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Perfect Partitions of Convex Sets in the Plane

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Abstract. For a region X in the plane, we denote by area(X) the area of X and by $\ell(\partial(X))$ the length of the boundary of X. Let S be a convex set in the plane, let $n \geq 2$ be an integer, and let $\alpha_1,\ \alpha_2, \ldots, \alpha_n$ be positive real numbers such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$ and $0 < \alpha_i \leq \frac{1}{2}$ for all $1 \leq i \leq n$. Then we shall show that S can be partitioned into n disjoint convex subsets $T_1,\ T_2, \ldots, T_n$ so that each T_i satisfies the following three conditions: (i) area$(T_i) = \alpha_i \times$ area(S); (ii) $\ell(T_i \cap \partial(S)) = \alpha_i \times \ell(\partial(S))$; and (iii) $T_i \cap \partial(S)$ consists of exactly one continuous curve.

1. Introduction

We begin with a motivation of the original problem related to our results. Some children attend a birthday party, and there is a big non-circular birthday cake. We want to divide the cake among all the children in such a way that each child gets the same amount of cake and the same amount of icing (exposed area) and holds it easily (i.e., each cake is convex and has exactly one icing side) [1]. If the height of the cake is constant, then the above problem can be said as follows. Let S be a convex set in the plane, which corresponds to the base of the cake. Then is it possible to partition S into n convex subsets so that each subset has the same area and has exactly one continuous part of the boundary of S with the same length (Fig. 1)? If such a partition exists, we say that S can be *perfectly partitioned* into n convex subsets, and call this partition a *perfect n-partition*.

It was proved in [2] that a perfect partition always exists for every $n \geq 3$, that is, the following theorem was obtained.

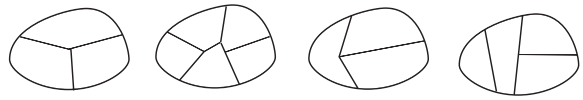

Fig. 1. (a), (b) Perfect partitions and (c), (d) non-perfect partitions.

Theorem 1 [2]. *Every convex set in the plane has a perfect n-partition for every integer* $n \geq 3$ (*Fig.* 1).

For a domain X in the plane, we denote by area(X) the area of X and by $\partial(X)$ the boundary of X. For a curve C in the plane, $\ell(C)$ denotes the length of C. In particular, $\ell(\partial(X))$ denotes the length of the boundary of X.

In this paper we prove the following Theorem 2, which is a generalization of Theorem 1, and the partition given in Theorem 2 is called a *generalized perfect n-partition*.

Theorem 2. *Let S be a convex set in the plane, let $n \geq 2$ be an integer, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers such that $\alpha_1+\alpha_2+\cdots+\alpha_n = 1$ and $0 < \alpha_i \leq \frac{1}{2}$ for all $1 \leq i \leq n$. Then S can be partitioned into n convex subsets $T_1, T_2, \ldots, T_n$ so that each T_i satisfies the following three conditions*: (i) area$(T_i) = \alpha_i \times$ area(S); (ii) $\ell(T_i \cap \partial(S)) = \alpha_i \times \ell(\partial(S))$; *and* (iii) $T_i \cap \partial(S)$ *consists of exactly one continuous curve* (*Fig.* 2).

If $\frac{1}{2} < \alpha_1 < 1$ and $\alpha_1 + \alpha_2 = 1$, then it is impossible to partition a circle C into two subsets satisfying the conditions of Theorem 2 since the area of a convex subset T_1 with $\ell(T_1 \cap \partial(C)) = \alpha_1 \times \ell(\partial(C))$ is always greater than $\alpha_1 \times$ area(C). Hence we need the condition that $\alpha_i \leq \frac{1}{2}$ for all i.

We now explain the relationship between a perfect n-partition and the result on balanced partitions of two sets of points in the plane. The following theorem was conjectured and proved for $n = 1, 2$ in [6] and [7], and was recently proved independently for every $n \geq 1$ in [4], [5], and [9]. Note that other interesting results related to our topic are found in [3], where partitions by fans are considered.

Theorem 3 [4], [5], [9]. *Let $m \geq 1$, $n \geq 1$ and $k \geq 2$ be positive integers. Let R be a set of mk red points and B a set of nk blue points in the plane such that no three points*

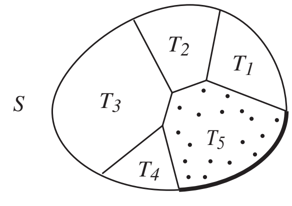

Fig. 2. A generalized perfect 5-partition of a convex set S with the emphasis of T_5 and $T_5 \cap \partial(S)$.

of $R \cup B$ *lie on the same line. Then* $R \cup B$ *can be partitioned into* k *disjoint subsets* $X_1, X_2, \ldots, X_k$ *so that every* X_i $(1 \leq i \leq k)$ *contains exactly* m *red points and* n *blue points, and* $\mathrm{conv}(X_i) \cap \mathrm{conv}(X_j) = \emptyset$ *for all* $i \neq j$, *where* $\mathrm{conv}(X_i)$ *denotes the convex hull of* X_i.

For a given convex set S in the plane, if we uniformly put a lot of red points on $\partial(S)$ and a lot of blue points on S, then by the above Theorem 3, we can partition S into k convex subsets $\{X_i\}$ so that each X_i contains the same number of red points and the same number of blue points, that is, the length of $X_i \cap \partial(S)$ is constant and the area of X_i is also constant. However, we cannot say that $X_i \cap \partial(S)$ consists of exactly one continuous curve (Fig. 1(d)). Thus even a perfect n-partition cannot be obtained directly from Theorem 3.

We conclude this section with a remark on Theorem 4 and a conjecture. When we consider a convex polygon in the plane instead of a convex set, we can similarly partition the convex polygon into some convex polygons under weaker conditions. This partition is given in Theorem 4. The following conjecture is a generalization of Theorem 3. Note that it is shown in [8] that if either $m_1 + m_2 + \cdots + m_k \leq 8$, or $1 \leq m_i \leq 2$ for every $1 \leq i \leq k$, then the conjecture holds.

Conjecture A. *Let* $k \geq 1$ *be a positive integer. Let* $m_1 \geq m_2 \geq \cdots \geq m_k \geq 1$ *and* $n_1 \geq n_2 \geq \cdots \geq n_k \geq 1$ *be positive integers such that* $m_1 \leq (m_1 + m_2 + \cdots + m_k)/3$ *and*

$$\frac{n_1}{m_1} = \frac{n_2}{m_2} = \cdots = \frac{n_k}{m_k}.$$

Let R *be a set of* $m_1 + m_2 + \cdots + m_k$ *red points and let* B *be a set of* $n_1 + n_2 + \cdots + n_k$ *blue points in the plane such that no three points of* $R \cup B$ *lie on the same line. Then* $R \cup B$ *can be partitioned into* k *disjoint subsets* $X_1, X_2, \ldots, X_k$ *so that each* X_i *contains exactly* m_i *red points and* n_i *blue points and* $\mathrm{conv}(X_i) \cap \mathrm{conv}(X_j) = \emptyset$ *for all* $i \neq j$.

2. Proof of Theorem 2

We define some notations. For two points X and Y in the plane, we denote by XY the *straight-line segment* joining X to Y and by $|XY|$ the length of XY, which is equal to the distance between X and Y.

A quadrilateral with consecutive vertices (P_1, P_2, P_3, P_4), a hexagon with consecutive vertices $(Q_1, Q_2, \ldots, Q_6)$, and an octagon with consecutive vertices $(R_1, R_2, \ldots, R_8)$ are denoted by $\mathrm{quad}(P_1P_2P_3P_4)$, $\mathrm{hex}(Q_1Q_2 \cdots Q_6)$, and $\mathrm{octagon}(R_1R_2 \cdots R_8)$, respectively.

We begin with a theorem on partitions of convex polygons, which might be of interest in itself and is used in the proof of Theorem 2.

Theorem 4. *Let* n *and* m *be integers such that* $3 \leq n$ *and* $1 \leq m \leq n$. *Let* P *be a convex polygon in the plane with* n *vertices, and let* $\beta_1, \beta_2, \ldots, \beta_m$ *be positive real numbers such that* $\beta_1 + \beta_2 + \cdots + \beta_m = \mathrm{area}(P)$. *Then for given* m *edges* $e_1, e_2, \ldots, e_m$

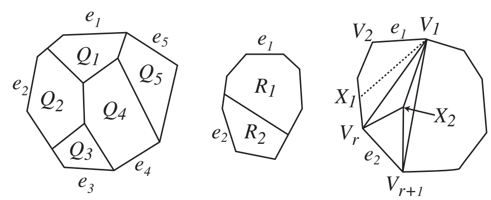

Fig. 3. A partition $\{Q_i\}$ of P, and the figure in the proof.

of P, P can be partitioned into m disjoint convex polygons $Q_1, Q_2, \ldots, Q_m$ so that each Q_i $(1 \le i \le m)$ contains the edge e_i and has area β_i (Fig. 3).

Proof. We prove the theorem by induction on m. If $m = 1$, then $Q_1 = P$ is the desired partition. If $m = 2$, then there exists a line that partition P into two sub-polygons R_1 and R_2 in such a way that R_1 contains e_1 but not e_2 and has area β_1, which gives us the desired partition. So we may assume that $m \ge 3$.

Let $V_1, V_2, \ldots, V_n$ be the consecutive vertices of P. By a new labeling of $\{V_i\}$ and $\{\beta_i\}$, we may assume that $e_1 = V_1V_2$, $e_2 = V_rV_{r+1}$ and no edges of P between V_2 and V_r are chosen in $\{e_i\}$.

Let P_1 be the sub-polygon with vertex set $\{V_1, V_2, \ldots, V_r\}$, which is obtained from P by dividing by the diagonal V_1V_r (Fig. 3). If $\text{area}(P_1) \ge \beta_1$, then we can find a point X_1 on the edges $V_2V_3 \cup \cdots \cup V_{r-1}V_r$ such that the area of the sub-polygon divided by V_1X_1 is equal to β_1. Then we can apply the inductive hypothesis to the remaining polygon. Therefore we may assume that $\text{area}(P_1) < \beta_1$.

Let P_2 be the sub-polygon with vertex set $\{V_1, V_2, \ldots, V_r, V_{r+1}\}$. If $\text{area}(P_2) \ge \beta_1 + \beta_2$, then $\text{area}(\triangle V_1V_rV_{r+1}) \ge \beta_1 - \text{area}(P_1) + \beta_2$, and so we can easily find a point X_2 in $\triangle V_1V_rV_{r+1}$ such that

$$\text{area}(\triangle X_2V_1V_r) = \beta_1 - \text{area}(P_1) \quad \text{and} \quad \text{area}(\triangle X_2V_rV_{r+1}) = \beta_2,$$

which implies that the convex polygon $P_1 + \triangle X_2V_1V_r$ has area β_1 (Fig. 3). Then we apply the inductive hypothesis to the remaining convex polygon, and get the desired partition of P. Hence we may assume that $\text{area}(P_2) < \beta_1 + \beta_2$.

Put $\gamma = \beta_1 + \beta_2 - \text{area}(P_2) > 0$. We consider the polygon $P - P_2$ together with the edges $\{V_1V_{r+1}, e_3, \ldots, e_m\}$ and the positive real numbers $\gamma, \beta_3, \ldots, \beta_m$. Then by the inductive hypothesis, $P - P_2$ can be partitioned into $m-1$ convex subsets $R, Q_3, \ldots, Q_m$. It is easy to see that $R \cup P_2$ is a convex polygon with area $\beta_1 + \beta_2$, and can be partitioned into two convex polygons that contain e_1 and e_2, respectively, and have areas β_1 and β_2, respectively. Consequently, the theorem is proved. □

In order to prove our theorem, we need some lemmas. The following lemma was proved in [2].

Lemma 5. *Let $\triangle ABC$ be a triangle in the plane, and let S be a convex set that is contained in $\triangle ABC$ and contains BC. Let* $\text{arc}(BC) = \partial(S) - BC$. *If $\angle B \ge \angle C$, then*

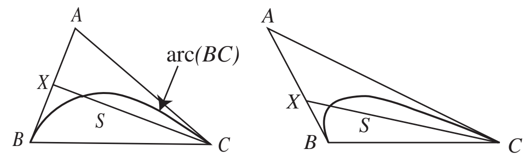

Fig. 4. Triangles $\triangle ABC$ and convex sets S.

for a point X on AB such that $|BX|+|XC| = \ell(\mathrm{arc}(BC))$, *it follows that* $\mathrm{area}(\triangle XBC) \le \mathrm{area}(S)$ (*Fig.* 4).

Lemma 6. *Let* $\triangle ABC$, *S and* $\mathrm{arc}(BC)$ *be the same as Lemma* 5 *above. Let h denote the height of* $\triangle ABC$ *relative to base AB or AC* (*Fig.* 5(*a*)–(*c*)). *Then*

$$\mathrm{area}(S) < \tfrac{1}{2}h \times \ell(\mathrm{arc}(BC)). \tag{1}$$

Proof. Without loss generality, we may assume that h is the height of $\triangle ABC$ relative to base AB. Let D be the foot of the perpendicular dropped from C to the line containing AB. Then $h = |CD|$. We first assume that $\angle B \le \pi/2$, that is, $\angle B$ is acute (Fig. 5(a),(b)).

If D is outside of $\triangle ABC$, then

$$\mathrm{area}(S) \le \mathrm{area}(\triangle ABC) = \tfrac{1}{2}|AB|h < \tfrac{1}{2}h \times \ell(\mathrm{arc}(BC)).$$

Thus we may assume that D lies on AB. Let E, if any, be the intersection of CD and $\mathrm{arc}(BC)$ (Fig. 5(d)). If E does not exist, then S is contained in $\triangle DBC$, and so $\mathrm{area}(S) \le \mathrm{area}(\triangle DBC) = (h/2)|BD| < (h/2) \times \ell(\mathrm{arc}(BC))$. Thus we may assume that the intersection E exists. Then S is divided into two subset $S_1 = S \cap \triangle ADC$ and $S_2 = S \cap \triangle DBC$ by the line CD. We have

$$\mathrm{area}(S_2) \le \triangle DBC = \tfrac{1}{2}h|DB| \le \tfrac{1}{2}h \times \ell(\mathrm{arc}(BE)).$$

Since S_1 is clearly contained in a rectangle R with edge CD and height $\ell(\mathrm{arc}(CE))/2$, it follows that

$$\mathrm{area}(S_1) < \mathrm{area}(R) = \tfrac{1}{2}h \times \ell(\mathrm{arc}(EC)).$$

Therefore we get the desired inequality in this case.

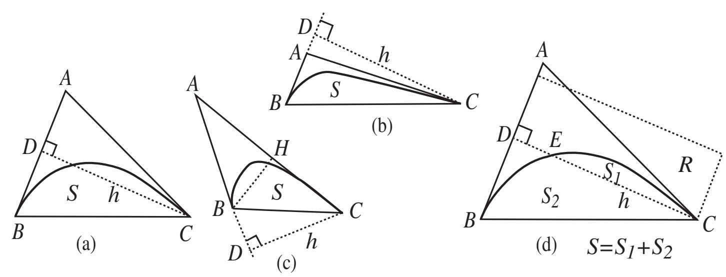

Fig. 5. Triangles $\triangle ABC$ and convex sets S.

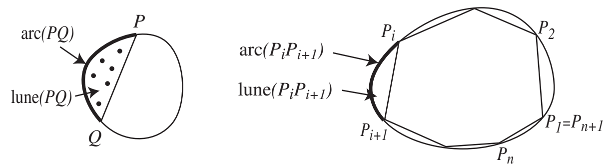

Fig. 6. Arc(PQ), lune(PQ), arc(P_iP_{i+1}) and lune(P_iP_{i+1}).

Next suppose that $\angle B > \pi/2$ (Fig. 5(c)). Let H be the foot of the perpendicular dropped from B to line AC. In this case we can show that the following inequality holds by the same argument as above:

$$\text{area}(S) < \tfrac{1}{2}|BH| \times \ell(\text{arc}(BC)).$$

Since $h = |CD| > |BH|$, the above inequality implies the desired inequality (1) of the lemma. $\square$

For two points P and Q on the boundary of a convex set S, the boundary of S is divided into two arcs by P and Q, and arc(PQ) denotes one of the arcs between P and Q that is easily determined from the context and is the shorter one in almost every case (Fig. 6). If it is not easily determined, we explain it more precisely. Moreover, we denote by lune(PQ) the *lune* surrounded by the arc arc(PQ) and by the line segment PQ (Fig. 6).

Lemma 7. *Let $n \ge 3$ be an integer, and let S and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the same as in Theorem* 2. *Let $P_1, P_2, \ldots, P_n$ be n points on $\partial(S)$ such that for every $1 \le i \le n$, $\ell(\text{arc}(P_iP_{i+1})) = \alpha_i \times \ell(\partial(S))$. Then* area(lune$(P_iP_{i+1})) < \alpha_i \times$ area(S) *for all* $1 \le i \le n$ *except at most one certain integer, where $P_{n+1} = P_1$ (Figure* 6).

Proof. Suppose that the lemma does not hold. By a new suitable labeling of $\{P_i\}$, we may assume that there exist n points $P_1, P_2, \ldots, P_n$ on $\partial(S)$ such that $\ell(\text{arc}(P_iP_{i+1})) = \alpha_i \times \ell(\partial(S))$ for all $1 \le i \le n$,

$$\text{area}(\text{lune}(P_1P_2)) \ge \alpha_1 \text{area}(S) \quad \text{and} \quad \text{area}(\text{lune}(P_rP_{r+1})) \ge \alpha_r \text{area}(S)$$

for some r, $2 \le r \le n$. We first consider the case that $3 \le r \le n-2$ (i.e., the case where arc(P_1P_2) and arc(P_rP_{r+1}) have no common vertex).

Since S is a convex set, there exist lines tangent to S at P_1, P_2, P_r and P_{r+1}, respectively. We first consider the case where these four lines makes a quadrilateral, that is, we first assume that the quadrilateral quad$(B_1B_2B_3B_4)$ given in Fig. 7 exists.

Consider the triangle $\triangle P_1P_{r+1}B_1$ and the convex subset $S \cap \triangle P_1P_{r+1}B_1$. Without loss of generality, we may assume that $\angle P_1 \le \angle P_{r+1}$ since otherwise we can apply the same argument to B_1P_1 instead of B_1P_{r+1}. Let Y be the point on B_1P_{r+1} such that $|P_1Y| + |YP_{r+1}| = \ell(\text{arc}(P_1P_{r+1}))$. Then by Lemma 5, we have

$$\text{area}(\triangle YP_1P_{r+1}) \le \text{area}(S \cap \triangle P_1P_{r+1}B_1).$$

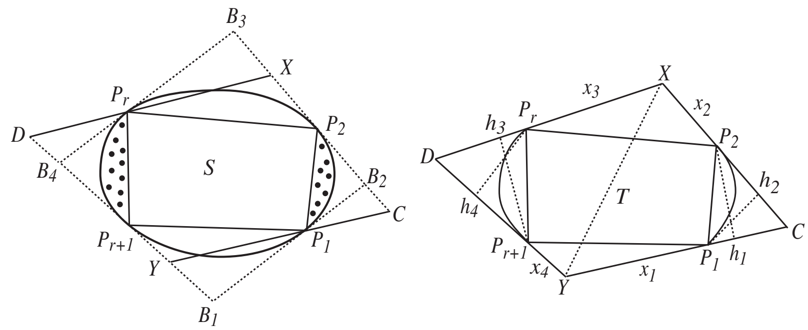

Fig. 7. Quad$(B_1B_2B_3B_4)$ and the convex set T.

By the same argument as above, for the convex subset $S \cap \triangle P_2B_3P_r$ and for the point X on P_2B_3 (or B_3P_r) with $|P_2X| + |XP_r| = \ell(\mathrm{arc}(P_2P_r))$, we have

$$\mathrm{area} \triangle XP_rP_2 \le \mathrm{area}(S \cap \triangle P_2B_3P_r).$$

Let

$$T := (S - (\mathrm{lune}(P_2P_r) \cup \mathrm{lune}(P_{r+1}P_1))) \cup \triangle YP_1P_{r+1} \cup \triangle XP_rP_2$$

(Fig. 7). Then T is a convex set with $\ell(\partial(T)) = \ell(\partial(S))$ and $\mathrm{area}(T) \le \mathrm{area}(S)$, and is contained in a quadrilateral quad$(CXDY)$. The following equalities and inequalities immediately hold:

$$\ell(\mathrm{arc}(P_1P_2)) = \alpha_1\ell(\partial(T)), \qquad \ell(\mathrm{arc}(P_rP_{r+1})) = \alpha_r\ell(\partial(T)),$$
$$\mathrm{area}(\mathrm{lune}(P_1P_2)) \ge \alpha_1\mathrm{area}(T), \qquad \mathrm{area}(\mathrm{lune}(P_rP_{r+1})) \ge \alpha_r\mathrm{area}(T).$$

Put $\ell^* = \ell(\partial(T))$, $x_1 = |YP_1|, x_2 = |XP_2|, x_3 = |XP_r|, x_4 = |YP_{r+1}|$ and $a = \ell(\mathrm{arc}(P_1P_2)) = \alpha_1\ell^*, b = \ell(\mathrm{arc}(P_rP_{r+1})) = \alpha_r\ell^*$. Let h_1 and h_2 the heights of $\triangle CP_2P_1$ relative to bases CP_1 and CP_2, respectively, and let h_3 and h_4 be the heights of $\triangle DP_{r+1}P_r$ relative to bases DP_r and DP_{r+1}, respectively. Then we obtain the following inequalities by Lemma 6:

$$\begin{aligned} \mathrm{area}(\mathrm{lune}(P_1P_2)) &< \tfrac{1}{2}ah_1, & \mathrm{area}(\mathrm{lune}(P_1P_2)) &< \tfrac{1}{2}ah_2, \\ \mathrm{area}(\mathrm{lune}(P_rP_{r+1})) &< \tfrac{1}{2}bh_3, & \mathrm{area}(\mathrm{lune}(P_rP_{r+1})) &< \tfrac{1}{2}bh_4, \end{aligned}$$

$$\begin{aligned} \mathrm{area}(\mathrm{quad}(XYP_1P_2)) &= \mathrm{area}(\triangle XP_1P_2) + \mathrm{area}(\triangle XYP_1) \\ &\ge \mathrm{area}(\triangle XP_1P_2) + \mathrm{area}(\triangle YP_1P_2) \\ &= \tfrac{1}{2}(x_2h_2 + x_1h_1), \end{aligned} \tag{2}$$

$$\begin{aligned} \mathrm{area}(\mathrm{quad}(XP_rP_{r+1}Y)) &\ge \mathrm{area}(\triangle XP_rP_{r+1}) + \mathrm{area}(\triangle YP_rP_{r+1}) \\ &= \tfrac{1}{2}(x_3h_3 + x_4h_4). \end{aligned}$$

By symmetry, we may assume that h_1 is the smallest among all the h_i's. Then

$$\begin{aligned}\mathrm{area}(\mathrm{hex}(XP_rP_{r+1}YP_1P_2)) &\geq \tfrac{1}{2}(x_1+x_2+x_3+x_4)h_1\\ &= \tfrac{1}{2}(1-\alpha_1-\alpha_r)\ell^*h_1,\end{aligned}$$

$$\begin{aligned}\mathrm{area}(\mathrm{hex}(XP_rP_{r+1}YP_1P_2)) &= \mathrm{area}(T)-\mathrm{area}(\mathrm{lune}(P_1P_2))-\mathrm{area}(\mathrm{lune}(P_rP_{r+1}))\\ &\leq (1-\alpha_1-\alpha_r)\mathrm{area}(T)\leq(1-\alpha_1-\alpha_r)\mathrm{area}(S).\end{aligned}$$

Hence $\ell^*h_1/2 \leq \mathrm{area}(S)$, and thus

$$\mathrm{area}(\mathrm{lune}(P_1P_2)) \geq \alpha_1\mathrm{area}(S) \geq \frac{\alpha_1\ell^*h_1}{2}.$$

However, this contradicts the fact that

$$\mathrm{area}(\mathrm{lune}(P_1P_2)) < \tfrac{1}{2}ah_1 = \frac{\alpha_1\ell^*h_1}{2} \qquad \text{(by (2))}.$$

We next assume that the quadrilateral quad$(B_1B_2B_3B_4)$ does not exist, that is, we assume that the configuration given in Figs. 8 or 9 occurs. We first consider the case of Fig. 8. Let B_1, B_2, B_3 be the intersections of lines tangent to S at P_2, P_r, P_{r+1}, P_1. We take two points X and Y on $P_2B_1\cup B_1P_r$ and $P_{r+1}B_3\cup B_3P_1$, respectively, which satisfy the conditions of Lemma 5. Let D be the intersection of the two lines containing XP_r and YP_{r+1}, respectively. Let h_1 and h_2 be the heights of $\triangle P_rDP_{r+1}$ relative to bases DP_{r+1} and DP_r, respectively. Then by Lemmas 5 and 6, we have

$$\begin{aligned}\mathrm{area}(\mathrm{lune}(P_rP_{r+1})) &< \tfrac{1}{2}h_1\ell(\mathrm{arc}(P_rP_{r+1})),\\ \mathrm{area}(\mathrm{lune}(P_rP_{r+1})) &< \tfrac{1}{2}h_2\ell(\mathrm{arc}(P_rP_{r+1})),\end{aligned}$$

$$\begin{aligned}|P_2X|+|XP_r| &= \ell(\mathrm{arc}(P_2P_r)), &\qquad \mathrm{area}(\triangle P_2XP_r)&\leq \mathrm{area}(\mathrm{lune}(P_2P_r)),\\ |P_{r+1}Y|+|YP_1| &= \ell(\mathrm{arc}(P_{r+1}P_1)), &\qquad \mathrm{area}(\triangle P_{r+1}YP_1)&\leq \mathrm{area}(\mathrm{lune}(P_{r+1}P_1)).\end{aligned}$$

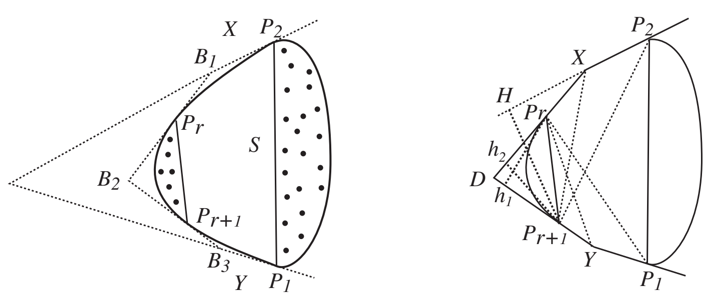

Fig. 8. The convex set S and hex$(P_2XP_rP_{r+1}YP_1)$.

Put $\ell^* = \ell(\partial(S))$. By the symmetry of h_1 and h_2, we may assume that $h_1 \le h_2$. Then we obtain

$$\begin{aligned}(1-\alpha_1-\alpha_r)\mathrm{area}(S) &\ge \mathrm{area}(\mathrm{hex}(P_2XP_rP_{r+1}YP_1)),\\ &> \mathrm{area}(\triangle P_2XP_{r+1}) + \mathrm{area}(\triangle XP_rP_{r+1})\\ &\quad + \mathrm{area}(\triangle P_1YP_r) + \mathrm{area}(\triangle YP_{r+1}P_r)\\ &\ge \tfrac{1}{2}|P_2X|h_2 + \tfrac{1}{2}|XP_r|h_2\\ &\quad + \tfrac{1}{2}|P_1Y|h_1 + \tfrac{1}{2}|YP_{r+1}|h_1 \qquad (\text{by } |P_{r+1}H| \ge h_2)\\ &= \tfrac{1}{2}\ell(\mathrm{arc}(P_2P_r))h_2 + \tfrac{1}{2}\ell(\mathrm{arc}(P_{r+1}P_1))h_1\\ &\ge \tfrac{1}{2}h_1(1-\alpha_1-\alpha_r)\ell^*.\end{aligned}$$

Therefore $\mathrm{area}(S) > \frac{1}{2}h_1\ell^*$. Then it follows from Lemma 5 that

$$\mathrm{area}(\mathrm{lune}(P_rP_{r+1})) < \tfrac{1}{2}h_1\ell(\mathrm{arc}(P_rP_{r+1})) = \tfrac{1}{2}h_1\alpha_r\ell^* < \alpha_r\mathrm{area}(S).$$

This contradicts the assumption that $\mathrm{area}(\mathrm{lune}(P_rP_{r+1})) \ge \alpha_r\mathrm{area}(S)$.

We next consider the case of Fig. 9, where KP_2, KP_r, BP_1 and BP_{r+1} are tangent to S at P_2, P_r, P_1 and P_{r+1}, respectively. By considering $\mathrm{lune}(P_2P_r)$ and $\triangle P_2KP_r$, we take a point R on $P_2K \cup KP_r$ which satisfies the conditions of Lemma 5. Let D be the intersection of the line passing through BR and $\partial(S)$. We draw a line AC tangent to S at D, and take two points X and Y on $DA \cup AP_1$ and $DC \cup CP_{r+1}$, respectively, which satisfy the conditions of Lemma 5 (Fig. 9).

Let E be the intersection of AB and a line containing RP_2, and let F be the intersection of BC and a line containing RP_r. Let k_1 and k_2 denote the heights of $\triangle P_1EP_2$ and $\triangle P_rFP_{r+1}$, respectively. Then we obtain the following inequalities, where we may assume that two lines passing through P_1P_{r+1} and RP_2 intersect at some point above BD since otherwise two lines passing through P_1P_{r+1} and RP_r intersect at some point below BD and the similar arguments given below can be applied:

$$\begin{aligned}\ell(\mathrm{arc}(P_1P_2)) &= \alpha_1\ell(\partial(S)), &\qquad \mathrm{area}(\mathrm{lune}(P_1P_2)) &\ge \alpha_1\mathrm{area}(S),\\ \ell(\mathrm{arc}(P_rP_{r+1})) &= \alpha_r\ell(\partial(S)), &\qquad \mathrm{area}(\mathrm{lune}(P_rP_{r+1})) &\ge \alpha_r\mathrm{area}(S),\end{aligned}$$

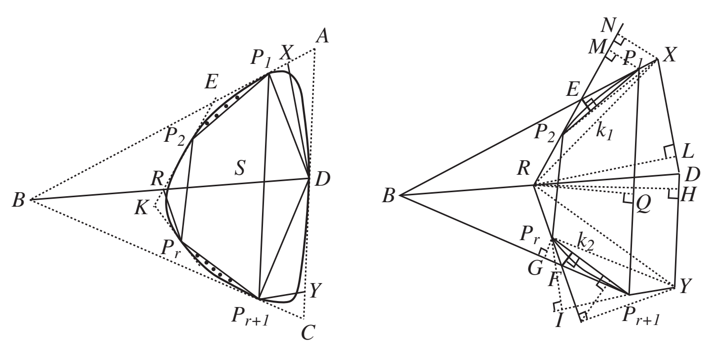

Fig. 9. The convex set S and octagon$(XP_1P_2RP_rP_{r+1}YD)$.

$$\ell(\text{arc}(P_2P_r)) = |P_2R| + |RP_r|, \qquad \text{area}(\text{lune}(P_2P_r)) \geq \text{area}(\triangle P_2RP_r),$$
$$\ell(\text{arc}(DP_1)) = |DX| + |XP_1|, \qquad \text{area}(\text{lune}(DP_1)) \geq \text{area}(\triangle DXP_1),$$
$$\ell(\text{arc}(P_{r+1}D)) = |P_{r+1}Y| + |YD|, \qquad \text{area}(\text{lune}(P_{r+1}D)) \geq \text{area}(\triangle P_{r+1}YD),$$

$$\text{area}(\triangle XP_1P_2) \geq \tfrac{1}{2}|XP_1|k_1, \qquad \text{area}(\triangle XP_2R) = \tfrac{1}{2}|P_2R||XN| > \tfrac{1}{2}|P_2R|k_1,$$

$$|RL| \geq |RQ| \geq |MP_1| > k_1, \qquad \text{area}(\triangle XRD) > \tfrac{1}{2}|XD|k_1,$$

$$|P_rI| > |P_rG| > k_2, \qquad \text{area}(\triangle YP_rP_{r+1}) \geq \tfrac{1}{2}|YP_{r+1}|k_2,$$

$$\text{area}(\triangle YRP_r) > \tfrac{1}{2}|RP_r|k_2,$$

$$|RH| \geq |RQ| \geq |MP_1| > k_1, \qquad \text{area}(\triangle YRD) > \tfrac{1}{2}|YD|k_1.$$

Hence we have

$$\begin{aligned}&\text{area}(\text{octagon}(XP_1P_2RP_rP_{r+1}YD))\\ &\qquad > \tfrac{1}{2}(|DX| + |XP_1| + |P_2R| + |YD|)k_1 + \tfrac{1}{2}(|RP_r| + |P_{r+1}Y|)k_2.\end{aligned}$$

We first assume that $k_1 \leq k_2$. Put $\ell^* = \ell(\partial(S))$. Then

$$\begin{aligned}(1 - \alpha_1 - \alpha_r)\text{area}(S) &\geq \text{area}(\text{octagon}(XP_1P_2RP_rP_{r+1}YD))\\ &> \tfrac{1}{2}(|DX| + |XP_1| + |P_2R| + |YD| + |RP_r| + |P_{r+1}Y|)k_1\\ &= (1 - \alpha_1 - \alpha_r)\ell^* k_1.\end{aligned}$$

Therefore area$(S) \geq \frac{1}{2}k_1\ell^*$, and thus

$$\frac{\alpha_1\ell^* k_1}{2} \leq \alpha_1\text{area}(S) \leq \text{area}(\text{lune}(P_1P_2)) < \tfrac{1}{2}\ell(\text{arc}(P_1P_2))k_1 = \frac{\alpha_1\ell^* k_1}{2}.$$

This is a contradiction.

We next assume $k_2 < k_1$. Then we can similarly show that area$(S) \geq \frac{1}{2}k_2\ell^*$. Thus we can derive a contradiction as follows:

$$\frac{\alpha_r\ell^* k_2}{2} \leq \alpha_r\text{area}(S) \leq \text{area}(\text{lune}(P_rP_{r+1})) < \tfrac{1}{2}\ell(\text{arc}(P_rP_{r+1}))k_2 = \frac{\alpha_r\ell^* k_2}{2}.$$

Consequently the proof of the case of Fig. 9 is complete.

We next consider the case that $r = 2$. In this case we have two configurations given in Figs. 10 and 11. Since these figures are very similar to Figs. 8 and 9, we can similarly derive a contradiction in each case by almost the same arguments given above. Consequently the lemma is proved. □

Lemma 8. *Let S, n and $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the same as in Theorem* 2. *Then there exist n points $P_1, P_2, \ldots, P_n$ on $\partial(S)$ such that $\ell(\text{arc}(P_iP_{i+1})) = \alpha_i\ell(\partial(S))$ and* area(lune$(P_i \times P_{i+1})) \leq \alpha_i$area$(S)$ *for every $1 \leq i \leq n$, where $P_{n+1} = P_1$.*

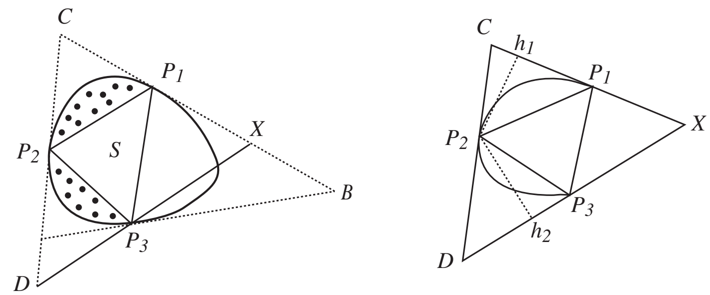

Fig. 10. The convex set S and hex$(DXP_1P_2P_3Y)$.

Proof. By Lemma 7 and by a new labeling of $\{P_i\}$, we may assume that there exist n points $Q_1, \ldots, Q_n$ on $\partial(S)$ such that $\ell(\text{arc}(Q_iQ_{i+1})) = \alpha_i\ell(\partial(S))$ for all $1 \le i \le n$, area(lune$(Q_1Q_2)) > \alpha_1$area(S) and area(lune$(Q_jQ_{j+1})) \le \alpha_j$area$(S)$ for all $2 \le j \le n$.

If there exist n points $R_1, R_2, \ldots, R_n$ such that $\ell(\text{arc}(R_iR_{i+1})) = \alpha_i\ell(\partial(S))$ for all $1 \le i \le n$ and area(lune$(R_1R_2)) \le \alpha_1$area(S), then by Lemma 7, when we continuously move $\{Q_i\}$ to $\{R_i\}$, we obtain the desired n points $\{P_i\}$ satisfying the conditions of the lemma. So it is sufficient to show the existence such n points $R_1, R_2, \ldots, R_n$. Moreover, if there exists two points Y_1 and Y_2 on $\partial(S)$ for which $\ell(\text{arc}(Y_1Y_2)) = \alpha_1\ell(\partial(S))$ and area(lune$(Y_1Y_2)) \le \alpha_1$area(S), then add the remaining $n-2$ points $Y_3, \ldots, Y_n$ on $\partial(S) - \text{arc}(Y_1Y_2)$ so that $\ell(\text{arc}(Y_iY_{i+1})) = \alpha_i\ell(\partial(S))$ for $2 \le i \le n$. Then by Lemma 7, these n points $Y_1, Y_2, \ldots, Y_n$ are the desired n points $\{R_i\}$.

We now show the existence of two such points Y_1 and Y_2. Since $\alpha_1 < \frac{1}{2}$, there exist at least three points $Z_1, Z_2, \ldots, Z_m$ $(m \ge 3)$ such that $\ell(\text{arc}(Z_iZ_{i+1})) = \alpha_1\ell(\partial(S))$ for all $1 \le i \le m-1$ and $\ell(\text{arc}(Z_mZ_1)) < \alpha_1\ell(\partial(S))$. By applying Lemma 7 to the points $Z_1, Z_2, \ldots, Z_m$, we can say that at least one of lune(Z_1Z_2) and lune(Z_2Z_3) has area less than $\alpha_1$area(S). Therefore the lemma is proved. □

Proof of Theorem 2. By Lemma 8, there exist n points $\{P_i\}$ on $\partial(S)$ such that $\ell(\text{arc}(P_i \times P_{i+1})) = \alpha_i\ell(\partial(S))$ and area(lune$(P_iP_{i+1})) \le \alpha_i$area$(S)$ for all $1 \le i \le n$. Let P^* be the polygon with vertex set $\{P_1, P_2, \ldots, P_n\}$, and let $\{e_1, e_2, \ldots, e_m\}$ be the set of edges

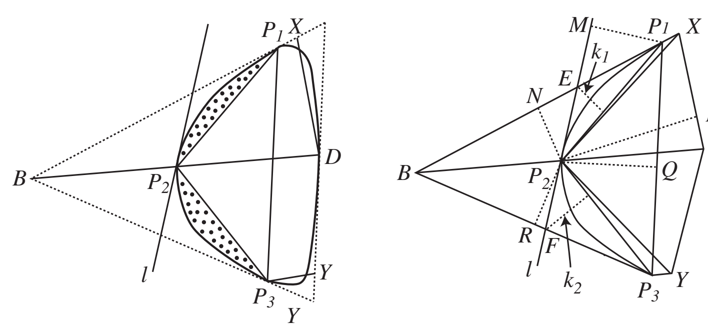

Fig. 11. The convex set S and hex$(DXP_1P_2P_3Y)$.

$e_k = P_j P_{j+1}$ such that area(lune($P_j P_{j+1}$)) $< \alpha_j$area(S) for some $1 \leq j \leq n$. Define positive real numbers $\beta_1, \beta_2, \ldots, \beta_m$ by

$$\beta_k = \alpha_j \text{area}(S) - \text{area}(\text{lune}(P_j P_{j+1})).$$

Then by Theorem 4, P^* can be partitioned into m convex subsets $Q_1, Q_2, \ldots, Q_m$ such that each Q_k has area β_k and contains e_j. Since S is a convex set, it is clear that lune($P_j P_{j+1}$) $\cup\, Q_k$ is a convex subset. It is also obvious that lune($P_j P_{j+1}$) $\cup\, Q_k$ has area $\alpha_j$area(S) and one continuous part of $\partial(S)$ with length $\ell(\text{arc}(P_j P_{j+1})) = \alpha_i \ell(\partial(S))$. Consequently the theorem is proved. □

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