Fayolle, R. Iasnogorodski, and V. Malyshev. Random walks in the quarter-plane, volume 40 of Applications of Mathematics (New York). Springer-Verlag, Berlin, 1999. Algebraic methods, boundary value problems and applications.
[13] P. Flajolet and R. Sedgewick. Analytic Combinatorics. Cambridge University Press, 2009.
[14] S. Garrabrant and I. Pak. Counting with irrational tiles. http://arxiv.org/abs/1407.8222.
[15] I. M. Gessel and D. Zeilberger. Random walk in a Weyl chamber. Proc. Amer. Math. Soc., 115(1):27-31, 1992.
[16] D. J. Grabiner. A combinatorial correspondence for walks in Weyl chambers. J. Combin. Theory Ser. A, 71(2):275-292, 1995.
[17] D. J. Grabiner and P. Magyar. Random walks in Weyl chambers and the decomposition of tensor powers. $J$. Algebraic Combin., 2(3):239-260, 1993.
[18] J. E. Humphreys. Introduction to Lie algebras and representation theory. Springer-Verlag, New York-Berlin, 1972. Graduate Texts in Mathematics, Vol. 9.
[19] E. J. Janse van Rensburg, T. Prellberg, and A. Rechnitzer. Partially directed paths in a wedge. J. Combin. Theory Ser. A, 115(4):623-650, 2008.
[20] M. Kauers, M. Jaroschek, and F. Johansson. Ore polynomials in sage. Lecture Notes in Computer Science, Computer Algebra and Polynomials, to appear.
[21] C. Koutschan. A fast approach to creative telescoping. Math. Comput. Sci., 4(2-3):259-266, 2010.
[22] L. Lipshitz. D-finite power series. J. of Algebra, 122(2):353-373, 1989.
[23] S. Melczer and M. Mishna. Singularity analysis via the iterated kernel method. Combinatorics, Probability \& Computing, 23:861-888, 2014.
[24] R. Pemantle and M.C. Wilson. Asymptotics of multivariate sequences: I. smooth points of the singular variety. J. Comb. Theory, Ser. A, 97(1):129-161, 2002.
[25] R. Pemantle and M.C. Wilson. Analytic Combinatorics in Several Variables. Cambridge University Press, 2013.
[26] A. Raichev. amgf documentation - release 0.8. https://github.com/araichev/amgf, 2012.
[27] A. Raichev and M.C. Wilson. Asymptotics of coefficients of multivariate generating functions: Improvements for smooth points. Electr. J. Comb., 15(1), 2008.
[28] A. Raichev and M.C. Wilson. Asymptotics of coefficients of multivariate generating functions: improvements for multiple points. Online Journal of Analytic Combinatorics, 6(0), 2011.
[29] J. Wimp and D. Zeilberger. Resurrecting the asymptotics of linear recurrences. J. Math. Anal. Appl., 111(1):162176, 1985.
[30] G. Xin. Determinant formulas relating to tableaux of bounded height. Adv. in Appl. Math., 45(2):197-211, 2010.
[31] D. Zeilberger. A holonomic systems approach to special functions identities. J. Comput. Appl. Math., 32(3):321368, 1990.
(S. Melczer) Cheriton School of Computer Science, University of Waterloo, Waterloo On Canada
\& U. Lyon, CNRS, ENS de Lyon, Inria, UCBL, Laboratoire LiP
E-mail address: smelczer@uwaterloo.ca
(M. Mishna) Department of Mathematics, Simon Fraser University, Burnaby BC, Canada, V5A 1S6

E-mail address: mmishna@sfu.ca


[^0]:    Key words and phrases. Lattice path enumeration, D-finite, diagonal, analytic combinatorics in several variables, Weyl chambers.
    ${ }^{1}$ A function is D-finite if it satisfies a linear differential equation with polynomial coefficients

