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Oriented Matroids and Associated Valuations*

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Abstract. It is possible to associate a valuation on the "orthant lattice" with each oriented matroid. In the case of uniform oriented matroids, it is not difficult to provide a characterization of the corresponding valuations. This is done here, thereby establishing a new characterization of the uniform oriented matroids themselves. Additionally, the connection between the valuations and the total polynomials associated with uniform oriented matroids is noted.

1. Introduction

Here we show how to associate with each oriented matroid a valuation on the "orthant lattice." The valuation associated with a given oriented matroid retains enough information to determine the oriented matroid. We are able to characterize the valuations which arise in this way from *uniform* oriented matroids in terms of certain linear inequalities and integrality constraints. This gives a new characterization of uniform oriented matroids.

The set $\{1, 2, \ldots, n\}$ is denoted by $[n]$. The *orthant lattice* $\mathcal{Q}^n$ is the collection of sets $Q(I, J) \subseteq R^n$, where for sets $I, J \subseteq [n]$,

$$Q(I, J) = \left\{ (x_1, \ldots, x_n) \in R^n \colon \begin{cases} x_i \geq 0 & \text{if} \quad i \in I \\ x_i \leq 0 & \text{if} \quad i \in J \end{cases} \right\}.$$

There are 4^n such sets. The intersection of two such sets is another; also, $Q(I_1, J_1) \subseteq Q(I_2, J_2)$ if and only if $I_2 \subseteq I_1$ and $J_2 \subseteq J_1$.

The word "orthant" as defined above has a more liberal meaning than is normally assigned to it. We use the phrase *pointed orthant* to designate the 3^n subsets $Q(I, J)$,

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where $I \cup J = [n]$, of R^n. The full-dimensional pointed orthants are the sets which are usually called (closed) "orthants." These are the 2^n orthants $Q(I, J)$ such that $I \cup J = [n]$ and $I \cap J = \emptyset$. Also, we call the 2^n coordinate subspaces $Q(I, I)$, for $I \subseteq [n]$, *linear orthants*. The dimension of the orthant $Q = Q(I, J)$ is $\dim(Q) = n - |I \cap J|$.

The coordinate hyperplanes are the linear orthants $Q(\{i\}, \{i\})$, which will also be denoted by H_i^0. The two closed halfspaces bounded by H_i^0 are $H_i^+ = Q(\{i\}, \emptyset)$ and $H_i^- = Q(\emptyset, \{i\})$.

2. Valuations on the Orthant Lattice

Let $\mathcal{C}$ denote a collection of subsets of R^n. Following [9], we define a *valuation* on $\mathcal{C}$ to be a function $\nu\colon \mathcal{C} \to A$, where A is an additive abelian group, such that whenever P, Q, $P \cap Q$, and $P \cup Q$ are elements of $\mathcal{C}$,

$$\nu(P \cup Q) + \nu(P \cap Q) = \nu(P) + \nu(Q).$$

Possibilities for $\mathcal{C}$ which are of interest include: $\mathcal{C} = \mathcal{Q}^n$; $\mathcal{C} =$ (the collection of closed, convex, polyhedral cones emanating from the origin in R^n); $\mathcal{C} =$ (the collection of closed, convex polyhedra in R^n); $\mathcal{C} =$ (the collection of convex polytopes in R^n); and $\mathcal{C} =$ (the collection of compact, convex sets in R^n).

Given a set $S \subseteq R^n$, we denote by $[S]$ its *indicator function*, defined on R^n by

$$[S](x) = \begin{cases} 1 & \text{if} \quad x \in S, \\ 0 & \text{if} \quad x \notin S. \end{cases}$$

We denote by $\mathcal{S}(\mathcal{C})$ the additive group of Z-valued functions on R^n which is generated by the indicator functions $[P]$ of $P \in \mathcal{C}$.

If P and Q are sets in $\mathcal{C}$ such that $P \cap Q$ and $P \cup Q$ are also in $\mathcal{C}$, then it is clear that $[P \cup Q] + [P \cap Q] = [P] + [Q]$. From this it follows that if $\bar{\nu}\colon \mathcal{S}(\mathcal{C}) \to A$ is a homomorphism of abelian groups, then the function $\nu\colon \mathcal{C} \to A$ defined by $\nu(P) = \bar{\nu}([P])$ for $P \in \mathcal{C}$ is a valuation on $\mathcal{C}$. Conversely, for any but the last of the possibilities listed above, if $\nu\colon \mathcal{C} \to A$ is a valuation, and (when $\emptyset \in \mathcal{C}$) $\nu(\emptyset) = 0$, then there is a unique homomorphism $\bar{\nu}\colon \mathcal{S}(\mathcal{C}) \to A$ such that $\bar{\nu}([P]) = \nu(P)$, for each $P \in \mathcal{C}$. For the class of convex polytopes, this was proven by Volland [13] and rediscovered by Perles and Sallee [11]. For the class of orthants, $\mathcal{Q}^n$, it is Corollary 2 of Theorem 1, below. It is not known if valuations on the class of compact, convex sets determine such homomorphisms; however, Groemer [5] has proven the existence of such homomorphisms in this case when an additional continuity assumption is imposed. When it exists, we say that the homomorphism $\bar{\nu}$ is the *homomorphism induced by* ν.

The collections $\mathcal{C}$ are sometimes closed under intersection, as in all the possibilities mentioned above. In this case there are useful extension results which are consequences of the existence of an induced homomorphism: If $\nu\colon \mathcal{C} \to A$ is a valuation and the induced homomorphism $\bar{\nu}$ exists, then ν has a unique extension to a valuation on the distributive lattice of finite unions of elements of $\mathcal{C}$; and, stronger when $\nu(\emptyset) = 0$, ν has a unique extension to a valuation on the boolean lattice generated by $\mathcal{C}$ under intersection, union, and complementation.

Let χ_0 be the function defined on closed convex polyhedra in R^n such that $\chi_0(\emptyset) = 0$ and $\chi_0(P) = 1$ if P is a nonempty, closed convex polyhedron. Clearly, χ_0 is a valuation. Denote its (unique) extension to a valuation on the distributive lattice of finite unions of closed convex polyhedra by χ. This will be referred to as the *Euler characteristic*.

Sallee, in [12], has used the phrase "weak valuation" to indicate a function $\nu\colon \mathcal{C} \to A$ such that the defining equation for valuations,

$$\nu(P \cup Q) + \nu(P \cap Q) = \nu(P) + \nu(Q),$$

holds, whenever P, Q, $P \cap Q$, and $P \cup Q$ are in $\mathcal{C}$, and, additionally, the relative interiors of P and Q have no points in common. He showed that if $\mathcal{C}$ is the collection of convex polytopes, then every weak valuation on $\mathcal{C}$ is a valuation.

We call a function $\nu\colon \mathcal{Q}^n \to A$ a *weak valuation* if, whenever $P \in \mathcal{Q}^n$ and $i \in [n]$, the equation

$$\nu(P) + \nu(P \cap H_i^0) = \nu(P \cap H_i^+) + \nu(P \cap H_i^-)$$

holds. We will see that every weak valuation on $\mathcal{Q}^n$ is a valuation.

For a given pointed orthant $Q \in \mathcal{Q}^n$, let $\nu_Q\colon \mathcal{Q}^n \to Z$ denote the function

$$\nu_Q(P) = \begin{cases} 1 & \text{if} \quad P \supseteq Q, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that ν_Q is a valuation. Indeed, if $Q = Q(I, J)$, where $I \cup J = [n]$, and letting $x = (x_1, \ldots, x_n) \in R^n$ be a point in the relative interior of Q, so that $x_i = 0$ if $i \in I \cap J$, $x_i > 0$ if $i \in I \backslash J$, and $x_i < 0$ if $i \in J \backslash I$, then for $P \in \mathcal{Q}^n$ we have

$$\nu_Q(P) = \begin{cases} 1 & \text{if} \quad x \in P, \\ 0 & \text{otherwise,} \end{cases}$$

from which it is clear that ν is a valuation.

Lemma 1. *If Q_1 and Q_2 are pointed orthants such that $Q_1 \subseteq Q_2$ then*

$$\sum_{Q\colon\, Q_1 \subseteq Q \subseteq Q_2} (-1)^{\dim(Q) - \dim(Q_1)} = \begin{cases} 1 & \textit{if} \quad Q_2 = Q_1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Suppose $Q_i = Q(I_i, J_i)$ $(i = 1, 2)$. Then $I_2 \subseteq I_1$, $J_2 \subseteq J_1$, and $I_2 \cup J_2 = I_1 \cup J_1 = [n]$. The summation is over orthants $Q = Q(I, J)$ such that $I_2 \subseteq I \subseteq I_1$ and $J_2 \subseteq J \subseteq J_1$. The collection of such orthants is in bijective correspondence with the boolean lattice of subsets S of $(I_1 \backslash I_2) \cup (J_1 \backslash J_2)$ by the mapping which takes $Q(I, J)$ to $S = (I \backslash I_2) \cup (J \backslash J_2)$. (This mapping reverses inclusion order.) Let $d = \dim(Q_2) - \dim(Q_1) = |(I_1 \backslash I_2) \cup (J_1 \backslash J_2)|$. We may rewrite the sum as

$$\sum_{S \subseteq (I_1 \backslash I_2) \cup (J_1 \backslash J_2)} (-1)^{d - |S|}.$$

This, in turn, equals

$$\sum_{k=0}^{d} \sum_{S\colon\, |S| = k} (-1)^{d-k} = \sum_{k=0}^{d} (-1)^{d-k} \binom{d}{k} = \begin{cases} 1 & \text{if} \quad d = 0, \\ 0 & \text{if} \quad d > 0. \end{cases} \qquad \square$$

Theorem 1. *Suppose $\nu\colon \mathcal{Q}^n \to A$ is a weak valuation and $P \in \mathcal{Q}^n$. Then*

$$\nu(P) = \sum_{Q,\ \text{pointed}} \alpha_Q \nu_Q(P),$$

where

$$\alpha_Q = \sum_{Q' \subseteq Q} (-1)^{\dim(Q)-\dim(Q')} \nu(Q') \in A,$$

for each pointed orthant Q.

Proof. Denote the right-hand side of the equation by $\mu(P)$:

$$\mu(P) = \sum_{Q,\ \text{pointed}} \alpha_Q \nu_Q(P).$$

Clearly, μ is a weak valuation. We must show that, for each $P \in \mathcal{Q}^n$, $\nu(P) = \mu(P)$.

Let $P = Q(I, J)$. We proceed by induction on the value of the parameter $k = |[n]\backslash(I \cup J)|$. First, suppose this is 0; that is, P is pointed. We have

$$\begin{aligned}\mu(P) &= \sum_{Q,\ \text{pointed}} \left(\sum_{Q' \subseteq Q} (-1)^{\dim(Q)-\dim(Q')} \nu(Q') \right) \nu_Q(P) \\ &= \sum_{Q',\ \text{pointed}} \left(\sum_{Q:\ Q' \subseteq Q \subseteq P} (-1)^{\dim(Q)-\dim(Q')} \right) \nu(Q').\end{aligned}$$

By the lemma,

$$\sum_{Q:\ Q' \subseteq Q \subseteq P} (-1)^{\dim(Q)-\dim(Q')} = \begin{cases} 1 & \text{if} \quad P = Q', \\ 0 & \text{otherwise}, \end{cases}$$

so this does indeed yield $\nu(P)$.

Now suppose $k > 0$ and that the result holds for orthants for which this value is $k - 1$. Let i be an element of $[n]\backslash(I \cup J)$. Then

$$\begin{aligned}\nu(P) &= \nu(P \cap H_i^+) + \nu(P \cap H_i^-) - \nu(P \cap H_i^0) \\ &= \nu(Q(I \cup \{i\}, J)) + \nu(Q(I, J \cup \{i\})) - \nu(Q(I \cup \{i\}, J \cup \{i\})).\end{aligned}$$

The inductive assumption applies to each of the three terms, so this equals

$$\mu(Q(I \cup \{i\}, J)) + \mu(Q(I, J \cup \{i\})) - \mu(Q(I \cup \{i\}, J \cup \{i\})),$$

which, since μ is a weak valuation, equals $\mu(P)$.

By induction, the result holds for all P. $\square$

Corollary 1. *If $\nu\colon \mathcal{Q}^n \to A$ is a weak valuation then it is a valuation.*

Proof. According to the theorem, ν is a combination of the ν_Q's of the form $\sum_Q \alpha_Q \nu_Q$, where the α_Q's are in A. Clearly, any such combination of valuations is again a valuation. □

Henceforth, we dispense with the adjective "weak."

Corollary 2. *If $\nu\colon \mathcal{Q}^n \to A$ is a valuation, then there is a unique homomorphism $\bar{\nu}\colon \mathcal{S}(\mathcal{Q}^n) \to A$ such that $\bar{\nu}([P]) = \nu(P)$, for each orthant P.*

Proof. Uniqueness follows from the fact that $\mathcal{S}(\mathcal{Q}^n)$ is generated additively by the functions $[P]$ (for $P \in \mathcal{Q}^n$).

It remains to establish the existence of $\bar{\nu}$. By the theorem, ν is a combination of the ν_Q's, so it will suffice to show that, for each of the valuations ν_Q ($Q \in \mathcal{Q}^n$), there exists such a homomorphism $\bar{\nu}_Q\colon \mathcal{S}(\mathcal{Q}^n) \to Z$. Let $x = (x_1, \ldots, x_n)$ be a point in the relative interior of Q. For $F \in \mathcal{S}(\mathcal{Q}^n)$, define $\bar{\nu}_Q(F) = F(x)$. This is clearly a homomorphism of $\mathcal{S}(\mathcal{Q}^n)$ to Z, and

$$\bar{\nu}_Q([P]) = \nu_Q(P) = \begin{cases} 1 & \text{if} \quad x \in P, \\ 0 & \text{otherwise,} \end{cases}$$

as required. □

Theorem 2. *The set of functions $\nu_Q\colon \mathcal{Q}^n \to Z$, where Q is a pointed orthant, forms a basis for the additive group (Z-module) of Z-valued valuations.*

Proof. It follows from Theorem 1 that this set generates the set of valuations as a Z-module. It remains to establish independence.

Suppose that this is not the case, so that we have an equation

$$\sum_{Q,\ \text{pointed}} \beta_Q \nu_Q(P) = 0$$

holding for each $P \in \mathcal{Q}^n$, where the β_Q's are integers, not all 0.

Choose a minimal set Q_0 such that $\beta_{Q_0} \neq 0$. Letting $P = Q_0$ yields

$$0 = \sum \beta_Q \nu_Q(Q_0) = \sum_{Q \subseteq Q_0} \beta_Q = \beta_{Q_0},$$

a contradiction; so the set is independent. □

3. Some Valuations on the Orthant Lattice

In this section we give some examples of valuations on $\mathcal{Q}^n$. The first two classes of these examples will serve as prototypes of valuations associated with lopsided sets and oriented matroids in later sections.

Suppose that P is a convex set in R^n. Define

$$\lambda_P(Q) = \begin{cases} 1 & \text{if} \quad P \cap Q \neq \emptyset, \\ 0 & \text{if} \quad P \cap Q = \emptyset. \end{cases}$$

Notice that if Q is an orthant and $i \in [n]$, then

$$\lambda_P(Q) = \max(\lambda_P(Q \cap H_i^+), \lambda_P(Q \cap H_i^-))$$

and also

$$\lambda_P(Q \cap H_i^0) = \min(\lambda_P(Q \cap H_i^+), \lambda_P(Q \cap H_i^-)).$$

Clearly, these properties imply that λ_P is indeed a valuation on the orthant lattice. A valuation satisfying the stronger properties is called a *max-min valuation*.

If χ is the Euler characteristic and the set P is a closed convex polyhedron, we can write $\lambda_P(Q) = \chi(P \cap Q)$. Then the valuation property follows from the fact that the Euler characteristic is a valuation.

Here are some properties of λ_P:

λ_P is a valuation; λ_P is monotone—that is, if Q_1, Q_2 are orthants and $Q_1 \subseteq Q_2$, then $\lambda_P(Q_1) \leq \lambda_P(Q_2)$; and λ_P is $\{0, 1\}$-valued. These properties are equivalent to the statement that λ_P is a $\{0, 1\}$-valued max-min valuation on $\mathcal{Q}^n$.

For small values of n these conditions characterize the valuations that can be obtained in this way from a convex set K. This is not the case, however, for $n \geq 7$. We will see in Section 5 that the $\{0, 1\}$-valued max-min valuations correspond to "lopsided sets." For an example of a lopsided subset of the 7-cube which is not realizable by a convex set as above, see [10].

Next, let W denote a linear subspace of R^n. Let ∂B^n denote the boundary of the n-dimensional cross-polytope,

$$B^n = \text{conv}\{e_1, -e_1, \ldots, e_n, -e_n\},$$

where $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ is the vector having 1 in the ith entry and zeros elsewhere. We define a valuation $\mu_W\colon \mathcal{Q}^n \to Z$ by

$$\mu_W(Q) = \chi(Q \cap W \cap \partial B^n).$$

That μ_W is a valuation on $\mathcal{Q}^n$ is immediate from the similar property of χ. Also it is clear that

$$\mu_W(Q) = \begin{cases} 1 & \text{if} \quad Q \cap W \text{ is not a linear subspace}, \\ 1 + (-1)^{d-1} & \text{if} \quad Q \cap W \text{ is a linear subspace of dimension } d, \end{cases}$$

for in one case $Q \cap W \cap \partial B^n$ is homeomorphic to a ball, $Q \cap W$ being a closed convex cone which is not a linear subspace, and in the other case $Q \cap W \cap \partial B^n$ is homeomorphic to a sphere of dimension $d - 1$, or, for $d = 0$, empty. It is easy to verify that μ_W is a valuation directly from this. Note, in particular, $\mu_W(\{0\}) = 0$.

We will see in Section 4 that any oriented matroid on $[n]$ yields a valuation on $\mathcal{Q}^n$.

Suppose we impose the further restrictions on W that

(a) $\dim(W) = r$ for some fixed r, and
(b) W is in *general position* with respect to the coordinate axes, so that, for each linear orthant Q of dimension at most $n - r$, $W \cap Q = \{0\}$.

These conditions imply also that

(c) if Q is an orthant which is not linear and $Q \cap W$ is a linear subspace then $Q \cap W = \{0\}$, and
(d) if Q is a linear orthant of dimension $n - m$, where $m \leq r$, then the dimension of the linear subspace $Q \cap W$ is $r - m$.

It is then possible to say much more about the function μ_W, as follows.

If Q is an orthant which is not linear then, by (c), $Q \cap W \cap \partial B^n$ is either empty or a topological ball. Therefore, $\mu_W(Q)$ is either 0 or 1 when Q is not linear, being 0 if and only if $Q \cap W = \{0\}$. When Q is linear, $Q \cap W \cap \partial B^n$ is either empty or a sphere, so in this case the possible values for $\mu_W(Q)$ are 0 and 2.

Furthermore, if Q_1 and Q_2 are orthants which are not linear, and $Q_1 \subseteq Q_2$, then $\mu_W(Q_1) \leq \mu_W(Q_2)$; that is, we cannot have $\mu_W(Q_1) = 1$ and $\mu_W(Q_2) = 0$. We say that μ_W is *partly monotone*.

For a linear orthant Q of dimension $d \geq n - r$, $Q \cap W$ is a linear subspace of dimension $r + d - n$, so $\mu_W(Q) = \chi(Q \cap W \cap \partial B^n) = 1 + (-1)^{r+d-n-1}$. If Q is an orthant of dimension $d \leq n - r$ then $Q \cap W = \{0\}$, so $\mu_W(Q) = 0$. In particular, $\mu_W(\{0\}) = 0$.

We will show in Section 6 that a subset of these properties delineate the valuations arising from uniform oriented matroids: such valuations are those which are symmetric, partly monotone, $\{0, 1\}$-valued on orthants which are not linear, and have value 0 on the smallest orthant. Since the oriented matroid is retrievable using knowledge of its corresponding valuation, this yields a characterization of the uniform oriented matroids.

Next, we consider four special valuations on $\mathcal{Q}^n$ having values in $\mathcal{S}(\mathcal{Q}^n)$.

A very simple example of such a valuation is $\iota\colon \mathcal{Q}^n \to \mathcal{S}(\mathcal{Q}^n)$, defined by $\iota(P) = [P]$. Another is $\rho\colon \mathcal{Q}^n \to \mathcal{S}(\mathcal{Q}^n)$, where $\rho(P) = [-P]$ for $P \in \mathcal{Q}^n$ and $-P$ denotes the orthant $Q(J, I)$, which is the image of P under reflection through the origin. It is easy to verify that ι and ρ are valuations.

Also, $\sigma\colon \mathcal{Q}^n \to \mathcal{S}(\mathcal{Q}^n)$ is defined by

$$\sigma(P) = \sum_{\substack{Q \subseteq P,\\ \text{pointed}}} (-1)^{\dim(Q)}[Q].$$

This maps P to the indicator of its relative interior, times $(-1)^{\dim(P)}$.

The valuation $\tau\colon \mathcal{Q}^n \to \mathcal{S}(\mathcal{Q}^n)$ is defined by

$$\tau(Q(I, J)) = [Q([n]\backslash I, [n]\backslash J)].$$

It maps the orthant $P = Q(I, J)$ to the indicator of its normal cone, $P^\perp = Q([n]\backslash I, [n]\backslash J)$.

Each of these valuations is the restriction to the orthant lattice of a valuation defined analogously on the closed, convex, polyhedral cones emanating from the origin. See [7]. For each, the induced homomorphism is an involutive isomorphism of the Z-module

$\mathcal{S}(\mathcal{Q}^n)$. The induced homomorphism $\bar{\sigma}$, when applied to the indicator of the relative interior of an orthant, yields the indicator of the (closed) orthant. The valuation σ is the restriction to $\mathcal{Q}^n$ of the Sallee–Shephard mapping, defined in [7].

The valuations μ_W arising from linear subspaces in general position with respect to the coordinate hyperplanes satisfy conditions that can be expressed utilizing the valuations ρ, σ, and τ. These conditions are given in the proposition below. For the third statement, we need two additional, special, Z-valued valuations, ε_1, ε_2, on $\mathcal{Q}^n$. On orthants Q which are not linear, each of these has value 1. For linear orthants Q, $\varepsilon_1(Q) = 1 + (-1)^d$, where d is the dimension of Q; and $\varepsilon_2(Q) = 1 + (-1)^{d+1}$, where d is the dimension of Q. Clearly, $\varepsilon_1(Q) + \varepsilon_2(Q) = 2$, for each orthant Q.

Proposition 1. *We have, for each orthant Q:*

$$\bar{\mu}_W(\rho(Q)) = \mu_W(Q);$$

$$\bar{\mu}_W(\sigma(Q)) = (-1)^{n-r-1}\mu_W(Q);$$

and

$$\bar{\mu}_W(\tau(Q)) = \varepsilon_i(Q) - \mu_{W^\perp}(Q),$$

where $W^\perp$ is the normal cone to W (its orthogonal complement), $i = 1$ if the dimension of W is odd, and $i = 2$ otherwise.

Proof. The first equation clearly holds, since W is symmetric with respect to the origin, and $\rho(Q) = [-Q]$.

For the second equation, note that, for a pointed orthant Q, using the fact that W is in general position, either the sets $W \cap \partial B^n \cap \mathrm{ri}(Q)$ and $W \cap \partial B^n \cap Q$ are both empty or the former is the relative interior of the latter, which is a nonempty convex polytope. If they are empty, their Euler characteristics are 0. Otherwise, the Euler characteristic of the former is $(-1)^m$, where $m = r + \dim(Q) - n - 1$ is its dimension, and that of the latter is 1. The desired equality now follows, since the left-hand side is

$$\bar{\mu}_W(\sigma(Q)) = (-1)^{\dim(Q)}\chi(W \cap \partial B^n \cap \mathrm{ri}(Q)),$$

and the right-hand side is $(-1)^{n-r-1}$. Equality holds in general, since both sides are valuations, and each valuation is determined by its values on the pointed orthants.

The third equation shows that $\mu_{W^\perp}$ is determined by μ_W. The choice of ε_1 or ε_2 ensures that equality holds for $Q = \{0\}$, both sides having value 0 in this instance. For other pointed orthants Q, $W \cap Q^\perp$ contains a nonzero point if and only if $W^\perp \cap Q$ does not (using that W is in general position with respect to the coordinate hyperplanes); so equality holds, for each pointed orthant Q. Equality holds in general, since both sides are valuations, and each valuation is determined by its values on the pointed orthants. □

4. Valuations from Arbitrary Oriented Matroids

Before giving this construction, we switch to more concise notation for dealing with the partially ordered set of orthants.

An *involuted set* consists of a set (here, finite), E, together with a function which maps E to itself, *: $E \to E$, the image of $e \in E$ under this function being denoted by e^*, and satisfying, for each $e \in E$,

(i) $(e^*)^* = e$, and
(ii) $e^* \neq e$.

Clearly, E must have even cardinality in order to admit such a function. Given $S \subseteq E$, we write $S^* = \{e^*\colon\ e \in S\}$. A set $S \subseteq E$ is *involuted* if $S^* = S$.

For E, we take the set of $2n$ symbols,

$$E = \{1^+, 1^-, 2^+, 2^-, \ldots, n^+, n^-\},$$

and we let

$$E^+ = \{1^+, 2^+, \ldots, n^+\}$$

and

$$E^- = \{1^-, 2^-, \ldots, n^-\};$$

the involution * then switches i^+ and i^-, for each $i \in [n]$.

Let φ: $E \to [n]$ denote the mapping which takes i^+ and i^- to i, for each $i \in [n]$. If $A \subseteq E$ then $\varphi(A)$ denotes its image under this map; if $A \subseteq [n]$ then $\varphi^{-1}(A)$ denotes the inverse image of A under this map.

The collection $\mathcal{Q}^n$ of orthants can be identified, as a partially ordered set, with the collection of subsets of E: For $S \subseteq E$ let

$$Q(S) = Q(\varphi(E^+ \backslash S), \varphi(E^- \backslash S)).$$

Then Q is an order-preserving bijection from 2^E to $\mathcal{Q}^n$. The $2n$ coordinate halfplanes H_i^+ and H_i^- are the sets of the form $Q(E \backslash \{e\})$, for $e \in E$; and

$$Q(S) = \bigcap_{e \in E \backslash S} Q(E \backslash \{e\}).$$

With this association, it is clear that an integer-valued valuation on $\mathcal{Q}^n$ corresponds to a function ν: $2^E \to Z$ such that, for $S \subseteq E$ and $p \in E \backslash (S \cup S^*)$,

$$\nu(S \cup \{p, p^*\}) + \nu(S) = \nu(S \cup \{p\}) + \nu(S \cup \{p^*\}).$$

(Notice that this equation holds, for any function ν, when $p \in S \cup S^*$.) We will see how to associate a particular valuation with each oriented matroid.

Recall, from [3], that an oriented matroid is a triple $\mathcal{O} = (E, h, ^*)$, where $(E, ^*)$ is an involuted set, and h: $2^E \to 2^E$ is a function satisfying:

(1) $A \subseteq h(A)$, for each subset $A \subseteq E$;
(2) if $A \subseteq B$ then $h(A) \subseteq h(B)$;
(3) $h(h(A)) = h(A)$, for each $A \subseteq E$;
(4) for $A \subseteq E$ and $p \in E$, if $p \in h(A \cup \{p^*\})$ then $p \in h(A)$;
(5) for $A \subseteq E$ and $p, q \in E$, if $q \in h(A \cup \{p^*\}) \backslash h(A)$ then $p \in h((A \cup \{q^*\}) \backslash \{p\})$; and
(6) for $A \subseteq E$, $h(A^*) = h(A)^*$.

If W is a linear subspace of R^n then, for $A \subseteq E$, letting $h(A)$ denote the set of elements $e \in E$ such that $Q(E \backslash \{e\})$ contains the intersection $Q(E \backslash A) \cap W$, $\mathcal{O}_W = (E, h, ^*)$ is an oriented matroid.

A *matroid* is a pair $\mathcal{M} = (F, g)$, where F is a finite set and g is an operator $g\colon 2^F \to 2^F$, satisfying the *closure axioms*

(a) if $A \subseteq F$ then $g(A) \supseteq A$,
(b) if $A \subseteq B \subseteq F$ then $g(A) \subseteq g(B)$,
(c) if $A \subseteq F$ then $g(g(A)) = g(A)$,

and the *exchange axiom*

(d) if $p, q \in F$, $A \subseteq F$, and $q \in g(A \cup \{p\}) \backslash g(A)$ then $p \in g(A \cup \{q\})$.

A *flat* of the matroid (F, g) is a subset $A \subseteq F$ such that $g(A) = A$. Then F is itself a flat and the collection $\mathcal{G}$ of flats is closed under intersection. The collection $\mathcal{G}$ therefore has the structure of a lattice under the inclusion ordering. This lattice is a geometric lattice, and, consequently, if O denotes the smallest element, $g(\emptyset)$, of $\mathcal{G}$, then for $A \in \mathcal{G}$ the lengths of any two maximal chains from O to A are equal. This is called the *rank*, $\operatorname{rank}(A)$, of A. The rank function is extended to all subsets $A \subseteq F$ by the rule $\operatorname{rank}(A) = \operatorname{rank}(g(A))$.

Every oriented matroid has an *underlying matroid*. For $\mathcal{O} = (E, h, ^*)$, where $E = \{i^+, i^-\colon i \in [n]\}$, as above, define, for $A \subseteq [n]$, $g(A) = \varphi(h(\varphi^{-1}(A)))$. Then $([n], g)$ is the underlying matroid of $\mathcal{O}$.

For a set $A \subseteq E$, the rank of A (in the oriented matroid) is defined to be $\operatorname{rank}(A) = \operatorname{rank}(\varphi(A))$.

If W is a linear subspace, $\mathcal{O}_W$ is the associated oriented matroid, and $([n], g)$ is its underlying matroid, then

$$g(A) = \left\{ i \in [n]\colon\ H_i^0 \supseteq W \cap \bigcap_{j \in A} H_j^0 \right\}$$

for $A \subseteq [n]$.

The oriented matroid $\mathcal{O} = (E, h, ^*)$ determines a valuation, as follows.

Theorem 3. *For $S \subseteq E$, define*

$$\rho(S) = \begin{cases} 0 & \text{if} \quad h(S) \neq h(S^*), \\ (-1)^{\operatorname{rank}(S)} & \text{if} \quad h(S) = h(S^*). \end{cases}$$

Then ρ is a valuation.

Proof. Clearly, for $S \subseteq E$, $\rho(S) = \rho(h(S))$; that is, the value of ρ is a function of $h(S)$.

We are required to show that, for $S \subseteq E$ and $p \in E$, $\rho(S \cup \{p, p^*\}) + \rho(S) = \rho(S \cup \{p\}) + \rho(S \cup \{p^*\})$.

This certainly holds if $p \in h(S)$, for then $h(S) = h(S \cup \{p\})$ and $h(S \cup \{p, p^*\}) = h(S \cup \{p^*\})$, so that $\rho(S) = \rho(S \cup \{p\})$ and $\rho(S \cup \{p, p^*\}) = \rho(S \cup \{p^*\})$. Similarly the equation holds if $p \in h(S^*)$.

Suppose $p \notin h(S) \cup h(S^*)$ and $h(S) \neq h(S^*)$. Then $\rho(S) = 0$. There is $q \in S$ such that $q^* \notin h(S)$. If $q^* \in h(S \cup \{p\})$ then by the exchange axiom (5), $p^* \in h(S \cup \{q\}) = h(S)$, contrary to our assumption. It follows that $q^* \notin h(S \cup \{p\})$. Similarly, $q^* \notin h(S \cup \{p^*\})$. It follows that $q^* \notin h(S \cup \{p, p^*\})$, from which we see that all four values $\rho(S)$, $\rho(S \cup \{p, p^*\})$, $\rho(S \cup \{p\})$, and $\rho(S \cup \{p, p^*\})$ are 0, and equality holds.

Finally, suppose $p \notin h(S) \cup h(S^*)$ and $h(S^*) = h(S)$. Then $h(S \cup \{p, p^*\}) = h(S^* \cup \{p, p^*\})$. It follows that $\rho(S) = (-1)^{\text{rank}(S)}$ and $\rho(S \cup \{p, p^*\}) = (-1)^{\text{rank}(S \cup \{p, p^*\})} = (-1)^{\text{rank}(S)+1}$, so the left-hand side in the equation is 0. Since $p \notin h(S)$, it follows that $p \notin h(S \cup \{p^*\})$, so $h(S^* \cup \{p^*\}) \neq h(S \cup \{p\})$, and therefore $\rho(S \cup \{p\}) = 0$. Similarly, $\rho(S \cup \{p^*\}) = 0$, and from these it follows that the right-hand side is also 0. □

We now define $\chi_{\mathcal{O}}\colon 2^E \to Z$ by

$$\chi_{\mathcal{O}}(S) = 1 - (-1)^{\text{rank}(E)} \rho(E \backslash S).$$

Clearly this is also a valuation, called the *valuation associated with* $\mathcal{O}$.

If $h(E \backslash S^*) = h(E \backslash S)$ (which certainly holds when $S^* = S$) then $\chi_{\mathcal{O}}(S)$ is 0 or 2, depending on whether the rank of $E \backslash S$ is even or odd, being 0 when $S = \emptyset$. If $h(E \backslash S^*) \neq h(E \backslash S)$ then $\chi_{\mathcal{O}}(S) = 1$.

The circuits of $\mathcal{O}$ are the minimal subsets $S \subseteq E$ such that S is not involuted and $h(S)$ is involuted. This property of S can certainly be checked, using only the associated valuation: such sets S are the minimal sets which are not involuted and satisfy $\chi_{\mathcal{O}}(S) \neq 1$. Therefore the structure of the oriented matroid is completely determined by the associated valuation.

A *uniform* oriented matroid is one such that if S is involuted and has cardinality less than twice the rank of E, then $h(S) = S$; that is, the underlying matroid is uniform. In the realizable case, the uniform oriented matroids are those which arise from vector spaces W which are in general position with respect to the coordinate axes. For a uniform oriented matroid $\mathcal{O}$, more can be said about the values of $\chi_{\mathcal{O}}$, as follows.

Proposition 2. *Suppose $\mathcal{O}$ is uniform of rank r. Recall $\frac{1}{2}|E| = n$. If S is a subset of E which is not involuted then*

$$\chi_{\mathcal{O}}(S) = \begin{cases} 0 & \text{if} \quad h(E \backslash S) = E, \\ 1 & \text{if} \quad h(E \backslash S) \neq E. \end{cases}$$

If S is a subset of E which is involuted and $m = \frac{1}{2}|S|$ then

$$\chi_{\mathcal{O}}(S) = \begin{cases} 0 & \text{if} \quad m \not\equiv r + n \pmod 2, \\ 2 & \text{if} \quad m \equiv r + n \pmod 2. \end{cases}$$

The valuation $\chi_{\mathcal{O}}$ is partly monotone (*so that if $A \subseteq B \subseteq E$ and A, B are not involuted, then $\chi_{\mathcal{O}}(A) \leq \chi_{\mathcal{O}}(B)$*).

Proof. For uniform oriented matroids, if $h(U^*) = h(U)$ then either $h(U) = E$ or U itself is involuted and $h(U) = U$. If $S \subseteq E$ is not involuted and $h(E \backslash S) = E$ then $\rho(E \backslash S) = (-1)^r$ so $\chi_{\mathcal{O}}(S) = 1 - (-1)^r \rho(E \backslash S) = 0$; if S is not involuted and $h(E \backslash S) \neq E$ then $h(E \backslash S)$ is not involuted so $\rho(E \backslash S) = 0$ and $\chi_{\mathcal{O}}(S) = 1$.

If S is involuted

$$\begin{aligned}\chi_{\mathcal{O}}(S) &= 1-(-1)^r\rho(E\backslash S)\\ &= 1-(-1)^{r+(n-m)}\\ &= \begin{cases}0 & \text{if} \quad m \not\equiv r+n \pmod 2,\\ 2 & \text{if} \quad m \equiv r+n \pmod 2.\end{cases}\end{aligned}$$

It is clear from the first assertion of the proposition that if the subsets $S_1 \subseteq S_2$ of E are not involuted, then $\chi_{\mathcal{O}}(S_1) \leq \chi_{\mathcal{O}}(S_2)$, so $\chi_{\mathcal{O}}$ is partly monotone. □

5. Valuations and Lopsided Sets

A *lopsided set* is a subset L of the vertex set $\{-1, 1\}^n$ of the n-cube $[-1, 1]^n$ such that whenever F and G are coordinate subspaces of R^n which are complementary, so that $F \cap G = \{0\}$ and $F + G = R^n$ (and we may write $F = Q(A, A)$, $G = Q(B, B)$, where A and B are complementary subsets of $[n]$), either there is a face of the cube which is parallel to F, all of whose vertices are in L or there is a face parallel to G, none of whose vertices are in L.

Given an involuted set $S = S^*$, a *crosscut* of S is a subset $C \subseteq S$ such that $C \cup C^* = S$ and $C \cap C^* = \emptyset$; then E^+ and E^- are complementary crosscuts of E. The collection of crosscuts of an involuted set S is denoted by $\mathcal{C}(S)$. We may identify the vertices of the cube with the elements of $\mathcal{C}(E)$: the vertex $(\delta_1, \ldots, \delta_n) \in \{-1, 1\}^n$ corresponds to the crosscut C, where, if $\delta_i = 1$ then $i^+ \in C$, and if $\delta_i = -1$ then $i^- \in C$. With this identification, we can now refer to "lopsided sets" of crosscuts of E.

The following definition is equivalent in the setting of involuted sets to the one already given: a set L of crosscuts of E is *lopsided* if for each partition of E into involuted sets U, V, either there is a crosscut X_0 of U such that, for each crosscut Y of V, $X_0 \cup Y \in L$, or there is a crosscut Y_0 of V such that, for each crosscut X of U, $X \cup Y_0 \notin L$. Note that both conditions cannot hold, considering the set $X_0 \cup Y_0$.

An equally simple, equivalent formulation of this definition can be given in terms of the indicator function ι_L of the set L of crosscuts:

$$\iota_L(C) = \begin{cases}1 & \text{if} \quad C \in L,\\ 0 & \text{if} \quad C \notin L.\end{cases}$$

A moment's reflection shows that a $\{0, 1\}$-valued function ι on crosscuts is the indicator function of a lopsided set of crosscuts if and only if the equation

$$\max_{\substack{\text{crosscuts}\\ X \text{ of } U}} \min_{\substack{\text{crosscuts}\\ Y \text{ of } V}} \iota(X \cup Y) = \min_{\substack{\text{crosscuts}\\ Y \text{ of } V}} \max_{\substack{\text{crosscuts}\\ X \text{ of } U}} \iota(X \cup Y)$$

holds for all partitions $\{U, V\}$ of E into involuted sets.

Given a lopsided set L with indicator function ι, define $\nu_L \colon 2^E \to Z$ by

$$\nu_L(S) = \min_{\substack{\text{crosscuts}\\ X \text{ of } E\backslash(S\cup S^*)}} \max_{\substack{\text{crosscuts}\\ Y \text{ of } S\cap S^*}} \iota(X \cup Y \cup (S\backslash S^*)).$$

Here the set $X \cup Y \cup (S\backslash S^*)$ is a crosscut of E.

Clearly, ν_L is $\{0, 1\}$-valued. Also, if $S_1 \subseteq S_2$, then $\nu_L(S_1) \leq \nu_L(S_2)$. We see therefore that ν_L is monotone and $\{0, 1\}$-valued. We will see (in Theorem 4 below) that it is a max-min valuation. Furthermore, it is clear that if S is a crosscut then $\nu_L(S) = \iota(S)$.

The following lemma shows that we may reverse the order of the max and min in the definition of ν_L.

Lemma 2. *We have the equality*

$$\nu_L(S) = \max_{\substack{\text{crosscuts}\\ Y \text{ of } S\cap S^*}} \min_{\substack{\text{crosscuts}\\ X \text{ of } E\backslash(S\cup S^*)}} \iota(X \cup Y \cup (S\backslash S^*)).$$

Proof. Suppose, on the contrary, that there is some set $S \subseteq E$ for which this equation fails. Choose such a set with $|S\backslash S^*|$ as small as possible. By definition of lopsidedness, the equation does not fail when S is involuted; therefore $S\backslash S^* \neq \emptyset$. Choose $s \in S\backslash S^*$.

Certainly

$$\begin{aligned}\nu_L(S) &= \min_{\substack{\text{crosscuts}\\ X \text{ of } E\backslash(S\cup S^*)}} \max_{\substack{\text{crosscuts}\\ Y \text{ of } S\cap S^*}} \iota(X \cup Y \cup (S\backslash S^*)) \\ &\geq \max_{\substack{\text{crosscuts}\\ Y \text{ of } S\cap S^*}} \min_{\substack{\text{crosscuts}\\ X \text{ of } E\backslash(S\cup S^*)}} \iota(X \cup Y \cup (S\backslash S^*)),\end{aligned}$$

so, since ι is $\{0, 1\}$-valued, we must have that

$$\nu_L(S) = 1 \tag{a}$$

and

$$\max_{\substack{\text{crosscuts}\\ Y \text{ of } S\cap S^*}} \min_{\substack{\text{crosscuts}\\ X \text{ of } E\backslash(S\cup S^*)}} \iota(X \cup Y \cup (S\backslash S^*)) = 0. \tag{b}$$

Clearly then also

$$\min_{\substack{\text{crosscuts } X\\ \text{of } (E\backslash(S\cup S^*))\cup\{s,s^*\}}} \max_{\substack{\text{crosscuts } Y\\ \text{of } S\cap S^*}} \iota(X \cup Y \cup (S\backslash(S^* \cup \{s\}))) = 1 \tag{c}$$

and

$$\max_{\substack{\text{crosscuts } Y \text{ of}\\ (S\cap S^*)\cup\{s,s^*\}}} \min_{\substack{\text{crosscuts } X\\ \text{of } E\backslash(S\cup S^*)}} \iota(X \cup Y \cup (S\backslash(S^* \cup \{s\}))) = 0. \tag{d}$$

By the minimality of $|S\backslash S^*|$, (c) yields

$$\max_Y \min_X \iota(X \cup Y \cup (S\backslash(S^* \cup \{s\}))) = 1 \tag{e}$$

and (d) yields

$$\min_X \max_Y \iota(X \cup Y \cup (S\backslash S^* \cup \{s\})) = 0. \tag{f}$$

Using (e), we see that there is a crosscut Y_0 of $(S \cap S^*) \cup \{s, s^*\}$ such that, for each crosscut X of $E\backslash(S \cup S^*)$, $\iota(X \cup Y_0 \cup (S\backslash S^* \cup \{s\})) = 1$. By (b), it is clear that $s^* \in Y_0$. Let $Y_0' = Y_0\backslash\{s^*\}$. Using (f), we see that there is a crosscut X_0 of $E\backslash(S \cup S^*) \cup \{s, s^*\}$ such that, for each crosscut Y of $S \cap S^*$, $\iota(X_0 \cup Y \cup (S\backslash(S^* \cup \{s\}))) = 0$. By (a), it is clear that $s^* \in X_0$. Let $X_0' = X_0\backslash\{s^*\}$.

The value $\iota(X_0' \cup Y_0' \cup (S\backslash(S^* \cup \{s\}) \cup \{s^*\}))$ can be neither 0 nor 1, a contradiction. $\square$

Theorem 4. *The function ν_L is a max-min valuation.*

Proof. Suppose $A \subseteq E$, $p \in E\backslash(A \cup A^*)$. Then

$$\nu_L(A) = \min_{\substack{\text{crosscuts } X \\ \text{of } E\backslash(A\cup A^*)}} \max_{\substack{\text{crosscuts } Y \\ \text{of } A\cap A^*}} \iota(X \cup Y \cup (A\backslash A^*))$$

which is the minimum of

$$\min_{\substack{\text{crosscuts } X \text{ of} \\ E\backslash(A\cup A^*\cup\{p,p^*\})}} \max_{\substack{\text{crosscuts } Y \\ \text{of } A\cap A^*}} \iota(X \cup Y \cup (A\backslash A^*) \cup \{p\})$$

and

$$\min_{\substack{\text{crosscuts } X \\ \text{of } E\backslash(A\cup A^*\cup\{p,p^*\})}} \max_{\substack{\text{crosscuts } Y \\ \text{of } A\cap A^*}} \iota(X \cup Y \cup (A\backslash A^*) \cup \{p^*\});$$

that is, $\nu_L(A) = \min\{\nu_L(A \cup \{p\}), \nu_L(A \cup \{p^*\})\}$.

From the lemma, similarly, we get $\nu_L(A\cup\{p, p^*\}) = \max\{\nu_L(A\cup\{p\}), \nu_L(A\cup\{p^*\})\}$. □

Obviously, ι (and L) can be retrieved from ν_L: ι is the restriction of ν_L to the collection of crosscuts.

Suppose we begin with a max-min, $\{0, 1\}$-valued valuation ν. We will see that it arises as above from a lopsided set. Clearly all that is needed for this is to show that the restriction to the collection of crosscuts is the indicator function of a lopsided set.

Theorem 5. *Suppose that ν is a $\{0, 1\}$-valued, max-min valuation. Then the collection L of crosscuts C such that $\nu(C) = 1$ is a lopsided set.*

Proof. We verify that the restriction ι of ν to crosscuts satisfies the defining condition. Suppose $\{U, V\}$ is a partition of E into involuted sets U, V. It is clear that by repeated application of the equation $\nu(S \cup \{p, p^*\}) = \min\{\nu(S \cup \{p\}), \nu(S \cup \{p^*\})\}$, we obtain

$$\nu(U) = \max_{\substack{\text{crosscuts} \\ X \text{ of } U}} \nu(X).$$

By repeated application of $\nu(S) = \min\{\nu(S\cup\{p\}), \nu(S\cup\{p^*\})\}$ (where $p, p^* \notin S$), we continue

$$= \min_{\substack{\text{crosscuts } X \\ \text{of } U}} \max_{\substack{\text{crosscuts } Y \\ \text{of } V}} \nu(X \cup Y).$$

Similarly, we also obtain

$$\nu(U) = \max_{\substack{\text{crosscuts } Y \\ \text{of } V}} \min_{\substack{\text{crosscuts } X \\ \text{of } U}} \nu(X \cup Y).$$

The desired conclusion follows. □

6. Valuations from Uniform Oriented Matroids

Let $\mathcal{O}$ be an oriented matroid and let ν be the associated valuation, as in Section 4. At the end of that section we saw that, under the assumption that $\mathcal{O}$ is uniform, the function ν has, among others, the following properties: the valuation ν is $\{0, 1, 2\}$-valued, and is $\{0, 1\}$-valued on sets $S \subseteq E$ which are not involuted; ν is partly monotone; and $\nu(S^*) = \nu(S)$ for all subsets S of E (ν is symmetric). We now show that any valuation ν having these properties arises as in Section 4 from a uniform oriented matroid. The following lemma will be of use.

Lemma 3. *Suppose that ν and ν' are valuations, $\nu, \nu'\colon 2^E \to Z$, which agree on sets which are not involuted and on $\emptyset$. Then $\nu' = \nu$.*

Proof. Suppose not. Let $S \subseteq E$ be a minimal set for which $\nu'(S) \neq \nu(S)$. Then S is involuted and $S \neq \emptyset$. Choose $p \in S$ and let $S_0 = S \backslash \{p, p^*\}$. Then

$$\nu'(S) = \nu'(S_0 \cup \{p, p^*\}) = \nu'(S_0 \cup \{p\}) + \nu'(S_0 \cup \{p^*\}) - \nu'(S_0)$$

and

$$\nu(S) = \nu(S_0 \cup \{p, p^*\}) = \nu(S_0 \cup \{p\}) + \nu(S_0 \cup \{p^*\}) - \nu(S_0).$$

By minimality of S, the right-hand sides are equal; so the left-hand sides are equal, contradicting our assumption. □

Recall that a *tope* of the oriented matroid $\mathcal{O}$ is a crosscut C of E such that $h(C) = C$.

Theorem 6. *The map $\mathcal{O} \to \chi_{\mathcal{O}}$ of Section 4, restricted to uniform oriented matroids, is a bijective correspondence between the set of uniform oriented matroids on E and the set of valuations on E which are partly monotone, symmetric, $\{0, 1\}$-valued on sets which are not involuted, and have value 0 on $\emptyset$.*

Proof. Given $\mathcal{O}$ we have seen how to obtain the valuation $\chi_{\mathcal{O}}$.

Let χ be a valuation having the properties listed. Let L be the set of all crosscuts $C \subseteq E$ such that $\chi(C) = 1$. For each $p \in E$ we define a function χ_p on $E \backslash \{p, p^*\}$ by the rule $\chi_p(S) = \chi(S \cup \{p\})$. Since in this expression $S \cup \{p\}$ cannot be symmetric, it follows that χ_p is a max-min valuation on $E \backslash \{p, p^*\}$ which has values in $\{0, 1\}$. By Theorem 5 the set L_p of crosscuts C of $E \backslash \{p, p^*\}$ such that $\chi_p(C) = 1$ is lopsided. Since this holds for each $p \in E$, it follows by Theorem 9 of [6] that L is the set of topes of a uniform oriented matroid $\mathcal{O}$.

Now it is clear that $\chi_{\mathcal{O}}$ and χ agree on crosscuts, each having value 1 on topes of $\mathcal{O}$. Since for each $p \in E$ and for $S \subseteq E \backslash \{p, p^*\}$, both $S \mapsto \chi(S \cup \{p\})$ and $S \mapsto \chi_{\mathcal{O}}(S \cup \{p\})$ are max-min valuations, agreeing on crosscuts of $E \backslash \{p, p^*\}$, we see that χ and $\chi_{\mathcal{O}}$ agree on sets of the form $S \cup \{p\}$, where $p^* \notin S$, that is, on sets which are not involuted. Since each has value 0 on $\emptyset$, the lemma applies, and the two valuations coincide. □

7. Valuations and Total Polynomials

The valuation $\chi_{\mathcal{O}}$ associated with the uniform oriented matroid $\mathcal{O}$ is closely related to the "total polynomial" $T_{\mathcal{O}}$ of $\mathcal{O}$, introduced in [8]. $T_{\mathcal{O}}$ is a sum of certain monomials associated with the nonzero covectors of the oriented matroid. With the terminology of Section 4, the nonzero covectors correspond to the sets $A \subseteq E$ such that $A \neq E$, $A \cup A^* = E$, and $h(A) = A$. Let $\mathcal{L}$ denote the set of such subsets. For each element $e \in E$ let x_e denote an indeterminate. For $A \subset E$ let m_A be the monomial $\prod_{e \in A} x_e$. Then the *total polynomial* $T_{\mathcal{O}}$ of the uniform oriented matroid $\mathcal{O} = (E, h, ^*)$ is

$$T_{\mathcal{O}} = \sum_{A \in \mathcal{L}} m_{E \backslash A}.$$

Suppose $A \subseteq E$, $A \neq E$, and $A \cup A^* = E$. Note that A is not involuted. If $h(A) = A$, so that $A \in \mathcal{L}$, then $h(A) = A$ is not involuted, so $\rho(A) = 0$, and $\chi_{\mathcal{O}}(E \backslash A) = 1$. If $h(A) \neq A$, then, since $\mathcal{O}$ is uniform, $h(A) = E$, so that $\chi_{\mathcal{O}}(E \backslash A) = 0$. Also $\chi_{\mathcal{O}}(\emptyset) = 0$, and we may write

$$T_{\mathcal{O}} = \sum_{A:\ A \cup A^* = E} \chi_{\mathcal{O}}(E \backslash A) m_{E \backslash A};$$

or, replacing A by its complement in this expression,

$$T_{\mathcal{O}} = \sum_{A:\ A \cap A^* = \emptyset} \chi_{\mathcal{O}}(A) m_A.$$

This shows that the total polynomial may be derived rather simply from the valuation. The reverse is also true: the values $\chi_{\mathcal{O}}(A)$, for A such that $A \cap A^* = \emptyset$, are the coefficients of $T_{\mathcal{O}}$; and, since $\chi_{\mathcal{O}}$ is a valuation, its values on other sets $A \subseteq E$ are determined by these.

The three equations of Proposition 1 hold for uniform oriented matroids in general.

Proposition 3. *We have, for each orthant P:*

$$\bar{\chi}_{\mathcal{O}}(\rho(P)) = \chi_{\mathcal{O}}(P);$$

$$\bar{\chi}_{\mathcal{O}}(\sigma(P)) = (-1)^{n-r+1} \chi_{\mathcal{O}}(P);$$

and

$$\bar{\chi}_{\mathcal{O}}(\tau(P)) = \varepsilon_i(P) - \chi_{\hat{\mathcal{O}}}(P),$$

where $\hat{\mathcal{O}}$ is the dual of $\mathcal{O}$, r is the rank of $\mathcal{O}$, $i = 1$ if r is odd, and $i = 2$ otherwise.

Proof. The first of these follows from the equation $\chi_{\mathcal{O}}(A^*) = \chi_{\mathcal{O}}(A)$; the second and third are equivalent to Theorems 1 and 2 (respectively) of [8]. $\square$

8. Notes

The above characterization of the valuations associated with uniform oriented matroids of fixed rank r can be viewed as follows. These valuations form a certain finite subset of the real vector space of functions η: $2^E \to Z$. According to the results above, this subset is determined by finite collections of linear equations and inequalities, together with integrality constraints. Additionally, many important invariants of the oriented matroids, for example, the number of faces of some fixed dimension of the oriented matroid polytope, are restrictions to this finite set of linear functions on the vector space. Therefore, problems of maximization or minimization of such invariants are seen to be large integer programming problems.

To illustrate, suppose n and r are given and it is desired to maximize or minimize some sufficiently nice function over uniform oriented matroids of rank r on E, where, as before, E is involuted and $|E| = 2n$. Producing variables x_A, one for each subset $A \subseteq E$, we can write the conditions given in Theorem 6 that with the x_A's as values, $\eta(A) = x_A$ for $A \subseteq E$, the function η is the valuation corresponding to some uniform oriented matroid on E. They are:

(1) $x_\emptyset = 0$;
(2) $x_{A^*} = x_A$, for each set $A \subseteq E$;
(3) if $A \subseteq B \subseteq E$, $A \neq A^*$, and $B \neq B^*$, then $x_A \leq x_B$;
(4) if $A \subseteq E$ and $p \in E\backslash(A \cup A^*)$, then

$$x_{A\cup\{p,p^*\}} + x_A - x_{A\cup\{p\}} - x_{A\cup\{p^*\}} = 0;$$

and
(5) if $A \subseteq E$ and $A \neq A^*$, then $x_A \in \{0, 1\}$.

To guarantee that the rank is r, fixing subsets $A_0, B_0 \subseteq E$ such that $A_0^* = A_0$, $B_0^* = B_0$, $|A_0| = 2(n - r + 1)$, $|B_0| = 2(r + 1)$, we can use

(6) $x_{A_0} = 2$ and $x_{B_0} = 0$.

Consider now the function $\sum x_A$, where the sum extends over, say, all crosscuts A of E such that $|A \cap E^+| = k$, where k is fixed. If the x_A's are the values of a valuation associated to a given uniform oriented matroid on E having rank r—that is, if conditions (1)–(6) are met—then this function counts the number of topes of the oriented matroid which have exactly k positive elements. Its maximization subject to the constraints (1)–(6) is an integer programming problem.

This is of course a huge integer programming problem, in any interesting case. Its usefulness suffers as well from the fact that its linear relaxation, obtained upon dropping the requirement (5) of integrality, is rather far from a characterization of the convex hull of the feasible (integer-valued) assignments of x_A's. Nevertheless, at least in the case of problems involving uniform oriented matroids which have rank 3 and are acyclic, there is an integer programming method which is computationally feasible for problems having n at most a dozen or so. This method will be presented elsewhere.

It would be nice to extend the characterization to oriented matroids in general. In this paper we have relied on results of [6] in our characterization of the valuations arising from uniform oriented matroids. Da Silva [2] has improved upon the results of [6] by giving a related characterization of the subsets of the vertex sets of the cubes which correspond to topes of oriented matroids in general. Perhaps this characterization could be used to extend the description of the valuations associated with uniform oriented matroids, to arbitrary oriented matroids.

It is worth noting that another useful characterization of the uniform oriented matroids is described by Gärtner and Welzl in [4]. Also in that paper, a connection between lopsided sets and the notion of "Vapnik–Chervonenkis dimension" is described.

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