On the Parameterized Complexity of Reconfiguration Problems

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Abstract. We present the first results on the parameterized complexity of reconfiguration problems, where a reconfiguration version of an optimization problem Q takes as input two feasible solutions S and T and determines if there is a sequence of *reconfiguration steps* that can be applied to transform S into T such that each step results in a feasible solution to Q. For most of the results in this paper, S and T are subsets of vertices of a given graph and a reconfiguration step adds or deletes a vertex. Our study is motivated by recent results establishing that for most NP-hard problems, the classical complexity of reconfiguration is PSPACE-complete.

We address the question for several important graph properties under two natural parameterizations: k, the size of the solutions, and ℓ, the length of the sequence of steps. Our first general result is an algorithmic paradigm, the *reconfiguration kernel*, used to obtain fixed-parameter algorithms for the reconfiguration versions of Vertex Cover and, more generally, Bounded Hitting Set and Feedback Vertex Set, all parameterized by k. In contrast, we show that reconfiguring Unbounded Hitting Set is $W[2]$-hard when parameterized by $k+\ell$. We also demonstrate the $W[1]$-hardness of the reconfiguration versions of a large class of maximization problems parameterized by $k+\ell$, and of their corresponding deletion problems parameterized by ℓ; in doing so, we show that there exist problems in FPT when parameterized by k, but whose reconfiguration versions are $W[1]$-hard when parameterized by $k+\ell$.

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1 Introduction

The reconfiguration version of an optimization problem asks whether it is possible to transform a source feasible solution S into a target feasible solution T by a sequence of *reconfiguration steps* such that every intermediate solution is also feasible; other variants return a (possibly minimum-length) *reconfiguration sequence* of solutions. Reconfiguration problems model real-life dynamic situations in which we seek to transform a solution into a more desirable one, maintaining feasibility during the process. The study of reconfiguration yields insights into the structure of the solution space of the underlying optimization problem, crucial for the design of efficient algorithms.

Motivated by real world situations as well as by trying to understand the structure of all feasible solutions, there has been a lot of recent interest in studying the complexity of reconfiguration problems. Problems for which reconfiguration has been studied include Vertex Colouring [3, 5–8], List Edge-Colouring [18], Independent Set [16, 17], Clique, Set Cover, Matching, Matroid Bases [17], Satisfiability [13], Shortest Path [4, 19], and Dominating Set [15, 26]. Most work has been limited to the problem of determining the existence of a reconfiguration sequence between two given solutions; for most NP-complete problems, this problem has been shown to be PSPACE-complete.

As there are typically exponentially many feasible solutions, the length of the reconfiguration sequence can be exponential in the size of the input instance. It is thus natural to ask whether reconfiguration problems become tractable if we allow the running time to depend on the length of the sequence; this approach suggests the use of the paradigm of parameterized complexity. In this work, we explore reconfiguration in the framework of parameterized complexity [10] under two natural parameterizations: k, a bound on the size of feasible solutions, and ℓ, the length of the reconfiguration sequence. One of our key results is that for most problems, the reconfiguration versions remain intractable in the parameterized framework when we parameterize by ℓ. It is important to note that when k is not bounded, the reconfiguration we study become easy.

We present fixed-parameter algorithms for problems parameterized by k by modifying known parameterized algorithms for the problems. The paradigms of bounded search tree and kernelization typically work by exploring minimal solutions. However, a reconfiguration sequence may necessarily include non-minimal solutions. Any kernel that removes solutions (non-minimal or otherwise) may render finding a reconfiguration sequence impossible, as the missing solutions might appear in every reconfiguration sequence; we must thus ensure that the kernelization rules applied retain enough information to allow us to determine whether a reconfiguration sequence exists. To handle these difficulties, we introduce a general approach for parameterized reconfiguration problems. We use a *reconfiguration kernel*, showing how to adapt Bodlaender's cubic kernel [2] for Feedback Vertex Set, and a special kernel by Damaschke and Molokov [9] for Bounded Hitting Set (where the cardinality of each input set is bounded) to obtain polynomial reconfiguration kernels, with respect to k. These results can be considered as interesting applications of kernelization, and a general approach for other similar reconfiguration problems.

As a counterpart to our result for Bounded Hitting Set, we show that reconfiguring Unbounded Hitting Set or Dominating Set is $W[2]$-hard parameterized by $k+\ell$ (Section 4). Finally, we show a general result on reconfiguration problems of hereditary properties and their 'parametric duals', implying the $W[1]$-hardness of reconfiguring Independent Set, Induced Forest, and Bipartite Subgraph parameterized by $k+\ell$ and Vertex Cover, Feedback Vertex Set, and Odd Cycle Transversal parameterized by ℓ.

2 Preliminaries

Unless otherwise stated, we assume that each input graph G is a simple, undirected graph on n vertices with vertex set $V(G)$ and edge set $E(G)$. To avoid confusion, we refer to *nodes* in reconfiguration graphs (defined below), as distinguished from *vertices* in the input graph. We use the modified big-Oh notation O^* that suppresses all polynomially bounded factors.

Our definitions are based on optimization problems, each consisting of a polynomial-time recognizable set of valid instances, a set of feasible solutions for each instance, and an objective function assigning a nonnegative rational value to each feasible solution.

Definition 1. *The* reconfiguration graph $R_Q(I, \mathrm{adj}, k)$*, consists of a node for each feasible solution to instance I of optimization problem Q, where the size of each solution is at least k for Q a maximization problem (of size at most k for Q a minimization problem, respectively), for positive integer k, and an edge between each pair of nodes corresponding to solutions in the binary adjacency relation* adj *on feasible solutions.*

We define the following *reconfiguration problems*, where S and T are feasible solutions for I: Q Reconfiguration determines if there is a path from S to T in $R_Q(I, \mathrm{adj}, k)$; the *search variant* returns a *reconfiguration sequence*, the sequence of feasible solutions associated with such a path; and the *shortest path variant* returns the reconfiguration sequence associated with a path of minimum length. For convenience, solutions paired by adj are said to be *adjacent*.

Using the framework developed by Downey and Fellows [10], a *parameterized reconfiguration problem* includes in the input a positive integer ℓ (an upper bound on the length of the reconfiguration sequence) and a parameter p (typically k or ℓ). For a parameterized problem Q with inputs of the form (x, p), $|x| = n$ and p a positive integer, Q is *fixed-parameter tractable* (or in *FPT*) if it can be decided in $f(p)n^c$ time, where f is an arbitrary function and c is a constant independent of both n and p. Q is in the class *XP* if it can be decided in $n^{f(p)}$ time. Q has a *kernel* of size $f(p)$ if there is an algorithm A that transforms the input (x, p) to (x', p') such that A runs in polynomial time (with respect to $|x|$ and p) and (x, p) is a yes-instance if and only if (x', p') is a yes-instance, $p' \leq g(p)$, and $|x'| \leq f(p)$. Each problem in *FPT* has a kernel, possibly of exponential (or worse) size.

We introduce the related notion of a *reconfiguration kernel*; it follows from the definition that a reconfiguration problem that has such a kernel is in FPT.

Definition 2. *A* reconfiguration kernel *of an instance of a parameterized reconfiguration problem* $(x, p) = (P, \mathrm{adj}, S, T, k, \ell, p)$ *is a set of* $h(p)$ *instances, for an arbitrary function* h*, such that for* $1 \leq i \leq h(p)$*:*

- *for each instance in the set,* $(x_i, p_i) = (P, \mathrm{adj}, S_i, T_i, k_i, \ell_i, p_i)$*, the values of* S_i*,* T_i*,* k_i*,* ℓ_i*, and* p_i *can all be computed in polynomial time,*
- *the size of each* x_i *is bounded by* $j(p)$*, for an arbitrary function* j*, and*
- (x, p) *is a yes-instance if and only if at least one* (x_i, p_i) *is a yes-instance.*

The main hierarchy of parameterized complexity classes is $FPT \subseteq W[1] \subseteq W[2] \subseteq \ldots \subseteq XP$, where W-hardness, shown using *FPT reductions*, is the analogue of NP-hardness in classical complexity. A parameterized problem Q *FPT reduces* to a parameterized problem Q' if there is an algorithm A that transforms an instance (I, p) of Q to an instance (I', p') of Q' such that A runs in time $f(p)poly(|I|)$ where f is a function of k, *poly* is a polynomial function, and $p' = g(p)$ for some function g. In addition, the transformation has the property that (I, p) is a yes-instance of Q if and only if (I', p') is a yes-instance of Q'. It is known that standard parameterized versions (are there p vertices that form the solution?) of Clique and Independent Set are complete for the class $W[1]$, and Dominating Set is $W[2]$-complete. The reader is referred to [12, 24] for more on parameterized complexity.

Most problems we consider can be defined using graph properties, where a *graph property* π is a collection of graphs, and is *non-trivial* if it is non-empty and does not contain all graphs. A graph property is *polynomially decidable* if for any graph G, it can be decided in polynomial time whether G is in π. For a subset $V' \subseteq V$, $G[V']$ is the *subgraph of* G *induced on* V', with vertex set V' and edge set $\{\{u, v\} \in E \mid u, v \in V'\}$. The property π is *hereditary* if for any $G \in \pi$, any induced subgraph of G is also in π. Examples of hereditary properties include graphs having no edges and graphs having no cycles. It is well-known [22] that every hereditary property π has a forbidden set $\mathcal{F}_\pi$, in that a graph has property π if and only if it does not contain any graph in $\mathcal{F}_\pi$ as an induced subgraph.

For a graph property π, we define two reconfiguration graphs, where solutions are sets of vertices and two solutions are adjacent if they differ by the addition or deletion of a vertex. The *subset reconfiguration graph of* G *with respect to* π, $R^{\pi}_{\text{SUB}}(G, k)$, has a node for each $S \subseteq V(G)$ such that $|S| \geq k$ and $G[S]$ has property π, and the *deletion reconfiguration graph of* G *with respect to* π, $R^{\pi}_{\text{DEL}}(G, k)$, has a node for each $S \subseteq V(G)$ such that $|S| \leq k$ and $G[V(G) \setminus S]$ has property π. We can obtain $R^{\pi}_{\text{DEL}}(G, |V(G)| - k)$ by replacing the set corresponding to each node in $R^{\pi}_{\text{SUB}}(G, k)$ by its (setwise) complement. The following is a consequence of the fact that two nodes can differ by the deletion or addition of a single vertex.

Fact 1 *The degree of each node in* $R^{\pi}_{\text{SUB}}(G, k)$ *and each node in* $R^{\pi}_{\text{DEL}}(G, k)$ *is at most* $|V(G)|$*.*

Definition 3. *For any graph property π, graph G, positive integer k, $S \subseteq V(G)$, and $T \subseteq V(G)$, we define the following decision problems:*
π-DELETION(G, k)*: Is there $V' \subseteq V(G)$ such that $|V'| \leq k$ and $G[V(G) \setminus V'] \in \pi$?*
π-SUBSET(G, k)*: Is there $V' \subseteq V(G)$ such that $|V'| \geq k$ and $G[V'] \in \pi$?*
π-DEL-RECONF(G, S, T, k, ℓ)*: For $S, T \in V(R^{\pi}_{\text{DEL}}(G, k))$, is there a path of length at most ℓ between the nodes for S and T in $R^{\pi}_{\text{DEL}}(G, k)$?*
π-SUB-RECONF(G, S, T, k, ℓ)*: For $S, T \in V(R^{\pi}_{\text{SUB}}(G, k))$, is there a path of length at most ℓ between the nodes for S and T in $R^{\pi}_{\text{SUB}}(G, k)$?*

We say that π-DELETION(G, k) and π-SUBSET(G, k) are *parametric duals* of each other. Note that in π-SUBSET(G, k), we seek a set of vertices of size *at least* k inducing a subgraph in π, whereas in π-DELETION(G, k), we seek a set of vertices of size *at most* k whose *complement set* induces a subgraph in π. We refer to π-DEL-RECONF(G, S, T, k, ℓ) and π-SUB-RECONF(G, S, T, k, ℓ) as *reconfiguration problems for* π; for example, for π the set of graphs with no edges, the former is VERTEX COVER RECONFIGURATION and the latter is INDEPENDENT SET RECONFIGURATION.

3 Fixed-Parameter Tractability Results

We first observe that for any polynomially decidable graph property, the π-DELETION and π-SUBSET reconfiguration versions are in XP when parameterized by ℓ; we conduct breadth-first search on the reconfiguration graph starting at S, stopping either upon discovery of T or upon completing the exploration of ℓ levels. Fact 1 implies a bound of at most n^{ℓ} vertices to explore in total.

Fact 2 *For any polynomially decidable graph property π, π-DEL-RECONF$(G, S, T, k, \ell) \in XP$ and π-SUB-RECONF$(G, S, T, k, \ell) \in XP$ when parameterized by ℓ.*

For an instance (G, S, T, k, ℓ), we partition $V(G)$ into the sets $C = S \cap T$ (vertices common to S and T), $S_D = S \setminus C$ (vertices to be deleted from S in the course of reconfiguration), $T_A = T \setminus C$ (vertices to be added to form T), and $O = V(G) \setminus (S \cup T) = V(G) \setminus (C \cup S_D \cup T_A)$ (all other vertices). Furthermore, we can partition C into two sets C_F and $C_M = C \setminus C_F$, where a vertex is in C_F if and only if it is in every feasible solution of size bounded by k.

The following fact is a consequence of the definitions above, the fact that π is hereditary, and the observations that $G[S_D]$ and $G[O]$ are both subgraphs of $G[V(G) \setminus T]$, and $G[T_A]$ and $G[O]$ are both subgraphs of $G[V(G) \setminus S]$.

Fact 3 *For an instance π-DEL-RECONF(G, S, T, k, ℓ) of a reconfiguration problem for hereditary property π, $G[O]$, $G[S_D]$, and $G[T_A]$ all have property π.*

In any reconfiguration sequence, each vertex in S_D must be deleted and each vertex in T_A must be added. We say that a reconfiguration sequence *touches* a vertex v if v is either added or deleted in at least one reconfiguration step. Any vertex that is not touched is *untouched*. In fact, since ℓ implies a bound on the total number of vertices that can be touched in a reconfiguration sequence, setting $\ell = |S_D| + |T_A|$ drastically simplifies the problem.

Observation 1 *For any polynomially decidable hereditary graph property π, if $|S_D| + |T_A| = \ell$, then π-*DEL-RECONF(G, S, T, k, ℓ) *and* π-SUB-RECONF(G, S, T, k, ℓ) *can be solved in $\mathcal{O}^*(2^\ell)$ time, and hence are in FPT when parameterized by ℓ.*

Proof. Since each vertex in T_A must be added and each vertex in S_D deleted, in ℓ steps we can touch each vertex in $S_D \cup T_A$ exactly once; all vertices in $V(G) \setminus (S_D \cup T_A)$ remain untouched.

Any node in the path between S and T in $R^\pi_{\text{SUB}}(G, k)$ represents a set $C \cup B$ where B is a subset of $S_D \cup T_A$. As $|S_D| + |T_A| = \ell$, there are only 2^ℓ choices for B. Our problem then reduces to finding the shortest path between S and T in the subgraph of $R^\pi_{\text{SUB}}(G, k)$ induced on the 2^ℓ relevant nodes; the bound follows from the fact that the number of edges is at most $2^\ell |V(G)|$, a consequence of Fact 1. The same argument holds for $R^\pi_{\text{DEL}}(G, k)$. □

In contrast, we show in the next section that for most hereditary properties, reconfiguration problems are hard when parameterized by ℓ.

3.1 Bounded Hitting Set

Here, we prove the parameterized tractability of reconfiguration for certain superset-closed k-subset problems when parameterized by k, where a *k-subset problem* is a parameterized problem Q whose solutions for an instance (I, k) are all subsets of size at most k of a domain set, and is *superset-closed* if any superset of size at most k of a solution of Q is also a solution of Q. For example, parameterized VERTEX COVER is a superset-closed problem.

Theorem 4. *If a k-subset problem Q is superset-closed and has an FPT algorithm to enumerate all its minimal solutions, the number of which is bounded by a function of k, then Q* RECONFIGURATION *parameterized by k is in FPT, as well as the search and shortest path variants.*

Proof. By enumerating all minimal solutions of Q, we compute the set M of all elements v of the domain set such that v is in a minimal solution to Q. For (I, S, T, k, ℓ) an instance of Q RECONFIGURATION, we show that there exists a reconfiguration sequence from S to T if and only if there exists a reconfiguration sequence from $S \cap M$ to $T \cap M$ that uses only subsets of M.

Each set U in the reconfiguration sequence from S to T is a solution, hence contains at least one minimal solution in $U \cap M$; $U \cap M$ is a superset of the minimal solution and hence also a solution. Moreover, since any two consecutive solutions U and U' in the sequence differ by a single element, $U \cap M$ and $U' \cap M$ differ by at most a single element. By replacing each subsequence of identical sets by a single set, we obtain a reconfiguration sequence from $S \cap M$ to $T \cap M$ that uses only subsets of M.

The reconfiguration sequence from $S \cap M$ to $T \cap M$ using only subsets of M can be extended to a reconfiguration sequence from S to T by transforming S to $S \cap M$ in $|S \setminus M|$ steps and transforming $T \cap M$ to T in $|T \setminus M|$ steps. In this sequence, each vertex in $C \setminus M$ is removed from S to form $S \setminus M$ and then readded to form T from $T \setminus M$. For each vertex $v \in C \setminus M$, we can choose instead to add v to each solution in the sequence, thereby decreasing ℓ by two (the steps needed to remove and then readd v) at the cost of increasing by one the capacity used in the sequence from $S \cap M$ to $T \cap M$. This choice can be made independently for each of these $\mathcal{E} = |C \setminus M|$ vertices.

Consequently, (I, S, T, k, ℓ) is a yes-instance for Q Reconfiguration if and only if one of the $\mathcal{E}+1$ reduced instances $(I, S \cap M, T \cap M, k-e, \ell-2(\mathcal{E}-e))$, for $0 \leq e \leq \mathcal{E}$ and $\mathcal{E} = |C \setminus M|$, is a yes-instance for Q' Reconfiguration: we define Q' as a k-subset problem whose solutions for an instance (I, k) are solutions of instance (I, k) of Q that are contained in M. To show that Q' Reconfiguration is in *FPT*, we observe that the number of nodes in the reconfiguration graph for Q' is bounded by a function of k: each solution of Q' is a subset of M, yielding at most $2^{|M|}$ nodes, and $|M|$ is bounded by a function of k. □

As a consequence, Bounded Hitting Set Reconfiguration, Feedback Arc Set Reconfiguration in Tournaments Reconfiguration, and Minimum Weight SAT in Bounded CNF Formulas Reconfiguation (where each solution is the set of variables that are set to true in a satisfying assignment, and the problem looks for a solution of cardinality at most k) are proved to be in FPT when parameterized by k:

Corollary 5. *Bounded Hitting Set Reconfiguration, Feedback Arc Set in Tournaments Reconfiguration, and Minimum Weight SAT in Bounded CNF Formulas Reconfiguation parameterized by k are in FPT.*

Proof. All these problems are superset-closed. Furthermore, standard techniques give FPT algorithms to enumerate their minimal solutions, and the number of minimal solutions is bounded by a function of k in all cases, as required by Theorem 4. We include the proofs for completeness.

We can devise a search tree algorithm that gradually constructs minimal hitting sets of instances of Bounded Hitting Set, producing all minimal hitting sets of size at most k in its leaves. Consider an instance of Bounded Hitting Set, where the cardinality of each set is bounded by a constant c. At each non-leaf node, the algorithm chooses a set that is not hit, and branches on all possible ways of hitting this set, including one of the (at most c) elements in the set in each branch. Since we are not interested in hitting sets of cardinality more than k, we do not need to search beyond depth k in the tree, proving an upper bound of c^k on the number of leaves, and an upper bound of $O^*(c^k)$ on the enumeration time.

For the problem Feedback Arc Set in Tournaments, a tournament is acyclic if and only if it has a directed cycle of length three [1], and a set of arcs is a minimal feedback arc set in a tournament if and only if reversing its arcs in the tournament results in an acyclic tournament [25]. Therefore, at each non-leaf node in a search tree for this problem, there is always a cycle C of length three and every feedback arc set shares at least one arc with C. The algorithm can thus branch on the three arcs in C, reversing one in each branch, and solve the problem recursively. As in the previous algorithm, since we are not interested in feedback arc sets of cardinality more than k, the search can be terminated at depth k, proving an upper bound of 3^k on the number of minimal k-feedback arc sets in tournaments, and an upper bound of $O^*(3^k)$ on the running time of this enumeration algorithm.

Finally, Misra et al. [23] give a search tree algorithm for bounded CNF formula instances of Minimum Weight SAT, where every clause has at most c literals for some constant c. At each node, the algorithm chooses a clause whose literals are all positive, and branches on all possible ways of satisfying the clause, setting one variable to true in each branch. If there is no such clause, the formula is satisfied with no increase in the number of true variables, by setting every non-assigned variable to false. As before, the algorithm stops the search when it reaches a depth of k, proving an upper bound of c^k on the number of satisfying assignments, and an upper bound of $O^*(c^k)$ on the enumeration time. □

For Bounded Hitting Set, the proof of Theorem 4 can be strengthened to develop a polynomial reconfiguration kernel. In fact, we use the ideas in Theorem 4 to adapt a special kernel that retains all minimal k-hitting sets in the reduced instances [9].

Theorem 6. Bounded Hitting Set Reconfiguration *parameterized by k has a polynomial reconfiguration kernel.*

Proof. We let (G, S, T, k, ℓ) be an instance of Bounded Hitting Set Reconfiguration: G is a family of sets of vertices of size at most r and each of S and T is a hitting set of size at most k, that is, a set of vertices intersecting each set in G. We form a reconfiguration kernel using the reduction algorithm $\mathcal{A}$ of Damaschke and Molokov [9]: $G' = \mathcal{A}(G)$ contains all minimal hitting set solutions of size at most k, and is of size at most $(r-1)k^r + k$.

Bounded Hitting Set is a k-subset problem that is superset-closed. Moreover, $V(G')$ includes all minimal k-hitting sets, and the k-hitting sets for G' are actually those k-hitting sets for G that are completely included in $V(G')$. Therefore, as in the proof of Theorem 4, (G, S, T, k, ℓ) is a yes-instance for Bounded Hitting Set Reconfiguration if and only if one of the $\mathcal{E}+1$ reduced instances $(G', S \cap V(G'), T \cap V(G'), k-e, \ell - 2(\mathcal{E}-e))$, for $0 \le e \le \mathcal{E}$, is a yes-instance for Bounded Hitting Set Reconfiguration.

Notice that unlike in the proof of Theorem 4, here we have access to an $f(k)$-bounded instance G' based on which we can solve Q' Reconfiguration. Another difference is that here the set containing all minimal solutions can be computed in polynomial time, whereas Theorem 4 guarantees only a fixed-parameter tractable procedure. □

Bounded Hitting Set generalizes Vertex Cover, Feedback Vertex Set in Tournaments, Cluster Deletion, and in general any deletion problem for a hereditary property with a finite forbidden set:

Corollary 7. *If π is a hereditary graph property with a finite forbidden set, then π-DEL-RECONF(G, S, T, k, ℓ) parameterized by k has a polynomial reconfiguration kernel.*

3.2 Undirected Feedback Vertex Set

Corollary 7 does not apply to Feedback Vertex Set, for which the associated hereditary graph property is the collection of all forests; the forbidden set is the set of all cycles and hence is not finite. Indeed, Theorem 4 does not apply to Feedback Vertex Set either, since the number of minimal solutions exceeds $f(k)$ if the input graph includes a cycle of length $f(k) + 1$, for any function f. While it maybe possible to adapt the compact enumeration of minimal feedback vertex sets [14] for reconfiguration, we develop a reconfiguration kernel for feedback vertex set by modifying a specific kernel for the problem.

We are given an undirected graph and two feedback vertex sets S and T of size at most k. We make use of Bodlaender's cubic kernel for Feedback Vertex Set [2], modifying reduction rules (shown in italics) to allow the reconfiguration sequence to use non-minimal solutions, and to take into account the roles of C, S_D, T_A, and O. In some cases we remove vertices from O only, as others may be needed in a reconfiguration sequence.

The reduction may introduce multiple edges, forming a multigraph. Bodlaender specifies that a double edge between vertices u and v consists of two edges with u and v as endpoints. Since we preserve certain degree-two vertices, we extend the notion by saying that there is a *double edge* between u and v if either there are two edges with u and v as endpoints, one edge between u and v and one path from u to v in which each internal vertex is of degree two, or two paths (necessarily sharing only u and v) from u to v in which each internal vertex is of degree two. Following Bodlaender, we define two sets of vertices, a feedback vertex set A of size at most $2k$ and the set B containing each vertex with a double edge to at least one vertex in A. A *piece* is a connected component of $G[V \setminus (A \cup B)]$, the *border* of a piece with vertex set X is the set of vertices in $A \cup B$ adjacent to any vertex in X, and a vertex v in the border *governs* a piece if there is a double edge between v and each other vertex in the border. We introduce $\mathcal{E}$ to denote how much capacity we can "free up" for use in the reduced instance by removing vertices and then readding them.

Bodlaender's algorithm makes use of a repeated initialization phase in which an approximate solution A is found and B is initialized; for our purposes, we set $A = C \cup S_D \cup T_A$ in the first round and thereafter remove vertices as dictated by the application of reduction rules. Although not strictly necessary, we preserve this idea in order to be able to apply Bodlaender's counting arguments. In the following rules, v, w, and x are vertices.

Rule 1 If v has degree 0, remove v from G. *If v is in $S_D \cup T_A$, subtract 1 from ℓ. If v is in C, increment $\mathcal{E}$ by 1.*

Rule 2 If v has degree 1, remove v and its incident edge from G. *If v is in $S_D \cup T_A$, subtract 1 from ℓ. If v is in C, increment $\mathcal{E}$ by 1.*

Rule 3 If there are three or more edges $\{v, w\}$, remove all but two.

Rule 4 If v has degree 2 *and v is in O*, remove v and its incident edges from G and add an edge between its neighbours w and x; add w (respectively, x) to B if a double edge is formed, w (respectively, x) is not in $A \cup B$, and x (respectively, w) is in A.

Rule 5 If v has a self-loop, remove v and all incident edges and decrease k by 1, then restart the initialization phase.

Rule 6 If there are at least $k + 2$ vertex-disjoint paths between $v \in A$ and any w and there is no double edge between v and w, add two edges between v and w, and if $w \notin A \cup B$, add w to B.

Rule 7 If for $v \in A$ there exist at least $k + 1$ cycles such that each pair of cycles has exactly $\{v\}$ as the intersection, remove v and all incident edges and decrease k by 1, then restart the initialization phase.

Rule 8 If v has at least $k + 1$ neighbours with double edges, remove v and all incident edges and decrease k by 1, then restart the initialization phase.

Rule 9 If $v \in A \cup B$ governs a piece with vertex set X and has exactly one neighbour w in X, then remove the edge $\{v, w\}$.

Rule 10 If $v \in A \cup B$ governs a piece with vertex set X and has at least two neighbours in X, then remove v and all incident edges and decrease k by 1, then restart the initialization phase. *Replaced by the following rule: If a piece with vertex set X has a border set Y such that there is a double edge between each pair of vertices in Y, remove X.*

Lemma 8. *The instance (G, S, T, k, ℓ) is a yes-instance if and only if one of the $\mathcal{E} + 1$ reduced instances $(G', S', T', k - e, \ell - 2(\mathcal{E} - e))$, for $0 \leq e \leq \mathcal{E}$, is a yes-instance.*

Proof. We show that no modification of a reduction rule removes possible reconfiguration sequences. This is trivially true for Rules 3 and 6.

The vertices removed by Rules 1, 2, and 4 play different roles in converting a reconfiguration sequence for a reduced instance to a reconfiguration sequence for the original instance. As there is no cycle that can be destroyed only by a vertex removed from O by Rule 1, 2, or 4, none of these vertices are needed. To account for the required removal (addition) of each such vertex in S_D (T_A), we remove all d such vertices and decrease ℓ by d. We can choose to leave a $v \in C_M$ in each solution in the sequence (with no impact on ℓ) or to remove and then readd v to free up extra capacity, at a cost of incrementing ℓ by two; in the reduced instance we thus remove v and either decrement k or subtract two from ℓ. Since this choice can be made for each of these vertices, $\mathcal{E}$ in total, we try to solve any of $\mathcal{E} + 1$ versions $(G', S', T', k - e, \ell - 2(\mathcal{E} - e))$ for $0 \leq e \leq \mathcal{E}$.

For each of Rules 5, 7, and 8, we show that the removed vertex v is in C_F; since the cycles formed by v must be handled by each solution in the sequence, the instance can be reduced by removing v and decrementing k. For Rule 5, $v \in C_F$ since every feedback arc set must contain v. For Rules 7 and 8, $v \in C_F$, since any feedback vertex set not containing v would have to contain at least $k+1$ vertices, one for each cycle.

For Rule 9, Bodlaender's Lemma 8 shows that the removed edge has no impact on feedback vertex sets.

For Rule 10, we first assume that Rule 9 has been exhaustively applied, and thus each vertex in the border has two edges to X. By Fact 3 for π the set of acyclic graphs, there cannot be a cycle in $G[O \cup \{v\}]$ for any $v \in S_D \cup T_A \cup O$, and hence each member of the border is in C. Lemma 9 in Bodlaender's paper shows that there is a minimum size feedback vertex set containing v: even if all the neighbours of v in the border are included in a feedback vertex set, at least one more vertex is required to break the cycle formed by v and X. There is no gain in capacity possible by replacing v in the reconfiguration sequence, and hence this particular piece is of no value in finding a solution. □

We first present the key points and lemmas in Bodlaender's counting argument and then show that, with minor modifications, the same argument goes through for our modified reduction rules and altered definition of *double edge*.

In Bodlaender's proof, the size of the reduced instance is bounded by bounding the sizes of A and B (Lemma 10), bounding the number of pieces (Lemma 12), and bounding the size of each piece. Crucial to the proof of Lemma 12 is Lemma 11, as the counting associates each piece with a pair of vertices in its border that are not connected by a double edge and then counts the number of pieces associated with each different type of pair. We use Lemma 9 in the discussion below.

Lemma 9. *[2] Suppose $v \in A \cup B$ governs a piece with vertex set X. Suppose there are at least two edges with one endpoints v and one endpoint in X. Then there is a minimum size feedback vertex set in G that contains v.*

Lemma 10. *[2] In a reduced instance, there are at most $2k$ vertices in A and at most $2k^2$ vertices in B.*

Lemma 11. *[2] Suppose none of the Rules 1–10 can be applied to G. Suppose $Y \subseteq V$ is the border of a piece in G. Then there are two disjoint vertices $v, w \in Y$ such that $\{v, w\}$ is not a double edge.*

Lemma 12. *[2] Suppose we have a reduced instance. There are at most $8k^3 + 9k^2 + k$ pieces.*

Lemma 13. *Each reduced instance has $O(k^3)$ vertices and $O(k^3)$ edges, and can be obtained in polynomial time.*

Proof. Our modifications to Rules 1–3 and 5–9 do not have an impact on the size of the kernel. Although our Rule 4 preserves some vertices in A of degree two, due to the initialization of A to be $C \cup S_D \cup T_A$, and hence of size at most $2k$, the bound on B and hence Lemma 10 follows from Rule 8. In essence, our extended definition of double edges handles the degree-two vertices that in Bodlaender's constructions would have been replaced by an edge.

To claim the result of Lemma 12, it suffices to show that Lemma 11 holds for our modified rules. Bodlaender shows that if there is a piece such that each pair of vertices in the border set is connected by a double edge, Rule 10 along with Rule 9 can be applied repeatedly to remove vertices from the border of the piece and thereafter Rules 2 and 1 to remove the piece entirely.

To justify Rule 10, Bodlaender shows in Lemma 9 that if $v \in A \cup B$ governs a piece with vertex set X and there are at least two edges between v and X, then there is a minimum size feedback vertex set in G that contains v. For our purposes, however, since there may be non-minimum size feedback vertex sets used in the reconfiguration sequence, we wish to retain v rather than removing it. Our modification to Rule 10 allows us to retain v, handling all the removals from the piece without changing the border, and thus establishing Lemma 11, as needed to prove Lemma 12.

In counting the sizes of pieces, our modifications result in extra degree-two vertices. Rule 4 removes all degree-two vertices in O, and hence the number of extra vertices is at most $2k$, having no effect on the asymptotic count. □

Theorem 14. Feedback Vertex Set Reconfiguration *and the search variant parameterized by k are in* FPT.

Proof. Since the number of reduced instances is $\mathcal{E} + 1 \leq |C| + 1 \leq k + 1$, as a consequence of Lemmas 8 and 13, we have a reconfiguration kernel, proving the first result.

For the search version, we observe that we can generate the reconfiguration graph of the reduced yes-instance and use it to extract a reconfiguration sequence. We demonstrate that we can form a reconfiguration sequence for (G, S, T, k, ℓ) from the reconfiguration sequence σ for the reduced yes-instance $(G', S', t', k - e, \ell - 2(\mathcal{E} - e))$. We choose an arbitrary partition of the vertices removed from G by Rules 1 and 2 into two sets, K (the ones to keep) of size e and M (the ones to modify) of size $\mathcal{E} - e$. We can modify σ into a sequence σ' in which all vertices in K are added to each set; clearly no set will have size greater than k. Our reconfiguration sequence then consists of $\mathcal{E} - e$ steps each deleting an element of M, the sequence σ', and $\mathcal{E} - e$ steps each adding an element of M, for a length of at most $(\mathcal{E} - e) + (\ell - (\mathcal{E} - e)) + (\mathcal{E} - e) \leq \ell$, as needed. □

4 Hardness Results

The reductions presented in this section make use of the forbidden set characterization of heredity properties. A π*-critical graph* H is a (minimal) graph in the forbidden set $\mathcal{F}_\pi$ that has at least two vertices; we use the fact that $H \notin \pi$, but the deletion of any vertex from H results in a graph in π. For convenience, we will refer to two of the vertices in a π-critical graph as *terminals* and the rest as *internal vertices*. We construct graphs from multiple copies of H. For a positive integer c, we let H_c^* be the ("star") graph obtained from each of c

copies H_i of H by identifying an arbitrary terminal v_i, $1 \leq i \leq c$, from each H_i; in H_c^* vertices v_1 through v_c are replaced with a vertex w, the *gluing vertex of* v_1 *to* v_c, to form a graph with vertex set $\cup_{1\leq i\leq c}(V(H_i) \setminus \{v_i\}) \cup \{w\}$ and edge set $\cup_{1\leq i\leq c}\{\{u,v\} \in E(H_i) \mid v_i \notin \{u,v\}\} \cup \cup_{1\leq i\leq c}\{\{u,w\} \mid \{u,v_i\} \in E(H_i)\}$. A terminal is *non-identified* if it is not used in forming a gluing vertex.

In Figure 1, H is a K_3 with terminals marked black and gray; H_4^* is formed by identifying all the gray terminals to form w.

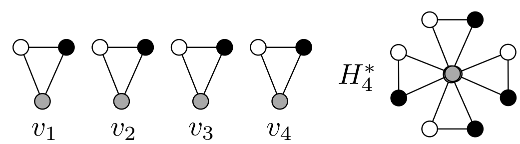

Fig. 1. An example H_c^*

Theorem 15. *Let π be any hereditary property satisfying the following:*

- *For any two graphs G_1 and G_2 in π, the graph obtained by their disjoint union is in π.*
- *There exists an $H \in \mathcal{F}_\pi$ such that if H_c^* is the graph obtained from identifying a terminal from each of c copies of H, then the graph $R = H_c^*[V(H_c^*) \setminus \{u_1, u_2, \ldots u_c\}]$ is in π, where $u_1, u_2, \ldots u_c$ are the non-identified terminals in the c copies of H.*

Then each of the following is at least as hard as π-SUBSET(G,k):

1. *π-DEL-RECONF(G,S,T,k,ℓ) parameterized by ℓ, and*
2. *π-SUB-RECONF(G,S,T,k,ℓ) parameterized by $k+\ell$.*

Proof. Given an instance of π-SUBSET(G,k) and a π-critical graph H satisfying the hypothesis of the lemma, we form an instance of π-DEL-RECONF$(G',S,T,|V(G)|+k,4k)$, with G', S, and T defined below. The graph G' is the disjoint union of G and a graph W formed from k^2 copies of H, where $H_{i,j}$ has terminals $\ell_{i,j}$ and $r_{i,j}$. We let a_i, $1 \leq i \leq k$, be the gluing vertex of $\ell_{i,1}$ through $\ell_{i,k}$, and let b_j, $1 \leq j \leq k$, be the gluing vertex of $r_{1,j}$ through $r_{k,j}$, so that there is a copy of H joining each a_i and b_j. An example W is shown in Figure 2, where copies of H are shown schematically as gray ovals. We let $A = \{a_i \mid 1 \leq i \leq k\}$, $B = \{b_j \mid 1 \leq j \leq k\}$, $S = V(G) \cup A$, and $T = V(G) \cup B$. Clearly $|V(G')| = |V(G)| + 2k + k^2(|V(H)| - 2)$ and $|S| = |T| = |V(G)| + k$. Moreover, each of $V(G') \setminus S$ and $V(G') \setminus T$ induce a graph in π, as each consists of k disjoint copies of H_k^* with one of the terminals removed from each H in H_k^*.

Suppose the instance of π-DEL-RECONF$(G',S,T,|V(G)|+k,4k)$ is a yes-instance. As there is a copy of H joining each vertex of A to each vertex of B, before deleting $a \in A$ from S the reconfiguration sequence must add all of B

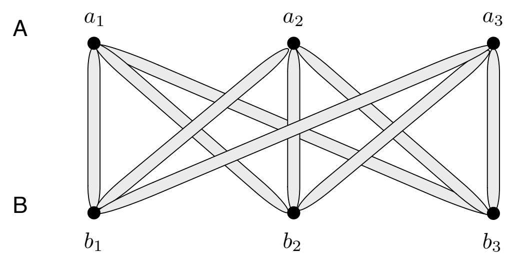

Fig. 2. An example W

to ensure that the complement of each intermediate set induces a graph in π. Otherwise, the complement will contain at least one copy of H as a subgraph and is therefore not in π. The capacity bound of $|V(G)| + k$ implies that the reconfiguration sequence must have deleted from S a subset $S' \subseteq V(G)$ of size at least k such that $V(G') \setminus (S \setminus S') = S' \cup B$ induces a subgraph in π. Thus, $G[S'] \in \pi$, and hence π-SUBSET(G, k) is a yes-instance.

Conversely if the instance of π-SUBSET(G, k) is a yes-instance, then there exists $V' \subseteq V(G)$ such that $|V'| = k$ and $G[V'] \in \pi$. We form a reconfiguration sequence between S and T by first deleting all vertices in V' from S to yield a set of size $|V(G)|$. $G'[V(G') \setminus (S \setminus V')]$ consists of the union of $G'[V'(G) \setminus S]$ and $G'[V'] = G[V']$, both of which are in π. Next we add one by one all vertices of B, then delete one by one all vertices of A and then add back one by one each vertex in the set V' resulting in a reconfiguration sequence of length $k+k+k+k = 4k$. It is clear that in every step, the complement of the set induces a graph in π.

Thus we have showed that π-SUBSET(G, k) is a yes-instance if and only if there is a path of length at most $4k$ between S and T in $R^{\pi}_{\text{DEL}}(G', |V(G)| + k)$. Since $|V(G')| - (|V(G)| + k) = k + k^2(|V(H)| - 2))$, this implies that π-SUBSET(G, k) is a yes-instance if and only if there is a path of length at most $4k$ between $V(G') \setminus S$ and $V(G') \setminus T$ in $R^{\pi}_{\text{SUB}}(G', k + k^2(|V(H)| - 2))$. Therefore, π-SUB-RECONF(G, S, T, k, ℓ) parameterized by $k + \ell$ is at least as hard as π-SUBSET(G, k), proving the second part. ☐

Corollary 16. VERTEX COVER RECONFIGURATION*,* FEEDBACK VERTEX SET RECONFIGURATION*, and* ODD CYCLE TRANSVERSAL RECONFIGURATION *parameterized by ℓ are all $W[1]$-hard and* INDEPENDENT SET RECONFIGURATION*,* FOREST RECONFIGURATION*, and* BIPARTITE SUBGRAPH RECONFIGURATION *parameterized by $k + \ell$ are all $W[1]$-hard.*

Proof. It is known that for any hereditary property π that consists of all edgeless graphs but not all cliques [20], π-SUBSET(G, k) is $W[1]$-hard. It is clear that the collections of all edgeless graphs, of all bipartite graphs, and of all forests satisfy this condition for hardness, as well as the hypothesis of Theorem 15.

For the collection of independent sets, the only $H \in \mathcal{F}_\pi$ is an edge both of whose endpoints are terminals. Here identifying multiple copies of H at a terminal forms a star, and deleting the non-identified terminal from each of the edges results in a single vertex, which is in π.

For the collection of forests, and bipartite graphs, we let $H \in \mathcal{F}_\pi$ be a triangle. When we identify multiple triangles at a vertex, and remove another vertex of each of the triangles, we obtain a tree, which is in π. □

We obtain further results for properties not covered by Theorem 15. Lemma 17 handles the collection of all cliques, which does not satisfy the first condition of the theorem and the collection of all *cluster graphs* (disjoint unions of cliques), which satisfies the first condition but not the second. Moreover, as π-SUBSET(G, k) is in *FPT* for π the collection of all cluster graphs [20], Theorem 15 provides no lower bounds.

Lemma 17. CLIQUE RECONFIGURATION *and* CLUSTER SUBGRAPH RECONFIGURATION *parameterized by* $k + \ell$ *are* $W[1]$*-hard.*

Proof. We first give an *FPT* reduction from t-CLIQUE, known to be $W[1]$-hard, to CLUSTER SUBGRAPH RECONFIGURATION. For (G, t) an instance of t-CLIQUE, $V(G) = \{v_1, \ldots, v_n\}$, we form a graph consisting of four K_t's (with vertex sets A, B, C, and D) and a subgraph mimicking G (with vertex set X), where there is an edge from each vertex in X to each vertex in each K_t, and each of subgraphs induced on the following vertex sets induce a K_{2t}: $A \cup B$, $A \cup C$, $B \cup D$, $C \cup D$. More formally, $G' = (X \cup A \cup B \cup C \cup D, E_X \cup E_T \cup E_C)$, where $X = \{x_1, \ldots, x_n\}$, $|A| = |B| = |C| = |D| = t$, $E_X = \{\{x_i, x_j\} \mid \{v_i, v_j\} \in E(G)\}$ corresponds to the edges in G, $E_T = \{\{a, a'\} \mid a, a' \in A, a \neq a'\} \cup \{\{b, b'\} \mid b, b' \in B, b \neq b'\} \cup \{\{c, c'\} \mid c, c' \in C, c \neq c'\} \cup \{\{d, d'\} \mid d, d' \in D, d \neq d'\}$ forms the K_t cliques, and $E_C = \{\{x, a\}, \{x, b\}, \{x, c\}, \{x, d\}, \{a, b\}, \{a, c\}, \{b, d\}, \{c, d\} \mid a \in A, b \in B, c \in C, d \in D, x \in X\}$ forms the connections among the vertex setes.

We let $(G', S, T, 2t, 6t)$ be an instance of CLUSTER SUBGRAPH RECONFIGURATION, where $S = A \cup B$ and $T = C \cup D$. Clearly $|S| = |T| = 2t$ and both S and T induce cluster graphs (in fact cliques). We claim that G has a clique of size t if and only if there is a path of length $6t$ from S to T.

If G has a clique of size t, then there exists a subset $Y \subseteq X$ forming a clique of size t. We form a reconfiguration sequence of length $6t$ as follows; add the vertices Y, remove the vertices in A, add the vertices in D, remove the vertices in B, add the vertices in C, and remove the vertices in Y, one by one. It is not hard to see that at every step in this sequence we maintain an induced clique in G' of size greater than or equal to $2t$ (and hence a cluster subgraph).

If there exists a path of length $6t$ from S to T, we make use of the fact that no cluster subgraph contains an induced path of length three to show that G has a clique of size t. Observe that before adding any vertex of C, we first need to remove (at least) all of B since otherwise we obtain an induced path of length three containing vertices in C, A, and B, respectively. Similarly, we cannot add any vertex of D until we have removed all of A. Therefore, before adding any vertex from T, we first need to delete at least t vertices from S. To do so without violating our minimum capacity of $2t$, at least t vertices must be added from X. Since every vertex in X is connected to all vertices in S and T, if any pair of those t vertices do not share an edge, we obtain an induced path on three vertices. Thus X, and hence G, must have a clique of size t.

Since in our reduction S and T are cliques and every reconfiguration step maintains an induced clique in G' of size greater than or equal to $2t$, the same applies to the Clique Reconfiguration problem. Consequently, both Clique Reconfiguration and Cluster Subgraph Reconfiguration parameterized by $k + \ell$ are $W[1]$-hard. □

As neither Dominating Set nor its parametric dual is a hereditary graph property, Theorem 15 is inapplicable; we instead use a construction specific to this problem in Lemma 18, which in turn leads to Corollary 19, since Dominating Set can be phrased as a hitting set of the family of closed neighborhood of the vertices of the graph.

Lemma 18. Dominating Set Reconfiguration *parameterized by* $k + \ell$ *is* $W[2]$*-hard.*

Proof. We give a reduction from t-Dominating Set; for (G, t) an instance of t-Dominating Set, we form G' as the disjoint union of two graphs G'_1 and G'_2.

We form G'_1 from $t+2$ $(t+1)$-cliques C_0 (the *outer clique*) and $C_1, \ldots, C_{t+1}$ (the *inner cliques*); $V(C_0) = \{o_1, \ldots, o_{t+1}\}$ and $V(C_i) = \{w_{(i,0)}, w_{(i,1)}, \ldots w_{(i,t)}\}$ for $1 \le i \le t+1$. The edge set of G'_1 contains not only the edges of the cliques but also $\{\{o_j, w_{(i,j)}\} \mid 1 \le i \le t+1, 0 \le j \le t\}$; the graph to the left in Figure 3 illustrates G'_1 for $t = 2$. Any dominating set that does not contain all vertices in the outer clique must contain a vertex from each inner clique.

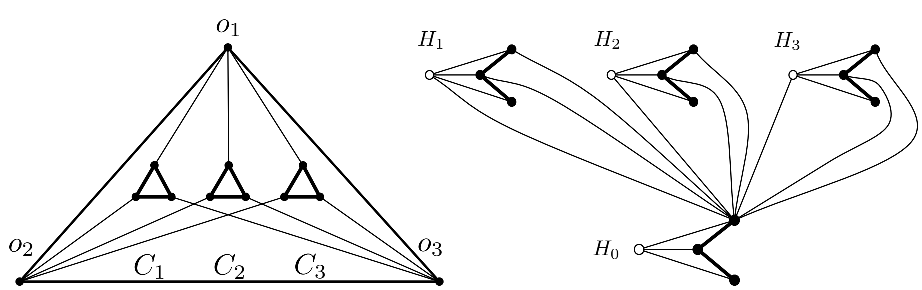

Fig. 3. Graphs used for the dominating set reduction

To create G'_2, we first define G^+ to be the graph formed by adding a universal vertex to G, where we assume without loss of generality that $V(G) = \{v_1, \ldots, v_{|V(G)|}\}$. We let $V(G'_2) = \cup_{0 \le i \le t} V(H_i)$, where $H_0, \ldots, H_t$ are $t+1$ copies of G^+; we use u_i to denote the universal vertex in H_i and $v_{(i,j)}$ to denote the copy of v_j in H_i, $1 \le j \le |V(G)|$, $0 \le i \le t$. The edge set consists of edges between each non-universal vertex $v_{(0,j)}$ in H_0 and, in each H_i, the universal vertex, its image, and the images of its neighbours in G, or more formally $E(G'_2) = \{\{v_{0,j}, u_i\} \mid 1 \leq j \leq |V(G)|, 1 \leq i \leq t\} \cup \{\{v_{0,j}, v_{i,j}\} \mid 1 \leq j \leq |V(G)|, 1 \leq i \leq t\} \cup \{\{(v_{0,j}, v_{i,k}\} \mid 1 \leq j \leq |V(G)|, 1 \leq i \leq t, (v_j, v_k) \in E(G)\}$. The graph to the right in Figure 3 illustrates part of G'_2, where universal vertices are shown in white and, for the sake of readability, the only edges outside of G^+ shown are those adjacent to a single vertex in H_0.

We form an instance $(G', S, T, 3t+2, 6t+4)$ of Dominating Set Reconfiguration, where $S = \{u_i \mid 0 \leq i \leq t\} \cup V(C_0)$ and $T = \{u_i \mid 0 \leq i \leq t\} \cup \{w_{i,i-1} \mid 1 \leq i \leq t+1\}$. Both S and T are dominating sets, as each universal vertex u_i dominates H_i as well as H_0 and $V(G'_1)$ is dominated by the outer clique in S and by one vertex from each inner clique in T. Clearly $|S| = |T| = 2t + 2$.

We claim that G has a dominating set of size t if and only if there is a path of length $6t+4$ from S to T. In G'_1, to remove any vertex from the outer clique, we must first add a vertex from each inner clique, for a total of $t+1$ additions; since $k = 3t+2$ and $|S| = 2t+2$, this can only take place after G'_2 has been dominated using at most t vertices. In G'_2, a universal vertex u_i cannot be deleted until H_i has been dominated. If G can be dominated with t vertices, then it is possible to add the dominating set in H_0 and remove all the universal vertices, thus making the required capacity available. If not, then none of the universal vertices, say u_i, can be removed without first adding at least $t + 1$ vertices to dominate H_i, for which there is not enough capacity. Therefore, there exists a reconfiguration sequence from S to some S' such that $S' \cap G'_2$ has t vertices if and only if G has a dominating set of size t. Moreover, the existence of a dominating set D of size t in G implies a path of length $6t + 4$ from S to T; we add D in H_0, remove all universal vertices, reconfigure G'_1, add all universal vertices, and then remove D. Consequently, there exists a reconfiguration sequence from S to T in $6t + 4$ steps if and only if G has a dominating set of size t. ☐

The following is a result of there being a polynomial-time parameter-preserving reduction from Dominating Set:

Corollary 19. Unbounded Hitting Set Reconfiguration *parameterized by* $k + \ell$ *is* $W[2]$*-hard.*

5 Conclusions and Directions for Further Work

Our results constitute the first study of the parameterized complexity of reconfiguration problems. We give a general paradigm, the reconfiguration kernel, for proving fixed-parameter tractability, and provide hardness reductions that apply to problems associated with hereditary graph properties. Our result on cluster graphs (Lemma 17) demonstrates the existence of a problem that is fixed-parameter tractable [20], but whose reconfiguration version is W-hard when parameterized by k; this clearly implies that fixed-parameter tractability of the underlying problem does not guarantee fixed-parameter tractability of reconfiguration when parameterized by k. Since there is unlikely to be a polynomial-sized kernel for the problem of determining whether a given graph has a cluster of size at least k [21], it is possible (though in our opinion, unlikely) that an underlying problem having a polynomial-sized kernel is sufficient for the reconfiguration problem to be fixed-parameter tractable when parameterized by k.

It remains open whether there exists an NP-hard problem for which the reconfiguration version is in FPT when parameterized by ℓ.

Our FPT algorithms for reconfiguration of Bounded Hitting Set and Feedback Vertex Set have running times of $O^*(2^{O(k \lg k)})$. Further work is needed to determine whether the running times can be improved to $O^*(2^{O(k)})$, or whether these bounds are tight under the *Exponential Time Hypothesis*.

We observe connections to another well-studied paradigm, local search [11], where the aim is to find an *improved solution* at distance ℓ of a given solution S. Not surprisingly, as in local search, the problems we study turn out to be hard even in the parameterized setting when parameterized by ℓ. Other natural directions to pursue (as in the study of local search) are the parameterized complexity of reconfiguration problems in special classes of graphs and of non-graph reconfiguration problems, as well as other parameterizations.

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