Embedding graphs into embedded graphs

Radoslav Fulek[*]

Abstract

A (possibly denerate) drawing of a graph G in the plane is approximable by an embedding if it can be turned into an embedding by an arbitrarily small perturbation. We show that testing, whether a straight-line drawing of a planar graph G in the plane is approximable by an embedding, can be carried out in polynomial time, if a desired embedding of G belongs to a fixed isotopy class. In other words, we show that c-planarity with embedded pipes is tractable for graphs with fixed embeddings.

To the best of our knowledge an analogous result was previously known essentially only when G is a cycle.

1 Introduction

In the theory of graph visualization a drawing of a graph $G = (V, E)$ in the plane is usually assumed to be free of degeneracies, i.e., edge overlaps and edges passing through a vertex. However, in practice degenerate drawings often arise and need to be dealt with.

Recent papers [1, 7] address a certain aspect of this problem for simple polygons which can be thought of as straight-line (rectilinear) embeddings of graph cycles. Chang et al. [7] gave an $O(n^2 \log n)$-time algorithm to detect if a given polygon with n vertices can be turned into a simple (non self-intersecting) one by small perturbations of its vertices, or in other words if the polygon is **weakly simple**. We mention that there exists an earlier closely related definition of weakly simple polygons by Toussaint [6, 27], however, as pointed out in [7] this notion is not well-defined for general polygons with "spurs", see [7] for an overview of attempts at combinatorial definitions of polygons not crossing itself.

An $O(n \log n)$ improvement on the running time of the algorithm by Chang et al. was announced very recently by Akitaya et al. [1]. The combinatorial formulation of this problem corresponds to the setting of **c-planarity with embedded pipes** introduced by Cortese et al. [10] well before the two aforementioned papers. Therein only an $O(n^3)$-time algorithm for the problem was given. Nevertheless, the algorithms in [1, 7] were built upon the ideas from [10]. Moreover, to the best of our knowledge the complexity status of the c-planarity with embedded pipes is essentially known only for cycles. Recently the problem was studied for general planar graphs by Angelini and Da Lozzo [3], but they gave only an FPT algorithm. The introduction of this problem was motivated by a more general and well known problem of **c-planarity** by Feng et al. [13, 14], whose tractability status was open since 1995 even in much more restricted cases than the one that we consider. Biedl [4] gave a polynomial-time algorithm for c-planarity with two clusters. Beyond two clusters a polynomial time algorithm for c-planarity was obtained only in special cases, e.g., [9, 18, 19, 21, 22], and most recently in [5, 8, 15].

There is, however, another tightly related line of research on approximability or realizations of maps pioneered by Sieklucki [25], Minc [23] and M. Skopenkov [26] that is completely independent from the aforementioned developments, and that is also a major source of inspiration for our work. It can be easily seen that the result [26, Theorem 1.5] implies that c-planarity is tractable for flat instances with three clusters or cyclic clustered graphs [17, Section 6] with a fixed isotopy class of a desired embedding. An algorithm with a better running time was given by the author in [15].

The aim of the present work is to show that c-planarity with embedded pipes is tractable for planar graphs with a fixed isotopy class of embeddings, which extends results of [2, 3, 15]. Our work also implies the tractability of deciding whether a drawing is approximable by an embedding in a fixed isotopy class, which extends results of [1, 7]. This also answers in the affirmative a question posed in [7, Section 8.2] if the isotopy class of an embedding of G is fixed. The combinatorial formulation of the problem, **c-planarity with embedded pipes** follows, see Fig. 1. We are given

(A) A planar graph $G = (V, E)$, whose vertex set is partitioned into k parts $V = V_{\nu_1} \uplus V_{\nu_2} \uplus \ldots \uplus V_{\nu_k}$ called **clusters** given by the isotopy class of an embedding of G in the plane;
(B) a planar graph $H = (V(H), E(H))$ that is straight-line embedded in the plane, where $V(H) = \{\nu_1, \ldots, \nu_k\}$.

Remark 1.1. Since H is straight-line embedded, H does not contain multiple edges. The assumption that H is given by a straight-line embedding as opposed to a piecewise linear/polygonal embedding is not crucial, since every planar graph admits a straight-line embedding in the plane by Fáry–Wagner theorem [12, 28], and its imposition is just a matter of convenience.

Let $\mathrm{dist}(\mathbf{p}, \mathbf{q})$ denote the Euclidean distance between $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$. Let $\mathrm{dist}(\mathbf{p}, S) = \min_{\mathbf{q} \in S} \mathrm{dist}(\mathbf{p}, \mathbf{q})$, where $S \subset \mathbb{R}^2$. Let $N_\varepsilon(S)$ for $S \subset \mathbb{R}^2$ denote the ε-neighborhood of S, i.e., $N_\varepsilon(S) = \{\mathbf{p} \in \mathbb{R}^2 | \mathrm{dist}(\mathbf{p}, S) \leq \varepsilon\}$. Let $\varepsilon, \varepsilon' > 0$ be small values as described later. The **thickening** $\mathcal{H}$ of H is the union of $N_\varepsilon(\nu_i)$, for all $\nu_i \in V(H)$ and $N_{\varepsilon'}(\rho)$, for all $\rho \in E(H)$ [1]. Let the **pipe** of $\rho \in E(H)$ be the closure of $N_{\varepsilon'}(\rho) \setminus (N_\varepsilon(\nu_i) \cup N_\varepsilon(\nu_j))$, where $\rho = \nu_i \nu_j$. Let the **valve** of ρ at ν_i be the curve obtained as the intersection of $N_\varepsilon(\nu_i)$ and the pipe of ρ. We put $\varepsilon' < \varepsilon := \varepsilon(H)$ so that the valves are pairwise disjoint in $\mathcal{H}$ and $\varepsilon > 0$ is smaller than $d/4$, where $d := d(H)$ is the minimum distance between a vertex ν of H and an edge ρ of H not incident to ν over all such edge-vertex pairs.

We want to decide if the given isotopy class of G contains an embedding contained in $\mathcal{H}$, where the vertices in V_{ν_i}, for every ν_i, are drawn in the interior of $N_\varepsilon(\nu_i)$ and every edge crosses the boundary of $N_\varepsilon(\nu_i)$, for every $\nu_i \in V(H)$, at most once. Such an embedding of G is **H-compatible**. Let $E_{\nu_i \nu_j} = \{uv \in E(G) | u \in V_{\nu_i}, v \in V_{\nu_j}\}$. An H-compatible embedding of G is encoded by G, H, and a set of total orders $(E_{\nu_i \nu_j}, <_\omega)$, for every $\nu_i \nu_j \in E(H)$ and a valve ω of $\nu_i \nu_j$, where $(E_{\nu_i \nu_j}, <_\omega)$ encodes the order of crossings of ω with edges along ω. The isotopy class of G is encoded by a choice of the outer face, a set of rotations at its vertices and a containment relation of its connected components as described in Section 2. Since we are interested only in combinatorial aspects of the problem, H is also given by the isotopy class of its embedding. Throughout the paper we assume that G and H are given as in **(A)** and **(B)**.

Theorem 1.2. *There exists $O(|V(G)|^2)$ algorithm that decides if the given isotopy class of G contains an H-compatible embedding. An H-compatible embedding of G can be also constructed in $O(|V(G)|^2)$ time if it exists. In other words, c-planarity with embedded pipes is tractable, when an isotopy class of a desired embedding of G is fixed.*

---

[1]Throughout the paper we denote vertices and edges of H by Greek letters.
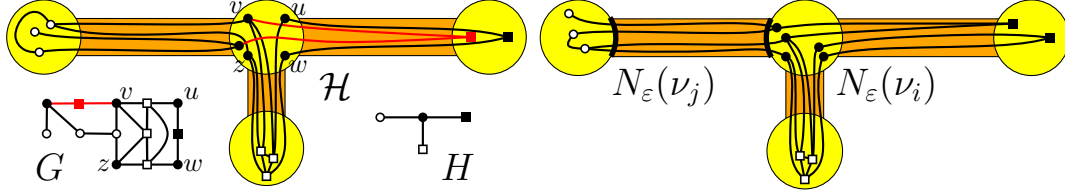
Figure 1: Instance of c-planarity with embedded pipes. The partition of the vertex set of G into clusters is encoded by the shape of vertices. An H-compatible embedding of a subgraph of G that cannot be extended to the whole G (left). An H-compatible embedding of G (right) inside $\mathcal{H}$. The valves of $\rho = \nu_i \nu_j$ at $N_\varepsilon(\nu_i)$ and $N_\varepsilon(\nu_j)$ are highlighted by bold arcs.

As a corollary of our result we obtain that we can test in polynomial time if a piecewise linear drawing of a graph in the plane is approximable by an embedding and construct such an embedding if it exists. We defer the definition of the approximability by an embedding to Section 3, where also the proof of the corollary can be found. As previously discussed this extends results in [1, 7] and also [26].

Corollary 1.3. *There exists an $O(n^4)$ time algorithm that decides if a piecewise linear drawing of a graph in the plane is approximable by an embedding, and constructs such an embedding if it exists, where n is the size of the representation of the drawing.*

Extensions of our results. By [24, Theorem 3.1], our result holds also in the setting of rectilinear, i.e., straight-line, drawings of graphs. To extend it further in this setting by allowing "forks" (see Section 3) seems to be just a little bit technical.

In a recent manuscript [16], we verified a conjecture of M. Skopenkov [26, Conjecture 1.6] implying that that our problem is tractable, when we lift the restriction on the isotopy class G. This does not imply that the problem with the restriction on the isotopy class G is tractable except when G is connected. The running time of the algorithm, that is implied by this work, is $O(|V|^{2\omega})$, where $O(n^\omega)$ is the running time of the fastest algorithm for multiplying a pair of n by n matrices. Since $\omega > 2$ due to the matrix size, this is much worse that the running time claimed by Theorem 1.2. Furthermore, the algorithm is not constructive.

As noted by Chang et al. [7], the technique of Cortese et al. [10] extends directly from the plane to any closed two-dimensional surface. The same holds for our method, but since considering general two-dimensional surfaces does not bring anything substantially new to our treatment of the problem, for the sake of simplicity we consider only the planar case.

Strategy of the proof of Theorem 1.2. Formally, the **input** of our algorithm is a triple (G, H, γ), where the partition of the vertex set of G corresponds to the map γ from the set of vertices of G to the set of vertices of H. Hence, for $v \in V_\nu$, where $\nu \in V(H)$, we have $\gamma(v) = \nu$. The input (G, H, γ) is **positive** if there exists an H-compatible embedding of G in the given isotopy class of G, and **negative** otherwise.

The most problems in constructing a polynomial time algorithm for our problem are caused by so called "spurs" such as the red vertex in Fig. 1 (left), i.e., connected components in subgraphs of G induced by clusters, whose all adjacent vertices belong to the same cluster. Due to the presence of spurs it is hard to see that our problem is tractable even in the case, when G is a path.

The centerpiece of our method is an extension of the definition of the derivative of maps of intervals/loops (corresponding to the case, when G is a path/cycle, in our terminology) in the plane introduced by Minc [23] to arbitrary graphs. We adapt this notion to the setting of c-planarity with embedded pipes. The derivative is an operator that takes (G, H, γ), and either

detects that there exists no H-compatible embedding of G in the given isotopy class of G, or outputs (G', H', γ'), that is also a valid input for our algorithm, such that (G, H, γ) is positive if and only if (G', H', γ') is positive. Intuitively, H' is reminiscent of the line graph of H and the subgraphs of G, that are mapped by γ to the edges of H, are turned into subgraphs of G' mapped by γ' into vertices of H'. This results in a shortening of problematic spurs, and zooming into the structure of the map γ. We show that by iterating the derivative $|E(G)|$ times we either detect that there exists no H-compatible embedding of G in the given isotopy class of G, or we arrive at an input without problematic spurs. Since it is fairly easy to solve the problem for the latter inputs; the derivative at every iteration can be computed in linear time in $|V(G)|$; and by derivating the size of the input is increased only by a little, the tractability follows.

The operation of node expansion and base contraction introduced in [10] resemble the derivative. The main difference is that these two operations affect only a single cluster or a pair of clusters in (G, H, γ), and therefore they are local, whereas the derivative changes the whole input. We are very positive that our method is applicable to other graph drawing problems related to c-planarity whose tractability is open. This is documented by our recent manuscript [16] in which a similar technique was applied.

The derivative is applied to an input (G, H, γ), in which every cluster V_{ν_i} induces in G an independent set. Such an input is in the **normal form**. The detailed description of the algorithm proving Theorem 1.2 is in Section 4. We show in Section 4.1 that an input can be assumed to be in the normal form. The definition of the derivative is given in Section 4.2, and sufficiently simplified inputs are dealt with in Section 4.3.

2 Preliminaries

Throughout the paper we tacitly use Jordan-Schönflies theorem.

Let $G = (V, E)$ denote a planar graph possibly with multiple edges and loops. For $V' \subseteq V$ we denote by $G[V']$ the sub-graph of G induced by V'. A **star** $St(v)$ of a vertex v in a graph G is the subgraph of G consisting of all the edges incident to v. Throughout the paper we use standard graph theoretical notions such as path, cycle, walk, vertex degree $deg(v)$ etc., see [11].

A **drawing** $\mathcal{D}(G)$ is a representation of G in the plane, where every vertex in V is represented by a point and every edge $e = uv$ in E is represented by a simple piecewise linear curve joining the points that represent u and v. Thus, a drawing can be thought of as a map from G understood as a topological space into the plane. In a drawing, we additionally require every pair of distinct curves representing edges to meet only in finitely many points each of which is a proper crossing or a common endpoint. In a **degenerate** drawing, we allow a pair of distinct vertices to be represented by the same point and a pair of edges to be represented by the same curve. A drawing in which every vertex is represented by a unique point and every edge by a unique curve is **non-degenerate**. In a non-degenerate drawing, multiple edges are mapped to distinct arcs meeting at their endpoints. In the paper we consider non-degenerate drawings, except in Section 3. An edge crossing-free non-degenerate drawing is an **embedding**. A graph given by an embedding in the plane is a **plane graph**. If it leads to no confusion, we do not distinguish between a vertex or an edge and its representation in the drawing and we use the words "vertex" and "edge" in both contexts.

The following lemma is well known.

Lemma 2.1. *Let G be a plane graph with n vertices such that G does not contain a pair of multiple edges joining the same pair of vertices that form a face of size two, i.e., a lens, except for the outer face. The graph G has $O(n)$ edges.*

4

The **rotation** at a vertex in an embedding of G is the counterclockwise cyclic order of the end pieces of its incident edges. The rotation at a vertex is stored as a doubly linked list of edges. Furthermore, we assume that for every edge of G we store a pointer to its preceding and succeeding edge in the rotation at both of its end vertices. The **interior** and **exterior** of a cycle in an embedded graph is the bounded and unbounded, respectively, connected component of its complement in the plane. Similarly, the **interior** of an inner face and outer face in an embedded connected graph is the bounded and unbounded, respectively, connected component of the complement of its facial walk in the plane bounded by the walk. An embedding of a connected graph G is up to an isotopy described by the rotations at its vertices and the choice of its outer (unbounded) face. If G is not connected the isotopy class of its embedding is described by isotopy classes of its connected components $G_1, \ldots, G_l$ and the containment relation $G_i \subset f$, for every G_i, where f is a face of G_j, $j \neq i$, such that G_i is embedded in the interior of f.

3   Approximation of maps by embeddings

The aim of this section is to derive Corollary 1.3 from Theorem 1.2. By treating a graph G as a one-dimensional topological space, a drawing $\mathcal{D}$ of G is understood as a continuous map $\mathcal{D}$ mapping every $x \in G$ to $\mathbb{R}^2$. Such a drawing is given by the finite set of pairs of real values representing the end points of line segments of polylines corresponding in the drawing to edges of G.

Let $G = (V, E)$ denote a planar graph. Let $\mathcal{D}$ be a (possibly degenerate) drawing corresponding of G. Note that we do not allow an edge to pass through a vertex by the definition of the drawing, or in other words, we do not allow a drawing to contain **forks** [7]. The previous restriction is not crucial, since we can subdivide edges at "fork" vertices while still having an input of a quadratic size in the size of the original input. This yields the claimed running time. An ϵ-**approximation** of a drawing $\mathcal{D}$ of a graph G is a drawing $\mathcal{D}'$ of G such that $\text{dist}(\mathcal{D}(x), \mathcal{D}'(x)) < \epsilon$, for all $x \in G$. A drawing $\mathcal{D}$ is **approximable by an embedding** if there exists $\varepsilon(\mathcal{D}) > 0$ such that for every ε, $0 < \varepsilon < \varepsilon(\mathcal{D})$, there exists an ε-approximation $\mathcal{D}$ that is an embedding. It is clear that a drawing with edge crossings is not approximable by an embedding, and thus, in the sequel we consider only drawings without edge crossings.

Given a drawing $\mathcal{D}$ of a graph G in the plane, in order to decide if $\mathcal{D}$ is approximable by an embedding in a fixed isotopy class of G, we construct an input (G_0, H, γ) for c-planarity with embedded pipes. The graph H is the embedded graph given by the image of $\mathcal{D}$, and G_0 is obtained from G by subdividing every edge e, that is not a drawn as a straight-line segment by $\mathcal{D}$, so that $\mathcal{D}$ is turned into a straight-line drawing of G_0. Then every end point of a line segment representing an edge of G_0 is turned into a vertex of H. We put $\gamma(v) := \mathcal{D}(v)$ for $v \in V(G_0)$.

The input (G_0, H, γ) is positive if and only if $\mathcal{D}$ is approximable by an embedding. The "only if" direction is easy. If (G_0, H, γ) is positive, then we can choose $\varepsilon(\mathcal{D}) := \varepsilon$, where ε is as in the definition of the thickening of H, witnessing that $\mathcal{D}$ is approximable by an embedding. Let ε' be as in the definition of the thickening of H. To prove the "if" direction, it is enough to show that an ε'-approximation of $\mathcal{D}$, that is an embedding, can be chosen such that for every $\nu_i \nu_j \in E(H)$, $v_i \in V_{\nu_i}$ and $v_j \in V_{\nu_j}$ we have $|\omega \cap \mathcal{D}(v_i v_j)| \leq 1$, where ω is a valve of $\nu_i \nu_j \in E(H)$. This can be achieved by an appropriate local deformation of the ε'-approximation as we show next.

Suppose that a valve ω at, let's say ν_i, crosses an edge at least two times in the ε'-approximation. We consider a pair of consecutive crossings with ω along an edge e such that the piece of e between the crossings in the pair is contained in the pipe. We choose the pair so that the distance between the crossings is minimal, and eliminate the crossings as illustrated in Fig. 2a. By repeating this procedure, we eventually obtain an H-compatible embedding of G_0.
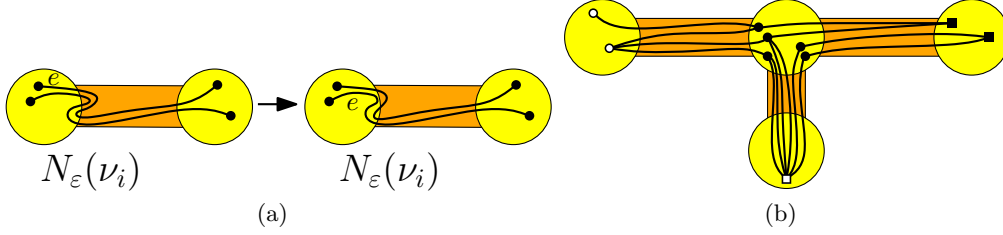
Figure 2: (a) Deforming an approximation so that every valve is crossed by an edge at most once. (b) The input (G^N, H^N, γ^N) in the normal form constructed from (G, H, γ) in Fig. 1.

4 Proof of Theorem 1.2

Let (G, H, γ) be the input of our algorithm, where the partition of the vertex set of G corresponds to the map γ of the vertices of G by vertices of H. Recall that for $v \in V_\nu$ we have $\gamma(v) = \nu$, where $\nu \in V(H)$. We also naturally extend γ to edges: $\gamma(v_i v_j) = \rho = \nu_i \nu_j$, for $v_i \in V_{\nu_i}$ and $v_j \in V_{\nu_j}$, and to subgraphs G_0 of G: $\gamma(G_0) = H_0 = (V(H_0), E(H_0))$ such that $V(H_0) = \{\nu \in V(H) | \gamma(v) = \nu, \ v \in V(G_0)\}$ and $E(H_0) = \{\rho \in E(H) | \ \gamma(e) = \rho, e \in E(G_0)\}$.

A vertex $\nu \in V(H)$ of degree two is **redundant** if $V_\nu \subseteq V(G)$ is an independent set consisting of vertices of degree two such that for every $v \in V_\nu$ we have $\gamma(vu) \neq \gamma(vw)$, where u and w are the two neighbors of v. We assume that every edge of H is used by at least one edge of G, i.e., for every $\rho \in E(H)$ there exists $e \in E(G)$ such that $\gamma(e) = \rho$.

4.1 The normal form

Similarly as in [15], the input (G, H, γ) is in the **normal form** if

(1) every cluster $V_\nu \subseteq V(G)$, for $\nu \in V(H)$, is an independent set without isolated vertices; and

(2) H does not contain a pair of redundant vertices joined by an edge.

We remark that (2) is required only due to the running time analysis. Then we do not forbid redundant vertices completely, since we do not allow H to contain multiple edges. In what follows we show how to either detect that no H-compatible embedding in the given isotopy class of G exists just by considering the subgraph of G induced by a single cluster V_ν, or construct an input (G^N, H^N, γ^N), see Fig. 2b, in the normal form, which is positive if and only if the input (G, H, γ) is positive. Clearly, (2) can be assumed without loss of generality. Before establishing the other condition we introduce a couple of definitions.

A **contraction** of an edge $e = uv$ in an embedding of a graph is an operation that turns e into a vertex by moving v along e towards u while dragging all the other edges incident to v along e. By a contraction we can introduce multiple edges or loops at the vertices. We will also use the following operation which can be thought of as the inverse operation of the edge contraction in an embedding of a graph. Note that a contraction can be carried out in $O(1)$ time, since it amounts to merging a pair of doubly linked lists, and redirecting at most four pointers. The same applies to the following operation. A **vertex split**, see Fig. 3a, in an embedding of a graph G is an operation that replaces a vertex v by two vertices u and w joined by a crossing free edge so that the neighbors of v are partitioned into two parts according to whether they are joined with u or w in the resulting drawing. The rotations at u and w are inherited from the rotation at v. When applied to G, the operations are meant to return a graph given by an isotopy class of its embedding; the same applies to vertex multisplit defined later.

In order to satisfy (1), by a series of successive edge contractions we contract each connected component of $G[V_{\nu_i}]$, for $\nu_i \in V(H)$, to a vertex. Since rotations are stored as doubly linked lists, contracting all such connected components can be carried out in linear time. We delete any created loop and isolated vertices. If a loop at a vertex from V_{ν_i} contains a vertex from a different cluster V_{ν_j}, $\nu_j \neq \nu_i$, in its interior we know that the input is negative, since for every ν_j all the vertices in V_{ν_j} must be contained in the outer face of $G[V_{\nu_i}]$ if the input is positive. All this can be easily checked in linear time in $|V(G)|$ by the the breadth-first or depth-first search algorithm. If a loop at a vertex from V_{ν_i} does not contain a vertex from a different cluster, such a contraction preserves the existence of an H-compatible embedding in the given isotopy class of G. Indeed, isolated vertices and deleted empty loops can be introduced in an H-compatible embedding of the resulting graph, and contracted edges recovered via vertex splits. Let (G^N, H^N, γ^N) denote the resulting input in the normal form. We proved the following.

Lemma 4.1. *If a loop at a vertex of G obtained during the previously described procedure contains a vertex of G in its interior the input (G, H, γ) is negative. Otherwise, the input (G, H, γ) is positive if and only if (G^N, H^N, γ^N) is positive.*

4.2 Derivative

We present the operation of the derivative that simplifies the input, and whose iterating results in an input that is easy to deal with. Such inputs are treated in Section 4.3. Before we describe the derivative we give a couple of definitions.

A **vertex multisplit**, see Fig. 3a, in an embedding of a graph G is an operation that replaces a vertex v with a crossing free star $(\{v, v_1, \ldots, v_l\}, \{vv_1, \ldots, vv_l\})$ so that the resulting underlying graph has vertex set $V(G) \cup \{v_1, \ldots, v_l\}$ and edge set $(E(G) \setminus \{vu_1, \ldots, vu_{deg(v)}\}) \cup \{v_{i_j} u_j \mid j = 1, \ldots, deg(v)\} \cup \{vv_1, \ldots, vv_l\}$, where $u_1, \ldots, u_{deg(v)}$ are neighbors of v in G and $1 \leq i_j \leq l$, for all j. The rotations at $v_1, \ldots, v_l$ are inherited from the rotation at v so that by contracting all the edges of $St(v)$ in the resulting graph we obtain the original embedding of G. Note that a vertex multisplit can be carried out in $O(deg(v))$ time.

The rotation of $\nu \in V(H)$ is **consistent** with the rotation of $v \in V_\nu$ if the rotation given by $(\gamma(vv_1), \ldots, \gamma(vv_{deg(v)}))$, where $(vv_1, \ldots, vv_{deg(v)})$ is the rotation at v in an embedding of G in the given isotopy class, is the rotation at $\nu \in V(H)$ in the embedding of H. Let the **potential** $p(G, H, \gamma) = |E(G)| - |E(H)|$. Obviously, $p(G, H, \gamma) \geq 0$ and $p(G, H, \gamma) = 0$, if G is isomorphic to H via γ, and if G is connected the opposite implication also holds. The input in the normal form (G, H, γ) is **locally injective** if

(i) the restriction of γ to $V(St(v))$ is injective, for all $v \in V(G)$; and

(ii) every vertex v of degree one in G is incident to an edge e such that $\gamma(e) = \gamma(f)$ implies $e = f$ for all $f \in E(G)$.

Given an input (G, H, γ), the vertex $v \in V(G)$ is **fixed** if the condition of property (i) holds for v, and v is alone in its cluster, i.e., $\gamma(u) = \gamma(v)$ implies $u = v$. If v is fixed then we call $\gamma(v) = \nu_i \in V(H)$ also **fixed**.

Given an input (G, H, γ) in the normal form that is not locally injective, we either detect that there does not exist an H-compatible embedding of G in the given isotopy class, or we construct the input (G', H', γ') having a smaller potential after being brought to the normal form, such that (G', H', γ') is positive if and only if (G, H, γ) is positive. The input (G', H', γ') is obtained as follows, see Fig. 3b.

First, we construct the graph G' by applying the following procedure to every vertex $v \in V(G)$ such that the star $\gamma(St(v))$ has at least two edges, and thus, v is not a "spur". By
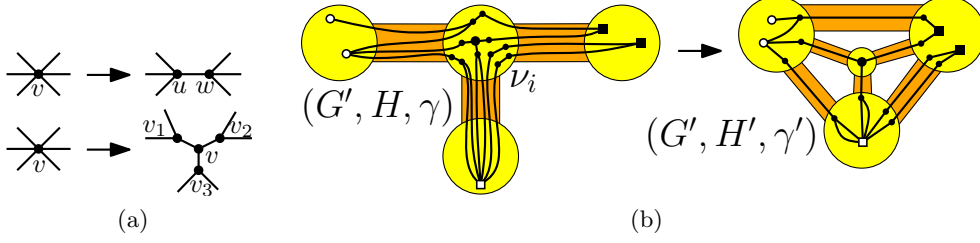
Figure 3: (a) Operation of vertex split (top) and multisplit (bottom). (b) The derivative of (G, H, γ) in the normal form from Fig. 2b. On the left the input after splitting vertices, and on the right the obtained derivative; in the example we have $H' = H'_{\nu_i}$, since every other H_{ν_j}, for $i \neq j$, is a trivial graph with one vertex.

slightly abusing the notation we will extended γ to take values on the vertices of G'. The input (G, H, γ) is clearly negative, if there exists a vertex v in G with four incident edges $vv_1, \ldots, vv_4 \in E(G)$ such that vv_1, vv_2, vv_3 and vv_4 appear in the rotation at v in the given order and $\gamma(vv_1) = \gamma(vv_3) \neq \gamma(vv_2), \gamma(vv_4)$. Otherwise, the following operations of vertex split and multisplit are applicable to G.

If $|E(\gamma(St(v)))| = 2$, we apply the operation of vertex split to v thereby turning it into an edge uw as follows. Let $E(\gamma(St(v))) = \{\rho_1, \rho_2\}$. Let $v_1, \ldots, v_{deg(v)}$ be the neighbors of v. Let $\{v_1 \ldots v_l\} \cup \{v_{l+1} \ldots v_{deg(v)}\}$ be the partition of the neighbors of v such that $\gamma(vv_1) = \ldots = \gamma(vv_l) = \rho_1$ and $\gamma(vv_{l+1}) = \ldots = \gamma(vv_{deg(v)}) = \rho_2$. We put $\gamma(u), \gamma(w) := \gamma(v)$, and join u by an edge with the vertices in $\{v_1 \ldots v_l\}$ and w with the vertices in $\{v_{l+1} \ldots v_{deg(v)}\}$. Let $E_2 \subset E(G')$ denote the set of edges in G' consisting of every edge uw obtained by splitting $v \in V(G)$ such that $|E(\gamma(St(v)))| = 2$.

If $|E(\gamma(St(v)))| \geq 3$, we analogously apply the operation of vertex multisplit to v so that we replace v with a star $(\{v, v_1, \ldots, v_l\}, \{vv_1, \ldots, vv_l\})$ with $|E(\gamma(St(v)))| = l$ edges, in which every leaf vertex v_i is incident to the edges mapped by γ to the same edge of H and $\gamma(v_i) = \gamma(v)$. Let $V_{\geq 3} \subset V(G)$ denote the set of vertices in G consisting of the vertices $v \in V(G)$ such that $|E(\gamma(St(v)))| \geq 3$. Note that $V_{\geq 3}$ can be treated also as a subset of $V(G')$. Let $\mathcal{C}$ denote the set of connected components of $G' \setminus E_2 \setminus V_{\geq 3}$.

Second, we construct H': $V(H') = \{\rho^* | \rho \in E(H)\} \cup \{\nu_v | v \in V_{\geq 3}\}$, and $E(H') = \{\nu_v \rho^* | \rho \in E(\gamma(St(v)))\} \cup \{\gamma(C)\gamma(D) | C, D \in \mathcal{C} \ s.t.$ there exists $e \in E_2$ joining C with $D\}$. We put $\gamma'(v) := \gamma(C)^*$, for $v \in V(C)$ where $C \in \mathcal{C}$; and $\gamma'(v) := \nu_v$, for $v \in V_{\geq 3}$. Note that ν_v and v are fixed in the latter.

Finally, the embedding of H', if it exists, is constructed as follows. By Fáry–Wagner theorem it is enough to give any embedding of H' in the plane in a desired isotopy class, which we describe next by constructing a particular embedding of H'. Let H'_{ν_i}, $\nu_i \in V(H)$, denote the subgraph of H' induced by $\{\rho^* | \rho = \nu_i \nu \in E(H)\} \cup \{\nu_v | \gamma(v) = \nu_i\}$. Let $\hat{H}'_{\nu_i}$ be obtained from H'_{ν_i} by adding to H'_{ν_i} (1) the missing edges of the cycle traversing $\{\rho^* | \rho = \nu_i \nu \in E(H)\}$ according to the rotation of ν_i, let us denote the cycle by C_{ν_i}; and (2) a new vertex joined by the edges exactly with all the vertices of C_{ν_i}. Note that $\hat{H}'_{\nu_i}$ is vertex three-connected, and hence, if $\hat{H}'_{\nu_i}$ is planar, then the rotations at vertices in its embedding are determined up to the choice of orientation. Note that the construction of (G', H', γ') can be carried out in $O\left(\sum_{v \in V(G')} deg(v)\right) = O(|V(G)|)$.

Suppose that every $\hat{H}'_{\nu_i}$, for $\nu_i \in V(H)$, is a planar graph. Let us fix for every $\nu_i \in V(H)$ an embedding of H'_{ν_i}, in which the cycle C_{ν_i} bounds the outer face and its orientation corresponds to the rotation of ν_i. Such an embedding is obtained as a restriction of an embedding of $\hat{H}'_{\nu_i}$. Note that for every i the graph H'_{ν_i} does not have multiple edges. Since H also does not have
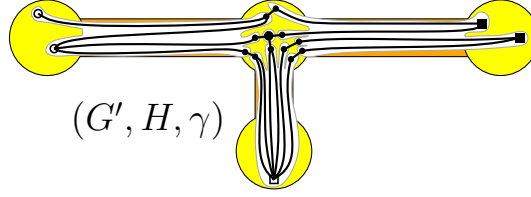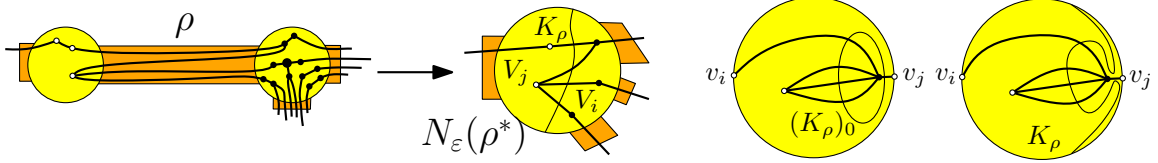
Figure 4: Discs D_ρ filled white.

Figure 5: The curve K_ρ splitting the cluster of ρ in the derivative on the left. The construction of the curve $(K_\rho)_0$ and its deformation into K_ρ on the right.

multiple edges, H'_{ν_i} and H'_{ν_j}, for $i \neq j$, are either disjoint (if $\nu_i \nu_j \notin E(H)$) or intersect in a single vertex $(\nu_i \nu_j)^*$ (if $\nu_i \nu_j \in E(H)$). It follows that H' does not have multiple edges. The desired embedding of H' is obtained by combining embeddings of H'_{ν_i}, for $\nu_i \in V(H)$, in the same isotopy class as the embeddings of H'_{ν_i}, that we fixed above, by identifying the corresponding vertices so that the restriction of the obtained embedding of H' to every H'_{ν_i} has the rest of H' in the interior of the outer face (of this restriction).

Lemma 4.2. *The input (G, H, γ) is negative if one of the following three conditions is satisfied. There exists a vertex v in G with four incident edges $vv_1, \ldots, vv_4 \in E(G)$ such that vv_1, vv_2, vv_3 and vv_4 appear in the rotation at v in the given order and $\gamma(vv_1) = \gamma(vv_3) \neq \gamma(vv_2), \gamma(vv_4)$. The graph $\hat{H}'_{\nu_i}$, for some $\nu_i \in V(H)$, is not planar. The rotation of a vertex $\nu_v \in V(H'_{\nu_i})$, for some $\nu_i \in V(H)$ and $v \in V(G')$, in the obtained embedding of H'_{ν_i} is not consistent with the rotation of v in G'.*

The input (G, H, γ) is positive if and only if the input (G', H', γ') is positive.

Proof. The first part of the claim is obvious. For the second part, we start with "only if" direction, which is easier.

To this end given an H-compatible embedding of G, we first easily construct an H-compatible embedding of G' with respect to the input (G', H, γ). In the second step, for every $\rho \in E(H)$, we construct a disc D_ρ containing the restriction to $G'_\rho = \bigcup_{C \in \mathcal{C}, \gamma(C) = \rho} C$ of the H-compatible embedding of G in its interior as follows. Let $(G'_\rho)_\times$ be a plane graph obtained from the embedding of G'_ρ by turning the crossings of edges of G'_ρ with both valves of ρ into vertices; and parts of the valves joining closest pairs of crossing (which were turned into vertices) into edges. The disc D_ρ, see Fig. 4, is a small neighborhood of the union of the inner faces in the embedding of $(G'_\rho)_\times$. Finally, we apply a homeomorphism of the plane that maps the union of the discs D_ρ's with G'_ρ into the thickening of H', so that every D_ρ is mapped onto $N_\varepsilon(\rho^*)$ and a small neighborhood of every $v \in V_{\geq 3}$ onto $N_\varepsilon(\nu_v)$. This concludes the proof of the "only if" direction.

It remains to prove the "if" direction. We show that by [17, Lemma 6] given an H'-embedding of G' in the given isotopy class, every $N_\varepsilon(\rho^*)$, for $\rho^* \in V(H')$, $\rho = \nu_i \nu_j$, can be split by a simple continuous curve K_ρ, see Fig. 5, disjoint from every edge mapped by γ' to an edge of H' into two parts as follows. The vertices in $\gamma^{-1}(\rho)$, for which $\gamma(v) = \nu_i$, are in one part and the vertices, for which $\gamma(v) = \nu_j$, are in the other part.
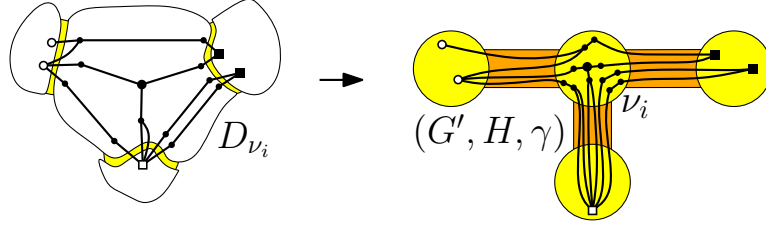
9

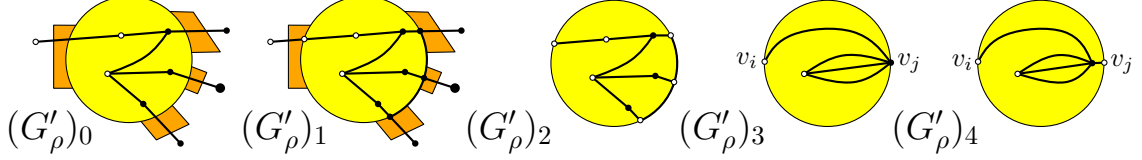Figure 6: Construction of an H-compatible embedding of G' from the H'-compatible embedding of G' in Fig. 3b.

Figure 7: Construction of $(G'_\rho)_4$ from G'_ρ in Fig. 5.

For a while suppose that K_ρ's exist. Then it follows that an H-compatible embedding of G in the given isotopy class exists. Analogously, to the previous paragraph, for every $\nu_i \in V(H)$, we construct a disc D_{ν_i}, see Fig. 6, containing the subgraph of G' induced by the vertex set $\gamma^{-1}(\nu_i)$. We construct D_{ν_i} so that (1) the intersection of the boundary of D_{ν_i} with the thickening of H' is $\bigcup_{\rho=\nu_i\nu\in E(H)} K_\rho$, which is intersected by the boundary in the order given by the rotation at ν_i; (2) $N_\varepsilon(\nu) \cap D_{\nu_i} = \emptyset$, for $\nu \notin V(H'_{\nu_i})$; and (3) every pair of discs D_{ν_i} and D_{ν_j}, for $i \neq j$, is internally disjoint. We perturb discs D_{ν_i}'s a little bit in order to make them pairwise disjoint. Then we apply a homeomorphism of the plane that maps the union of the discs D_{ν_i}'s with G' into the thickening of H, so that every D_{ν_i} is mapped onto $N_\varepsilon(\nu_i)$. Finally, we contract the edges incident to the vertices in $V_{\geq 3}$ and contract edges in E_2 in order to obtain a desired H-compatible embedding of G. It remains to show that K_ρ's exist, which is rather simple, but a detailed argument requires some work. The claim essentially follows due to the fact that given an embedded bipartite graph in the plane there exists a simple closed curve that crosses every edge of the graph exactly once.

Let G'_ρ be as above. We construct an auxiliary graph $(G'_\rho)_4$ in five steps, see Fig. 7. The graph G'_ρ has the bipartition $V_i \cup V_j = V(G'_\rho)$ such that $\gamma(V_i) = \nu_i$ and $\gamma(V_j) = \nu_j$, where $\rho = \nu_i\nu_j$. Let $(G'_\rho)_0$ be the union of G'_ρ with its incident edges in G'. Let $(G'_\rho)_1$ be a plane graph obtained from the embedding of $(G'_\rho)_0$ by turning the crossings of edges of G'_ρ with valves into vertices; and parts of the boundary of $N_\varepsilon(\rho^*)$ joining consecutive pairs of crossings into edges as follows. A consecutive pair of vertices both of which are joined by an edge with a vertex of V_i (or V_j), is joined by an edge contained in the boundary of $N_\varepsilon(\rho^*)$ so that the edge is disjoint from every valve of an edge in H'_{ν_j} (or H'_{ν_i}). Let $(G'_\rho)_2$ be the subgraph of $(G'_\rho)_1$ contained in $N_\varepsilon(\rho^*)$. Let $(G'_\rho)_3$ be the plane graph obtained from $(G'_\rho)_2$ by contracting all the edges that do not join a vertex of V_i with a vertex of V_j. Let v_i and v_j denote the vertices that resulted from the contractions in the construction of $(G'_\rho)_3$. We assume that v_i is joined by an edge with vertices in V_j and v_j with vertices in V_i. Note that v_i or v_j might not exist. If none of v_i and v_j exist we simply have $(G'_\rho)_3 = G'_\rho$. Finally, let $(G'_\rho)_4$ be the plane graph obtained from $(G'_\rho)_3$ by applying the vertex split to v_j, if exists, so that the newly created edge is incident to a vertex of degree one denoted by v_j. We can assume that $(G'_\rho)_4$ is drawn in $N_\varepsilon(\rho^*)$ such that v_i is contained in the valve of an edge of H'_{ν_j} and v_j in the valve of an edge of H'_{ν_i}.

By taking the bipartition $V_i \cup \{v_i\}$ and $V_i \cup \{v_j\}$ of $(G'_\rho)_4$, it follows by [17, Lemma 6] that there exists a simple curve closed curve $(K_\rho)_0 \subset N_\varepsilon(\rho^*)$ intersecting every edge of $(G'_\rho)_4$ exactly

once. Finally, we construct K_ρ by cutting and deforming $(K_\rho)_0$ as follows, see Fig. 5 right. We distinguish two cases depending on whether v_j exists.

First, suppose that v_j exists. The desired curve K_ρ is obtained by cutting $(K_\rho)_0$ at its crossing point with the edge incident to v_j, and applying a homeomorphism of $N_\varepsilon(\rho^*)$ that takes the severed end points very close to a pair of the boundary points of $N_\varepsilon(\rho^*)$ that split the boundary into two parts, one of which contains the valves of the edges in H'_{ν_i} and the other the valves of the edges in H'_{ν_j}. Second, if v_j does not exist, we cut $(K_\rho)_0$ at its arbitrary point in the outer face of $(G'_\rho)_4$, and apply a similar homeomorphism of $N_\varepsilon(\rho^*)$.

In the end, we extend K_ρ a little bit so that both of its end points are contained in the boundary of $N_\varepsilon(\rho^*)$ and split the contracted vertices in $(G'_\rho)_4$ thereby recovering G'_ρ. ∎

4.3 Locally injective inputs

The following lemma implies that by iterating the derivative at most $|E(G)| = O(|V(G)|)$ many times we obtain an input that is locally injective.

Lemma 4.3. *If (G, H, γ) is in the normal form then $p((G')^N, (H')^N, (\gamma')^N) \leq p(G, H, \gamma)$. If additionally (G, H, γ) is not locally injective then the inequality is strict, i.e., $p((G')^N, (H')^N, (\gamma')^N) < p(G, H, \gamma)$. Moreover, $p((G')^N, (H')^N, (\gamma')^N) \leq p(G, H, \gamma) - \frac{1}{2}t$, where t is the number of vertices in G that do not satisfy the condition in property (i) or (ii) of locally injective inputs.*

Proof. Note that edges incident to fixed vertices in $((G')^N, (H')^N, (\gamma')^N)$ do not contribute towards $p((G')^N, (H')^N, (\gamma')^N)$, and thus, we will deal only with the remaining edges. We consider vertices in $V_{\geq 3} \subseteq V(G')$ to be their corresponding vertices in $V((G')^N)$. Since suppressing the vertices of degree two in G' and H' violating property (2) of the normal form in order to make the property satisfied does not increase the value of the potential, for the purpose of the proof of the lemma by somewhat abusing the notation we assume that we keep such vertices in $(G')^N$ and $(H')^N$.

Let H'_0 be the subgraph of $(H')^N = H'$ induced by its vertex subset $\{\rho^* |\ \rho \in E(H)\}$. Every connected graph on n vertices has at least $n - 1$ edges. It follows that ($\diamond$) the number of edges in H'_0 is at least $|V(H'_0)| - c = |E(H)| - c$, where c is the number of connected components of H'_0 that are trees. We use this fact together with the following observation to prove the claim.

Suppose for a while that H'_0 is connected. The set of edges of $(G')^N$ not incident to any $v \in V_{\geq 3}$, where the vertices in $V_{\geq 3}$ are now fixed, forms a matching M' whose edges are in one-to-one correspondence with edges in E_2 in G'. Note that none of the end vertices of edges in E_2 is of degree one in G'. Let $I_2 = \{(v, e_2)\ | v \in e_2 \in E_2\}$. By using the natural one-to-one correspondence between the edges of $G' \setminus E_2 \setminus V_{\geq 3}$ and the edges of G, it follows that the size of M' is upper bounded by the size of $E(G)$, since $2|M'| = 2|E_2| = |I_2| \leq |\{(v, e)|\ e \cap e_2 = \{v\}, e_2 \in E_2, e \in E(G')\}| \leq 2|E(G)|$. Hence, it follows that

$$|M'| = |E_2| \leq |E(G)| \tag{1}$$

Furthermore, $|M'| = |E(G)|$, only if (G, H, γ) is locally injective, and H'_0 contains a cycle. Indeed, if H'_0 does not contain a cycle, it is either a trivial graph consisting of a single vertex, or it contains a pair of vertices $(\rho_0)^*$ and $(\rho_1)^*$ of degree one such that $\gamma'(v_0) = (\rho_0)^*$ and $\gamma'(v_1) = (\rho_1)^*$, where $v_0 \in e_0 \in E_2, v_0 \in f_0 = v_0 u_0 \notin E_2$, and $v_1 \in e_1 \in E_2, v_1 \in f_1 = v_1 u_1 \notin E_2$. It holds that $|M'| < |E(G)|$, because (u_0, f_0) and (u_1, f_1) are not in the image of the injective map μ from I_2 taking $(v, e_2), v \in e_2 \in E_2$ to a pair $(v, e), e \in E(G')$, such that $e \cap e_2 = \{v\}$. Note that there exists at least two such pairs also if H'_0 is trivial (which is a fact that we will need later). Namely, (u, uv) and (v, uv), for some $\gamma'(uv) = \rho^* \in V(H'_0)$. By the same token, we
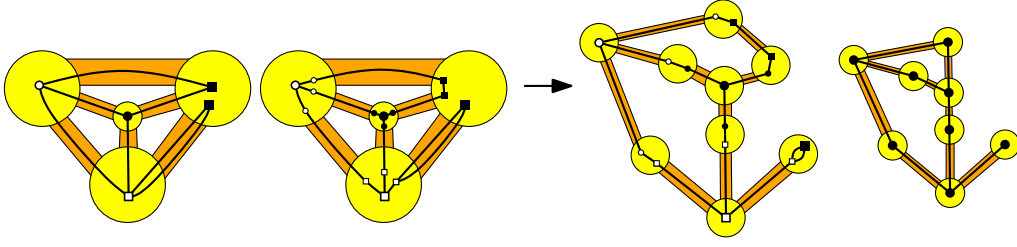
Figure 8: Constructing the normal form and derivating one more time the derivative from Fig. 3b on the left, we obtain an input that is strongly locally injective in the normal form on the right.

have that $|M'| < |E(G)|$, if (G, H, γ) is not locally injective. In fact, if $|M'| = |E(G)|$ then every connected component of G' must be a cycle.

If H'_0 has more connected components, we then have $|M'| \leq |E(G)| - c$, where the inequality is strict if (G, H, γ) is not locally injective. Indeed, if $|M'| = |E(G)| - c$, then there exist exactly $2c$ pairs (v, e), $v \in e \in E(G') \setminus E_2 \setminus V_{\geq 3}$, that are not in the image of the map μ. However, we showed in the previous paragraph that there are at least $2c$ such pairs (u, f), where both u and f are mapped by γ' to a vertex of degree at most one in H'_0. Hence, if $|M'| = |E(G)| - c$ then all the pairs, that are not contained in the image of μ, are accounted for by such (u, f)'s, in which case $V_{\geq 3}$ is exactly the subset of $V(G')$ of vertices of degree at least three. This establishes property (i) of locally injective inputs. Finally, to establish also (ii) we consider the natural correspondence of the vertices of degree one in G with the vertices of degree one in G'. Note that none of the pairs (u, f), $u \in f \in E(G')$, where u is of degree one, is in the image of ν. Thus, if $|M'| = |E(G)| - c$, then every leaf $u \in V(G')$ is mapped by γ' to some ρ^*, $\rho \in E(H)$, which is an isolated vertex or a leaf of H'_0. We need to show that there is no other edge besides $f \ni u$ mapped by γ' to ρ^*. If ρ^* is an isolated vertex of H'_0 this is immediate, since otherwise $|M'| < |E(G)| - c$. If ρ^* is a leaf of H'_0, every other edge $g \neq f$ such that $\gamma'(g) = \rho^*$ must share both end vertices with an edge of E_2, but then ρ^* has degree at least two in H'_0 (contradiction).

Putting it together, we have $|M'| \leq |E(G)| - c$ and $(\diamond)$ $|E(H)| - c \leq |E(H'_0)|$, where the first inequality is strict if (G, H, γ) is not locally injective as we just showed. Since the remaining edges of $(G')^N$ and $(H')^N$ contributes together zero towards $p((G')^N, (H')^N, (\gamma')^N)$, summing up the inequalities concludes the proof.

The "moreover" part follows immediately due to the fact that every vertex of G not satisfying (i) or (ii) causes the slack of $\frac{1}{2}$ in (1) as shown by the previous analysis. ∎

Given an input (G, H, γ) in the normal form. Similarly as in Section 4.2, let $V_{\geq 3} \subseteq V(G)$ denote the set of vertices in G consisting of the vertices $v \in V(G)$ such that $|E(\gamma(St(v)))| \geq 3$. The input is **strongly locally injective** if it is locally injective and

(iii) every vertex in $V_{\geq 3}$ is fixed.

For convenience, we would like work with strongly locally injective inputs, see Fig. 8. The following lemma shows that if the input (G, H, γ) is locally injective, but not strongly, we just derivate it one more time in order to arrive at a strongly locally injective input.

Lemma 4.4. *Suppose that (G, H, γ) in the normal form is locally injective. Then in $((G')^N, (H')^N, (\gamma')^N)$, every vertex $v \in V((G')^N)$, such that $|E((\gamma')^N(St(v)))| \geq 3$ is fixed. Moreover, $((G')^N, (H')^N, (\gamma')^N)$ is still locally injective.*

Proof. The lemma follows directly from the definition of the derivative. ∎

Deciding in, roughly, quadratic time in $p(G, H, \gamma)$, which is sufficient for us due to the bottleneck discussed in Section 4.4, whether the strongly locally injective input (G, H, γ) is positive, is quite straightforward. The reason is that in this case the order of crossings of a valve with edges, that are incident to the same vertex v of G, along the valve in an H-compatible embedding of G is determined by the rotation at v. In order to decide if a desired H-compatible embedding of G exists, we just detect if for every valve ω such an order of all the edges crossing ω exists, such that together the orders are compatible. To this end we consider relations between unordered pairs of edges of G such that the edges in a pair are mapped by γ to the same edge of H, and two pairs are related if they intersect in a pair of vertices. In the following we assume that (G, H, γ) is strongly locally injective.

Let $\Xi = \{\{e, f\} \mid e, f \in E(G) \text{ s.t. } e \neq f \text{ and } \gamma(e) = \gamma(f)\}$. Two elements $\{e_1, f_1\} \in \Xi$ and $\{e_2, f_2\} \in \Xi$ are **neighboring** if $|e_1 \cap e_2| = 1$, $|f_1 \cap f_2| = 1$ and $\gamma(e_1 \cap e_2) = \gamma(f_1 \cap f_2)$; we write $\{e_1, f_1\} \sim \{e_2, f_2\}$. An element $\{e_1, f_1\} \in \Xi$ is a **boundary pair** if there exists at most one $\{e_2, f_2\} \in \Xi$ such that $\{e_1, f_1\}$ and $\{e_2, f_2\}$ are neighboring. Let $\Xi_1, \ldots \Xi_l$ be equivalence classes of the transitive closure of the relation $\sim$. A boundary pair $\{e_1, f_1\} \in \Xi$ is **determined** if there exists a pair of edges e_2 and f_2 such that $|e_1 \cap e_2| = 1$, $|f_1 \cap f_2| = 1$, $\gamma(e_1 \cap e_2) = \gamma(f_1 \cap f_2)$ and $\gamma(e_2) \neq \gamma(f_2)$. By properties (i) and (iii), the subgraph G_Ξ of G induced by $\bigcup_{\{e,f\} \in \Xi} \{e, f\}$ has maximum degree two. First, we consider the case when a connected component of G_Ξ does not contain a vertex of degree one.

Lemma 4.5. *If there exists an equivalence class Ξ_c, such that the subgraph G_{Ξ_c} of G induced by $\bigcup_{\{e,f\} \in \Xi_c} \{e, f\}$ is a cycle, then (G, H, γ) is a negative input.*

Proof. Let $\Xi_c = \{\{e_0, f_0\}, \ldots \{e_{m-1}, f_{m-1}\}\}$, where $\{e_p, f_p\} \sim \{e_{p+1 \mod m}, f_{p+1 \mod m}\}$. Since $\bigcup_{\{e,f\} \in \Xi_c} \{e, f\}$ induces a cycle C of G, there exists the minimum value $p_0 < m$ such that $e_{p_0} = f_0$ or $f_{p_0} = e_0$. Note that $p_0 | m$, since $\gamma(e_q) = \gamma(e_{q+a p_0 \mod m})$, for every $0 \leq q, a < m$, and that $C = e_0, \ldots, e_{m-1}$. Due to the fact that the plane is orientable, it follows that the cycle C does not admit an H-compatible emebedding, since in an H-compatible embedding C must wind around a point in the plane more than once. $\blacksquare$

Note that Lemma 4.5 does not cover only the case when G_{Ξ_c} is a union of two cycles. By (ii), it must be that if Ξ_c contains a boundary pair, then it, in fact, contains exactly two boundary pairs, both of which are determined. Hence, in the following we assume that every Ξ_c either gives rise to a pair of cycles, or contains exactly two determined boundary pairs. We construct for every valve ω of $\rho \in E(H)$ the relation $(E_\rho, <_\omega)$, where $E_\rho = \{e \in E(H) \mid \gamma(e) = \rho\}$. We define relations $(E_\rho, <_\omega)$ by propagating relations enforced by the determined boundary pairs, for every determined pair contained in Ξ. We assume that $(E_\rho, <_\omega)$ encodes the increasing order of the crossing points of edges with ω as encountered when traversing $\omega \subset N_\varepsilon(\nu)$ in the direction inherited from the counterclockwise orientation of the boundary of $N_\varepsilon(\nu)$.

Let $\{e_1, f_1\} \in \Xi_c \subseteq \Xi$ be determined. Let $\Xi_c = \{\{e_1, f_1\}, \ldots \{e_m, f_m\}\}$ such that $\{e_p, f_p\} \sim \{e_{p+1}, f_{p+1}\}$. Let $\gamma(e_1) = \gamma(f_1) = \nu_i \nu_j$, $\gamma(e_0) = \nu_i \nu_{j'}$, $\gamma(f_0) = \nu_i \nu_{j''}$, where $\nu_{j'} \neq \nu_{j''}$ and $|e_0 \cap e_1| = 1$ and $|f_0 \cap f_1| = 1$. W.l.o.g. we suppose that $\nu_i \nu_j, \nu_i \nu_{j'}$ and $\nu_i \nu_{j''}$ appear in the rotation of ν_i in this order counterclockwise. Let ω_1 be the valve of $\nu_i \nu_j$ at ν_i. Let ω_2 be the valve of $\nu_i \nu_j$ at ν_j. We put the relation $f_1 <_{\omega_1} e_1$ into $(E_{\nu_i \nu_j}, <_{\omega_1})$ and $e_1 <_{\omega_2} f_1$ into $(E_{\nu_i \nu_j}, <_{\omega_2})$. Recursively, we put $f_{p+1} <_{\omega_{2p+1}} e_{p+1}$ into $(E_{\nu_i \nu_j}, <_{\omega_{2p+1}})$ and $e_{p+1} <_{\omega_{2(p+1)}} f_{p+1}$ into $(E_{\nu_i \nu_j}, <_{\omega_{2(p+1)}})$, if $f_p <_{\omega_{2p-1}} e_p$ and $e_p <_{\omega_{2p}} f_p$, and vice-versa, where ω_{2p} and ω_{2p+1} are valves contained in the boundary of the same disc.

If G_{Ξ_c} is a union of two disjoint cycles we add $f_p <_{\omega_{2p}} e_p$ and $e_p <_{\omega_{2p-1}} f_p$, or $f_p >_{\omega_{2p}} e_p$ and $e_p >_{\omega_{2p-1}} f_p$ for every p, in correspondence with the isotopy class of G.

Lemma 4.6. *Suppose that every equivalence class Ξ_c contains exactly two determined boundary pairs or G_{Ξ_c} is a union of two disjoint cycles. We can test in $O((p(G,H,\gamma))^2 + |V(G)|)$ time if (G,H,γ) is positive or negative.*

Proof. The relations $(E_\rho, <_\omega)$ can be clearly constructed in $O((p(G,H,\gamma)^2)$ time, since only edges not incident to fixed vertices are contained in pairs of Ξ. If the constructed $(E_\rho, <_\omega)$ is a total order for all $\rho \in E(H)$ and its valve ω, the isotopy class of every H-compatible embedding of G is determined by an embedding constructed as follows. We first draw the crossings of valves with edges of G according to the orders $(E_\rho, <_\omega)$; join every pair of consecutive crossing on the same edge of G by a straight-line segment contained in a pipe of an edge of H; and finish by drawing the straight-line segments joining vertices of G with the already drawn parts of edges contained in pipes. It is enough to check if the obtained embedding is in the desired isotopy class of G, which can be easily done in $O(|V(G)|)$ time by traversing orders $(E_\rho, <_\omega)$. Note that the only thing that can make the input negative is the containment of connected components of G in the interiors of its faces.

If $(E_\rho, <_\omega)$, for some $\rho \in E(H)$, contains a cyclic chain of inequalities, the input is clearly negative. ∎

4.4 Algorithm

We start with a description of the decision algorithm proving the first part of the theorem.

Decision Algorithm. Let $(G,H,\gamma) = (G_0, H_0, \gamma_0)$ be the input. We work with inputs in which G contains multiple edges and loops. However, w.l.o.g we assume that G does not contain a pair of multiple edges joining the same pair of vertices that form a face of size two, i.e., a lens, except for the outer face. Moreover, we assume that whenever a lens is created during the execution of the algorithm, the lens is eliminated by deleting one of its edges.

An execution of the algorithm is divided into steps. During the s-th step we process (G_s, H_s, γ_s) and output $(G_{s+1}, H_{s+1}, \gamma_{s+1})$ as follows.

First, by following the procedure described in Section 4.1 we either construct an instance $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ in the normal form that is positive if and only if (G_s, H_s, γ_s) is positive, or output that (G,H,γ) is negative, if the hypothesis of the first part of Lemma 4.1 is satisfied.

Second, if $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ is not strongly locally injective we proceed as follows. If (G_s, H_s, γ_s) satisfies the hypothesis of the first part of Lemma 4.2 with $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ playing the role of (G,H,γ) we output that (G,H,γ) is negative; otherwise we construct the derivative $((G_s^N)', (H_s^N)', (\gamma_s^N)') = (G_{s+1}, H_{s+1}, \gamma_{s+1})$ defined in Section 4.2 and proceed to the $(s+1)$-st step. Otherwise, $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ is strongly locally injective and we construct equivalence classes $\Xi_1, \ldots \Xi_l$ from Section 4.3 defined by $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ and proceed as follows.

We check if there exists a class Ξ_c satisfying the hypothesis of Lemma 4.5. If this is the case, then we output that (G,H,γ) is negative. Otherwise, we construct relations $(E_\rho, <_\omega)$, for every $\rho \in (H_s)^N$ and its valve ω. If there exists $(E_\rho, <_\omega)$ that is not a total order we output that (G,H,γ) is negative; otherwise we check if the isotopy class of an H-compatible embedding of G_s enforced by relations $(E_\rho, <_\omega)$ is the same as the given one and output that (G,H,γ) is positive if and only if this is the case.

The correctness of the algorithm follows directly from Lemma 4.1,4.2,4.5, and 4.6.

Running time analysis. By Lemma 4.3, Lemma 4.4, and Lemma 2.1 the number of steps of our algorithm is $O(|V(G)|)$. Furthermore, we show that $|V((G_s)^N)| = O(|V(G)|)$, for every s.

By Lemma 4.3, $p((G_{s+1})^N, (H_{s+1})^N, (\gamma_{s+1})^N) \leq p((G_s)^N, (H_s)^N, (\gamma_s)^N) - \frac{1}{2}t$, where t is the number of vertices in $(G_s)^N$ that do not satisfy the condition in property (i) or (ii) of locally injective inputs. The number of newly created vertices in $(G_{s+1})^N$ of degree at least three satisfying the condition in property (i) during the s-th step of the algorithm is at most $t = 2(p((G_s)^N, (H_s)^N, (\gamma_s)^N) - p((G_{s+1})^N, (H_{s+1})^N, (\gamma_{s+1})^N))$. Note that a vertex of degree $d \geq 3$ in $((G_s)^N, (H_s)^N, (\gamma_s)^N)$ satisfying the condition of property (i) becomes a fixed vertex of degree d in $((G_{s+1})^N, (H_{s+1})^N, (\gamma_{s+1})^N)$.

Let $V_{\geq 3}$ denote the set of fixed vertices of degree at least three in $(G_s)^N$. By the previous paragraph, $|V_{\geq 3}| \leq |V(G)| + \sum_{s \geq 0} 2(p((G_s)^N, (H_s)^N, (\gamma_s)^N) - p((G_{s+1})^N, (H_{s+1})^N, (\gamma_{s+1})^N)) \leq |V(G)| + 2p((G_0)^N, (H_0)^N, (\gamma_0)^N) = O(|E(G)|) = O(|V(G)|)$, for every s, due to the definition of the potential. The fact $|V((G_s)^N)| = O(|V(G)|)$, for every s, then follows by (2) in the definition of the normal form. Indeed, the number of vertices of degree two in $(G_s)^N$ mapped to redundant vertices in $(H_s)^N$ is linear in the number of remaining vertices in $(G_s)^N$ due to Lemma 2.1. Hence, the number of vertices in $(G_s)^N$ that are not fixed vertices of degree at least three is linear in $p(G, H, \gamma) + |V((H_s)^N)| = p(G, H, \gamma) + |V(H_0')| + |V_{\geq 3}|$ due to Lemma 2.1, where H_0' is defined as in the proof of Lemma 4.3 with $((G_{s-1})^N, (H_{s-1})^N, (\gamma_{s-1})^N)$ playing the role of (G, H, γ). It follows that the number of vertices in $|V((G_s)^N)|$, for every s, is linear in $p(G, H, \gamma) + |V(G)|$, since the size of the subset of $V(H_0')$ of non-redundant vertices is upper bounded by $|E(G)| = O(|V(G)|)$. This more-or-less follows inductively from (1) in the proof of Lemma 4.3, except that in every step we consider the subgraph of $(G_{s'})^N$, $s' < s$, induced by the set E_2, where E_2 is defined with $((G_{s'})^N, (H_{s'})^N, (\gamma_{s'})^N)$ playing the role of (G, H, γ). Indeed, the rest of the edges in $G_{s'+1}$ are incident to fixed vertices and are mapped by $\gamma_{s'+1}$ to a redundant vertex of $H_{s'+1}$. Formally, we show by induction on s' that $|E((G_{s'})^N \setminus V_{s'})| \leq |E(G)|$, where $V_{s'}$ is the set of fixed vertices of degree at least two in $V((G_{s'})^N)$ for all s'. In the base case we have $|E((G_1)^N \setminus V_1)| \leq |E_2| \leq |E(G)|$ by (1). For $s' > 0$, we have $|E((G_{s'+1})^N \setminus V_{s'+1})| \leq |E((G_{s'})^N \setminus V_{s'})|$ by the argument that we used to prove (1).

After having shown that $O(|V((G_{s-1})^N)|) = O(|V(G)|)$, it is easy to see that the s-th step of the algorithm can be easily carried out in $O(|V(G)|)$ time, since the planarity testing and embedding construction of all $\hat{H}_{\nu_i}'$'s needed in the construction of the derivative can be done in linear time in $O(|V(G_s)|) = O(|V((G_{s-1})^N)|)$ [20], and the construction of the instance in the normal form from the given one takes the same running time. The last step of the algorithm in which we construct orders $(E_\rho, <_\omega)$ can be easily done in $O(|V(G)|^2)$, since the number of pairs in Ξ is $O(|V(G)|^2)$ due to $|V(G_s)| = O(|V(G)|)$.

Algorithm constructing an embedding. The construction of an H-compatible embedding of G for strongly locally injective inputs is given by the set of total orders $(E_\rho, <_\omega)$, for every $\rho \in H$ and a valve ω of ρ. Therefore in order to construct a desired H-compatible embedding of G we need to reverse the order of steps in the decision algorithm. To this end we make the proof of the second part of Lemma 4.2 algorithmic. In order words, we need to construct the order in which a curve K_ρ intersect edges of G'. Since we can construct a desired order in linear time by the following lemma, the overall quadratic running time follows.

Lemma 4.7. *Given a plane bipartite graph G, we can construct in $O(|V(G)|)$ time a cyclic order $\mathcal{O}$ of edges of G such that there exists a simple closed curve in the plane properly crossing every edge of G exactly once, but otherwise disjoint from G, in the order given by $\mathcal{O}$.*

Proof. Let $V_1 \uplus V_2 = V(G)$ be the bipartition of $V(G)$. We construct a plane graph $G \cup T$ such that $V(T) = V_1$, $E(T) \cap E(G) = \emptyset$, and T is a spanning tree of $(G \cup T)[V_1] = T$. The tree T is constructed in linear time as follows. We subdivide every face of G by as many edges as possible in $\binom{V_1}{2}$, while keeping the resulting graph G_0 plane and its subgraph $G_0[V_1]$ without multiple

edges, e.g., we perform the subdivision so that every subgraph of $G_0[V_1]$ subdiving a face of G is a star containing all the vertices incident to the face. Since rotations at vertices are stored in doubly linked lists, G_0 can be constructed in time $O(|V(G)|)$. Note that $G_0[V_1]$ is connected, since every face of G is incident to a vertex in V_1. The tree T is obtained as a spanning tree of $G_0[V_1]$.

We contract all the edges of T in $G \cup T$. Let $\mathcal{O}'$ be the rotation at the vertex, that T was contracted into, in the resulting graph. The desired order $\mathcal{O}$ is obtained by substituting in $\mathcal{O}'$ for every edge its corresponding edge in G. ∎

References

[1] Hugo A. Akitaya, Greg Aloupis, Jeff Erickson, and Csaba Tóth. Recognizing weakly simple polygons. In *32nd International Symposium on Computational Geometry (SoCG 2016)*, volume 51 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 8:1–8:16, Dagstuhl, Germany, 2016. Schloss Dagstuhl–Leibniz-Zentrum für Informatik.

[2] Patrizio Angelini, Giordano Da Lozzo, Giuseppe Di Battista, and Fabrizio Frati. Strip planarity testing for embedded planar graphs. *Algorithmica*, 77(4):1022–1059, 2017.

[3] Patrizio Angelini and Giordano Da Lozzo. Clustered Planarity with Pipes. In Seok-Hee Hong, editor, *27th International Symposium on Algorithms and Computation (ISAAC 2016)*, volume 64 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 13:1–13:13, 2016.

[4] Therese C. Biedl. Drawing planar partitions III: Two constrained embedding problems. *Rutcor Research Report 13-98*, 1998.

[5] Thomas Bläsius and Ignaz Rutter. *A New Perspective on Clustered Planarity as a Combinatorial Embedding Problem*, pages 440–451. Springer Berlin Heidelberg, Berlin, Heidelberg, 2014.

[6] David Dylan Bremner. *Point visibility graphs and restricted-orientation polygon covering*. PhD thesis, Simon Fraser University, 1993.

[7] Hsien-Chih Chang, Jeff Erickson, and Chao Xu. Detecting weakly simple polygons. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1655–1670, 2015.

[8] Markus Chimani, Giuseppe Di Battista, Fabrizio Frati, and Karsten Klein. Advances on testing c-planarity of embedded flat clustered graphs. In Christian Duncan and Antonios Symvonis, editors, *Graph Drawing*, Lecture Notes in Computer Science, pages 416–427. 2014.

[9] Pier Francesco Cortese, Giuseppe Di Battista, Fabrizio Frati, Maurizio Patrignani, and Maurizio Pizzonia. C-planarity of c-connected clustered graphs. *J. Graph Algorithms Appl.*, 12(2):225–262, 2008.

[10] Pier Francesco Cortese, Giuseppe Di Battista, Maurizio Patrignani, and Maurizio Pizzonia. On embedding a cycle in a plane graph. *Discrete Mathematics*, 309(7):1856 – 1869, 2009.

[11] Reinhard Diestel. *Graph Theory*. Springer, New York, 2010.

[12] István Fáry. On straight line representation of planar graphs. *Acta Univ. Szeged. Sect. Sci. Math.*, 11:229–233, 1948.

[13] Qing-Wen Feng, Robert F. Cohen, and Peter Eades. How to draw a planar clustered graph. In Ding-Zhu Du and Ming Li, editors, *Computing and Combinatorics*, volume 959 of *Lecture Notes in Computer Science*, pages 21–30. Springer Berlin Heidelberg, 1995.

[14] Qing-Wen Feng, Robert F. Cohen, and Peter Eades. Planarity for clustered graphs. In Paul Spirakis, editor, *Algorithms ESA '95*, volume 979 of *Lecture Notes in Computer Science*, pages 213–226. Springer Berlin Heidelberg, 1995.

[15] Radoslav Fulek. C-planarity of embedded cyclic c-graphs. In *International Symposium on Graph Drawing and Network Visualization*, pages 94–106. Springer, 2016.

[16] Radoslav Fulek and Jan Kynčl. Hanani–Tutte for approximating maps of graphs. manuscript.

[17] Radoslav Fulek, Jan Kynčl, Igor Malinovic, and Dömötör Pálvölgyi. Clustered planarity testing revisited. *Electronic Journal of Combinatorics*, 22, 2015.

[18] Michael T. Goodrich, George S. Lueker, and Jonathan Z. Sun. *C-Planarity of Extrovert Clustered Graphs*, pages 211–222. Springer Berlin Heidelberg, Berlin, Heidelberg, 2006.

[19] Carsten Gutwenger, Michael Jünger, Sebastian Leipert, Petra Mutzel, Merijam Percan, and René Weiskircher. *Advances in C-Planarity Testing of Clustered Graphs*, pages 220–236. Springer Berlin Heidelberg, Berlin, Heidelberg, 2002.

[20] John Hopcroft and Robert Tarjan. Efficient planarity testing. *J. ACM*, 21(4):549–568, October 1974.

[21] Vít Jelínek, Eva Jelínková, Jan Kratochvíl, and Bernard Lidický. *Clustered Planarity: Embedded Clustered Graphs with Two-Component Clusters*, pages 121–132. Springer Berlin Heidelberg, Berlin, Heidelberg, 2009.

[22] Eva Jelínková, Jan Kára, Jan Kratochvíl, Martin Pergel, Ondřej Suchý, and Tomáš Vyskočil. Clustered planarity: Small clusters in cycles and Eulerian graphs. *J. Graph Algorithms Appl.*, 13(3):379–422, 2009.

[23] Piotr Minc. Embedding simplicial arcs into the plane. *Topol. Proc. 22*, pages 305–340, 1997.

[24] Ares Ribó Mor. Ph.d. thesis : Realization and counting problems for planar structures: Trees and linkages, polytopes and polyominoes. 2006. Freie U., Berlin.

[25] K Sieklucki. Realization of mappings. *Fundamenta Mathematicae*, 65(3):325–343, 1969.

[26] Mikhail Skopenkov. On approximability by embeddings of cycles in the plane. *Topology and its Applications*, 134(1):1–22, 2003.

[27] Godfried Toussaint. On separating two simple polygons by a single translation. *Discrete & Computational Geometry*, 4(3):265–278, 1989.

[28] Klaus Wagner. Bemerkungen zum vierfarbenproblem. *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 46:26–32, 1936.