# Improved approximation algorithms for path vertex covers in regular graphs

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#### Abstract

Given a simple graph G = (V, E) and a constant integer  $k \ge 2$ , the k-path vertex cover problem (PkVC) asks for a minimum subset  $F \subseteq V$  of vertices such that the induced subgraph G[V-F] does not contain any path of order k. When k = 2, this turns out to be the classic vertex cover (VC) problem, which admits a  $\left(2 - \Theta\left(\frac{1}{\log|V|}\right)\right)$ -approximation. The general PkVC admits a trivial k-approximation; when k = 3 and k = 4, the best known approximation results for P3VC and P4VC are a 2-approximation and a 3-approximation, respectively. On d-regular graphs, the approximation ratios can be reduced to min  $\left\{2 - \frac{5}{d+3} + \epsilon, 2 - \frac{(2-o(1))\log\log d}{\log d}\right\}$  for VC (*i.e.*, P2VC),  $2 - \frac{1}{d} + \frac{4d-2}{3d|V|}$  for P3VC,  $\frac{\lfloor d/2 \rfloor (2d-2)}{\lfloor (\lfloor d/2 \rfloor + 1)(d-2)}$  for P4VC, and  $\frac{2d-k+2}{d-k+2}$  for PkVC when  $1 \le k - 2 < d \le 2(k-2)$ . By utilizing an existing algorithm for graph defective coloring, we first present a  $\frac{\lfloor d/2 \rfloor (2d-k+2)}{\lfloor (\lfloor d/2 \rfloor + 1)(d-k+2)}$ -approximation results for PkVC on d-regular graphs when  $1 \le k - 2 < d$ . This beats all the best known approximation results for PkVC on d-regular graphs for  $k \ge 3$ , except for P4VC it ties with the best prior work and in particular they tie at 2 on cubic graphs and 4-regular graphs. We then propose a 1.875-approximation and a 1.852-approximation algorithm for P4VC on cubic graphs and 4-regular graphs, respectively. We also present a better approximation algorithm for P4VC on d-regular bipartite graphs.

**Keywords:** Path vertex cover; regular graph; defective coloring; maximum independent set; approximation algorithm

## 1 Introduction

We investigate a vertex deletion problem called the *minimum k-path vertex cover* problem, denoted as PkVC, which is a generalization of the classic minimum vertex cover (VC) problem [14]. The PkVC problem has been studied for more than three decades in the literature, and it has applications in wireless sensor networks such as constructing optimal connectivity paths and in networking security such as monitoring the message traffic and detecting malicious attack [1].

Given a simple graph G = (V, E) and a constant integer  $k \ge 2$ , a k-path (or, a path of order k) is a simple path containing k vertices; the PkVC problem asks for a minimum subset  $F \subseteq V$ 

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of vertices such that the induced subgraph G[V - F] (set minus operation) does not contain any k-path [7, 17, 16]. When k = 2, this turns out to be VC. In the literature, a k-path vertex cover is also called a vertex k-path cover [6], or a vertex cover  $P_k$  [26, 25], or a  $P_k$  vertex cover [10], or a k-observer [1, 24]. Also, when  $F \subseteq V$  is a (2-path) vertex cover, V - F is an independent set, and when  $F \subseteq V$  is a 3-path vertex cover, V - F is a dissociation set. The maximum independent set (MIS) problem is another classic NP-hard problem [14]; the maximum dissociation set problem is also classic, was introduced more than three decades ago by Yannakakis [27], and is NP-hard even on bipartite graphs.

The concept of k-path vertex covers, and many related ones, form a line of research in graph theory. The minimum cardinality of a k-path vertex cover, for  $k \geq 2$ , in the graph G = (V, E) is denoted by  $\psi_k(G)$  [6]. Clearly,  $\psi_2(G) = |V| - \alpha(G)$ , where  $\alpha(G)$  is the independent number, that is, the maximum cardinality of an independent set in G. The maximum cardinality of a dissociation set in G, also known as the 1-dependence number [12, 13], is denoted as diss(G) [23], and we have  $\psi_3(G) = |V| - diss(G)$ . Let  $P_n$ ,  $C_n$  and  $K_n$  denote a simple path, a simple cycle and a complete graph with n vertices, respectively, then  $\psi_k(P_n) = \lfloor \frac{n}{k} \rfloor$ ,  $\psi_k(C_n) = \lceil \frac{n}{k} \rceil$  and  $\psi_k(K_n) = n - k + 1$  [1, 6]. When G is some special graph [7, 6, 17, 5, 16], the exact value of  $\psi_k(G)$ , for some values of k, can also be computed in polynomial time. For the cases where the exact value of  $\psi_k(G)$  is unable to be computed in polynomial time, there are works proving several lower and/or upper bounds on  $\psi_k(G)$ ; to name a few, Brešar et al. [7] proved that  $\psi_3(G) \leq \frac{2|V|+|E|}{6}$  and  $\psi_k(G) \leq |V| - \frac{k-1}{k} \sum_{v \in V} \frac{2}{1+d(v)}$ , where d(v) is the degree of v in G; Brešar et al. [6] showed that  $\psi_k(G) \geq \frac{d-k+2}{2d-k+2}|V|$  for d-regular graphs when  $d \geq k - 1$ .

The PkVC problem is NP-hard for every  $k \ge 2$  [7, 1]. From the inapproximability (hardness of approximation) perspective, the VC (*i.e.*, P2VC) problem is APX-complete even on cubic graphs [2]; it cannot be approximated within  $10\sqrt{5}-21 \approx 1.3606$  unless P = NP [11] and it cannot be approximated within any constant factor less than 2 [20] under the unique game conjecture [19]. Brešar et al. [7] proved that for any  $\rho \ge 1$ , a  $\rho$ -approximation for the PkVC problem implies a  $\rho$ -approximation for the VC problem. It follows that it is NP-hard to approximate the PkVC problem, for every  $k \ge 3$ , within 1.3606 too, unless P = NP. Recall that the maximum dissociation set problem is NP-hard even on (some sub-classes of) planar bipartite graphs [27, 4, 23]; the P3VC problem is shown NP-hard on cubic planar graphs with girth 3 [25]. The P4VC problem is proven APX-complete on cubic bipartite graphs and on  $K_{1,4}$ -free graphs [10].

From the approximation algorithm perspective, the simple greedy algorithm, that iteratively takes all the k vertices from a k-path in the remaining graph until there is no k-path left, is a k-approximation for the PkVC problem, for every  $k \ge 2$ . Furthermore, the VC (*i.e.*, P2VC) problem admits a  $\left(2 - \Theta\left(\frac{1}{\log|V|}\right)\right)$ -approximation [18]; the P3VC problem admits a primal-dual 2-approximation [26]; the P4VC problem admits a primal-dual 3-approximation [8].

On *d*-regular graphs, where  $d \ge 3$ , the PkVC problem can be approximated better, for every  $k \ge 2$ . The approximation ratio for the VC (*i.e.*, P2VC) problem can be reduced to min  $\left\{2 - \frac{5}{d+3} + \epsilon \left[3\right], 2 - \frac{(2-o(1))\log\log d}{\log d} \left[15\right]\right\}$ ; Ries et al. [24] gave a  $\left(2 - \frac{1}{d} + \frac{4d-2}{3d|V|}\right)$ -approximation for P3VC and Devi et al. [10] presented a greedy  $\frac{\lfloor d/2 \rfloor (2d-2)}{(\lfloor d/2 \rfloor + 1)(d-2)}$ -approximation for P4VC (using the lower bound  $\frac{d-k+2}{2d-k+2}|V|$  given by Brešar et al. [6]). Ries et al. [24] also proposed a  $\frac{2d-k+2}{d-k+2}$ -approximation for PkVC when  $1 \le k-2 < d \le 2(k-2)$ .

More specifically on cubic graphs, Tu and Yang [25] gave a  $\frac{25}{16}$  (= 1.5625)-approximation for the P3VC problem; Ries et al. [24] claimed that their approximation algorithm for the P3VC problem can be reduced to 1.389 + o(1). Table 1 summarizes the best approximation ratios prior this work

for the PkVC problem, for  $k \ge 2$ .

PkVC	General graphs	Cubic	4-regular	d-regular $(d \ge 5)$
k = 2	$2 - \Theta\left(\frac{1}{\log V }\right) [18]$	$\min\left\{2-\frac{1}{d}\right\}$	$\frac{5}{+3} + \epsilon \; [3], \; 2 - \epsilon$	$\frac{(2-o(1))\log\log d}{\log d} \ [15] \bigg\}$
k = 3	2 [26]	1.389 + o(1) [24]	$\frac{7}{4} + o(1)$ [24]	$2 - \frac{1}{d} + \frac{4d-2}{3d V }$ [24]
k = 4	3 [8]	2 [10]	2 [10]	$\frac{\lfloor d/2 \rfloor (2d-2)}{(\lfloor d/2 \rfloor+1)(d-2)} \ [10]$
$k \ge 5$	k	n.a.	$\frac{2d-k+2}{d-k+2}$ , when	$k-2 < d \le 2(k-2)$ [24]

Table 1: The best prior approximation ratios for the PkVC problem.

In this paper, we aim to design improved approximation algorithms for the PkVC problem, for  $k \geq 3$ , on *d*-regular graphs. To this purpose, in Section 2 we first employ an existing polynomial time graph defective coloring algorithm to design a simple yet effective approximation algorithm for the PkVC problem, and we are able to show that its approximation ratio is  $\frac{|d/2|(2d-k+2)}{(|d/2|+1)(d-k+2)}$ , when  $1 \leq k-2 < d$ . This beats all the best prior approximation results except for P4VC, where it ties with the approximation by Devi et al. [10] and in particular at 2 on cubic graphs and on 4-regular graphs. In Section 3, we first prove a lower bound on  $\psi_4(G)$  when G is d-regular, then integrate the graph defective coloring algorithm and the current best approximation algorithm for the MIS problem on degree-bounded graphs, to design a  $\frac{15}{8}$ -approximation for P4VC on cubic graphs. When  $d \geq 4$  is even, we show in Section 4 how to compute  $\frac{d}{2}$  4-path vertex covers in G and selecting the minimum one gives a  $\frac{(3d-2)(2d-2)}{(3d+4)(d-2)}$ -approximation for P4VC. This turns out to be a  $\frac{15}{8}$ -approximation for P4VC on 4-regular graphs. Also in Section 4, we are able to provide a better analysis to show that the algorithm is actually a 1.852-approximation on 4-regular graphs, and construct an instance to show that the ratio 1.852 is almost tight for APPROX2 4-regular graphs. Lastly, in Section 5, we propose a  $\frac{d^2}{d^2-d+1}$ -approximation algorithm for P4VC on d-regular graphs. We conclude the paper in Section 6 with some remarks.

# **2** An approximation for PkVC on *d*-regular graphs

We consider the PkVC problem for  $k \ge 3$ , for which the best approximation algorithms prior our work are summarized in Table 1.

As a consequence of Lovász's graph decomposition [22], Cowen and Jesurum [9] have proved the following result on *defective coloring* for any graph of maximum degree  $\Delta$ , where a *defective* (p,q)-coloring colors the vertices of the graph using p colors such that each vertex is adjacent to at most q the same colored neighbors. A (p, 0)-coloring is the classic vertex coloring using p colors.

**Theorem 1** [9] Any graph G = (V, E) of maximum degree  $\Delta$  can be  $\left(p, \left\lfloor \frac{\Delta}{p} \right\rfloor\right)$ -colored in  $O(\Delta|E|)$  time.

Let G = (V, E) be a *d*-regular graph. Using Theorem 1 by setting  $p = \lfloor \frac{d}{2} \rfloor + 1$ , we have a defective (p, 1)-coloring for the graph G. In this defective (p, 1)-coloring, suppose these p colors are  $1, 2, \ldots, p$ ; let  $V^i$  denote the subset of the vertices colored i. Then clearly,

1)  $\{V^1, V^2, \dots, V^p\}$  is a partition of the vertex set V, and

2) the subgraph induced on  $V^i$ ,  $G[V^i]$ , does not contain any 3-path, suggesting  $F^i = V - V^i$  is a 3-path vertex cover (and thus it is also a k-path vertex cover for any  $k \ge 3$ ), for every i = 1, 2, ..., p.

It follows that the minimum among  $F^1, F^2, \ldots, F^p$ , denoted as  $F^{\min}$ , has size

$$|F^{\min}| \le \left(1 - \frac{1}{p}\right)|V|. \tag{1}$$

Recall that when  $d \ge k - 1$ , Brešar et al. [6] have proved the following lower bound on  $\psi_k(G)$ :

$$\psi_k(G) \ge \frac{d-k+2}{2d-k+2} |V|.$$
(2)

Therefore,  $F^{\min}$  turns out to be an approximate solution within ratio

$$\frac{|F^{\min}|}{\psi_k(G)} \le \frac{(p-1)(2d-k+2)}{p(d-k+2)} = \frac{\lfloor d/2 \rfloor (2d-k+2)}{(\lfloor d/2 \rfloor + 1)(d-k+2)}.$$

That is, using the existing graph defective coloring algorithm, we can design an algorithm, denoted as DC, to first compute in  $O(d^2|V|)$  time a defective  $(\lfloor \frac{d}{2} \rfloor + 1, 1)$ -coloring for the input d-regular graph G = (V, E), then in O(d) time find the color i with the most vertices, and last return F containing all the vertices not colored i. See Figure 1 for a high-level description of the algorithm DC. We conclude with Theorem 2.

**Theorem 2** The algorithm DC for the PkVC problem on d-regular graphs, where  $1 \le k - 2 < d$ , is an  $O(d^2|V|)$ -time  $\frac{\lfloor d/2 \rfloor (2d-k+2)}{(\lfloor d/2 \rfloor+1)(d-k+2)}$ -approximation.

The algorithm DC for PkVC on d-regular graphs:
Step 1. Set p = ⌊d/2 ⌋ + 1 and compute a defective (p, 1)-coloring;
Step 2. let i denote the color with the most vertices, and V<sup>i</sup> denote the set of vertices colored i; return F = V − V<sup>i</sup>.

Figure 1: The approximation algorithm DC for PkVC on *d*-regular graphs.

On *d*-regular graphs, our algorithm DC for the P*k*VC problem for  $k \ge 3$  beats all the best prior approximation results except for k = 4 ours ties with the approximation by Devi et al. [10]. Table 2 summarizes the improvement over the corresponding entries in Table 1. From this table, we see that currently there is no published approximation result for P*k*VC on *d*-regular graphs such that  $d \le k-2$  (and  $k \ge 5$ ). For P4VC, the approximation ratios are constants strictly less than 2 for all  $d \ge 5$ , while they are 2 for both d = 3, 4. In the next two sections, we design a  $\frac{15}{8}$ -approximation for P4VC on cubic graphs and a 1.852-approximation for P4VC on 4-regular graphs, respectively.

PkVC	Cubic	4-regular	$d$ -regular ( $d \ge 5$ )
k = 3	$\frac{5}{4}$ (Theorem 2)	$\frac{14}{9}$ (Theorem 2)	$\frac{\lfloor d/2 \rfloor (2d-1)}{(\lfloor d/2 \rfloor+1)(d-1)} $ (Theorem 2)
k = 4	2 [10]	2 [10]	$\frac{\lfloor d/2 \rfloor (2d-2)}{(\lfloor d/2 \rfloor + 1)(d-2)} \ [10]$
$k \ge 5$	n.a.	$\frac{\lfloor d/2 \rfloor (2d-k+2)}{(\lfloor d/2 \rfloor+1)(d-k+2)},$	when $k - 2 < d$ (Theorem 2)

Table 2: The updated approximation ratios for PkVC,  $k \ge 3$ , on regular graphs.

# **3** P4VC on cubic graphs

In this and the next sections, we will design improved approximation algorithms for P4VC on cubic graphs and on 4-regular graphs, with performance ratios  $\frac{15}{8}$  and 1.852, respectively. With them, all the approximation ratios for P4VC on *d*-regular graphs become strictly less than 2.

Let G = (V, E) denote the input *d*-regular graph (we set d = 3 later after we develop a lower bound on general *d*). We first examine some structural properties associated with the optimal 4-path vertex covers in *G*.

**Lemma 1** Let G = (V, E) be a d-regular graph. Then,  $\psi_4(G) \ge \frac{d-1}{2d}|V| - \frac{1}{d^2}f$ , where f denotes the total number of vertices on the 3-cycles in  $G[V - F^*]$  and  $F^*$  is an optimal 4-path vertex cover in G.

PROOF. Let  $F^*$  be an optimal 4-path vertex cover in G; then the subgraph  $G[V - F^*]$  does not contain any 4-path. In other words, all the connected components of  $G[V - F^*]$  can be classified into the following five kinds:

- A) a 1-path (also called a *singleton*),
- B) a 2-path,
- C) a 3-path,
- E) a  $K_{1,\ell}$  claw, for  $\ell = 3, 4, \ldots, d$ , and
- F) a  $C_3$  cycle (also called a *triangle*);

let a, b, c, e, f denote the total numbers of vertices in these five kinds of components, respectively. It follows that

$$|V| = |F^*| + a + b + c + e + f.$$
(3)

Because G is d-regular, the number of edges connecting a vertex of  $F^*$  and a vertex of  $V - F^*$  is at least

$$da + (d-1)b + \left(d - \frac{4}{3}\right)c + \left(\min_{3 \le \ell \le d} \left\{d - 2 + \frac{2}{\ell+1}\right\}\right)e + (d-2)f.$$

Given that each vertex of  $F^*$  can be incident with at most d such edges, and that  $\min_{3 \le \ell \le d} \left\{ d - 2 + \frac{2}{\ell+1} \right\}$  achieves at  $\ell = d$ , we have

$$d|F^*| \ge \left(d-2+\frac{2}{d+1}\right)(a+b+c+e) + (d-2)f = \frac{d(d-1)}{d+1}(|V|-|F^*|) - \frac{2}{d+1}f.$$

It follows from Eq. (3) that, when G is d-regular,

$$\psi_4(G) = |F^*| \ge \frac{d-1}{2d} |V| - \frac{1}{d^2} f.$$
(4)

This proves the lemma.

We now consider only d = 3, that is, G is cubic.

In this case, Lemma 1 (or Eq. (4)) states that  $\psi_4(G) \ge \frac{1}{3}|V| - \frac{1}{9}f$ . Therefore, one sees that when the number of triangles in  $G[V - F^*]$  is small, we can expect this new lower bound to be more effective, compared against the lower bound  $\frac{1}{4}|V|$  stated in Eq. (2). For example, when  $f \le \frac{3}{5}|V|$ , we have

$$\psi_4(G) \ge \frac{1}{9} \left(3 - \frac{3}{5}\right) |V| = \frac{4}{15} |V|.$$

It follows that the defective (2, 1)-coloring for G gives a 4-path vertex cover  $F^{\min}$ , see Eq. (1), satisfying

$$\frac{|F^{\min}|}{\psi_4(G)} \le \frac{1}{2} \times \frac{15}{4} = \frac{15}{8}.$$
(5)

On the other hand, when  $f > \frac{3}{5}|V|$ , that is, there are a considerable number of triangle components in  $G[V - F^*]$ , we will construct a new graph denoted as  $\mathcal{G} = (T, R)$  from the input graph G = (V, E) as follows. For every triangle in G, we create a distinct vertex of T in  $\mathcal{G}$ ; two vertices of T are adjacent if and only if the corresponding two triangles of G share a common edge or they are connected by an edge in G.<sup>1</sup> One may easily verify that the graph  $\mathcal{G} = (T, R)$  can be constructed in O(|V|) time,  $|T| \in O(|V|)$ , and the maximum degree of the vertices of  $\mathcal{G}$  is at most 3. Moreover, a subgraph of G that is a collection of triangle components one-to-one corresponds to an independent set of  $\mathcal{G}$ .

Recall the best approximation algorithm for the MIS problem on degree- $\Delta$  graphs by Berman and Fujito [3] has a performance ratio  $\frac{\Delta+3}{5} + \epsilon$  for any small positive  $\epsilon$ . We next run this approximation algorithm on  $\mathcal{G}$  to obtain an independent set I of  $\mathcal{G}$ , and therefore (roughly, by ignoring  $\epsilon$ )

$$|I| \ge \frac{5}{6}\alpha(\mathcal{G}) \ge \frac{5}{18}f,$$

where  $\alpha(\mathcal{G})$  is the independence number of  $\mathcal{G}$ , which corresponds to the maximum number of nonadjacent triangles in G. Let F denote the set of the vertices of V not on the triangles of I; then Fis a solution to the P4VC problem on G, and its cardinality is

$$|F| \le |V| - \frac{5}{6}f = \frac{5}{8}|V| + \frac{3}{8}|V| - \frac{5}{6}f \le \frac{5}{8}|V| - \frac{5}{24}f,$$
(6)

where the last inequality is due to  $f > \frac{3}{5}|V|$ .

Combining Eq. (6) and the lower bound in Eq. (4), we have

$$\frac{|F|}{\psi_4(G)} \le \frac{5}{8} \times 3 = \frac{15}{8}.$$
(7)

From Eqs. (5) and (7), we can design an algorithm, denoted as APPROX1, to first compute in O(|V|) time a defective (2,1)-coloring for the input cubic graph G = (V, E), then in O(1) time

<sup>&</sup>lt;sup>1</sup>We remark that two distinct triangles of the cubic graph G either share exactly one edge, or have no vertex in common.

find the color with less vertices and set  $F^{\min}$  to contain all these vertices. It also constructs the triangle graph  $\mathcal{G} = (T, R)$  from G = (V, E) and applies the best approximation algorithm for MIS to compute an independent set I in  $\mathcal{G}$ , then it sets F to contain all the vertices of V not on the triangles of I. Lastly, it returns the smaller one between  $F^{\min}$  and F as the final solution. See Figure 2 for a high-level description of the algorithm APPROX1, of which the running time is dominated by the running time of the best approximation algorithm for the MIS problem on degree-3 graphs. We thus conclude with Theorem 3.

The algorithm APPROX1 for P4VC on cubic graphs:
Step 1. Compute a defective (2, 1)-coloring for the input graph G;
Step 2. find the color with less vertices and set F<sup>min</sup> to contain them;
Step 3. construct the triangle graph G from G;
Step 4. call the <sup>6</sup>/<sub>5</sub>-approximation to compute an independent set I in G, and set F to contain all the vertices not on the triangles of I;
Step 5. return the smaller one between F<sup>min</sup> and F.

Figure 2: A high-level description of the approximation algorithm APPROX1 for P4VC on cubic graphs.

**Theorem 3** The algorithm APPROX1 is a  $\frac{15}{8}$ -approximation for the P4VC problem on cubic graphs.

# 4 P4VC on 4-regular graphs

The design ideas in the above algorithm APPROX1 for cubic graphs do not trivially extend to 4regular graphs, for one of the most important reasons that there are many more configurations for two triangles being adjacent (due to degree 4) and the maximum degree of the similarly constructed triangle graph  $\mathcal{G} = (T, R)$  can be as high as 9. Such a high maximum degree voids the effectiveness of the best approximation algorithm for MIS on degree-9 graphs.

We present next an approximation algorithm, denoted as APPROX2, for P4VC on *d*-regular graphs when  $d \ge 4$  is even, and show that its performance ratio is  $\frac{(3d-2)(2d-2)}{(3d+4)(d-2)}$ . When d = 4, the ratio is  $\frac{15}{8} = 1.875$ . We are able to provide a better but slightly more complex analysis for d = 4to show that the approximation ratio is actually no greater than 1.852; we also use an instance to show that the ratio 1.852 is almost tight for APPROX2 when d = 4.

## 4.1 An approximation algorithm when $d \ge 4$ is even

One of the design ideas in our algorithm is borrowed from Devi et al. [10]. Let G = (V, E) denote the input *d*-regular graph, where  $d \ge 4$  is even.

In the algorithm APPROX2, we first compute a subset  $V_1$  of vertices by iteratively adding to it a degree-*d* vertex until no more degree-*d* exists in the remaining graph; then similarly and sequentially compute a subset  $V_i$  of vertices by iteratively adding to it a degree-(d - i + 1) vertex until no more

degree-(d - i + 1) exists in the remaining graph, for  $i = 2, 3, \ldots, d - 2$ . Denote

$$V_{d-1} = V - \bigcup_{i=1}^{d-2} V_i.$$

The last remaining graph is  $G[V_{d-1}]$ , which has maximum degree 2 and thus an optimal (*i.e.*, minimum) 4-path vertex cover, denoted as  $U_{d-1}$ , can be computed in  $O(|V_{d-1}|)$  time.

We will prove in Theorem 4 that  $U_{d-1} \cup \bigcup_{i=1}^{d-2} V_i$  is a 4-path vertex cover of the input graph G, so is  $V - (V_{2i-1} \cup V_{2i})$ , for each  $i = 1, 2, \ldots, \frac{d}{2} - 1$ . The algorithm APPROX2 outputs the smallest among these  $\frac{d}{2}$  covers as the final solution. A high-level description of APPROX2 is depicted in Figure 3, and we prove in Theorem 4 that APPROX2 is an  $O(d^2|V|)$ -time  $\frac{(3d-2)(2d-2)}{(3d+4)(d-2)}$ -approximation for P4VC on d-regular graphs, when  $d \ge 4$  is even.

> The algorithm APPROX2 for P4VC on d-regular graphs  $(d \ge 4 \text{ is even})$ : **Step 1.** Compute  $V_1$  by iteratively adding to it a degree-d vertex until no more degree-d exists in  $G[V - V_1]$ ; **Step 2.** for i = 2, 3, ..., d - 2, similarly and sequentially compute  $V_i$  by iteratively adding to it a degree-(d - i + 1) vertex until no more degree-(d - i + 1) exists in  $G[V - \cup_{j=1}^{i}V_j)]$ ; **Step 3.** set  $V_{d-1} = V - \bigcup_{i=1}^{d-2}V_i$ ; find an optimal 4-path vertex cover  $U_{d-1}$  in  $G[V_{d-1}]$ ; **Step 4.** return the smallest one among  $U_{d-1} \cup \bigcup_{i=1}^{d-2}V_i$  and  $V - (V_{2i-1} \cup V_{2i})$ ,  $i = 1, 2, \ldots, \frac{d}{2} - 1$ .

Figure 3: A high-level description of the approximation algorithm APPROX2 for the P4VC problem on *d*-regular graphs when  $d \ge 4$  is even.

**Theorem 4** For the P4VC problem on d-regular graphs, when  $d \ge 4$  is even, the algorithm AP-PROX2 is an  $O(d^2|V|)$ -time  $\frac{(3d-2)(2d-2)}{(3d+4)(d-2)}$ -approximation.

PROOF. First of all, since G = (V, E) is *d*-regular, we have  $|E| = \frac{d}{2}|V|$ ; therefore,  $V_i$  is computed in O(d|V|) time, for each i = 1, 2, ..., d-2. Computing  $U_{d-1}$  needs only O(|V|) time since each connected component of  $G[V_{d-1}]$  is either a simple cycle or a simple path. That is, the running time of APPROX2 is in  $O(d^2|V|)$ .

Next, we conclude that the vertices of  $V_i$ , for each *i*, are pairwise non-adjacent to each other. Also, in the induced subgraph graph  $G[V - \bigcup_{j=1}^{2i-2} V_j]$  which has maximum degree d - 2i + 2, since every vertex of the computed subset  $V_{2i-1}$  has degree exactly d - 2i + 2, a vertex of  $V_{2i}$  can be adjacent to at most one vertex of  $V_{2i-1}$ . These two properties suggest that the longest path in the subgraph induced on  $V_{2i-1} \cup V_{2i}$ ,  $G[V_{2i-1} \cup V_{2i}]$ , contains at most three vertices (two of  $V_{2i}$  and one of  $V_{2i-1}$ ), that is,  $V - (V_{2i-1} \cup V_{2i})$  is a 4-path vertex cover in G, for each  $i = 1, 2, \ldots, \frac{d}{2} - 1$ . On the other hand, since  $U_{d-1}$  is a 4-path vertex cover in  $G[V_{d-1}]$ ,  $U_{d-1} \cup \bigcup_{i=1}^{d-2} V_i$  is a 4-path vertex cover in G. This proves that the solution returned by APPROX2 is feasible.

Recall that each connected component of  $G[V_{d-1}]$  is either a simple cycle or a simple path. The optimal 4-path vertex cover  $U_{d-1}$  in  $G[V_{d-1}]$  contains exactly  $\lfloor \frac{k}{4} \rfloor$  vertices from each k-path, and

contains exactly  $\lceil \frac{k}{4} \rceil$  vertices from each k-cycle [1, 6]. From the fact that  $\lfloor \frac{k}{4} \rfloor \leq \frac{1}{4}k$  and  $\lceil \frac{k}{4} \rceil \leq \frac{2}{5}k$  for all  $k \geq 4$ , we conclude that

$$|U_{d-1}| \le \frac{2}{5} |V_{d-1}|$$

Consequently, the size of  $U_{d-1} \cup \bigcup_{i=1}^{d-2} V_i$  is

$$|U_{d-1}| + \sum_{i=1}^{d-2} |V_i| \le \frac{2}{5} |V| + \frac{3}{5} \sum_{i=1}^{d-2} |V_i|.$$
(8)

It follows that the minimum cardinality of these  $\frac{d}{2}$  4-path vertex covers is at most

$$\frac{6}{3d+4}\sum_{i=1}^{\frac{d}{2}-1}\left(|V|-|V_{2i-1}|-|V_{2i}|\right) + \frac{10}{3d+4}\left(\frac{2}{5}|V| + \frac{3}{5}\sum_{i=1}^{d/2-1}\left(|V_{2i-1}|+|V_{2i}|\right)\right) = \frac{3d-2}{3d+4}|V|.$$

From Eq. (2) we obtain the lower bound of  $\frac{d-2}{2d-2}|V|$  on  $\psi_4(G)$  using k = 4 [6], and therefore we prove that the algorithm APPROX2 has an approximation ratio of  $\frac{(3d-2)(2d-2)}{(3d+4)(d-2)}$ . Note that due to  $\frac{3d-2}{3d+4} < \frac{d/2}{d/2+1}$ , the above ratio is strictly less than that stated in Theorem 2 (or the one by Devi et al. [10]).

**Corollary 1** For the P4VC problem on 4-regular graphs, the algorithm APPROX2 is an O(|V|)-time  $\frac{15}{8}$ -approximation.

#### 4.2 APPROX2 is a 1.852-approximation

In this section, we present a better analysis for our algorithm APPROX2 than what is done in the proof of Theorem 4. Theorem 4 leads to the conclusion in Corollary 1 that APPROX2 is a  $\frac{15}{8}$ -approximation for PkVC on 4-regular graphs. Our better analysis shows that the performance ratio of APPROX2 is actually at most 1.852.

Recall that when d = 4, our algorithm APPROX2 (see Figure 3) computes  $V_1$  and  $V_2$ , and computes an optimal 4-path vertex  $U_3$  in  $G[V_3]$ , where  $V_3 = V - (V_1 \cup V_2)$ . Both  $V_3$  and  $U_3 \cup (V_1 \cup V_2)$ are feasible 4-path vertex covers in the input graph G. APPROX2 returns the smaller one between  $V_3$  and  $U_3 \cup (V_1 \cup V_2)$ , denoted as A. In the following we denote  $V_1 \cup V_2$  as  $V_{1,2}$ .

#### 4.2.1 An outline of the analysis

Throughout the analysis, we fix an arbitrary optimal 4-path vertex cover B in G for discussion.

We color the vertices of *B* black and color the other vertices of *G* (that is, V - B) white. An edge of *G* is black (respectively, white) if both of its endpoints are black (respectively, white); an edge of *G* neither black nor white is bicolor. Using this coloring scheme, a bicolor  $(V_{1,2}, V_3)$ -edge has its endpoint in  $V_{1,2}$  white and its endpoint in  $V_3$  black. Bicolor  $(V_{1,2}, V_{1,2})$ -edges,  $(V_3, V_{1,2})$ -edges, and  $(V_3, V_3)$ -edges are defined similarly.

Recall that each connected component of G[V - B] is one of the following five kinds:

- A) a 1-path,
- B) a 2-path,
- C) a 3-path,

- E) a  $K_{1,\ell}$  claw, for  $\ell = 3, 4$ , and
- F) a triangle (*i.e.*, a  $C_3$  cycle).

We merge the first four kinds and name them uniformly a *star*. The *center* vertex of a star S is the one with the maximum degree (tie broken arbitrarily), and all the other vertices (can be 0, 1, 2, 3, or 4 of them) are referred to as the *satellites* of S. This way, each connected component of G[V-B] is either a triangle or a star.

Our goal is to show that  $|A| \leq 1.852|B|$ . Let  $F = U_3 \cup V_{1,2}$ . Since  $|A| = \min\{|V_3|, |F|\} \leq (1 - \alpha)|V_3| + \alpha|F|$  for any coefficient  $0 \leq \alpha \leq 1$ , it suffices to show that there is a constant  $0 \leq \alpha \leq 1$  such that  $(1 - \alpha)|V_3| + \alpha|F| \leq 1.852|B|$ . In the remainder of this section, we show that  $\alpha = 0.556$  satisfies this inequality. To reach our goal, we first make two important observations summarized in the next two lemmas, respectively.

**Lemma 2** Let  $b_e$  be the number of black edges in G, and  $s_c$  be the number of star components in G[V-B]. Then,  $|B| = \frac{1}{3}|V| + \frac{1}{3}b_e + \frac{1}{3}s_c$ .

PROOF. We apply a similar counting as in the proof of Lemma 1. Let  $x_i$  denote the number of star components in which the star has *i* satellites, for i = 0, 1, 2, 3, 4; and *y* denote the number of triangle components. It follows that

$$|V| = |B| + \sum_{i=0}^{4} (i+1)x_i + 3y_i$$

Because G is 4-regular and there are  $b_e$  black edges, the number of bicolor edges (each connecting a vertex of B and a vertex of V - B) is exactly

$$4|B| - 2b_e = \sum_{i=0}^{4} 2(i+2)x_i + 6y.$$

By eliminating y from the above two equalities, we have

$$2|V| - 2|B| - \sum_{i=0}^{4} 2(i+1)x_i = 4|B| - 2b_e - \sum_{i=0}^{4} 2(i+2)x_i.$$

Using  $s_c = \sum_{i=0}^{4} x_i$ , we have  $3|B| = |V| + b_e + s_c$ . This proves the lemma.

**Lemma 3** Let  $p_{3\downarrow}$  (respectively,  $p_{4\uparrow}$ ) be the total number of vertices in those connected components of  $G[V_3]$  each is a path of order at most 3 (respectively, at least 4). Let  $c_3$  (respectively,  $c_{4,6\uparrow}$ ) be the total number of vertices in those connected components of  $G[V_3]$  each is a cycle of order exactly 3 (respectively, exactly 4 or at least 6). Then,  $|U_3| \leq \frac{2}{5}|V_3| - \frac{2}{5}(p_{3\downarrow} + c_3) - \frac{3}{20}p_{4\uparrow} - \frac{1}{15}c_{4,6\uparrow}$ .

PROOF. It is known that  $\psi_4(P_\ell) = \lfloor \frac{\ell}{4} \rfloor$  and  $\psi_4(C_\ell) = \lceil \frac{\ell}{4} \rceil$ , where  $P_\ell$  and  $C_\ell$  are a simple path and a simple cycle of order  $\ell$ , respectively [1, 6]. Let  $c_5$  be the total number of vertices in those connected components of  $G[V_3]$  each is a cycle of order exactly 5. Therefore,

$$|U_3| \le \frac{1}{4}p_{4\uparrow} + \frac{1}{3}c_{4,6\uparrow} + \frac{2}{5}c_5.$$

Using  $|V_3| = p_{3\downarrow} + p_{4\uparrow} + c_3 + c_{4,6\uparrow} + c_5$  to cancel out  $c_5$ , we achieve the inequality stated in the lemma. 

By the above Lemmas 2 and 3, it remains to show the following inequality:

$$\frac{\alpha |V_1| + \alpha |V_2| + (1 - \frac{3\alpha}{5})|V_3| - \frac{2\alpha}{5}(p_{3\downarrow} + c_3) - \frac{3\alpha}{20}p_{4\uparrow} - \frac{\alpha}{15}c_{4,6\uparrow}}{\frac{1}{3}|V| + \frac{1}{3}b_e + \frac{1}{3}s_c} \le 1.852.$$
(9)

In  $|B| = \frac{1}{3}|V| + \frac{1}{3}b_e + \frac{1}{3}s_c$  (the denominator in Eq. (9)), we call  $\frac{1}{3}|V|$  the basic lower bound on

 $|B| \text{ and call } \frac{1}{3}b_e + \frac{1}{3}s_c \text{ the extra lower bound on } |B|.$ Similarly, in  $|A| \le \alpha |V_1| + \alpha |V_2| + (1 - \frac{3\alpha}{5})|V_3| - \frac{2\alpha}{5}(p_{3\downarrow} + c_3) - \frac{3\alpha}{20}p_{4\uparrow} - \frac{\alpha}{15}c_{4,6\uparrow}$  (the numerator in Eq. (9)), we call  $\alpha |V_1| + \alpha |V_2| + (1 - \frac{3\alpha}{5})|V_3|$  the basic upper bound on |A| and call  $\frac{2\alpha}{5}(p_{3\downarrow} + c_3) + \frac{2\alpha}{5}(p_{3\downarrow} + c_3)$  $\frac{3\alpha}{20}p_{4\uparrow} + \frac{\alpha}{15}c_{4,6\uparrow}$  the saving on |A|.

Roughly speaking, we used only the basic lower bound on |B| (as in Eq. (2) and the basic upper bound |A| (with  $\alpha = 1$ , as in Eq. (8)) in the proof of Theorem 4 (when d = 4). In other words, Lemma 2 gives a better lower than Eq. (2) when k = 4 and d = 4, and Lemma 3 gives a better estimation than Eq. (8) when d = 4. The extra lower bound and the saving will help us get a better analysis.

It seems difficult to verify Eq. (9) if we consider the graph G as a whole. So, to ease the proof of Eq. (9), we consider the following two kinds of subgraphs of G and want to verify Eq. (9) on each of these subgraphs.

- Type-1: The subgraphs of this type one-to-one correspond to the connected components of G[V-B], and they are constructed as follows. Consider a connected component K of G[V-B], which is either a  $C_3$  cycle or a star, and all its vertices are white. Let K' be the subgraph of G induced by the vertices of K and their black neighbors in G. Let H be the graph obtained from K' by deleting all black edges. Then, H is the type-1 subgraph corresponding to K.
- Type-2: The subgraphs of this type one-to-one correspond to the black edges in G. That is, the subgraph corresponding to a black edge e consists of only e and its two ending black vertices.

Let  $\mathcal{G}_1$  (respectively,  $\mathcal{G}_2$ ) be the collection of type-1 (respectively, type-2) subgraphs in G. Obviously, each white vertex of G appears in exactly one subgraph in  $\mathcal{G}_1 \cup \mathcal{G}_2$ . In contrast, a black vertex of G can appear in one or more subgraphs in  $\mathcal{G}_1 \cup \mathcal{G}_2$ . Nevertheless, each edge of G appears in exactly one subgraph in  $\mathcal{G}_1 \cup \mathcal{G}_2$ .

To prove Eq. (9), we proceed as follows:

- Step 1: Distribute the numerator and the denominator of the left hand side of the inequality to the subgraphs in  $\mathcal{G}_1 \cup \mathcal{G}_2$ .
- Step 2: Prove that for each subgraph H in  $\mathcal{G}_1 \cup \mathcal{G}_2$ ,  $\frac{n_H}{d_H} \leq 1.852$ , where  $n_H$  (respectively,  $d_H$ ) is the portion of the numerator (respectively, denominator) of the left hand side of the inequality distributed to H.

The next three subsections are devoted to detailing the above two steps, respectively. After these two steps, we are done because for any sequence of positive numbers  $x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k$  $\frac{x_i}{y_i} \le 1.852$  for all  $i = 1, 2, \dots, k$  implies  $\frac{\sum_{i=1}^k x_i}{\sum_{i=1}^k y_i} \le 1.852$ .

#### 4.3 Distributing the denominator

Initially, we distribute the basic lower bound (namely,  $\frac{1}{3}|V|$ ) evenly to the edges in G so that each edge holds a basic lower bound of  $\frac{1}{6}$ ; we further distribute the extra lower bound (namely,  $\frac{1}{3}b_e + \frac{1}{3}s_c$ ) evenly to the black edges in G and the star components of G[V - B] so that each black edge holds an extra lower bound of  $\frac{1}{3}$  and so does each star component of G[V - B].

If a 5-cycle  $C_5$  in  $G[V_3]$  has at least one black edge, then it is good; otherwise, it is bad. Consider a bad 5-cycle  $C_5$  in  $G[V_3]$ . Since B is a solution (*i.e.*, 4-path vertex cover),  $C_5$  must have exactly two black vertices and a unique white edge. Let e be the white edge in  $C_5$ , and v be the white vertex of  $C_5$  that is not an endpoint of e. We call v the *independent white vertex* in  $C_5$ . If v or e appears in a star component of G[V - B] or at least one vertex of  $C_5$  is incident to a black edge in G, then  $C_5$  is *slightly bad*; otherwise,  $C_5$  is *very bad*. A simple but important observation is that no star component of G[V - B] can contain both v and e. A  $(V_{1,2}, V_3)$ -edge of G is good if its black endpoint either is an endpoint of a black edge in  $G[V_3]$  or appears in a good or slightly bad 5-cycle of  $G[V_3]$ .

First, consider a connected component K of  $G[V_3]$  that has at least one black edge. Let p be the number of black edges in K, and q be the number of good  $(V_{1,2}, V_3)$ -edges in G whose black endpoints appear in K. We collect the extra lower bounds held by the black edges of K; the total is obviously  $\frac{1}{3}p$ . From this total, we distribute 0.015q evenly to the q good  $(V_{1,2}, V_3)$ -edges so that each of them receives 0.015, and then distribute the remaining (namely,  $\frac{1}{3}p - 0.015q$ ) to the p black edge so that each of them receives  $\frac{1}{3} - \frac{0.015q}{p}$ . Since  $\frac{q}{p} \leq 6$ , each black edge in K still holds an extra lower bound of  $\geq \frac{73}{300}$ .

Next, consider a slightly bad 5-cycle  $C_5$  in  $G[V_3]$ . Let e be the white edge in  $C_5$ , and v be the independent white vertex of  $C_5$ . We transfer a portion of the extra lower bound (namely,  $\frac{1}{3}$ ) held by  $C_5$  as follows (three possible cases):

- Suppose that *e* appears in a star component *K* of G[V-B]. Then, we say that  $C_5$  is of *type-1*. Among the extra lower bound (namely,  $\frac{1}{3}$ ) held by *K*, we transfer 0.015 to each good  $(V_{1,2}, V_3)$ -edge whose black endpoint appears in  $C_5$ . Obviously,  $C_5$  is the unique bad 5-cycle whose white edge appears in *K*. Moreover, there are exactly 4 good  $(V_{1,2}, V_3)$ -edges whose black endpoints appear in  $C_5$ . Thus, the extra lower bound still held by *K* is  $\frac{1}{3} - 4 \times 0.015 = \frac{41}{150}$ .
- Suppose that v appears in a star component K of G[V-B] but e does not. Then, we say that  $C_5$  is of type-2. Among the extra lower bound (namely,  $\frac{1}{3}$ ) held by K, we transfer 0.015 to each good  $(V_{1,2}, V_3)$ -edge whose black endpoint appears in  $C_5$ . Obviously, there are at most 4 bad 5-cycles whose independent white vertices appear in K. Moreover, each bad 5-cycle contains the black endpoints of exactly 4 good  $(V_{1,2}, V_3)$ -edges. Thus, the extra lower bound still held by K is  $\frac{1}{3} 4 \times 4 \times 0.015 = \frac{7}{75}$ .
- Suppose that at least one vertex of  $C_5$  is incident to a black edge in G. Then, we say that  $C_5$  is of *type-3*. We choose an arbitrary black edge g such that one endpoint of g is in  $C_5$ . Among the extra lower bound (namely,  $\frac{1}{3}$ ) held by g, we transfer 0.015 to each good  $(V_{1,2}, V_3)$ -edge whose black endpoint appears in  $C_5$ . Obviously, there are at most 3 good  $(V_{1,2}, V_3)$ -edges whose black endpoints appear in  $C_5$ . Thus, the extra lower bound still held by g is at least  $\frac{1}{3} 3 \times 0.015 = \frac{173}{600}$ .

Table 3 summarizes how the basic and the extra lower bounds have been distributed to the edges of G and the star components of G[V - B].

Table 3: Fractional lower bounds distributed to the edges of G and the star components of G[V-B].

From lower bound	basic: $\frac{1}{3} V $		extra: $\frac{1}{3}b_e + \frac{1}{3}s_c$	
Entity	an edge	a black edge	a good $(V_{1,2}, V_3)$ -edge	a star component
Fraction received	$\frac{1}{3}$	$\geq \frac{73}{300}$	0.015	$\geq \frac{7}{75}$

#### 4.4 Distributing the numerator

Initially, we distribute the basic upper bound on A to the vertices of G so that each  $v \in V_{1,2}$  receives  $\alpha$  while each  $v \in V_3$  receives  $1 - \frac{3\alpha}{5}$ . Moreover, we distribute the saving on |A| to the vertices in  $V_3$  so that each vertex in a 3-cycle or a path of order at most 3 in  $G[V_3]$  receives  $\frac{2}{5}\alpha$ , each vertex in a path of order at least 4 vertices in  $G[V_3]$  receives  $\frac{3}{20}\alpha$ , and each vertex in a cycle of order exactly 4 or at least 6 in  $G[V_3]$  receives  $\frac{1}{15}\alpha$ .

Since a black vertex of G may belong to two or more subgraphs in  $\mathcal{G}_1 \cup \mathcal{G}_2$ , we next distribute the basic upper bound (namely,  $\alpha$ ) held by each black  $v \in V_1 \cup V_2$  evenly to the edges incident to v in G. For the same reason, we distribute the basic upper bound (namely,  $1 - \frac{3\alpha}{5}$ ) held by each black vertex  $v \in V_3$  evenly to the edges incident to v in G. Table 4 summarizes how the basic upper bounds have been distributed.

Table 4: Fractional basic upper bounds distributed to the entities in G.

		From basic upper bound: $\alpha  V_1  + \alpha  V_2  + (1 - \frac{3\alpha}{5})  V_3 $						
Entity	a white vertex		a black vertex	a bicolor edge		a black edge		
	in	in	or	black end	black end	both ends	both ends	others
	$V_{1,2}$	$V_3$	a white edge	in $V_{1,2}$	in $V_3$	in $V_{1,2}$	in $V_3$	
Received	α	$\frac{5-3\alpha}{5}$	0	$\frac{1}{4}\alpha$	$\frac{5-3\alpha}{20}$	$\frac{1}{2}\alpha$	$\frac{5-3\alpha}{10}$	$\frac{5+2\alpha}{20}$

We further consider transferring the savings held by the vertices in  $V_3$ . In some cases, we might discard all or a portion of the saving held by a vertex. In other words, we will show that Eq. (9) still holds even if we discard a portion of the saving on  $|A|^2$ 

A  $(V_{1,2}, V_3)$ -edge of G is saving if its black endpoint not only has no black neighbor in  $G[V_3]$  but also appears in a connected component of  $G[V_3]$  that is not a 5-cycle. Consider such a connected component K of  $G[V_3]$  (that is,  $K \neq C_5$ ). For each white vertex v of K, we perform the following:

• If v has no black neighbor in K (case 1), then we discard the whole saving held by v; otherwise (case 2), we distribute the whole saving held by v evenly to its neighbors in K.

After we have done the above for all the white vertices of K, for each black vertex u of K, if u is incident to no saving  $(V_{1,2}, V_3)$ -edge in G, then we discard the whole saving held by u; otherwise, we perform the following:

• When K is a cycle, u is incident to one or two saving  $(V_{1,2}, V_3)$ -edges in G and u now holds a saving of at least  $\frac{1}{15}\alpha + 2 \times \frac{1}{30}\alpha = \frac{2}{15}\alpha$ , in which the original saving held by u was at least  $\frac{1}{15}\alpha$  and u received at least  $\frac{1}{30}\alpha$  from each of its white neighbors in K. We distribute the

<sup>&</sup>lt;sup>2</sup>This suggests our final ratio 1.852 is not strictly tight, but only almost tight.

whole saving held by u evenly to the saving  $(V_{1,2}, V_3)$ -edges incident to u in G. Note that each saving  $(V_{1,2}, V_3)$ -edge incident to u receives a saving of at least  $\frac{1}{15}\alpha$ .

- When K is a singleton u, u is incident to at most four saving  $(V_{1,2}, V_3)$ -edges in G and the saving held by u is  $\frac{2}{5}\alpha$ . We distribute the whole saving held by u evenly to the saving  $(V_{1,2}, V_3)$ -edges incident to u in G. Each saving  $(V_{1,2}, V_3)$ -edge incident to u receives a saving of at least  $\frac{1}{10}\alpha$ .
- When K is a path of order at least 2, u is incident to at most three saving  $(V_{1,2}, V_3)$ -edges in G and u now holds a saving of at least  $\frac{3}{20}\alpha + \frac{3}{40}\alpha = \frac{9}{40}\alpha$ , in which the original saving held by u was at least  $\frac{3}{20}\alpha$  and u received at least  $\frac{3}{40}\alpha$  from its white neighbors in K. We distribute the whole saving held by u evenly to its incident saving  $(V_{1,2}, V_3)$ -edges. Each saving  $(V_{1,2}, V_3)$ -edge incident to u receives a saving of at least  $\frac{3}{40}\alpha$ .

Table 5 summarizes how the savings have been distributed.

Table 5: Savings distributed to the entities in G.

	From saving: $\frac{2\alpha}{5}(p_{3\downarrow}+c_3)+\frac{3\alpha}{20}p_{4\uparrow}+\frac{\alpha}{15}c_{4,6\uparrow}$			
Entity	saving $(V_{1,2}, V_3)$ -edge	any other edge	any vertex	
Received	$\geq \frac{1}{15}\alpha$	0	0	

## 4.5 Verifying Eq. (9) on subgraphs

For a subgraph H of G, we define the following notations:

- $u_H$ : the total basic upper bound distributed to the vertices and edges of H;
- $s_H$ : the total saving distributed to the saving  $(V_{1,2}, V_3)$ -edges of H;
- $\ell^b_H$ : the total basic lower bound distributed to the edges of H;
- $\ell_H^e$ : the total extra lower bound distributed to subgraphs of *H*.

Our goal is to show that, for each  $H \in \mathcal{G}_1 \cup \mathcal{G}_2$ ,  $\frac{u_H - s_H}{\ell_H^b + \ell_H^e} \leq 1.852$ .

Unfortunately, we may fail to do this for some subgraphs  $H \in \mathcal{G}_1$ , because of very bad 5-cycles. We call a  $(V_{1,2}, V_3)$ -edge very bad if its black endpoint appears in a very bad 5-cycle in  $G[V_3]$ . Compared with other bicolor edges, very bad  $(V_{1,2}, V_3)$ -edges hold large basic upper bounds but neither savings nor extra lower bounds.

For each very bad 5-cycle C in  $G[V_3]$ , the white edge of C appears in a 3-cycle in G[V - B]while the independent white edge of C appears in another 3-cycle C' in G[V - B]. We call C'the companion 3-cycle of C (see Figure 4(a)). From Figure 4(a), we can see that each very bad 5-cycle can contain the black endpoints of at most four very bad  $(V_{1,2}, V_3)$ -edges in G; we also see that C' can accommodate the white endpoints of up to four very bad  $(V_{1,2}, V_3)$ -edges in G. Thus,  $|X| \leq |Y|$ , where X is the set of very bad  $(V_{1,2}, V_3)$ -edges in G whose black endpoints are in C but whose white endpoints are not in C', and Y is the set of  $(V_{1,2}, V_3)$ - or  $(V_{1,2}, V_{1,2})$ -edges in G whose white endpoints are in C' but whose black endpoints are not in C. So, we can find an injection ffrom X to Y. As long as we are only concerned with the proof of Eq. (9), it is no harm to exchange

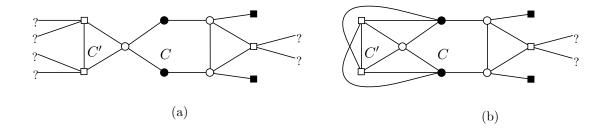


Figure 4: A very bad 5-cycle C and its companion 3-cycle C', where circle vertices are in  $V_3$ , square vertices are in  $V_{1,2}$ , and each question mark is a black circle or a black square.

the black endpoint of each  $x \in X$  with the black endpoint of f(x), because the exchange does not change the total basic or extra lower bound, the total basic upper bound, or the total saving held by x and f(x). Therefore, we may assume without loss of generality that:

• (Assumption 1.) For every very bad 5-cycle C in  $G[V_3]$ , each black vertex of C is adjacent to the two white vertices of the companion 3-cycle C' (see Figure 4(b)).

Now, we are ready to show that  $\frac{u_H - s_H}{\ell_H^b + \ell_H^e} \leq 1.852$ , for each  $H \in \mathcal{G}_1 \cup \mathcal{G}_2$ . We remark that all these four quantities  $u_H, s_H, \ell_H^b, \ell_H^e \geq 0$  are non-negative, for every  $H \in \mathcal{G}_1$ .

First, consider an  $H \in \mathcal{G}_2$ , which is a black edge. From Table 3,  $\ell_H^b = \frac{1}{6}$  and  $\ell_H^e \geq \frac{73}{300}$ . From Table 4,  $u_H \leq 2 \times \frac{5-3\alpha}{10} = \frac{5-3\alpha}{5}$  because  $\alpha = 0.556 < \frac{5}{8}$ . It follows that  $\frac{u_H - s_H}{\ell_H^b + \ell_H^e} \leq 1.852$ .

Next, consider an  $H \in \mathcal{G}_1$  corresponding to a star component K of G[V - B]. Note that K can have up to 4 satellites and both  $V_{1,2}$  and  $V_3$  may contain zero or more vertices of K. Since  $\alpha = 0.556$ , a tedious but easy analysis shows that  $\frac{u_H - s_H}{\ell_H^b + \ell_H^e}$  is maximized when

• K is a star with 5 vertices, the center of K is in  $V_1 \cup V_3$ , and each satellite of K appears in a bad 5-cycle of  $G[V_3]$ .

In this worst case, from Table 4,  $u_H \leq \alpha + 4 \times \frac{5-3\alpha}{5} + 4 \times \left(2 \times \frac{5-3\alpha}{20} + \frac{1}{4}\alpha\right) = \frac{30-8\alpha}{5}$ . Moreover, from Table 3,  $\ell_H^b = 16 \times \frac{1}{3} = \frac{8}{3}$ , and  $\ell_H^e \geq \frac{7}{75}$ . Thus,  $\frac{u_H - s_H}{\ell_H^b + \ell_H^e} \leq 1.852$ .

Next consider an  $H \in \mathcal{G}_1$  corresponding to a 3-cycle  $\tilde{K}$  of G[V-B]. Note that H has 6 black vertices. We distinguish the following three cases:

- Case 1: All vertices of K belong to  $V_3$ . In this case, all black vertices of H belong to  $V_{1,2}$ . Thus,  $u_H \leq 3 \times \frac{5-3\alpha}{5} + 6 \times \frac{1}{4}\alpha = \frac{30-3\alpha}{10}$  from Table 4,  $s_H \geq 3 \times \frac{1}{15}\alpha = \frac{1}{5}\alpha$  from Table 5, and  $\ell_H^b = 9 \times \frac{1}{6} = \frac{3}{2}$  from Table 3. So,  $\frac{u_H - s_H}{\ell_H^b + \ell_H^e} \leq 1.852$ .
- Case 2: Exactly one vertex of K belongs to  $V_3$ . In this case, both  $V_{1,2}$  and  $V_3$  may contain some black vertices of H. Since  $\alpha = 0.556$ , a tedious but easy analysis shows that  $\frac{u_H s_H}{\ell_H^b + \ell_H^e}$  is maximized when all black vertices of H belong to  $V_3$ . In this worst case,  $u_H \leq \frac{5-3\alpha}{5} + 2\alpha + 6 \times \frac{5-3\alpha}{20} = \frac{5+\alpha}{2}$  from Table 4, and  $\ell_H^b = 9 \times \frac{1}{6}$  from Table 3. So,  $\frac{u_H s_H}{\ell_H^b + \ell_H^e} \leq 1.852$ .
- Case 3: Exactly one vertex u of K belongs to  $V_{1,2}$ . In this case, at least two black vertices of H belong to  $V_{1,2}$ . Moreover, by Assumption 1, each bicolor edge incident to u in H is either a  $(V_{1,2}, V_{1,2})$ -edge or a good or saving  $(V_{1,2}, V_3)$ -edge in K. Since  $\alpha = 0.556$ , a tedious but easy analysis shows that  $\frac{u_H s_H}{\ell_H^b + \ell_H^e}$  is maximized when exactly four black vertices of H belong

to  $V_{1,2}$ . In this worst case,  $u_H \leq 2 \times \frac{5-3\alpha}{5} + \alpha + 2 \times \frac{5-3\alpha}{20} + 4 \times \frac{1}{4}\alpha = \frac{10+\alpha}{5}$  from Table 4, while  $\ell_H^b = 9 \times \frac{1}{6}$  from Table 3. So,  $\frac{u_H - s_H}{\ell_H^b + \ell_H^e} \leq 1.852$ .

In summary, we have proved the following theorem.

**Theorem 5** For the PkVC problem on 4-regular graphs, the algorithm APPROX2 actually is a 1.852-approximation.

Table 6 summarizes the improvement we have made for approximating the PkVC problem on regular graphs. Comparing against Table 1, all previously shown approximation ratios have been reduced except those for P4VC on *d*-regular graphs where  $d \ge 5$  is odd.

Table 6: The improved approximation ratios for PkVC,  $k \geq 3$ , on regular graphs.

$\mathbf{P}k\mathbf{VC}$	Cubic	4-regular	d-regular ( $d \ge 5$ is odd)	d-regular ( $d \ge 6$ is even)
k = 3			$\frac{\lfloor d/2 \rfloor (2d-1)}{(\lfloor d/2 \rfloor + 1)(d-1)} $ (Thm 2)	
k = 4	$\frac{15}{8}$ (Thm 3)	1.852 (Thm 5)	$(\lfloor a/2 \rfloor + 1)(a/2)$	$\frac{(3d-2)(2d-2)}{(3d+4)(d-2)} $ (Thm 4)
$k \ge 5$	n.a.	$\frac{\lfloor d/2 \rfloor (2d-k+2)}{(\lfloor d/2 \rfloor+1)(d-k+2)}, \text{ when } k-2 < d \text{ (Thm 2)}$		

#### 4.6 Almost tight instances

The analysis of our algorithm that leads to Theorem 5 gives us a constant  $\alpha = 0.556$  such that  $\frac{(1-\alpha)|V_3|+\alpha|V_{1,2}\cup U_3|}{|B|} \leq 1.852$ . One may wonder whether we can change the value of  $\alpha$  so that a smaller upper bound can be achieved. To answer this question, let us consider the two example graphs  $G_1$  and  $G_2$  in Figure 5.

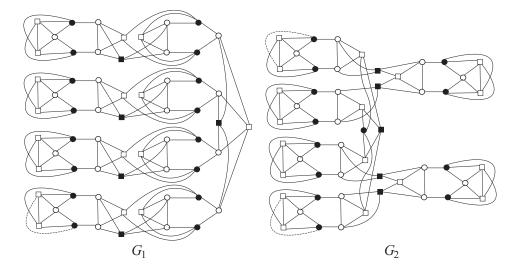


Figure 5: Two example graphs for showing the ratio 1.852 is almost tight for our algorithm AP-PROX2. In each graph, the circle vertices are in  $V_3$ , the square vertices are in  $V_{1,2}$ , respectively, and the filled (black) vertices are in the optimal 4-path vertex cover B.

For  $G_1$ , |B| = 21,  $|V_3| = 40$ , and  $|V_{1,2} \cup U_3| = 38$ ; for  $G_2$ , |B| = 18,  $|V_3| = 31$ , and  $|V_{1,2} \cup U_3| = 35$ . Our algorithm outputs  $V_{1,2} \cup U_3$  for  $G_1$  and outputs  $V_3$  for  $G_2$ . Solving the equation  $\frac{40(1-\alpha)+38\alpha}{21} = \frac{31(1-\alpha)+35\alpha}{18}$  yields  $\alpha = \frac{23}{40} = 0.575$ , we thus cannot get a better upper bound on  $\frac{(1-\alpha)|V_3|+\alpha|V_{1,2}\cup U_3|}{|B|}$  than  $\frac{31\times(1-\frac{23}{40})+35\times\frac{23}{40}}{18} = 1.85$ . Changing the value of  $\alpha$  could possibly achieve a smaller upper bound than 1.852, but only marginally.

In fact, using the above  $G_1$  and  $G_2$ , we can show that 1.85 is a lower bound on the worst-case performance ratio of our algorithm APPROX2. We make a copy of  $G_1$ , denoted as  $G_1^c$ , and create a new graph G by letting  $G_1$ ,  $G_1^c$ , and  $G_2$  be its three connected components; or furthermore we can modify G to be connected by exchanging the black endpoint of the dashed edge in  $G_1$  (respectively, in  $G_1^c$ ) with that of a dashed edge in  $G_2$  (see Figure 5). Either way, for the new graph G we have  $|B| = 21 \times 2 + 18 = 60$ ,  $|V_3| = 40 \times 2 + 31 = 111$ , and  $|V_{1,2} \cup U_3| = 38 \times 2 + 35 = 111$ . Therefore, the approximation ratio achieved by our algorithm APPROX2 on G is  $\frac{111}{60} = 1.85$ .

## 5 P4VC on regular bipartite graphs

Devi et al. [10] have shown that P4VC is APX-complete on cubic bipartite graphs. On the positive side, the 2-approximation for P3VC on (any) bipartite graphs by Kumar et al. [21] is also a 2-approximation for P4VC on (any) bipartite graphs. In this section, we present a  $\frac{d^2}{d^2-d+1}$ approximation for P4VC on *d*-regular bipartite graphs, for any  $d \ge 3$ . Note that this ratio is (much) smaller than the corresponding ratio for P4VC on *d*-regular graphs as listed in Table 6, and it approaches 1 with increasing *d*.

Recall that Eq. (4) provides a lower bound on  $\psi_4(G)$  when G is d-regular, using the total number f of vertices in all the triangle components of the induced subgraph  $G[V - F^*]$  where  $F^*$  is an optimal 4-path vertex cover. When G is also bipartite, there is no triangle in G, that is, f = 0. Thus, Eq. (4) becomes

$$\psi_4(G) \ge \frac{d-1}{2d} |V|. \tag{10}$$

Let A and B denote the two parts in the graph, *i.e.*, G = (A, B, E) is d-regular bipartite, where each edge of E connects a vertex of A and a vertex of B. One clearly sees that  $|A| = |B| = \frac{1}{2}|V|$ .

In the following algorithm for P4VC, denoted as APPROX3, we compute a subset  $A_1 \subset A$  by iteratively adding to it a degree-*d* vertex in *A*, followed by removing the vertex and all its *d* neighbors (in *B*) from the graph. The algorithm terminates when there is no degree-*d* vertex in *A* in the remaining graph, and returns  $A - A_1$  as the solution.

**Theorem 6** The algorithm APPROX3 is an  $O(d^2|V|)$ -time  $\frac{d^2}{d^2-d+1}$ -approximation for P4VC on *d*-regular bipartite graphs.

PROOF. Each iteration of adding a degree-*d* vertex in *A* to  $A_1$  and removing the vertex and all its *d* neighbors from the graph takes only  $O(d^2)$  time and there are at most |V| iterations. Therefore, the overall running time of APPROX3 is  $O(d^2|V|)$ .

Note that G is d-regular and bipartite. In the subgraph  $G[A_1 \cup B]$  induced on  $A_1 \cup B$ , each connected component is a  $K_{1,d}$  claw if it contains a vertex of  $A_1$ , or otherwise it is a 1-path (singleton). That is,  $G[A_1 \cup B]$  does not contain any 4-path and thus  $A - A_1$  is a 4-path vertex cover; and in  $G[A_1 \cup B]$ , there are exactly  $|A_1| K_{1,d}$  claws and exactly  $|B| - |A_1|d$  singletons.

Since each vertex of  $A - A_1$  can be adjacent to at most d - 1 1-path components of  $G[A_1 \cup B]$ , we have

$$(|A| - |A_1|)(d - 1) \ge (|B| - |A_1|d)d = (|A| - |A_1|d)d$$

It follows that

$$|A| - |A_1| \le \frac{d^2 - d}{d^2 - d + 1} |A| = \frac{d^2 - d}{2(d^2 - d + 1)} |V|.$$
(11)

Using the lower bound in Eq. (10), the above Eq. (11) leads to

$$\frac{|A| - |A_1|}{\psi_4(G)} \le \frac{d^2 - d}{2(d^2 - d + 1)} \times \frac{2d}{d - 1} = \frac{d^2}{d^2 - d + 1}.$$

This proves the theorem.

# 6 Conclusions and remarks

In this paper we considered the minimum k-path vertex cover problem (PkVC), which generalizes the classic vertex cover (VC) problem. We designed several improved approximation algorithms, including a  $\frac{\lfloor d/2 \rfloor (2d-k+2)}{(\lfloor d/2 \rfloor+1)(d-k+2)}$ -approximation for PkVC on d-regular graphs when  $1 \le k-2 < d$ . This beats all the best known approximation results for PkVC on d-regular graphs for  $k \ge 3$ , except for P4VC it ties with the best prior work and in particular they tie at 2 on cubic graphs and 4-regular graphs. We then designed a better 1.875-approximation for P4VC on cubic graphs, a better 1.852-approximation for P4VC on 4-regular graphs, and a better  $\frac{(3d-2)(2d-2)}{(3d+4)(d-2)}$ -approximation for P4VC on d-regular graphs when  $d \ge 4$  is even. We also presented a  $\frac{d^2}{d^2-d+1}$ -approximation for P4VC on d-regular bipartite graphs.

We thus improved all the best prior approximation algorithms for PkVC on *d*-regular graphs for  $k \ge 3$ , except for P4VC on *d*-regular graphs when  $d \ge 5$  is odd, where we tie with the best prior work. We leave it open on how to take advantage of the odd degrees.

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