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The Space of Unordered Real Line Arrangements

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Abstract. The set of all unordered real line arrangements of given degree in the real projective plane is known to have a natural semialgebraic structure. The nonreduced arrangements are singular points of this structure. We show that the set of all unordered real line arrangements of given degree also has a natural structure of a smooth compact connected affine real algebraic variety. In fact, as such, it is isomorphic to a real projective space. As a consequence, we get a projectively linear structure on the set of all real line arrangements of given degree. We also show that the universal family of unordered real line arrangements of given degree is not algebraic.

1. Introduction

Let K be any field. An *unordered line arrangement* over K is a closed subscheme $V(F)$ of the projective plane $\mathbb{P}^2 = \mathbb{P}^2_K$, defined by a nonzero homogeneous polynomial $F \in K[X, Y, Z]$ that is equal to the product of its linear factors in $K[X, Y, Z]$. Equivalently, an unordered line arrangement over K is a proper closed subscheme of $\mathbb{P}^2$ that is the scheme-theoretic union of finitely many projective lines in $\mathbb{P}^2$. Note that, with the current definition, unordered line arrangements are not necessarily reduced or nonempty (see [3]).

Since the set of projective lines in $\mathbb{P}^2$ is parametrized by the set $(\mathbb{P}^2)^\curlyvee(K)$ of all K-rational points of the dual projective plane $(\mathbb{P}^2)^\curlyvee$, the set of all unordered line arrangements of degree d over K is naturally parametrized by the symmetric power

$$\mathcal{A}_d = ((\mathbb{P}^2)^\curlyvee(K))^{(d)},$$

where d is a natural integer.

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Now, the set $\mathcal{A}_d$ has two bad properties:

1. $\mathcal{A}_d$ is not, in a natural way, the set of K-rational points of an algebraic variety over K, and
2. $\mathcal{A}_d$ contains singularities, as a subset of the algebraic variety $((\mathbb{P}^2)^\curlyvee)^{(d)}$.

Indeed, as for property 1, $\mathcal{A}_d$ is a subset of the set $((\mathbb{P}^2)^\curlyvee)^{(d)}(K)$ of all K-rational points of the symmetric power $((\mathbb{P}^2)^\curlyvee)^{(d)}$. One has a strict inclusion

$$\mathcal{A}_d \subsetneq ((\mathbb{P}^2)^\curlyvee)^{(d)}(K)$$

if and only if the field K admits a nontrivial extension of degree $\leq d$. For example, if K is algebraically closed then $\mathcal{A}_d$ *is* equal to the set of K-rational points of $((\mathbb{P}^2)^\curlyvee)^{(d)}$. However, if K is not algebraically closed, $\mathcal{A}_d$ is strictly contained in $((\mathbb{P}^2)^\curlyvee)^{(d)}(K)$ for all $d \geq d_0$, for some natural integer d_0. For example, if K is the field $\mathbb{R}$ of real numbers then $\mathcal{A}_d$ is a strict semialgebraic subset of $((\mathbb{P}^2)^\curlyvee)^{(d)}(\mathbb{R})$ for all $d \geq 2$.

As for property 2, since $(\mathbb{P}^2)^\curlyvee$ is two-dimensional, the symmetric power $((\mathbb{P}^2)^\curlyvee)^{(d)}$ is singular along the so-called big diagonal Δ for all $d \geq 2$.

While seemingly nothing can be done to resolve property 1, one can resolve property 2 by resolution of singularities. This has, however, the following drawback. Let $\tilde{\mathcal{A}}_d$ be a resolution of singularities of $\mathcal{A}_d$. Let $\mathcal{A}_\star$ be the disjoint union of $\mathcal{A}_d$, for $d \in \mathbb{N}$ and let, similarly, $\tilde{\mathcal{A}}_\star$ be the disjoint union of $\tilde{\mathcal{A}}_d$, for $d \in \mathbb{N}$. Then the scheme-theoretic union of unordered line arrangements over K, which is a monoid law

$$\mathcal{A}_\star \times \mathcal{A}_\star \longrightarrow \mathcal{A}_\star$$

on $\mathcal{A}_\star$, does not extend to a map

$$\tilde{\mathcal{A}}_\star \times \tilde{\mathcal{A}}_\star \longrightarrow \tilde{\mathcal{A}}_\star.$$

Therefore, even if K is algebraically closed, property 2 erects serious obstacles.

The object of this paper is to show that, when K is the field $\mathbb{R}$ of real numbers, both bad properties 1 and 2 can be resolved. More precisely, we show that $\mathcal{A}_d$ can be identified, in a natural way, with the *whole* set of real points of a proper *smooth* algebraic variety over $\mathbb{R}$. In fact, there is a natural bijection between $\mathcal{A}_d$ and the set of real points of a real projective space (see Theorem 2.1). In particular, $\mathcal{A}_d$ has a natural structure of a smooth compact connected affine real algebraic variety in the sense of [1]. Moreover, with respect to this structure, the scheme-theoretic union of real line arrangements

$$\mathcal{A}_\star \times \mathcal{A}_\star \longrightarrow \mathcal{A}_\star$$

is an algebraic map (see Corollary 2.3).

In Section 3 we show that the universal family U_d of unordered real line arrangements over $\mathcal{A}_d$ only is semialgebraic, and is *not* algebraic (see Theorem 3.2).

Convention. All line arrangements will be unordered, unless specified otherwise.

2. A Real Algebraic Structure on the Space of Real Line Arrangements

Let I be the homogeneous ideal of $\mathbb{R}[X, Y, Z]$ generated by the polynomial $X^2 + Y^2 + Z^2$. Let d be a natural integer. Let $\mathbb{R}[X, Y, Z]_d$ denote the real vector subspace of $\mathbb{R}[X, Y, Z]$ of all homogeneous polynomials of degree d. Let I_d be the real vector subspace of $\mathbb{R}[X, Y, Z]_d$ defined by

$$I_d = I \cap \mathbb{R}[X, Y, Z]_d,$$

i.e., I_d is the real vector subspace of $\mathbb{R}[X, Y, Z]_d$ consisting of all polynomial multiples of $X^2 + Y^2 + Z^2$. Define

$$S_d = \mathbb{R}[X, Y, Z]_d / I_d.$$

Since $\mathbb{R}[X, Y, Z]_d$ is of dimension $\frac{1}{2}(d+2)(d+1)$, for each d, the dimension of I_d is equal to $\frac{1}{2}d(d-1)$ and

$$\dim(S_d) = \tfrac{1}{2}(d+2)(d+1) - \tfrac{1}{2}d(d-1) = 2d + 1.$$

The object of this section is to show that there is a natural bijection from the set $\mathcal{A}_d$ of all real line arrangements of degree d onto the real projective space $\mathbb{P}(S_d)$. One can already observe that $\mathbb{P}(S_d)$ has the right dimension.

Let

$$\pi_d \colon \mathbb{R}[X, Y, Z]_d \longrightarrow S_d$$

be the quotient map. Since $\ker(\pi_d) = I_d$, the map π_d induces a projectively linear projection

$$\mathbb{P}(\pi_p) \colon \mathbb{P}(\mathbb{R}[X, Y, Z]_d) - - \rightarrow \mathbb{P}(S_d)$$

with center $\mathbb{P}(I_d)$. Note that $\mathbb{P}(\mathbb{R}[X, Y, Z]_d)$ can—and will—be identified with the space of all real algebraic plane curves of degree d, and, as such, contains the set $\mathcal{A}_d$ of all real line arrangements of degree d. Since $X^2 + Y^2 + Z^2$ is irreducible, the intersection of $\mathcal{A}_d$ and $\mathbb{P}(I_d)$ is empty. Hence, the restriction of $\mathbb{P}(\pi_d)$ to $\mathcal{A}_d$ is a true map

$$\rho_d \colon \mathcal{A}_d \longrightarrow \mathbb{P}(S_d).$$

Theorem 2.1. *Let $d \in \mathbb{N}$. The map ρ_d is a bijection. In particular, $\mathcal{A}_d$ has a natural structure of a smooth real algebraic variety of dimension $2d$. In fact, with respect to this structure, $\mathcal{A}_d$ is isomorphic to $\mathbb{P}^{2d}(\mathbb{R})$.*

For the proof of Theorem 2.1 it is convenient to have a more geometric description of ρ_d. Let Q be the real algebraic plane curve defined by the equation $X^2 + Y^2 + Z^2 = 0$. Since Q is a rational normal curve, $\mathbb{P}(S_d)$ can—and will—be identified with the complete linear system of all effective real divisors on Q of degree $2d$. If A is a real line arrangement of degree d, the intersection product $A \cdot Q$ is well defined since Q is a nondegenerate conic. The map ρ_d maps the real line arrangement A to the real effective divisor $A \cdot Q$.

Proof of Theorem 2.1. Let D be an effective real divisor on Q of degree $2d$. We have to show that there is a unique real line arrangement A of degree d such that $A \cdot Q = D$. Since D is an effective real divisor on Q of degree $2d$,

$$D = \sum_{i=1}^{n} m_i P_i,$$

where P_i are nonreal closed points of Q and m_i are nonzero natural integers satisfying $\sum m_i = d$. Here we have used that Q does not have any real points. Since each P_i is a nonreal closed point of Q, there is a unique real projective line L_i that contains P_i. Since Q is a conic, $L_i \cdot Q = P_i$. Let A be the real line arrangement defined by

$$A = \sum_{i=1}^{n} m_i L_i.$$

Then A is a real line arrangement of degree d and $A \cdot Q = D$. In order to show that A is unique, let B be another real line arrangement of degree d such that $B \cdot Q = D$. Since $B \cdot Q \geq m_i P_i$ and since B is a scheme-theoretic union of real projective lines, one has $B \geq m_i L_i$, for all i. Hence $B \geq A$ and, therefore, $B = A$. □

Remark 2.2. By Theorem 2.1, $\mathcal{A}_d$ is isomorphic to $\mathbb{P}^{2d}(\mathbb{R})$, with respect to its natural real algebraic structure. In particular, one gets a projectively linear structure on the set $\mathcal{A}_d$. For example, given two distinct real line arrangements A and B of degree d, there is a unique real projective line of real line arrangements of degree d that contains A and B!

For example, let $A = V(X^2)$ be the nonreduced real line arrangement of degree 2 defined by the polynomial X^2, and let $B = V(Y^2)$ be the nonreduced real line arrangement of degree 2 defined by the polynomial Y^2. Let us determine the real projective line $L \subseteq \mathcal{A}_2$ that contains A and B. Let $W \subseteq \mathbb{R}[X, Y, Z]_2$ be the real vector subspace generated by X^2, Y^2, Z^2. Since I_2 is the real vector subspace generated by $X^2+Y^2+Z^2$,

$$\mathbb{P}(\pi_2)^{-1}(\rho_2(L)) = \mathbb{P}(W) \backslash \mathbb{P}(I_2).$$

The elements of $\mathbb{P}(W) \backslash \mathbb{P}(I_2)$ are of the form $\lambda X^2 + \mu Y^2 + \nu Z^2$, where $\lambda, \mu, \nu \in \mathbb{R}$ are not all equal. If the real number λ is between μ and ν, i.e., if $\mu \leq \lambda \leq \nu$ or $\nu \leq \lambda \leq \mu$, then

$$\lambda X^2 + \mu Y^2 + \nu Z^2 \equiv (\mu - \lambda) Y^2 + (\nu - \lambda) Z^2 \pmod{I_2}.$$

Since $\mu - \lambda$ and $\nu - \lambda$ have opposite signs, $(\mu - \lambda)Y^2 + (\nu - \lambda)Z^2$ is a product of linear polynomials in $\mathbb{R}[X, Y, Z]$. Hence,

$$\mathbb{P}(\pi_2)(\lambda X^2 + \mu Y^2 + \nu Z^2) = \rho_2(V(\alpha Y^2 - \beta Z^2)),$$

for some nonnegative real numbers α, β, not both zero. One has a similar description of $\mathbb{P}(\pi_2)(\lambda X^2 + \mu Y^2 + \nu Z^2)$ when μ is between λ and ν, or when ν is between λ and

μ. It follows that the real projective L containing A and B is a union $L = L_1 \cup L_2 \cup L_3$, where

$$\begin{aligned} L_1 &= \left\{V(\alpha Y^2 - \beta Z^2) \mid (\alpha,\beta) \in (\mathbb{R}^2_{\geq 0})\backslash\{(0,0)\}\right\}, \\ L_2 &= \left\{V(\alpha X^2 - \beta Z^2) \mid (\alpha,\beta) \in (\mathbb{R}^2_{\geq 0})\backslash\{(0,0)\}\right\}, \\ L_3 &= \left\{V(\alpha X^2 - \beta Y^2) \mid (\alpha,\beta) \in (\mathbb{R}^2_{\geq 0})\backslash\{(0,0)\}\right\}. \end{aligned}$$

We conclude this section by stating a consequence of Theorem 2.1. Let $\mathbb{P}(S_\star)$ be the disjoint union of the real projective spaces $\mathbb{P}(S_d)$, where d runs through all natural numbers. The multiplication of the graded $\mathbb{R}$-algebra S induces a structure of a graded monoid on $\mathbb{P}(S_\star)$. In fact, $\mathbb{P}(S_\star)$ is a graded monoid in the category of real algebraic varieties, i.e., $\mathbb{P}(S_\star)$ is a graded real algebraic monoid. Let

$$\rho\colon\, \mathcal{A}_\star \longrightarrow \mathbb{P}(S_\star)$$

be the disjoint union of the maps ρ_d, for $d \in \mathbb{N}$.

Corollary 2.3. *The map ρ is an isomorphism of graded monoids. In particular, $\mathcal{A}_\star$ is a graded real algebraic monoid*, i.e., *the scheme-theoretic union on the set of all real line arrangements $\mathcal{A}_\star$ is real algebraic with respect to the natural real algebraic structure on $\mathcal{A}_\star$.*

3. The Universal Family of Real Line Arrangements

Let us make precise what we mean by an algebraic family of real line arrangements of degree d. Let T be a real algebraic variety in the sense of [1]. An *algebraic family* of real line arrangements of degree d over T is a closed subscheme A of the projective plane $\mathbb{P}^2$ over the ring $\mathcal{R}(T)$ of regular functions on T such that, Zariski-locally on T, A is defined by a homogeneous polynomial in X, Y, Z of degree d that is a product of its linear factors. If A is an algebraic family of real line arrangements of degree d over T, then, for each $t \in T$, the fiber A_t of A at t is a real line arrangement of degree d.

For example, the subscheme $A = V(\alpha^2X^2 - \beta^2Y^2)$ of $\mathbb{P}^2_{\mathcal{R}(T)}$ is an algebraic family of real line arrangements of degree 2 over the real algebraic variety $T = \mathbb{R}^2\backslash\{(0,0\}$. Its fiber $A_{(1,1)}$ at $(1,1)$ is the real line arrangement given by the polynomial $X^2 - Y^2$.

Another example of an algebraic family of real line arrangements that will be of interest to us is the following. Let $\tilde{\mathcal{A}}_d$ be the product of d copies of the dual real projective plane $\mathbb{P}^2(\mathbb{R})^\curlyvee$. Then $\tilde{\mathcal{A}}_d$ is the real algebraic variety of all ordered real line arrangements of degree d. Let $\tilde{U}_d$ be the algebraic family of real line arrangements of degree d over $\tilde{\mathcal{A}}_d$ whose fiber over $(L_1,\dots,L_d) \in \tilde{\mathcal{A}}_d$ is the real line arrangement defined by the polynomial $L_1 \cdots L_d$. It is the universal family of *ordered* real line arrangements of degree d. It is an algebraic family of real line arrangements over $\tilde{\mathcal{A}}_d$.

A weaker notion is the notion of a *semialgebraic family* of real line arrangements over a real algebraic variety T. Such a family is a closed subscheme A of $\mathbb{P}^2$ over the ring $\mathcal{S}(T)$ of semialgebraic continuous functions $\mathcal{S}(T)$ on T such that, semialgebraically locally on T, A is defined by a homogeneous polynomial in X, Y, Z of degree d that is a product of its linear factors.

The motivating example of a semialgebraic family of real line arrangements is the following. The symmetric group Σ_d acts on the d-fold product $\tilde{\mathcal{A}}_d$ of dual real projective planes by permutation of the factors. The quotient $\tilde{\mathcal{A}}_d/\Sigma_d$ can—and will—be identified with the set $\mathcal{A}_d$ of real line arrangements of degree d. The quotient $\tilde{\mathcal{A}}_d/\Sigma_d$ has a natural semialgebraic structure, as it follows from general theory [2], [4], that coincides with the semialgebraic structure on $\mathcal{A}_d$ that underlies the real algebraic structure on $\mathcal{A}_d$. There is a natural action of Σ_d on the scheme $\tilde{U}_d$ lying over the action of Σ_d on the ring $\mathcal{R}(\tilde{\mathcal{A}}_d)$. The quotient U_d of $\tilde{U}_d$ by the action of Σ_d is a semialgebraic family of real line arrangements over $\mathcal{A}_d$. It is the *universal family* of real line arrangements of degree d.

The object of this section is to prove that the universal family U_d of all real line arrangements of degree d is not algebraic, whenever $d \geq 2$. The following lemma implies that it suffices to show that statement for $d = 2$.

Lemma 3.1. *Let $d \geq 2$ be a natural integer. If U_d is algebraic then U_2 is algebraic.*

Proof. Choose three real line arrangements A_1, A_2, A_3 of degree $d-2$ such that the intersection $A_1 \cap A_2 \cap A_3$ is empty. Let

$$\sigma_i\colon \mathcal{A}_2 \longrightarrow \mathcal{A}_d$$

be defined by $\sigma(A) = A + A_i$, for $i = 1, 2, 3$, where, as usual, the sum $A + A_i$ represents the scheme-theoretic union. By hypothesis, U_d is a a closed subscheme of $\mathbb{P}^2$ over the ring $\mathcal{R}(A_d)$ of regular functions of $\mathcal{A}_d$. By Corollary 2.3, the map σ_i is real algebraic. Hence, we get an induced closed subscheme $\sigma_i^\star U_d$ of $\mathbb{P}^2$ over $\mathcal{R}(A_2)$, for $i = 1, 2, 3$. Since the intersection of the arrangements A_1, A_2, A_3 is empty, one has

$$U_2 = (\sigma_1^\star U_d) \cap (\sigma_2^\star U_d) \cap (\sigma_3^\star U_d).$$

It follows that U_2 is algebraic. □

Theorem 3.2. *Let $d \in \mathbb{N}$. The universal family U_d of real line arrangements of degree d is semialgebraic. If $d \geq 2$ then U_d is not algebraic.*

Proof. By Lemma 3.1, it suffices to show that U_d is not algebraic for $d = 2$. Suppose that U_2 is algebraic. Recall, from Remark 2.2, the explicit description of the real projective line L in $\mathcal{A}_2$ that contains the real line arrangements $A = V(X^2)$ and $B = V(Y^2)$. Since U_2 is supposed to be algebraic over $\mathcal{A}_2$, the restriction U of U_2 over L would be algebraic as well. Now, observe that the point $[0:0:1]$ belongs to the fiber U_t for all $t \in L_3 \subseteq L$. Since L_3 is Zariski-dense in L, the point $[0:0:1]$ belongs to all fibers of U. However, the real line arrangement $C = V(Z^2)$ is a fiber of U and does not contain the point $[0:0:1]$. Contradiction. □

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