# Complexity results for generating subgraphs

Vadim E. Levit Department of Computer Science Ariel University, ISRAEL levitv@ariel.ac.il

David Tankus Department of Software Engineering Sami Shamoon College of Engineering, ISRAEL davidt@sce.ac.il

#### Abstract

A graph G is *well-covered* if all its maximal independent sets are of the same cardinality. Assume that a weight function w is defined on its vertices. Then G is *w-well-covered* if all maximal independent sets are of the same weight. For every graph G, the set of weight functions w such that G is *w*-well-covered is a vector space, denoted WCW(G).

Let B be a complete bipartite induced subgraph of G on vertex sets of bipartition  $B_X$  and  $B_Y$ . Then B is generating if there exists an independent set S such that  $S \cup B_X$  and  $S \cup B_Y$  are both maximal independent sets of G. In the restricted case that a generating subgraph B is isomorphic to  $K_{1,1}$ , the unique edge in B is called a *relating edge*.

Deciding whether an input graph G is well-covered is **co-NP**-complete. Therefore finding WCW(G) is **co-NP**-hard. Deciding whether an edge is relating is **NP**-complete. Therefore, deciding whether a subgraph is generating is **NP**-complete as well.

In this article we discuss the connections among these problems, provide proofs for **NP**-completeness for several restricted cases, and present polynomial characterizations for some other cases.

**Keywords:** weighted well-covered graph; maximal independent set; relating edge; generating subgraph; vector space.

### 1 Introduction

### **1.1** Basic definitions and notation

Throughout this paper G is a simple (i.e., a finite, undirected, loopless and without multiple edges) graph with vertex set V(G) and edge set E(G).

Cycles of k vertices are denoted by  $C_k$ . When we say that G does not contain  $C_k$  for some  $k \ge 3$ , we mean that G does not admit subgraphs isomorphic to

 $C_k$ . Note that these subgraphs are not necessarily induced. Let  $\mathcal{G}(\widehat{C_{i_1}}, ..., \widehat{C_{i_k}})$  be the family of all graphs which do not contain  $C_{i_1}, ..., C_{i_k}$ .

Let u and v be two vertices in G. The *distance* between u and v, denoted d(u, v), is the length of a shortest path between u and v, where the length of a path is the number of its edges. If S is a non-empty set of vertices, then the *distance* between u and S, is defined as  $d(u, S) = \min\{d(u, s) : s \in S\}$ .

For every positive integer i, denote

$$N_i(S) = \{ x \in V(G) : d(x, S) = i \},\$$

and

$$N_i[S] = \{x \in V(G) : d(x, S) \le i\}.$$

If S contains a single vertex, v, then we abbreviate  $N_i(\{v\})$ ,  $N_i[\{v\}]$  to be  $N_i(v)$ ,  $N_i[v]$ , respectively. We denote by G[S] the subgraph of G induced by S. For every two sets, S and T, of vertices of G, we say that S dominates T if  $T \subseteq N_1[S]$ .

#### 1.2 Well-covered graphs

Let G be a graph. A set of vertices S is *independent* if its elements are pairwise nonadjacent. An independent set of vertices is *maximal* if it is not a subset of another independent set. An independent set of vertices is *maximum* if the graph does not contain an independent set of a higher cardinality.

The graph G is well-covered if every maximal independent set is maximum [12]. Assume that a weight function  $w: V(G) \longrightarrow \mathbb{R}$  is defined on the vertices of G. For every set  $S \subseteq V(G)$ , define

$$w(S) = \sum_{s \in S} w(s).$$

Then G is *w*-well-covered if all maximal independent sets of G are of the same weight.

The problem of finding a maximum independent set is **NP**-complete. However, if the input is restricted to well-covered graphs, then a maximum independent set can be found in polynomial time using the *greedy algorithm*. Similarly, if a weight function  $w : V(G) \longrightarrow \mathbb{R}$  is defined on the vertices of G, and Gis *w*-well-covered, then finding a maximum weight independent set is a polynomial problem. There is an interesting application, where well-covered graphs are investigated in the context of distributed *k*-mutual exclusion algorithms [18].

The recognition of well-covered graphs is known to be **co-NP**-complete. This is proved independently in [5] and [15]. In [4] it is proven that the problem remains **co-NP**-complete even when the input is restricted to  $K_{1,4}$ -free graphs. However, the problem can be solved in polynomial time for  $K_{1,3}$ -free graphs [16, 17], for graphs with girth 5 at least [6], for graphs with a bounded maximal degree [3], for chordal graphs [13], and for graphs without cycles of lengths 4 and 5 [7]. For every graph G, the set of weight functions w for which G is w-wellcovered is a vector space [3]. That vector space is denoted WCW(G) [2]. Since recognizing well-covered graphs is **co-NP**-complete, finding the vector space WCW(G) of an input graph G is **co-NP**-hard. However, finding WCW(G)can be done in polynomial time when the input is restricted to graphs with a bounded maximal degree [3], to graphs without cycles of lengths 4, 5 and 6 [11], and to chordal graphs [1].

### 1.3 Generating subgraphs and relating edges

Further we make use of the following notions, which have been introduced in [8]. Let B be an induced complete bipartite subgraph of G on vertex sets of bipartition  $B_X$  and  $B_Y$ . Assume that there exists an independent set S such that each of  $S \cup B_X$  and  $S \cup B_Y$  is a maximal independent set of G. Then B is a generating subgraph of G, and the set S is a witness that B is generating. We observe that every weight function w such that G is w-well-covered must satisfy the restriction  $w(B_X) = w(B_Y)$ .

If the generating subgraph B contains only one edge, say xy, it is called a *relating edge*. In such a case, the equality w(x) = w(y) is valid for every weight function w such that G is w-well-covered.

Recognizing relating edges is known to be **NP**-complete [2], and it remains **NP**-complete even when the input is restricted to graphs without cycles of lengths 4 and 5 [9]. Therefore, recognizing generating subgraphs is also **NP**-complete when the input is restricted to graphs without cycles of lengths 4 and 5. However, recognizing relating edges can be done in polynomial time if the input is restricted to graphs without cycles of lengths 4 and 6 [9], and to graphs without cycles of lengths 5 and 6 [11].

It is also known that recognizing generating subgraphs is a polynomial problem when the input is restricted to graphs without cycles of lengths 4, 6 and 7 [8], to graphs without cycles of lengths 4, 5 and 6 [11], and to graphs without cycles of lengths 5, 6 and 7 [11].

#### 1.4 Introducing the problems under consideration

The subject of this article is the following four problems and their interconnections.

• WC problem:

Input: A graph G. Question: Is G well-covered?

- WCW problem: Input: A graph G. Output: The vector space WCW(G).
- GS problem:

Input: A graph G, and an induced complete bipartite subgraph B of G.

Question: Is B generating?

• RE problem:

Input: A graph G, and an edge  $xy \in E(G)$ . Question: Is xy relating?

If we know the output of the **WCW** problem for a graph G, then we know the output of the **WC** problem for the same G: The graph G is well-covered if and only if  $w \equiv 1$  belongs to WCW(G). Therefore, the **WC** problem is not harder than the **WCW** problem. Let  $\Psi$  be a family of graphs. If the **WCW** problem can be solved in polynomial time, when its input is restricted to  $\Psi$ , then also the **WC** problem is polynomial, when its input is restricted to  $\Psi$ . On the other hand, if the **WC** problem is **co-NP-**complete, when its input is restricted to  $\Psi$ , then the **WCW** problem is **co-NP-**hard, when its input is restricted to  $\Psi$ .

A similar connection exists between the **GS** problem and the **RE** problem, since the **RE** problem is a restricted case of the **GS** problem. Therefore, for every family  $\Psi$  of graphs, if the **GS** problem can be solved in polynomial time, then the **RE** problem can be solved in polynomial time as well, and if the **RE** problem is **NP**-complete then the **GS** problem is also **NP**-complete.

This article considers bipartite graphs, graphs with girth 6 at least, and  $K_{1,4}$ -free graphs. Although for bipartite graphs and graphs with girth 6 at least, the **WC** problem is known to be solvable in polynomial time, we prove that the **GS** problem is **NP**-complete. For bipartite graphs, even the **RE** problem is **NP**-complete. Additionally, **NP**-completeness of the **GS** problem for  $K_{1,4}$ -free graphs is proved. We also present polynomial algorithms for the **RE** problem, the **GS** problem, and the **WCW** problem in the case that the maximum degree of the input graph is bounded.

### 2 NP-complete cases

A binary variable is a variable whose value is either 0 or 1. If x is a binary variable, then its negation is denoted by  $\overline{x}$ . Each of x and  $\overline{x}$  are called *literals*. Let  $X = \{x_1, ..., x_n\}$  be a set of binary variables. A clause c over X is a set of literals belonging to  $\{x_1, \overline{x_1}, ..., x_n, \overline{x_n}\}$  such that c does not contain both a variable and its negation. A truth assignment is a function

$$\Phi: \{x_1, \overline{x_1}, \dots, x_n, \overline{x_n}\} \longrightarrow \{0, 1\}$$

such that

$$\Phi(\overline{x_i}) = 1 - \Phi(x_i)$$
 for each  $1 \le i \le n$ 

A truth assignment  $\Phi$  satisfies a clause c if c contains at least one literal l such that  $\Phi(l) = 1$ .

### 2.1 Relating edges in bipartite graphs

In this subsection we consider the following problems:

#### • SAT problem:

Input: A set X of binary variables and a set C of clauses over X. Question: Is there a truth assignment for X which satisfies all clauses of C?

#### • BWSAT problem:

Input: A set X of binary variables and two sets,  $C_1$  and  $C_2$ , of clauses over X, such that all literals of the clauses belonging to  $C_1$  are variables, and all literals of clauses belonging to  $C_2$  are negations of variables. *Question*: Is there a truth assignment for X, which satisfies all clauses of  $C_1 \cup C_2$ ?

By Cook-Levin's Theorem, the **SAT** problem is **NP**-complete. We prove that the same holds for the **BWSAT** problem.

Lemma 2.1 The BWSAT problem is in NP-complete.

**Proof.** Obviously, the **BWSAT** problem is in **NP**. We prove its **NP**-completeness by showing a reduction from the **SAT** problem. Let

$$I_1 = (X = \{x_1, ..., x_n\}, C = \{c_1, ..., c_m\})$$

be an instance of the **SAT** problem. Define  $Y = \{x_1, ..., x_n, y_1, ..., y_n\}$ , where  $y_1, ..., y_n$  are new variables. For every  $1 \le j \le m$ , let  $c'_j$  be the clause obtained from  $c_j$  by replacing  $\overline{x_i}$  with  $y_i$  for each  $1 \le i \le n$ . Let  $C' = \{c'_1, ..., c'_m\}$ . For each  $1 \le i \le n$  define two new clauses,  $d_i = \{x_i, y_i\}$  and  $e_i = \{\overline{x_i}, \overline{y_i}\}$ . Let  $D = \{d_1, ..., d_n\}$  and  $E = \{e_1, ..., e_n\}$ . Obviously, all literals of  $C' \cup D$  are variables, and all literals of E are negations of variables. Hence,  $I_2 = (Y, C' \cup D, E)$  is an instance of the **BWSAT** problem, see Example 2.2. It remains to prove that  $I_1$  and  $I_2$  are equivalent.

Assume that  $I_1$  is a positive instance of the **SAT** problem. There exists a truth assignment

$$\Phi_1: \{x_1, \overline{x_1}, \dots, x_n, \overline{x_n}\} \longrightarrow \{0, 1\}$$

which satisfies all clauses of C. Extend  $\Phi_1$  to a truth assignment

$$\Phi_2: \{x_1, \overline{x_1}, \dots, x_n, \overline{x_n}, y_1, \overline{y_1}, \dots, y_n, \overline{y_n}\} \longrightarrow \{0, 1\}$$

by defining  $\Phi_2(y_i) = 1 - \Phi_1(x_i)$  for each  $1 \le i \le n$ . Clearly,  $\Phi_2$  is a truth assignment which satisfies all clauses of  $C' \cup D \cup E$ . Hence,  $I_2$  is a positive instance of the **BWSAT** problem.

Assume  $I_2$  is a positive instance of the **BWSAT** problem. There exists a truth assignment

$$\Phi_2: \{x_1, \overline{x_1}, \dots, x_n, \overline{x_n}, y_1, \overline{y_1}, \dots, y_n, \overline{y_n}\} \longrightarrow \{0, 1\}$$

that satisfies all clauses of  $C' \cup D \cup E$ . For every  $1 \leq i \leq n$  it holds that  $\Phi_2(y_i) = 1 - \Phi_2(x_i)$ , or otherwise one of  $d_i$  and  $e_i$  is not satisfied. Therefore,  $I_1$  is a positive instance of the **SAT** problem.

**Example 2.2** The following contains both an instance of the **SAT** problem and an equivalent instance of the **BWSAT** problem.

$$\begin{split} &I_1 = (X,C), \ where \ X = \{x_1,x_2,x_3,x_4,x_5\}, \\ &C = \{\{x_1,\overline{x_2},x_3\},\{x_1,x_3,x_4,x_5\},\{\overline{x_1},x_2,\overline{x_3},x_4\},\{x_1,x_2,\overline{x_4},\overline{x_5}\}\}, \\ &I_2 = (Y,C_1,C_2), \ where \ Y = \{x_1,x_2,x_3,x_4,x_5,y_1,y_2,y_3,y_4,y_5\}, \\ &C_1 = \{\{x_1,y_2,x_3\},\{x_1,x_3,x_4,x_5\},\{y_1,x_2,y_3,x_4\},\{x_1,x_2,y_4,y_5\},\{x_1,y_1\}, \\ &\{x_2,y_2\},\{x_3,y_3\},\{x_4,y_4\},\{x_5,y_5\}\}. \\ &C_2 = \{\{\overline{x_1},\overline{y_1}\},\{\overline{x_2},\overline{y_2}\},\{\overline{x_3},\overline{y_3}\},\{\overline{x_4},\overline{y_4}\},\{\overline{x_5},\overline{y_5}\}\} \end{split}$$

The following theorem is the main result of this section.

**Theorem 2.3** The **RE** problem is **NP**-complete even if its input is restricted to bipartite graphs.

**Proof.** The problem is obviously in **NP**. We prove **NP**-completeness by showing a reduction from the **BWSAT** problem. Let

$$I_1 = (X = \{x_1, ..., x_n\}, C_1, C_2)$$

be an instance of the **BWSAT** problem, where  $C_1 = \{c_1, ..., c_m\}$  is a set of clauses which contain only variables, and  $C_2 = \{c'_1, ..., c'_{m'}\}$  is a set of clauses which contain only negations of variables. Define a graph B as follows:

$$V(B) = \{x, y\} \cup \{v_j : 1 \le j \le m\} \cup \{v'_j : 1 \le j \le m'\} \cup \{u_i : 1 \le i \le n\} \cup \{u'_i : 1 \le i \le n\},\$$

$$E(B) = \{xy\} \cup \{xv_j : 1 \le j \le m\} \cup \{yv'_j : 1 \le j \le m'\} \cup \{v_ju_i : x_i \text{ appears in } c_j\} \cup \{v'_iu'_i : \overline{x_i} \text{ appears in } c'_i\} \cup \{u_iu'_i : 1 \le i \le n\}.$$

Clearly, B is bipartite, and the vertex sets of its bipartition are

$$\{u_i : 1 \le i \le n\} \cup \{x\} \cup \{v'_j : 1 \le j \le m'\}$$

and

$$\{v_j : 1 \le j \le m\} \cup \{y\} \cup \{u'_i : 1 \le i \le n\}$$

Consider the instance  $I_2 = (B, xy)$  of the **RE** problem. It is necessary to prove that  $I_1$  and  $I_2$  are equivalent.

Assume that  $I_1$  is a positive instance of the **BWSAT** problem. Let

$$\Phi: \{x_1, \overline{x_1}, \dots, x_n, \overline{x_n}\} \longrightarrow \{0, 1\}$$

be a truth assignment which satisfies all clauses of  $C_1 \cup C_2$ . Let

$$S = \{u_i : \Phi(x_i) = 1\} \cup \{u'_i : \Phi(x_i) = 0\}.$$

Obviously, S is independent. Since  $\Phi$  satisfies all clauses of  $C_1 \cup C_2$ , every vertex of

$$\{v_j : 1 \le j \le m\} \cup \{v'_j : 1 \le j \le m'\}$$

is adjacent to a vertex of S. Hence,  $S \cup \{x\}$  and  $S \cup \{y\}$  are maximal independent sets. Therefore, S is a witness that xy is a relating edge, and  $I_2$  is a positive instance of the **RE** problem.

On the other hand, assume that  $I_2$  is a positive instance of the **RE** problem. Let S be a witness of xy. Since S is a maximal independent set of

$$\{u_i : 1 \le i \le n\} \cup \{u'_i : 1 \le i \le n\}$$

exactly one of  $u_i$  and  $u'_i$  belongs to S, for every  $1 \le i \le n$ . Let

$$\Phi: \{x_1, \overline{x_1}, \dots, x_n, \overline{x_n}\} \longrightarrow \{0, 1\}$$

be a truth assignment defined by:  $\Phi(x_i) = 1 \iff u_i \in S$ . The fact that S dominates

$$\{v_j : 1 \le j \le m\} \cup \{v'_j : 1 \le j \le m'\}$$

implies that all clauses of  $C_1 \cup C_2$  are satisfied by  $\Phi$ . Therefore,  $I_1$  is a positive instance of the **BWSAT** problem.

**Example 2.4** Let  $I_1 = (X, C_1, C_2)$  be an instance of the **BWSAT** problem, where  $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ ,  $C_1 = \{\{x_1, x_2, x_3\}, \{x_2, x_4\}, \{x_1, x_4\}, \{x_1, x_5, x_6\}, \{x_3, x_5, x_6\}\}$ , and  $C_2 = \{\{\overline{x_1}, \overline{x_2}, \overline{x_3}\}, \{\overline{x_2}, \overline{x_3}, \overline{x_4}, \overline{x_5}\}, \{\overline{x_2}, \overline{x_4}, \overline{x_5}, \overline{x_6}\}\}$ 

Then  $I_2 = (G, xy)$  is an equivalent instance of the **RE** problem, where G is the graph shown in Figure 1. The instance  $I_1$  is positive because of the satisfying assignment  $\Phi$  defined by  $\Phi(x_i) = 0$  if  $i \in \{2, 6\}$ , and  $\Phi(x_i) = 1$  otherwise. The corresponding witness that  $I_2$  is positive is the set  $\{u_1, u'_2, u_3, u_4, u_5, u'_6\}$ .

**Corollary 2.5** The **GS** problem is **NP**-complete when its input is restricted to bipartite graphs.

#### 2.2 Graphs with girth 6 at least

In this subsection we consider the following problems:

• 3-SAT problem:

Input: A set X of binary variables and a set C of clauses over X such that every clause contains exactly 3 literals.

Question: Is there a truth assignment for X satisfying all clauses of C?



Figure 1: An example of the reduction from the **USAT** problem to the **RE** problem.

#### • DSAT problem:

Input: A set X of binary variables and a set C of clauses over X such that the following holds:

- Every clause contains 2 or 3 literals.
- Every two distinct clauses have at most one literal in common.
- If two clauses,  $c_1$  and  $c_2$ , have a common literal  $l_1$ , then there does not exist a literal  $l_2$  such that  $c_1$  contains  $l_2$  and  $c_2$  contains  $\overline{l_2}$ .

Question: Is there a truth assignment for X satisfying all clauses of C?

Let I = (X, C) be an instance of the **3-SAT** problem. A bad pair of clauses is a set of two clauses  $\{c_1, c_2\} \subseteq C$  such that there exist literals,  $l_1, l_2, l_3, l_4, l_5$ , and:

- $c_1 = \{l_1, l_2, l_3\}$  and  $c_2 = \{l_1, l_4, l_5\};$
- either  $l_2 = l_4$  or  $l_2 = \overline{l_4}$ .

Clearly, an instance of the **3-SAT** problem with no bad pair of clauses is also an instance of the **DSAT** problem. The **3-SAT** problem is known to be in **NP**-complete. We prove that the same holds for the **DSAT** problem.

#### Lemma 2.6 The DSAT problem is NP-complete.

**Proof.** Obviously, the **DSAT** problem is in **NP**. We prove its **NP**-completeness by showing a reduction from the **3-SAT** problem. Let

$$I_1 = (X = \{x_1, ..., x_n\}, C = \{c_1, ..., c_m\})$$

be an instance of the **3-SAT** problem.

Assume that there exists a bad pair of clauses,  $\{c_{j_1}, c_{j_2}\} \subseteq C$ , i.e. there exist literals,  $l_1, l_2, l_3, l_4, l_5$ , such that:

- $c_{j_1} = \{l_1, l_2, l_3\}$  and  $c_{j_2} = \{l_1, l_4, l_5\};$
- either  $l_2 = l_4$  or  $l_2 = \overline{l_4}$ .

Define a new binary variable  $x_{n+1}$ , and new clauses  $c_{j_2}^1 = \{l_1, x_{n+1}, l_5\}, c_{j_2}^2 = \{\overline{l_4}, x_{n+1}\}, \text{ and } c_{j_2}^3 = \{l_4, \overline{x_{n+1}}\}.$  Then

$$I'_1 = (X \cup \{x_{n+1}\}, (C \setminus \{c_{j_2}\}) \cup \{c^1_{j_2}, c^2_{j_2}, c^3_{j_2}\})$$

is an instance of the **SAT** problem.

We prove that  $I_1$ ,  $I'_1$  are equivalent. Assume that  $I_1$  is a positive instance of the **3-SAT** problem. There exists a truth assignment

$$\Phi_1: \{x_1, \overline{x_1}, \dots, x_n, \overline{x_n}\} \longrightarrow \{0, 1\}$$

which satisfies all clauses of C. Extend  $\Phi_1$  to a truth assignment

$$\Phi_2: \{x_1, \overline{x_1}, \dots, x_{n+1}, \overline{x_{n+1}}\} \longrightarrow \{0, 1\}$$

by defining  $\Phi_2(x_{n+1}) = \Phi_1(l_4)$ . Clearly,  $\Phi_2$  satisfies all clauses of  $I'_1$ . On the other hand, assume that there exists a truth assignment

$$\Phi_2: \{x_1, \overline{x_1}, \dots, x_{n+1}, \overline{x_{n+1}}\} \longrightarrow \{0, 1\}$$

which satisfies all clauses of  $I'_1$ . Clauses  $c^2_{j_2}$ , and  $c^3_{j_2}$  imply that  $\Phi_2(x_{n+1}) = \Phi_2(l_4)$ . Therefore,  $I_1$  is a positive instance of the **3-SAT** problem.

The new clauses we added contain a new binary variable. Hence, they do not belong to bad pairs of clauses. Moreover, the clause  $c_{j_2}$  which belongs to a bad pair in  $I_1$  was omitted in  $I'_1$ . Hence, the number of bad pairs of clauses in  $I'_1$  is smaller than the one in  $I_1$ .

Repeat that process until an instance without bad pairs of clauses is obtained, and denote that instance  $I_2$ . Clearly, every clause of  $I_2$  has 2 or 3 literals. Hence,  $I_2$  is an instance of the **DSAT** problem, and  $I_1$  and  $I_2$  are equivalent.

**Example 2.7** The following contains an instance of the **3-SAT** problem and an equivalent instance of the **DSAT** problem.

$$\begin{split} &I_1 = (X_1, C_1) \ \text{where} \ X_1 = \{x_1, x_2, x_3, x_4, x_5\} \ \text{and} \ C_1 = \{\{x_1, \overline{x_2}, x_3\}, \\ &\{x_1, x_3, x_4\}, \{x_1, x_3, x_5\}, \{\overline{x_3}, \overline{x_4}, x_5\}, \{x_2, \overline{x_3}, \overline{x_4}\}, \{\overline{x_1}, \overline{x_4}, \overline{x_5}\}\}. \\ &I_2 = (X_2, C_2) \ \text{where} \ X_2 = \{x_1, x_2, x_3, x_4, x_5, z_3, y_3, y_4, y_5\} \ \text{and} \\ &C_2 = \{\{x_1, \overline{x_2}, x_3\}, \{x_1, y_3, x_4\}, \{x_1, z_3, x_5\}, \{\overline{x_3}, \overline{x_4}, x_5\}, \{x_2, \overline{x_3}, y_4\}, \\ &\{\overline{x_1}, \overline{x_4}, y_5\}, \{\overline{x_3}, y_3\}, \{x_3, \overline{y_3}\}, \{\overline{x_3}, z_3\}, \{x_3, \overline{z_3}\}, \{x_4, y_4\}, \{\overline{x_4}, \overline{y_4}\}, \{x_5, y_5\}, \\ &\{\overline{x_5}, \overline{y_5}\}\}. \end{split}$$

**Theorem 2.8** The following problem is **NP**-complete: Input: A graph  $G \in \mathcal{G}(\widehat{C}_3, \widehat{C}_4, \widehat{C}_5)$  and an induced complete bipartite subgraph B of G.

Question: Is B generating?

**Proof.** The problem is obviously in **NP**. We prove its **NP**-completeness by showing a reduction from the **DSAT** problem. Let

$$I = (X = \{x_1, ..., x_n\}, C = \{c_1, ..., c_m\})$$

be an instance of the **DSAT** problem. Define a graph G as follows.

$$V(G) = \{y\} \cup \{a_j : 1 \le j \le m\} \cup \{v_j : 1 \le j \le m\} \cup \{u_i : 1 \le i \le n\} \cup \{u_i' : 1 \le i \le n\}.$$

 $E(G) = \{ya_j : 1 \le j \le m\} \cup \{a_jv_j : 1 \le j \le m\} \cup \{v_ju_i : x_i \text{ appears in } c_j\} \cup \{v_iu'_i : \overline{x_i} \text{ appears in } c_i\} \cup \{u_iu'_i : 1 \le i \le n\}.$ 

Since a clause can not contain both a variable and its negation, G does not contain  $C_3$ . The fact that there are no pairs of bad clauses implies that G does not contain  $C_4$  and  $C_5$ . Hence,  $G \in \mathcal{G}(\widehat{C}_3, \widehat{C}_4, \widehat{C}_5)$ . Let  $B = G[\{y\} \cup \{a_j : 1 \leq j \leq m\}]$ . Obviously, B is complete bipartite. Then J = (G, B) is an instance of the **GS** problem. It remains to prove that I and J are equivalent.

Assume that I is positive, and let

$$\Phi: \{x_1, \overline{x_1}, \dots, x_n, \overline{x_n}\} \longrightarrow \{0, 1\}$$

be a truth assignment which satisfies all clauses of C. Define

$$S = \{u_i : \Phi(x_i) = 1\} \cup \{u'_i : \Phi(x_i) = 0\}.$$

Obviously, S is independent. Since  $\Phi$  satisfies all clauses of C, the set S dominates  $\{v_j : 1 \leq j \leq m\}$ . Hence, S is a witness that B is generating, i.e., J is positive.

Assume that J is positive. Let S be a witness that B is generating, and let  $S^*$  be a maximal independent set of  $\{u_i : 1 \le i \le n\} \cup \{u'_i : 1 \le i \le n\}$  which contains S. For every  $1 \le i \le n$ , it holds that  $|S^* \cap \{u_i, u'_i\}| = 1$ . Define

$$\Phi: \{x_1, \overline{x_1}, \dots, x_n, \overline{x_n}\} \longrightarrow \{0, 1\}$$

by  $\Phi(x_i) = 1 \iff u_i \in S^*$  for every  $1 \le i \le n$ . Since  $S^*$  dominates  $\{v_j : 1 \le j \le m\}$ , the function  $\Phi$  satisfies all clauses of C, and I is a positive instance.



Figure 2: An example of the reduction from the **DSAT** problem to the **GS** problem.

**Example 2.9** Let  $I_1 = (X, C)$  be an instance of the **DSAT** problem, where  $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$  and  $C = \{\{x_1, \overline{x_2}, x_3\}, \{\overline{x_1}, x_2, x_4\}, \{x_1, \overline{x_4}, x_6\}, \{x_2, \overline{x_5}, \overline{x_6}\}, \{\overline{x_3}, x_4, x_5\}\}$ . Then  $I_2 = (G, B)$  is an equivalent instance of the **GS** problem, where G and B are the graphs shown in Figure 2. The instance  $I_1$  is positive because of the satisfying assignment  $\Phi$  defined by  $\Phi(x_i) = 1$  if  $i \in \{1, 2, 4\}, and \Phi(x_i) = 0$  otherwise. The corresponding witness that  $I_2$  is positive is the set  $\{u_1, u_2, u'_3, u_4, u'_5, u'_6\}$ .

#### **2.3** $K_{1,4}$ -free graphs

**Theorem 2.10** [4] The following problem is co-NP-complete: Input: A  $K_{1,4}$ -free graph G. Question: Is G well-covered?

We use Theorem 2.10 to prove the following.

**Theorem 2.11** The **GS** problem is **NP**-complete even when its input is restricted to  $K_{1,4}$ -free graphs.

**Proof.** Let G be a  $K_{1,4}$ -free graph. An induced complete bipartite subgraph of G is isomorphic to  $K_{i,j}$ , for  $1 \leq i \leq j \leq 3$ . Hence, the number of these subgraphs is  $O(n^6)$ , which is polynomial. Every unbalanced induced complete bipartite subgraph of G is a copy of  $K_{1,2}$  or  $K_{1,3}$  or  $K_{2,3}$ . The number of these subgraphs is  $O(n^5)$ .

Assume, on the contrary, that there exists a polynomial algorithm solving the **GS** problem for  $K_{1,4}$ -free graphs. The following algorithm decides in polynomial time whether a  $K_{1,4}$ -free graph G is well-covered. For each induced complete bipartite unbalanced subgraph B of G on vertex sets of bipartition  $B_X$  and  $B_Y$ , decide in polynomial time whether B is generating. Once an unbalanced generating subgraph is discovered, the algorithm terminates announcing G is not well-covered. If the algorithm checked all induced complete bipartite unbalanced subgraphs of G, and none of them is generating, then G is well-covered. Hence, the **WC** problem can be solved in polynomial time when its input is restricted to  $K_{1,4}$ -free graphs, but that contradicts Theorem 2.10. Thus the **GS** problem is **NP**-complete, when its input is a  $K_{1,4}$ -free graph.

## **3** Polynomial algorithms when $\Delta$ is bounded

In this section G will be a graph with n vertices and of maximum degree  $\Delta$ . The main findings of this section are polynomial algorithms for the **RE** problem and the **GS** problem in the restricted case, when  $\Delta$  is bounded. Our motivation here is the following.

**Theorem 3.1** [3] Let  $k \in N$ . The following problem is polynomial. Input: A graph G with  $\Delta_G \leq k \cdot (\log_2 n)^{\frac{1}{3}}$ , and a function  $w : V(G) \longrightarrow \mathbb{R}$ . Question: Is G w-well-covered?

We prove that the **GS** problem is polynomial, when  $\Delta$  is bounded using the same technique as in Theorem 3.1.

**Theorem 3.2** Let  $k \in N$ . The following problem can be solved in  $O(n^{2+2k^3})$  time.

Input: A graph G such that  $\Delta \leq k \cdot (\log_2 n)^{\frac{1}{3}}$ , and an induced complete bipartite subgraph B of G.

Question: Is B generating?

**Proof.** Let B be an induced complete bipartite subgraph of G on vertex sets of bipartition  $B_X$  and  $B_Y$ . For every  $V \in \{X, Y\}$ , let  $U \in \{X, Y\} - \{V\}$ , and define:

$$M_1(B_V) = N(B_V) \cap N_2(B_U), \ M_2(B_V) = N(M_1(B_V)) \cap N_2(B_V).$$

Then  $|M_1(B_V)| \leq k^2 (\log_2 n)^{2/3}$  and  $|M_2(B_V)| \leq k^3 \log_2 n$ . Obviously, *B* is generating if and only if there exists an independent set in  $M_2(B_X) \cup M_2(B_Y)$  that dominates  $M_1(B_X) \cup M_1(B_Y)$ .

The following algorithm decides whether B is generating. For each subset S of  $M_2(B_X) \cup M_2(B_Y)$ , check whether S is independent and dominates  $M_1(B_X) \cup M_1(B_Y)$ . Once an independent set  $S \subseteq M_2(B_X) \cup M_2(B_Y)$  is found such that  $M_1(B_X) \cup M_1(B_Y) \subseteq N[S]$ , the algorithm terminates announcing the instance at hand is positive. If all subsets of  $M_2(B_X) \cup M_2(B_Y)$  were checked,

and none of them is independent and dominates  $M_1(B_X) \cup M_1(B_Y)$ , then the algorithm returns a negative answer.

The number of subsets the algorithm checks is

$$O(2^{|M_2(B_X) \cup M_2(B_Y)|}) = O(2^{2k^3 \log_2 n}) = O(n^{2k^3}).$$

For each subset S, the decision whether S is both independent and dominates  $M_1(B_X) \cup M_1(B_Y)$  can be done in  $O(n^2)$ . Therefore, the algorithm terminates in  $O(n^{2+2k^3})$  time, which is polynomial.

We next prove that the **RE** problem is polynomial for the less restrictable bound in comparison with its counterpart from Theorem 3.2.

**Theorem 3.3** Let  $k \in N$ . The following problem can be solved in  $O(n^{2+2k^2})$  time.

Input: A graph G such that  $\Delta \leq k \cdot (\log_2 n)^{\frac{1}{2}}$ , and an edge  $xy \in E$ . Question: Is xy relating?

**Proof.** For every  $v \in \{x, y\}$ , let  $u \in \{x, y\} - \{v\}$ . Define:  $M_1(v) = N(v) \cap N_2(u)$ ,  $M_2(v) = N(M_1(v)) \cap N_2(v)$ . Then  $|M_1(v)| \leq k \cdot (\log_2 n)^{\frac{1}{2}}$  and  $|M_2(v)| \leq k^2 \log_2 n$ . Clearly, xy is relating if and only if there exists an independent set in  $M_2(x) \cup M_2(y)$ , which dominates  $M_1(x) \cup M_1(y)$ .

The following algorithm decides whether xy is relating. For each subset S of  $M_2(x) \cup M_2(y)$ , check whether S is independent and dominates  $M_1(x) \cup M_1(y)$ . Once an independent set  $S \subseteq M_2(x) \cup M_2(y)$  is found such that  $M_1(x) \cup M_1(y) \subseteq N[S]$ , the algorithm terminates announcing the instance at hand is positive. If all subsets of  $M_2(x) \cup M_2(y)$  were checked, and none of them is both independent and dominates  $M_1(x) \cup M_1(y)$ , then the algorithm returns a negative answer.

The number of subsets the algorithm checks is

$$O(2^{|M_2(x)\cup M_2(y)|}) = O(2^{2k^2\log_2 n}) = O(n^{2k^2}).$$

For each subset S, the decision whether S is both independent and dominates  $M_1(x) \cup M_1(y)$  can be done in  $O(n^2)$  time. Therefore, the algorithm terminates in  $O(n^{2+2k^2})$  time.

In what follows, our purpose is both to formalize and to give a detailed proof of a claim mentioned in [3].

**Theorem 3.4** Let  $k \in N$ . The following problem can be solved in  $O(n^{3+2k^2+2k^3})$  time.

Input: A graph G such that  $\Delta \leq k \cdot (\log_2 n)^{\frac{1}{3}}$ . Output: The vector space WCW(G).

**Proof.** Let G be a graph such that  $\Delta \leq k \cdot (\log_2 n)^{\frac{1}{3}}$ . For every vertex  $v \in V$ , let  $L_v$  be the vector space of all weight functions  $w : V(G) \longrightarrow \mathbb{R}$  which satisfy all restrictions of all generating subgraphs which contain the vertex v. Clearly,

 $WCW(G) = \bigcap_{v \in V(G)} L_v$ . Hence, we first present an algorithm for finding  $L_v$  for every  $v \in V$ .

Let  $v \in V$ . Since the diameter of every complete bipartite graph is at most 2, every complete bipartite subgraph of G which contains v is a subgraph of  $N_2[v]$ . However,

$$|N_2(v)| \le \Delta^2 \le k^2 (\log_2 n)^{\frac{2}{3}},$$

and

$$|N_2[v]| \le 2 |N_2(v)| \le 2k^2 (\log_2 n)^{\frac{2}{3}}.$$

Therefore, the number of induced complete bipartite subgraphs which contain v cannot exceed

$$2^{2k^2(\log_2 n)^{\frac{2}{3}}} < n^{2k^2}.$$

The following algorithm finds  $L_v$ :

- For each induced complete bipartite subgraph  $B = (B_X, B_Y)$  of G containing v:
  - Decide whether B is generating;
  - If B is generating add the restriction  $w(B_X) = w(B_Y)$  to the list of equations defining  $L_v$ .

We have proved that the number of induced complete bipartite subgraphs of G containing v cannot exceed  $n^{2k^2}$ . By Theorem 3.2, deciding for each subgraph whether it is generating can be done in  $O(n^{2+2k^3})$  time. Therefore, the algorithm for finding  $L_v$  terminates in  $O(n^{2+2k^2+2k^3})$  time. In order to find WCW(G), the algorithm for finding  $L_v$  should be invoked n times. Therefore, finding WCW(G) can be completed in  $O(n^{3+2k^2+2k^3})$  time.

### 4 Conclusions and future work

The following table presents complexity results concerning the four major problems presented in this paper. The empty table cells correspond to unsolved cases. In particular, we want to find the complexity status of the **WCW** problem for bipartite graphs and for graphs with girth 6 at least. For these families of graphs the **GS** problem is **NP**-complete while the **WC** problem is polynomial. Hence, either we obtain a family of graphs for which the **WC** problem is polynomial while the **WCW** problem is **co-NP**-hard, or we obtain a family of graphs for which the **GS** problem is **NP**-complete while the **WCW** problem is polynomial.

In addition, we are interested in finding some polynomial relaxations of the bipartite case, if any. For instance, can recognizing well-covered graphs belonging to  $\mathcal{G}(\widehat{C}_3, \widehat{C}_5)$  be done polynomially?

Let us emphasize that we do not know whether there exists a family of graphs for which the **RE** problem can be solved in polynomial time, but the **GS** problem is **NP**-complete.

Input	WC	WCW	RE	GS
	co-NPC	co-NPH	NPC	NPC
general	[5, 15]	[5, 15]	[2]	[2]
	Р	Р	Р	Р
$K_{1,3}$ -free	[16]	[10]	[17]	[17]
	co-NPC	co-NPH		NPC
$K_{1,4}$ -free	[4]	[4]		this paper
	Р		NPC	$\mathbf{NPC}$
$\mathcal{G}(\widehat{C_4},\widehat{C_5})$	[7]		[9]	[9]
			Р	
$\mathcal{G}(\widehat{C_4},\widehat{C_6})$			[9]	
			Р	
$\mathcal{G}(\widehat{C_5},\widehat{C_6})$			[11]	
			Р	Р
$\mathcal{G}(\widehat{C_5},\widehat{C_6},\widehat{C_7})$			[11]	[11]
	Р	Р	Р	Р
$\mathcal{G}(\widehat{C_4},\widehat{C_5},\widehat{C_6})$	[7]	[11]	[11]	[11]
			Р	Р
$\mathcal{G}(\widehat{C_4},\widehat{C_6},\widehat{C_7})$			[8]	[8]
	Р		NPC	NPC
bipartite	[14]		this paper	this paper
	Р			NPC
$\mathcal{G}(\widehat{C_3},\widehat{C_4})$	[6]			this paper
	Р			NPC
$\mathcal{G}(\widehat{C_3},\widehat{C_4},\widehat{C_5})$	[6]			this paper
	Р	Р	Р	Р
$\Delta \le k (\log_2 n)^{\frac{1}{3}}$	[3]	[3]	[3] and this paper	[3] and this paper
			Р	
$\Delta \leq k (\log_2 n)^{\frac{1}{2}}$			[3] and this paper	

Table 1: Complexity results on the 4 problems.

Another interesting open question is whether there exists a family of graphs for which the **GS** problem is polynomial and its corresponding **WCW** problem is **co-NP-**hard.



Figure 3: The failure of the naive algorithm.

The naive algorithm for the **GS** problem, receives as its input an instance  $I = (G, B = (B_X, B_Y))$ . Then it finds WCW(G). If there exists a weight function  $w \in WCW(G)$  such that  $w(B_X) \neq w(B_Y)$ , then B is not generating, and consequently, I is negative. Otherwise, I is positive. For every family  $\Psi$  of graphs, if the **WCW** problem can be solved polynomially, then the naive algorithm for the **GS** problem terminates polynomially.

However, the naive algorithm fails, when its input is (G, B), where G is the graph shown in Figure 3, and B is the subgraph induced by  $\{v_1, v_2, v_3\}$ . A function  $w : V(G) \longrightarrow \mathbb{R}$  belongs to WCW(G) if and only if the following conditions hold:

- $w(v_7) = w(v_9)$
- $w(v_8) = w(v_{10})$
- $w(v_6) = w(v_9) + w(v_{10})$
- $w(v_i) = 0$  for every  $1 \le i \le 5$ .

Hence,  $w(v_1) = w(v_2) + w(v_3)$  for every  $w \in WCW(G)$ , and the naive algorithm decides that B is generating, although it is not.

## References

 J. I. Brown, R. J. Nowakowski, Well covered vector spaces of graphs, SIAM Journal on Discrete Mathematics 19 (2006) 952–965.

- [2] J. I. Brown, R. J. Nowakowski, I. E. Zverovich, The structure of well-covered graphs with no cycles of length 4, Discrete Mathematics 307 (2007) 2235-2245.
- [3] Y. Caro, N. Ellingham, G. F. Ramey, Local structure when all maximal independent sets have equal weight, SIAM Journal on Discrete Mathematics 11 (1998) 644-654.
- [4] Y. Caro, A. Sebő, M. Tarsi, *Recognizing greedy structures*, Journal of Algorithms **20** (1996) 137-156.
- [5] V. Chvatal, P. J. Slater, A note on well-covered graphs, Quo Vadis, Graph Theory?, Annals of Discrete Mathematics 55, North Holland, Amsterdam (1993) 179-182.
- [6] A. Finbow, B. Hartnell, R. Nowakowski, A characterization of well-covered graphs of girth 5 or greater, Journal of Combinatorial Theory B 57 (1993) 44-68.
- [7] A. Finbow, B. Hartnell, R. Nowakowski, A characterization of well-covered graphs that contain neither 4- nor 5-cycles, Journal of Graph Theory 18 (1994) 713-721.
- [8] V. E. Levit, D. Tankus Weighted well-covered graphs without  $C_4$ ,  $C_5$ ,  $C_6$ ,  $C_7$ , Discrete Applied Mathematics **159** (2011) 354-359.
- [9] V. E. Levit, D. Tankus, On relating edges in graphs without cycles of length 4, Journal of Discrete Algorithms 26 (2014) 28-33.
- [10] V. E. Levit, D. Tankus, Weighted well-covered claw-free graphs, Discrete Mathematics 338 (2015) 99-106.
- [11] V. E. Levit, D. Tankus, Well-covered graphs without cycles of lengths 4, 5 and 6, Discrete Applied Mathematics 186 (2015) 158-167.
- [12] M. D. Plummer, Some covering concepts in graphs, Journal of Combinatorial Theory 8 (1970) 91-98.
- [13] E. Prisner, J. Topp and P. D. Vestergaard, Well-covered simplicial, chordal and circular arc graphs, Journal of Graph Theory 21 (1996) 113–119.
- [14] G. Ravindra, Well-covered graphs, Journal of Combinatorics, Information and System Sciences 2 (1977) 20-21.
- [15] R. S. Sankaranarayana, L. K. Stewart, Complexity results for well-covered graphs, Networks 22 (1992) 247-262.
- [16] D. Tankus, M. Tarsi, Well-covered claw-free graphs, Journal of Combinatorial Theory B 66 (1996) 293-302.

- [17] D. Tankus, M. Tarsi, The structure of well-covered graphs and the complexity of their recognition problems, Journal of Combinatorial Theory B 69 (1997) 230-233.
- [18] M. Yamashita, T. Kameda, Modeling k-coteries by well-covered graphs, Networks 34 (1999) 221–228.