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ε-Unit Distance Graphs

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Abstract. We consider a variation on the problem of determining the chromatic number of the Euclidean plane and define the **ε-unit distance graph** to be the graph whose vertices are the points of E^2, in which two points are adjacent whenever their distance is within ε of 1. For certain values of ε we are able to show that the chromatic number is exactly 7. For some smaller values we show the chromatic number is at least 5. We offer a conjecture, with some supporting evidence, that for any $\varepsilon > 0$ the chromatic number is 7.

1. Introduction

The chromatic number of the Euclidean plane is defined to be the minimum number of colors that can be used to color the points of E^2 so that no two points at distance 1 receive the same color. This problem is often discussed in the language of graph theory, wherein we define the **unit distance graph**, denoted G_0 (notation justified below), as the graph whose vertices are the points of E^2 and in which two points are adjacent if their distance is exactly 1. The problem is to determine the chromatic number of this graph. It is known that $4 \leq \chi(G_0) \leq 7$, bounds that have remained unchanged for 50 years [2].

The current lower bound of 4 follows from the Moser graph, a 4-chromatic subgraph of G_0 which is shown in Fig. 1. The upper bound of 7 follows from the periodic coloring obtained from a regular hexagonal tiling of the plane as shown in Fig. 2. Let s be the length of a side of a hexagon. Then two points in the same hexagon are at most $2s$ apart, whereas two points in different hexagons of the same color are at least $s\sqrt{7}$ apart (the length of the heavy line in Fig. 2). Therefore, if

$$1/\sqrt{7} < s < \tfrac{1}{2},$$

then no two points at distance 1 receive the same color.

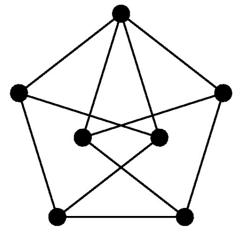

Fig. 1. The Moser graph.

This flexibility in the value of s suggests that we can add more edges to G_0 and still bound the chromatic number by 7. So we investigate supergraphs of the unit distance graph whose chromatic numbers are bounded above by 7. Define the ε**-unit distance graph**, denoted G_ε, to be the graph whose vertices are the points of E^2, in which two points are adjacent whenever their distance d satisfies

$$1 - \varepsilon \leq d \leq 1 + \varepsilon.$$

Our bounds on s imply that whenever

$$\varepsilon < \frac{\sqrt{7}-2}{\sqrt{7}+2} = 0.13899825\ldots$$

we have $\chi(G_\varepsilon) \leq 7$. We denote the value of the right-hand side of the above inequality by ε_1.

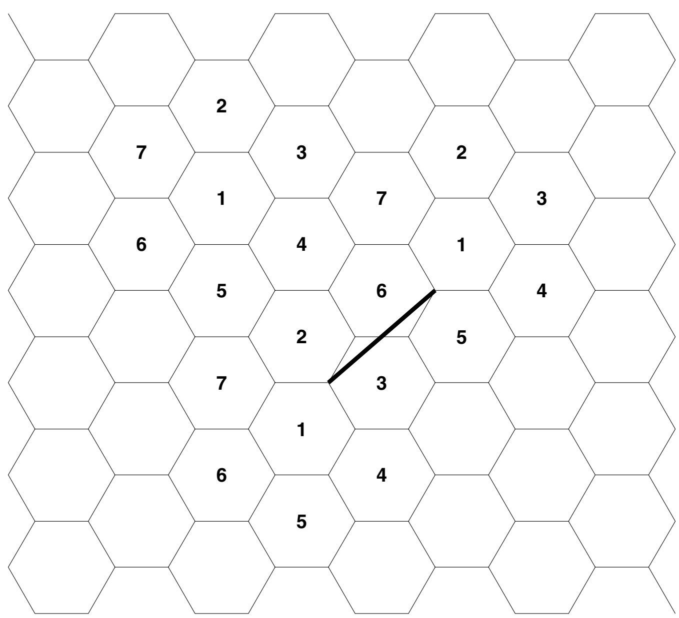

Fig. 2. The hexagon tiling.

2. Graphs

First we show that for certain values of ε, $\chi(G_\varepsilon)$ can be determined exactly, and then that for a larger range of values of ε, we can improve the lower bound on the chromatic number to 5. We note that the difficulty lies not so much in finding graphs with certain chromatic numbers, but more in finding examples that are small enough so that we can prove that their chromatic numbers are as claimed. In both cases the proofs are computational.

Theorem 1.

(a) $\chi(G_\varepsilon) = 7$, *for* ε *satisfying*

$$\frac{\sqrt{43}-5}{\sqrt{43}+5} < \varepsilon < \frac{\sqrt{7}-2}{\sqrt{7}+2}.$$

(b) $\chi(G_\varepsilon) \geq 5$ *for*

$$\varepsilon > \frac{\sqrt{149}-12}{\sqrt{149}+12}.$$

Numerically, the bounds in the theorem are

$$0.134756\cdots < \varepsilon < 0.138998\cdots$$

and

$$\varepsilon > 0.008533\ldots$$

Proof. (a) To show that $\chi(G_\varepsilon) = 7$ for the required values of ε, we use the regular triangular lattice graph with side length s. The vertices of our graph correspond to lattice points in the closed region bounded by the x-axis, the y-axis, the line $x = 13s$, and the line $y = 7\sqrt{3}s$. We take $s = 2/(5+\sqrt{43})$ and $\varepsilon > (\sqrt{43}-5)/(\sqrt{43}+5)$. In Fig. 3 we show the vertices of the graph and the neighbors of one vertex, v. The neighbors of v (the enlarged vertex in the center) are the circled vertices. This graph has order 203 and chromatic number 7. It turns out that some vertices of this graph can be removed without affecting the chromatic number. If four vertices are removed from the upper left and lower right corners of the graph (a total of eight vertices) and one vertex is removed from the upper right and lower left corners, we are left with a graph on 193 vertices which also has chromatic number 7. The computations that establish these values are discussed in the last section of the paper.

(b) To prove the second assertion, we construct an ε-unit distance graph on 475 vertices whose chromatic number is 5 for some $\varepsilon < 0.01$. Begin with the square grid graph with side length s. To construct a 5-chromatic ε-unit distance graph for small values of ε we look for small ranges of integers which can be expressed at the sums of squares in several different ways. This will give us edges joining a given lattice point to several other lattice points. Our choice is to use the range 144–149. The following table

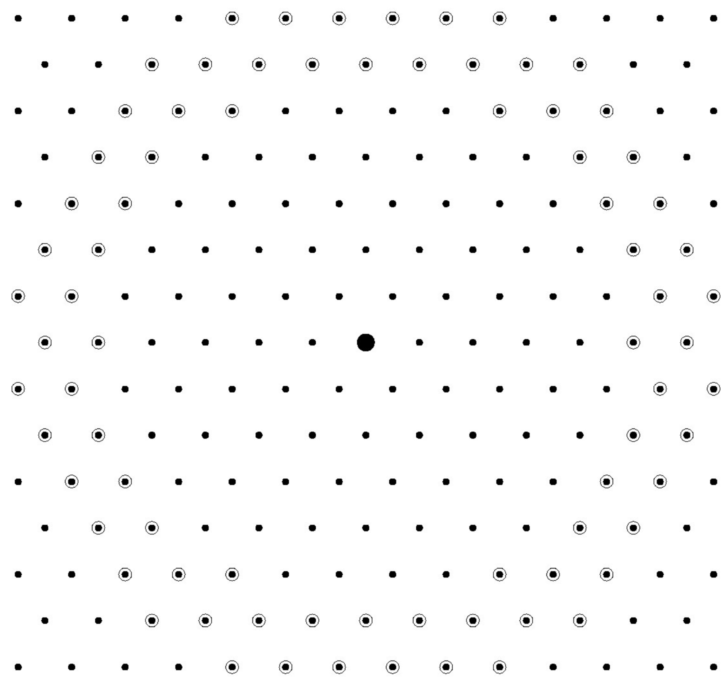

Fig. 3. An ε-unit distance graph with chromatic number 7.

shows how each of these values can be expressed as the sum of two squares:

144	$12^2 + 0^2$
145	$12^2 + 1^2$ and $9^2 + 8^2$
146	$11^2 + 5^2$
147	
148	$12^2 + 2^2$
149	$10^2 + 7^2$

If we choose our values of s and ε so that $1 - \varepsilon < 144s$ and $149s < 1 + \varepsilon$, we ensure that our graph will have a reasonable number of edges. Specifically, for each pair (a, b) of integers such that $144 \le a^2 + b^2 \le 149$, and for each grid point (i, j) in our graph, we have edges to the vertices $(i \pm a, j \pm b)$ and $(i \pm b, j \pm a)$. So the vertices in our graph can have degrees as large as 44 (since we will use a finite subgraph of the infinite grid, vertices near the border will have degrees less than 44). Consider a rectangular region of the grid, 25 vertices by 19 vertices. The induced subgraph on such a set of 475 vertices has chromatic number 5. The graph is an ε-unit distance graph whenever

$$s = \frac{2}{\sqrt{149} + 12} = 0.08262\ldots$$

and

$$\varepsilon > \frac{\sqrt{149} - 12}{\sqrt{149} + 12} = 0.008533\ldots.$$

This completes the proof. $\square$

We also tried to find the smallest subgraph of a G_ε, for $\varepsilon < \varepsilon_1$, whose chromatic number was more than 4 (the lower bound for $\chi(G_0)$). Our best effort has 24 vertices and chromatic number 5. It is constructed by taking a small section of the infinite triangular lattice generated by $(s, 0)$ and $(s/2, s\sqrt{3}/2)$. We take

$$s = \frac{2}{2\sqrt{3} + \sqrt{7}}$$

and

$$\varepsilon > \frac{2\sqrt{3} - \sqrt{7}}{2\sqrt{3} + \sqrt{7}} = 0.133939\ldots.$$

The 24 vertices are chosen as depicted in Fig. 4. In the figure the neighbors of one vertex **v** are circled.

For completeness, we note that for all $\varepsilon < 1$, $\chi(G_\varepsilon)$ is finite. Of course, for $\varepsilon \geq 1$, $\chi(G_\varepsilon)$ is infinite since any closed disk of radius less than 2 constitutes a clique.

Theorem 2. *$\chi(G_\varepsilon)$ is finite whenever $\varepsilon < 1$. In fact, if $r \geq 1$ is a positive integer, such that*

$$\varepsilon < \frac{\sqrt{9r^2 - 3r + 1} - 2}{\sqrt{9r^2 - 3r + 1} + 2}$$

we have

$$\chi(G_\varepsilon) \leq 3r^2 + 3r + 1.$$

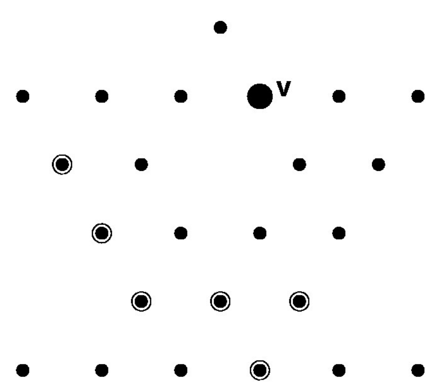

Fig. 4. An ε-unit distance graph with chromatic number 5.

Proof. Consider the regular hexagon tiling, $\mathcal{H}$, of the plane wherein each hexagon has side s. We want to define a distance between hexagons. To do this, consider the graph whose vertices correspond to the hexagons, with two vertices adjacent if the corresponding hexagons share a border. Then we define the distance between two hexagons, $h_1, h_2 \in \mathcal{H}$, to be equal to the graph theoretical distance between corresponding vertices. Consider the set $S(h, r)$ of hexagons whose distance from some fixed hexagon, h, is at most r. The number of hexagons in this set is

$$1 + \sum_{i=1}^{r} 6i = 3r^2 + 3r + 1.$$

If we color each hexagon in S with a different color, then we can use S to construct a periodic tiling of the plane with $3r^2 + 3r + 1$ colors. In fact, the coloring depicted in Fig. 2 is exactly this coloring for the case $r = 1$.

Next observe that with such a coloring, the distance between two points in the same hexagon is at most $2s$. On the other hand, by elementary geometry, we see that the distance between points in different hexagons of the same color is at least $\sqrt{9r^2 - 3r + 1}$. The theorem follows. □

3. A Conjecture

The graph used in Theorem 1(a) was chosen because it gave the smallest value of ε for any graph whose chromatic number we could establish definitely. We were able to find considerably large graphs, which are ε-unit distance graphs for smaller values of ε, whose chromatic numbers we believe to be 7. However, these graphs are too large to allow a computational proof of their chromatic numbers. On a number of such graphs we tried several of the best known heuristic coloring algorithms, and they all failed to find good colorings using six colors. In fact, we believe there is sufficient evidence for the following conjecture:

Conjecture. *For any* $\varepsilon > 0$, $\chi(G_\varepsilon) = 7$.

4. Chromatic Number Computations

The results in this paper depend on assertions concerning the chromatic numbers of our graphs. The computations that establish the chromatic numbers are lengthy and would probably be extremely tedious to duplicate by hand. Some discussion of the method is in order.

Certain algorithms for computing the chromatic number begin with the following simple greedy algorithm. Given an ordering of the vertices $v_1, \ldots, v_n$ of a graph G, and a list of colors $c_1, c_2, \ldots,$ one proceeds to color the graph in n steps. In step i, one colors v_i with the first color not already used on one of its neighbors. This procedure can then be combined with a simple backtrack to produce an algorithm that works well on small graphs.

An important enhancement, called **DSATUR** (for **degree saturation**), was developed by Brélaz [1]. The idea here is to reorder the vertices dynamically according to the number of neighbors that have already been colored. At each stage, the vertex chosen for coloring is one that has the maximum number of neighbors that are already colored. In many cases this provides a significant improvement over the simple greedy algorithm.

We used a method that is similar in spirit to DSATUR, but tailored to the specific structure of our graphs. We used a fixed ordering of the vertices, but one which was constructed based on an estimated degree saturation.

Consider the graph in Fig. 3 and imagine the vertices partitioned into four quadrants, delimited by horizontal and vertical lines through the center of the figure, and numbered counterclockwise beginning with the upper right quadrant. Our ordering of the vertices begins by choosing a vertex in the middle of quadrant 1, followed by a vertex in the middle of quadrant 2, followed by one in quadrant 3, and one in quadrant 4. The four vertices are chosen so that the distances between vertices 1 and 3, and between vertices 2 and 4 are as close to unit distance as possible. The next four vertices again represent each of the four quadrants, in order, slightly displaced from the first four vertices. We continue choosing vertices in groups of four, one from each quadrant. The subsequent groups are chosen so that if we look at the vertex ordering in any one quadrant, we see the vertices spiraling outward from the initial vertex.

We make no claim that our method is optimal, or even especially good—merely that it seems to work better on the graphs described in this paper than the well-known general purpose coloring algorithms.

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