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ON THE RECONSTRUCTION OF CONVEX SETS FROM RANDOM NORMAL MEASUREMENTS

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ABSTRACT. We study the problem of reconstructing a convex body using only a finite number of measurements of outer normal vectors. More precisely, we suppose that the normal vectors are measured at independent random locations uniformly distributed along the boundary of our convex set. Given a desired Hausdorff error η, we provide an upper bounds on the number of probes that one has to perform in order to obtain an η-approximation of this convex set with high probability. Our result rely on the stability theory related to Minkowski's theorem.

1. INTRODUCTION

Surface reconstruction is now a classical and rather well-understood topic in computational geometry. The input of this problem is a finite set of points P measured on (or close to) an underlying unknown surface S, and the goal is to reconstruct a Hausdorff approximation of this surface. In this article, we deal with a surface reconstruction question. However, our input is not a set of points, but a set of (unit outer) normal vectors measured at various unknown locations on S. This question stemmed from a collaboration with CEA-Leti, which has developed sensors that embed an accelerometer and a magnetometer, and can return their own orientation, but not their position. Since these sensors are small and rather inexpensive it is possible to use many of them to monitor the deformations of a known surface [17]. Can such sensors be used for surface reconstruction ? One cannot expect to be able to reconstruct a surface from a finite number of normal measurements, without assumptions on the surface or on the distribution of the points. Here, we study the case where the underlying surface is convex, and where the normals are measured at random and uniformly distributed locations.

Related work. Minkowski's theorem asserts that a convex set is uniquely determined, up to translation, by the distribution of its normals on the sphere. In the case of polyhedron, the precise statement is as follows: given a set of normal vectors $\mathbf{n}_1, \dots, \mathbf{n}_N$ in the unit sphere, and a set of positive numbers $a_1, \dots, a_N$ such that (i) $\sum_{i=1}^N a_i \mathbf{n}_i = 0$ and (ii) the set of normals spans $\mathbb{R}^d$, there exists a convex polytope with exactly N faces and such that the area of the face with normal $\mathbf{n}_i$ is a_i [15]. Moreover, this polytope is unique up to translation. The computational aspects related to Minkowski's theorem have been studied using variational techniques on the primal problem [14] or on the dual problem [3, 13], but also from the viewpoint of complexity theory [10].

Minkowski theorem can be interpreted as a reconstruction result, and such a result comes with a corresponding stability question: if the distribution of normals of two convex bodies are close to each other (in a sense to be made precise), are the corresponding bodies also close in the Hausdorff sense up to translation ? This question has been studied extensively in the convex geometry literature, using the theory surrounding the Brunn-Minkowski inequality, starting from two articles by Diskant [6, 7, 16, 12]. Our probabilistic convergence theorem bears some resemblance with [9], which studies a different inverse problem in convex geometry, namely reconstructing a convex set from its brightness function.

Contributions. In the present paper, we suppose the existence of an underlying convex body K, which is not necessarily a polytope, and from which a probing device measures unit outer normals. Our input data is a set of N unit normals $(\mathbf{n}_i)_{1\leq i\leq N}$, which have been measured at N locations on the boundary ∂K of K. These locations have been chosen randomly and independently, and are uniformly distributed with respect to the surface area on ∂K. Note that from now on, we assume that *only the measured normals are known to us*, and not the locations they were measured at. The question we consider is the following: given $\eta > 0$, what is the minimum number of such measurements needed so as to be able to reconstruct with high probability a convex set L_N which is η Hausdorff-close to K up to translation? Denoting $\mathrm{d_H}$ the Hausdorff distance, we prove the following theorem:

Theorem (Theorem 4.1). *Let K be a bounded convex set with non-empty interior and whose boundary has $(d-1)$-area one. Given $p \in (0,1)$ and $\eta > 0$, and*

$$N \geq \mathrm{const}\,(K,d) \cdot \eta^{\frac{d(1-d)}{2} - 2d} \log(1/p)$$

random normal measurements, it is possible to construct a convex body L_N such that

$$\mathbb{P}\left(\min_{x\in\mathbb{R}^d} \mathrm{d_H}(x+K, L_N) \leq \eta\right) \geq 1-p.$$

In the course of proving this theorem, we introduce a very weak notion of distance between measures on the unit sphere, which we call the "convex-dual distance". This distance is weaker than usual distances between measures, such as the bounded-Lipschitz[1] or the total variation distances. Surprisingly, it is nonetheless sufficiently strong to control the Hausdorff distance between two convex bodies in term of the convex-dual distances between their distribution of normals, as shown in Theorem 3.1. This theorem weakens the hypothesis in the stability results of Diskant [6, 7] and Hug–Schneider [12].

Notation. The Euclidean norm and scalar product on $\mathbb{R}^d$ are denoted $\|.\|$ and $. \cdot .$ respectively. The unit sphere of $\mathbb{R}^d$ is denoted $\mathcal{S}^{d-1}$, and $\mathrm{B}(x,r)$ is the ball centered at a point x with radius r. We call *convex body* a compact convex subsets of the Euclidean space $\mathbb{R}^d$ with non-empty interior. The boundary of a convex body K is denoted ∂K. Also, we denote $\mathcal{H}^d(A)$ the volume of a set A, and $\mathcal{H}^{d-1}(B)$ the $(d-1)$-Hausdorff measure of B. These notions coincide with the intuitive notions of volume and surface area in dimension three. A *(non-negative) measure* μ over a metric space X associates to any (Borel) subset B a non-negative number $\mu(B)$. It should enjoy

[1]The bounded-Lipschitz distance coincides with the Wasserstein (or Earthmover) distance with exponent one when the two measures have the same total mass.

also the following additivity property: if (B_i) is a countable family of disjoint subsets, then $\mu(\cup_i B_i) = \sum_i \mu(B_i)$. We call $\mu(X)$ the *total mass* of μ. The measure μ on X is a *probability measure* if $\mu(X) = 1$. The unit Dirac mass at a point x of X is the probability measure δ_x defined by $\delta_x(B) = 1$ if x belongs to B and $\delta_x(B) = 0$ if not.

2. Minkowski problem and the convex-dual distance

The problem originally posed by Minkowski concerns the reconstruction of a convex polyhedron P from its facet areas $(a_i)_{1\leq i\leq N}$ and unit outer normals $(\mathbf{n}_i)_{1\leq i\leq N}$. This data can be summarized by a measure on the unit sphere, and more precisely by a linear combination of Dirac masses: $\mu_P = \sum_{1\leq i\leq N} a_i \delta_{x_i}$. Minkowski's problem has been generalized to more general convex bodies by Alexandrov using the notion of *surface area measure*.

Recall that given a convex body K and a point x on its boundary, a unit vector v is a *unit outer normal* if for every point y in K, $(x-y)\cdot v \geq 0$. For $\mathcal{H}^{d-1}$-almost every point in ∂K, there is a single outer unit normal, which we denote $\mathbf{n}_K(x)$. The *Gauss map* of K is the map $\mathbf{n}_K : \partial K \to \mathcal{S}^{d-1}$.

Definition 2.1. The *surface area measure of* K is a measure μ_K on the unit sphere. The measure $\mu_K(B)$ of a (Borel) subset B of the sphere $\mathcal{S}^{d-1}$ is the $(d-1)$-area of the subset of ∂K whose normals lie in B. In other words,

$$\mu_K(B) := \mathcal{H}^{d-1}\left(\{x \in \partial K; \mathbf{n}_K(x) \in B\}\right) = \mathcal{H}^{d-1}(\mathbf{n}_K^{-1}(B)). \tag{1}$$

By definition, the total mass $\mu_K(\mathcal{S}^{d-1})$ of the surface area measure is equal to the $(d-1)$-volume of the boundary ∂K. In particular, the surface area of K is a probability measure if and only if K has *unit surface area*, i.e. $\mathcal{H}^{d-1}(\partial K) = 1$.

For instance, if P is a convex polyhedron with k d-dimensional facets $F_1, \ldots, F_k$, the unit exterior normal $\mathbf{n}_P(x)$ is well defined at any point x that lies on the relative interior of one of these facets. As noted earlier, the surface area measure of the polyhedron P can then be written as a finite weighted sum of Dirac masses, $\mu_P = \sum_{i=1}^N \mathcal{H}^{d-1}(F_i)\delta_{\mathbf{n}_{F_i}}$, where the unit normal to the ith face is denoted $\mathbf{n}_{F_i}$.

Alexandrov's theorem [1] generalizes the reconstruction theorem of Minkowski mentioned in the introduction. It shows that a convex body is uniquely determined, up to translation, by its surface area measure. It also gives a characterization of the measures on the sphere that can occur as surface area measures of convex bodies.

Definition 2.2. Given a measure μ on the unit sphere $\mathcal{S}^{d-1}$,

(i) the *mean* of μ is the point of $\mathbb{R}^d$ defined by $\mathrm{mean}(\mu) := \int_{\mathcal{S}^{d-1}} x \mathrm{d}\mu_K(x)$. The measure μ has *zero mean* if this point lies at the origin.

(ii) we say that the measure μ has *non-degenerate support* if for every hyperplane $H \subseteq \mathbb{R}^d$, the inequality $\mu_K(\mathcal{S}^{d-1} \setminus H) > 0$ holds. Equivalently, μ has non-degenerate support if and only its mass is not entirely contained on a single great circle of the sphere.

Theorem (Alexandrov)**.** *Given any measure μ on $\mathcal{S}^{d-1}$ with zero mean and non-degenerate support, there exists a convex body K whose surface area measure μ_K coincides with μ. Moreover, this convex body is unique up to translation.*

2.1. **Convex-dual distance.** One of our goals in this article is to refine existing quantitative estimates of uniqueness in Alexandrov's theorem. In other words, we want to be able to express the fact that if the surface area measures μ_K and μ_L are close to each other, then the convex bodies K and L are also close to each other. For this purpose, we introduce the convex-dual distance, a very weak notion of distance between measures on the unit sphere.

The *support function* of a convex body $K \subseteq \mathbb{R}^d$ is a function $\mathrm{h}_K : \mathcal{S}^{d-1} \to \mathbb{R}$ on the unit sphere defined by the formula $h_K(u) := \max_{x\in K} x \cdot u$. We will use the following known fact of convex geometry, whose proof is included for convenience.

Lemma 2.3. *If $K \subseteq \mathrm{B}(0,r)$, the support function h_K is r-Lipschitz and $|h_K| \leq r$.*

Proof. Consider u in the unit sphere, and x in K such that $\mathrm{h}_K(u) = u \cdot x$. For any vector v in the unit sphere,

$$\begin{aligned}\mathrm{h}_K(v) = \max_{y\in K} v \cdot y \geq v \cdot x &= u \cdot x + (v-u) \cdot x \\ &\geq \mathrm{h}_K(u) - \|u-v\|\|x\| \\ &\geq \mathrm{h}_K(u) - r\|u-v\|.\end{aligned}$$

Swapping u and v gives the Lipschitz bound. Moreover, for v in $\mathcal{S}^{d-1}$, we get by the Cauchy-Schwartz inequality

$$|\mathrm{h}_K(v)| = \max_{y\in K} |v\cdot y| \leq \|v\| \max_{y\in Y} \|y\| \leq r \qquad \square$$

Definition 2.4. Given two measures μ, ν on $\mathcal{S}^{d-1}$, their *convex-dual distance* is defined by:

$$\mathrm{d_C}(\mu,\nu) = \max_{K\subseteq \mathrm{B}(0,1)} \left| \int_{\mathcal{S}^{d-1}} \mathrm{h}_K \mathrm{d}\mu - \int_{\mathcal{S}^{d-1}} \mathrm{h}_K \mathrm{d}\nu \right|, \tag{2}$$

where the maximum is taken over the set of convex bodies included in the unit ball.

The function $\mathrm{d_C}$ defined this way is non-negative and symmetric, and it is easily seen to satisfy the triangle inequality on the space of measures on the sphere $\mathcal{S}^{d-1}$. However, nothing forbids *a priori* that for general measures the distance $\mathrm{d_C}(\mu,\nu)$ vanishes while $\mu \neq \nu$. The restriction of $\mathrm{d_C}$ to the space of surface area measures of convex sets satisfies the third axiom of a distance, i.e. given two convex bodies K and L, the distance $\mathrm{d_C}(\mu_K, \mu_L)$ vanishes if and only if $\mu_K = \mu_L$. The proof of this fact needs additional tools from convex geometry and is postponed to Lemma 3.5.

2.2. **Comparison with other distances.** There are many notions of distances on spaces of measures. In this paragraph, we compare the convex-dual distance with two of them. The *total variation* distance between two measures μ and ν on $\mathcal{S}^{d-1}$ is defined by

$$\mathrm{d_{TV}}(\mu,\nu) = \sup_{B\subseteq \mathcal{S}^{d-1}} |\mu(B) - \nu(B)|,$$

where the supremum is taken on all Borel subsets. The *bounded-Lipschitz* distance defined by the following supremum, where BL_1 denotes the set of functions on the unit sphere that are 1-Lipschitz and whose absolute value is bounded by one:

$$\mathrm{d}_{\mathrm{bL}}(\mu, \nu) = \sup_{f \in \mathrm{BL}_1} \left| \int_{\mathcal{S}^{d-1}} f \mathrm{d}\mu - \int_{\mathcal{S}^{d-1}} f \mathrm{d}\nu \right|.$$

Lemma 2.5 shows that the convex-dual distance is the weakest of these three distances. This implies that a stability result with respect to this distance is stronger than a stability result with respect to d_{TV} or d_{bL}. The main advantage for using the convex-dual distance over the bounded-Lipschitz distance comes from the fact that the set of support functions of convex sets included in $\mathrm{B}(0,1)$ is *much smaller* than the set BL_1. We will show in Section 4 the implications of this fact on the speed of convergence of random sampling.

Lemma 2.5. *Given two measures μ, ν on $\mathcal{S}^{d-1}$, $\mathrm{d}_{\mathrm{C}}(\mu,\nu) \leq \mathrm{d}_{\mathrm{bL}}(\mu,\nu) \leq \mathrm{const}(d)\mathrm{d}_{\mathrm{TV}}(\mu,\nu)$.*

Proof. By Lemma 2.3, the support function h_K of a convex set K contained in the ball $\mathrm{B}(0,1)$ is 1-Lipschitz and $|h_K|$ is bounded by one. This implies that h_K lies in BL_1, and therefore

$$\left| \int_{\mathcal{S}^{d-1}} \mathrm{h}_K \mathrm{d}_\mu - \int_{\mathcal{S}^{d-1}} \mathrm{h}_K \mathrm{d}_\nu \right| \leq \mathrm{d}_{\mathrm{bL}}(\mu, \nu).$$

Taking the maximum over all such support functions gives $\mathrm{d}_{\mathrm{C}}(\mu,\nu) \leq \mathrm{d}_{\mathrm{bL}}(\mu,\nu)$. The second inequality follows from e.g. [8], Theorem 6.15. □

3. Stability in Minkowski Problem

In this section, we refine existing stability results for Minkowski's problem so as to obtain a stability result with respect to the convex-dual distance between surface area measures. We rely and improve upon existing stability results due to Diskant and Hug–Schneider, using our definition of convex-dual distance and using a $\mathcal{C}^0$ regularity estimate for Minkowski's problem due to Cheng and Yau.

The following stability theorem is Theorem 3.1 in [12], and is deduced from earlier results of Diskant [6, 7], see also [16]. The *inradius* of a convex body K is the maximum radius of a ball contained in K and the *circumradius* is the minimum radius of a ball containing K.

Theorem (Diskant, Hug–Schneider; Theorem 3.1 in [12])**.** *Let K and L be convex bodies with inradius at least $r > 0$ and circumradius at most $R < +\infty$. Then,*

$$\min_{x \in \mathbb{R}^d} \mathrm{d}_{\mathrm{H}}(K + x, L) \leq \mathrm{const}(r, R, d)\mathrm{d}_{\mathrm{bL}}^{1/d}(\mu_K, \mu_L). \tag{3}$$

The main drawback for applying this theorem in the setting of geometric inference is that one makes an assumption regarding the inradius and circumradius of the underlying set K *but also* a similar assumption on the reconstructed set L. The second drawback is that the right-hand side involves the bounded-Lipschitz distance d_{bL} instead of the weaker convex-dual distance d_{C}. Our improvements to the previous stability results can be summarized as the the following theorem that obtains the same conclusions with weaker hypothesis:

Theorem 3.1. *Given a convex body K in $\mathbb{R}^d$, and for any measure μ on $\mathcal{S}^{d-1}$ with zero mean and such that $\mathrm{d_C}(\mu_K, \mu) \leq \epsilon_0$, there is a convex set L whose surface area measure coincides with μ and*

$$\min_{x \in \mathbb{R}^d} \mathrm{d_H}(K + x, L) \leq c\mathrm{d}_{\mathrm{C}}^{1/d}(\mu_K, \mu), \tag{4}$$

where c and ϵ_0 are two positive constants depending on d and K only.

As we will see later, the constants in the theorem above depend on the dimension, on the weak rotundity of the surface area measure μ_K, defined in the next paragraph, and on the area $\mathcal{H}^{d-1}(\partial K)$. The exponent in the right-hand side of (4) is very likely not optimal, but the optimal exponent is bounded from below by $\frac{1}{d-1}$, as noted in [12].

The remainder of this section is organised as follows. We introduce in §3.1 the notion of weak rotundity of the surface area measure of K, and show how the lower and upper bounds on the inradius and circumradius in Diskant's theorems can be replaced by a lower bound on the weak rotundity using a lemma of Cheng and Yau. Then, we recall some known facts from the theory of stability in Minkowski's theorem in §3.2. Finally, we combine these results in §3.3 to get a proof of Theorem 3.1

3.1. **Weak rotundity.** In this paragraph, we use a lemma of Cheng and Yau in order to remove the assumption on the inradius and circumradius of one of the two convex sets. We call *weak rotundity* of a measure μ on the unit sphere the following quantity

$$\mathrm{rotund}(\mu) := \min_{y \in \mathcal{S}^{d-1}} \left(\int_{\mathcal{S}^{d-1}} \max(y \cdot v, 0) \mathrm{d}\mu(v) \right)$$

Note that the positivity of $\mathrm{rotund}(\mu)$ is equivalent to the hypothesis that μ has non-degenerate support.

Lemma 3.2. *Given a measure on the sphere $\mathcal{S}^{d-1}$,* $\mathrm{rotund}(\mu) > 0$ *if and only if for any hyperplane $H \subseteq \mathbb{R}^d$ one has $\mu(\mathcal{S}^{d-1} \setminus H) > 0$.*

Proof. If there was a hyperplane $H = \{y\}^\perp$ such that the support of μ is included in $\mathcal{S}^{d-1} \cap H$, one would have

$$\mathrm{rotund}(\mu) \leq \int_{\mathcal{S}^{d-1}} \max(x, y\cdot, 0) \mathrm{d}\mu(x) = \int_{\mathcal{S}^{d-1} \cap H} \max(x, y\cdot, 0) \mathrm{d}\mu(x) = 0.$$

Therefore, if $\mathrm{rotund}(\mu) > 0$, the measure μ must have non-degenerate support. □

More interestingly, Cheng and Yau [4] established a quantitative lower bound on the inradius and an upper bound on the circumradius of K in term of weak rotundity of the surface area measure of K. Note that in their statement, the boundary ∂K is assumed to be of class $\mathcal{C}^4$, but their proof does not use this fact and can be extended verbatim to the non-smooth case. A simpler proof of these bounds using John's ellipsoid is presented in [11, §1.1].

Proposition 3.3 (Cheng-Yau lemma)**.** *Let K be a convex body of $\mathbb{R}^d$. Then, the inradius r and circumradius R of K satisfy the inequalities:*

$$R \leq \mathrm{const}(d) \left[\mu_K(\mathcal{S}^{d-1})\right]^{\frac{d}{d-1}} \mathrm{rotund}(\mu_K)^{-1},$$

$$r \geq \mathrm{const}(d) \left[\mu_K(\mathcal{S}^{d-1})\right]^{-d} \mathrm{rotund}(\mu_K)^{d}.$$

The advantage of the weak rotundity of μ_K over the inradius and circumradius of K is that this quantity is stable with respect to the convex-dual distance between measures on the sphere.

Lemma 3.4. *Let μ, ν be two measures on the unit sphere. Then,*

$$|\mathrm{rotund}(\mu) - \mathrm{rotund}(\nu)| \leq \mathrm{d_C}(\mu, \nu), \tag{5}$$

$$\left|\mu(\mathcal{S}^{d-1}) - \nu(\mathcal{S}^{d-1})\right| \leq \mathrm{d_C}(\mu, \nu). \tag{6}$$

Proof. We prove Eq. (5) first. Given a point y on the unit sphere, let S_y denote the line segment joining the origin to y. Then,

$$\mathrm{h}_{S_y}(v) = \max_{x \in S_y} v \cdot x = \max(v \cdot y, v \cdot 0) = \max(v \cdot y, 0).$$

Define $f_\mu(y) := \int_{\mathcal{S}^{d-1}} \max(y \cdot v, 0) \mathrm{d}\mu(v)$ and define f_ν similarly. As a consequence of the definition of the convex-dual distance, we obtain

$$\begin{aligned}|f_\mu(y) - f_\nu(y)| &= \left| \int_{\mathcal{S}^{d-1}} \max(y \cdot v, 0) \mathrm{d}\mu(v) - \int_{\mathcal{S}^{d-1}} \max(y \cdot v, 0) \mathrm{d}\mu(v) \right| \\ &= \left| \int_{\mathcal{S}^{d-1}} \mathrm{h}_{S_y}(v) \mathrm{d}\mu(v) - \int_{\mathcal{S}^{d-1}} \mathrm{h}_{S_y}(v) \mathrm{d}\mu(v) \right| \leq \mathrm{d_C}(\mu, \nu).\end{aligned}$$

We have just shown that the uniform distance between the functions f_μ and f_ν is bounded by $\mathrm{d_C}(\mu, \nu)$. In particular, the difference between the minimum of those functions is bounded by the same quantity, i.e. $|\mathrm{rotund}(\mu) - \mathrm{rotund}(\nu)| \leq \mathrm{d_C}(\mu, \nu)$. Inequality (6) is obtained simply by plugging the support function of the unit ball, $\mathrm{h}_{\mathrm{B}(0,1)} = 1$, in the definition of the convex-dual distance:

$$\left|\mu(\mathcal{S}^{d-1}) - \nu(\mathcal{S}^{d-1})\right| = \left| \int_{\mathcal{S}^{d-1}} \mathrm{h}_{\mathrm{B}(0,1)} \mathrm{d}(\mu - \nu) \right| \leq \mathrm{d_C}(\mu, \nu) \qquad \square$$

3.2. **Background on stability theory.** We need to introduce some tools from convex geometry in order to prove Theorem 3.1. We make use of the following representations for the volume $V(K)$ of a convex body K and the first mixed volume $\mathrm{V}_1(K, L)$ of K with another convex body L. The reader can consider these formulas as definitions. More details on mixed volumes can be found in e.g. [16, Chapter 5].

$$\mathrm{V}(K) = \frac{1}{d} \int_{\mathcal{S}^{d-1}} h_K(u) d\mu_K(u) \qquad \mathrm{V}_1(K, L) = \frac{1}{d} \int_{\mathcal{S}^{d-1}} h_L(u) d\mu_K(u).$$

The following inequality is called Minkowski's isoperimetric inequality:

$$\mathrm{V}_1^d(K, L) \geq \mathrm{V}^{d-1}(K) \mathrm{V}(L). \tag{7}$$

Equality holds in (7) if and only if the convex sets K and L are equal up to homothety and translation. With L equal to the unit ball, one recovers the usual isoperimetric inequality since then, $\mathrm{V}_1(K,L) = \frac{1}{d}\mathcal{H}^{d-1}(\partial K)$. Minkowski's inequality implies that the convex-dual distance introduced in Section 2 is indeed a distance between surface area measures.

Lemma 3.5. $\mathrm{d_C}(\mu_K, \mu_L) = 0$ *if and only if* $\mu_K = \mu_L$.

Proof. The hypothesis $\mathrm{d_C}(\mu_K, \mu_L) = 0$ implies that for any compact convex set M contained in the unit ball, one has

$$\int_{\mathcal{S}^{d-1}} \mathrm{h}_M \mathrm{d}\mu_K = \int_{\mathcal{S}^{d-1}} \mathrm{h}_M \mathrm{d}\mu_L. \tag{8}$$

If one replaces M by λM, with $\lambda > 0$, the two sides of this equality are multiplied by λ. Thus, Eq. (8) holds for any convex body M, regardless of the assumption that M is contained in $\mathrm{B}(0,1)$. Taking $M = L$ in Eq. (8) we get

$$\mathrm{V}_1(K,L) = \frac{1}{d}\int_{\mathcal{S}^{d-1}} \mathrm{h}_L(u)\mathrm{d}\mu_K(u) = \frac{1}{d}\int_{\mathcal{S}^{d-1}} \mathrm{h}_L(u)\mathrm{d}\mu_L(u) = \mathrm{V}(L). \tag{9}$$

Combining this with Minkowski's inequality implies

$$\mathrm{V}(L)^d = \mathrm{V}_1^d(K,L) \geq \mathrm{V}^{d-1}(K)\mathrm{V}(L). \tag{10}$$

Exchanging the role of K and L, we see that the volumes of K and L agree, and the inequality (10) becomes an equality. Using the equality case in Minkowski's inequality, this implies that K and L are equal up to homothety and translation. Using again the equality of volumes of K and L, we see that the factor of the homothety has to be one. Consequently, K and L are translate of each other, and the surface area measures μ_K and μ_L are equal. □

Minkowski's isoperimetric inequality is at the heart of Diskant's stability results. Instead of using Diskant's theorems directly, we will use the following consequence [16, Theorem 7.2.2].

Theorem 3.6 (Diskant, Schneider)**.** *Given two positive numbers $r < R$, there exists a positive constant $c = \mathrm{const}(r,R,d)$ such that for any pair of convex bodies K, L with inradii at least r and circumradii at most R, and*

$$\epsilon := \max(|V(K) - V_1(K,L)|, |V(L) - V_1(L,K)|), \tag{11}$$

the following inequality holds:

$$\min_{x\in\mathbb{R}^d} \mathrm{d_H}(K, x+L) \leq c\epsilon^{\frac{1}{d}}. \tag{12}$$

Note that there is one difference between the statement of Theorem 7.2.2 there and the statement given in Theorem 3.6 here however. We replace the strong assumption that the surface area measures of K and L are close in the total variation sense by a consequence of this fact, namely Eq. (7.2.6) there and Eq. (11) here. This weaker assumption is sufficient for the proof to work, as noted by Hug and Schneider in [12, Theorem 3.1].

3.3. **Proof of Theorem 3.1.** Assume that

$$\mathrm{d_C}(\mu_K, \mu) \leq \epsilon_0 := \min(\frac{1}{2}\mathrm{rotund}(\mu_K), \frac{1}{2}\mu_K(\mathcal{S}^{d-1})).$$

Then, the stability results of Lemma 3.4 imply

$$0 < \frac{1}{2}\mathrm{rotund}(\mu_K) \leq \mathrm{rotund}(\mu) \leq 2\mathrm{rotund}(\mu_K), \tag{13}$$

$$0 < \frac{1}{2}\mu_K(\mathcal{S}^{d-1}) \leq \mu(\mathcal{S}^{d-1}) \leq 2\mu_K(\mathcal{S}^{d-1}). \tag{14}$$

In particular, by Lemma 3.4, the measure μ has non-degenerate support. Applying Alexandrov's theorem, there exists a convex body L such that $\mu = \mu_L$. Cheng and Yau's lemma and Equations (13) and (14) imply that the inradii r_K and r_L of K and L are bounded from below by a constant r. Similarly, the circumradii R_K and R_L are bounded by a constant R. These constants r and R depend only on $\mathrm{rotund}(\mu_K)$ and $\mathcal{H}^{d-1}(\partial K)$. Now, by definition of the mixed volumes,

$$\begin{aligned}|\mathrm{V}(K) - \mathrm{V}_1(K,L)| &= \left|\int_{\mathcal{S}^{d-1}} (\mathrm{h}_K - h_L)\mathrm{d}\mu_K\right| \\ &\leq \left|\int_{\mathcal{S}^{d-1}} \mathrm{h}_K \mathrm{d}(\mu_K - \mu_L)\right| + \left|\int h_L \mathrm{d}(\mu_K - \mu_L)\right|.\end{aligned} \tag{15}$$

Since the stability theorem we are proving is up to translations, we can translate K and L if necessary. The circumradii R_K and R_L are bounded by R, and we therefore assume that K and L are included in the ball $\mathrm{B}(0, R)$. This means that the bodies $K' = \frac{1}{R}K$ and $L' = \frac{1}{R}L$ are included in the unit ball. Note also that $\mathrm{h}_{L'} = R\mathrm{h}_L$. Putting the definition of the convex-dual distance into Eq. (15), this gives

$$|\mathrm{V}(K) - \mathrm{V}_1(K,L)| \leq R\left|\int_{\mathcal{S}^{d-1}} \mathrm{h}_{K'}\mathrm{d}(\mu_K - \mu_L)\right| + R\left|\int h_{L'}\mathrm{d}(\mu_K - \mu_L)\right| \leq 2R\mathrm{d_C}(\mu_K, \mu_L).$$

The same inequality where L and K have been exchanged also holds, and this allows us to apply Theorem 3.6 with $\epsilon = 2R\mathrm{d_C}(\mu_K, \mu_L)$. Note that the constant that occurs in Eq. (12) depends on quantities that depend on $\mathrm{rotund}(K)$, $\mu_K(\mathcal{S}^{d-1}) = \mathcal{H}^d(\partial K)$ and d.

4. Random sampling

Let K be a convex body and μ_K its surface area measure. Note that by measuring normals only, one cannot determine the area of ∂K. Therefore, we assume that K has unit surface area, i.e. μ_K is a probability measure. We call *random normal measurements* a family of unit vectors $(\mathbf{n}_i)_{1\leq i\leq N}$ that are obtained by measuring the unit outer normal at N random independent locations on ∂K, whose distribution is given by the surface area on ∂K. Equivalently, the vectors $(\mathbf{n}_i)_{1\leq i\leq N}$ are obtained by i.i.d. sampling from the probability measure μ_K. The *empirical measure* associated to μ_K is therefore defined by the formula $\mu_{K,N} := \frac{1}{N}\sum_{i=1}^{N} \delta_{\mathbf{n}_i}$. The main result of the article is the following theorem.

Theorem 4.1. *Let K be a convex body with unit surface area. Given a desired probability $p \in (0,1)$, a desired error $\eta > 0$, and given*

$$N \geq \mathrm{const}\,(K,d) \cdot \eta^{\frac{d(1-d)}{2} - 2d} \log(1/p)$$

random normal measurements it is possible to construct a convex body L_N such that

$$\mathbb{P}\left(\min_{x \in \mathbb{R}^d} \mathrm{d_H}(x + K, L_N) \leq \eta\right) \geq 1 - p.$$

The exponents that we obtain are $N = \Omega(\eta^{-5})$ in dimension two and $N = \Omega(\eta^{-9})$ in dimension three, and are most likely not optimal.

4.1. **Zero-mean assumption.** Note that even if the mean of the the empirical measure $\mu_{K,N}$ will be close to zero with high probability, it will usually not be exactly zero. However, this is a necessary condition for the existence of a convex polytope L such that $\mu_L = \mu_{K,N}$. The following proposition shows that this equality can be enforced without perturbing μ_N too much in the sense of the convex-dual distance $\mathrm{d_C}$.

Proposition 4.2. *Given any convex body K with unit surface area, and any probability measure ν on $\mathcal{S}^{d-1}$, there exists a probability measure $\overline{\nu}$ on $\mathcal{S}^{d-1}$ with zero mean such that $\mathrm{d_C}(\overline{\nu}, \mu_K) \leq 3\mathrm{d_C}(\nu, \mu_K)$.*

Lemma 4.3. *Given any probability measure ν on $\mathcal{S}^{d-1}$ with mean m, there exists a probability measure $\overline{\nu}$ on $\mathcal{S}^{d-1}$ with zero mean such that $\mathrm{d_C}(\nu, \overline{\nu}) \leq 2\|m\|$.*

Proof. We only deal with the case of a probability measure on a finite set $\nu = \frac{1}{N}\sum_{1 \leq i \leq N} \delta_{x_i}$. The general case can be obtained using the density of these measures in the space of probability measures. Let m denote the mean of ν, i.e. $m = \frac{1}{N}\sum_{1 \leq i \leq N} x_i$. By convexity of $\mathrm{B}(0,1)$, the point m always lies inside the ball $\mathrm{B}(0,1)$. Moreover, by strict strict convexity of the ball, $\|m\| = 1$ occurs only when $\nu = \delta_m$. In this case, one can check that if $\overline{\nu}$ is the uniform probability measure on $\mathcal{S}^{d-1}$ then $\mathrm{d_C}(\nu, \overline{\nu}) \leq 2$. We will assume from now on that $\|m\| < 1$ and set $\overline{\nu} = \frac{1}{N}\sum_{1 \leq i \leq N} \lambda a_i \cdot \delta_{m_i}$, where

$$m_i = \frac{x_i - m}{\|x_i - m\|}, a_i = \|x_i - m\|, \lambda = \left(\frac{1}{N}\sum_{1 \leq i \leq N} a_i\right)^{-1}.$$

By construction, the measure $\overline{\nu}$ is a probability measure; and it has zero mean:

$$\frac{1}{N}\sum_{1 \leq i \leq N} \lambda a_i m_i = \frac{1}{N}\sum_{1 \leq i \leq N} \lambda a_i \frac{x_i - m}{a_i} = \left(\frac{1}{N}\lambda \sum_{1 \leq i \leq N} x_i\right) - \lambda m = 0.$$

Second, we want to bound the convex-dual distance between ν and $\overline{\nu}$. For that purpose, we consider a convex set M included in the unit ball $\mathrm{B}(0,1)$ and h_M its support function. We let $\hbar_M$ be the extension of the support function to $\mathbb{R}^d$ by the same formula $\hbar_M(x) = \sup_{p \in M} x \cdot p$. This function is positively homogeneous, i.e. for $\lambda > 0$,

$\hbar_M(\lambda v) = \lambda \hbar_M(v)$. We have:

$$\begin{aligned}\left|\int_{\mathcal{S}^{d-1}} \mathrm{h}_M(v)\mathrm{d}(\nu - \overline{\nu})(v)\right| &= \frac{1}{N}\left|\sum_{1\le i\le N} \mathrm{h}_M(x_i) - \sum_{1\le i\le N} \lambda a_i \mathrm{h}_M\left(\frac{x_i - m}{a_i}\right)\right| \\ &\le \frac{1}{N}\sum_{1\le i\le N}\left|\hbar_M(x_i) - \lambda a_i \hbar_M\left(\frac{x_i - m}{a_i}\right)\right| \\ &\le \frac{1}{N}\sum_{1\le i\le N}\left|\hbar_M(x_i) - \lambda \hbar_M\left(x_i - m\right)\right|\end{aligned} \tag{16}$$

From the first to the second line we used the triangle inequality, and from the second to the third line we used the homogeneity of $\hbar$. Finally, since M is contained in the unit ball, the function $\hbar_M$ is 1-Lipschitz (this follows from the same proof as in Lemma 2.3). Combining with $\hbar_M(0) = 0$, we get:

$$\begin{aligned}\left|\hbar_M(x_i) - \lambda \hbar_M\left(x_i - m\right)\right| &\le \left|\hbar_M(x_i) - \hbar_M\left(x_i - m\right)\right| + \left|(1-\lambda)\hbar_M(x_i - m)\right| \\ &\le \|m\| + |1-\lambda|\,\|x_i - m\|\end{aligned}$$

Summing these inequalities, and using the definition of λ gives us

$$\begin{aligned}\frac{1}{N}\sum_{1\le i\le N}\left|\hbar_M(x_i) - \lambda \hbar_M\left(x_i - m\right)\right| &\le \|m\| + |1-\lambda|\left(\frac{1}{N}\sum_{i=1}^{N}\|x_i - m\|\right) \\ &= \|m\| + \left|1 - \frac{1}{N}\sum_{i=1}^{N}\|x_i - m\|\right| \le 2\|m\|\end{aligned} \tag{17}$$

We conclude using the definition of the convex-dual distance and Eqs (16)–(17). $\square$

Proof of Proposition 4.2. We need to show that the mean m of the measure ν is not too far from zero. Given any point x on $\mathcal{S}^{d-1}$ and $K_x = \{x\}$ the convex set consisting of only x, one has $\mathrm{h}_{K_x}(v) := \max_{z\in K_x} z\cdot v = x\cdot v$. Therefore, using the definition of the convex-dual distance and the fact that μ_K has zero mean we obtain

$$\begin{aligned}\mathrm{d_C}(\mu_K,\nu) &\ge \left|\int_{\mathcal{S}^{d-1}} x\cdot v\mathrm{d}\mu_K(v) - \int_{\mathcal{S}^{d-1}} x\cdot v\mathrm{d}\nu(v))\right| \\ &= \left|\int_{\mathcal{S}^{d-1}} x\cdot v\mathrm{d}\nu(v)\right| = |m\cdot v|\,.\end{aligned}$$

Taking $x = m/\|m\|$ in this inequality proves that $\|m\|$ is bounded by $\mathrm{d_C}(\mu_K,\nu)$. We can then apply Lemma 4.3 to construct $\overline{\nu}$. Using the triangle inequality for $\mathrm{d_C}$ and $\mathrm{d_C}(\overline{\nu},\nu) \le 2\|m\|$, we get.

$$\mathrm{d_C}(\overline{\nu},\mu_K) \le \mathrm{d_C}(\overline{\nu},\nu) + \mathrm{d_C}(\nu,\mu_K) \le 2\|m\| + \mathrm{d_C}(\nu,\mu_K) \le 3\mathrm{d_C}(\nu,\mu_K). \qquad \square$$

4.2. **Convergence of the empirical measure.** We consider a probability measure μ on the unit sphere, and we denote by μ_N the empirical measure constructed from μ, i.e. $\mu_N = \frac{1}{N}\sum_{1\le i\le N}\delta_{X_i}$ where X_i are i.i.d random vectors with distribution μ. The following probabilistic statement determines the speed of convergence of μ_N to μ for the convex-dual distance.

Proposition 4.4. *Let μ be a probability measure on $\mathcal{S}^{d-1}$, and μ_N the corresponding empirical measure. Then, μ_N converges to μ for the convex dual distance with high probability. More precisely, for any positive $\epsilon \leq \mathrm{const}(d)$ and any N, the following inequality holds:*

$$\mathbb{P}\left[\mathrm{d_C}(\mu_N, \mu_K) \leq \epsilon\right] \geq 1 - 2\exp\left(\mathrm{const}(d)\epsilon^{\frac{1-d}{2}} - N\epsilon^2/2\right). \tag{18}$$

The proof of this proposition relies on the combination of a Theorem of Bronshtein [2, Theorem 5] with Chernoff's bound. Recall that the *ϵ-covering number* $\mathcal{N}(X, \epsilon)$ of a metric space X is the minimal number of closed balls of radius ϵ needed to cover X. Let C_1 be the set of convex bodies contained in the unit ball $\mathbb{R}^d$, endowed with the Hausdorff distance.

Theorem (Bronshtein)**.** *Assuming $\epsilon \leq \epsilon_d := 10^{-12}/(d-1)$, the following bound holds:*

$$\log_2(\mathcal{N}(\mathrm{C}_1, \epsilon)) \leq \mathrm{const}(d)\epsilon^{\frac{1-d}{2}}.$$

Proof of Proposition 4.4. By the theorem of Bronshtein, given any positive number ϵ smaller than ϵ_d, there exists n and n convex body $K_1, \dots, K_n$ included in the unit ball such that for any convex body $M \subseteq \mathrm{B}(0,1)$ one has $\min_{1\leq i\leq n} \mathrm{d_H}(K_i, M) \leq \epsilon$. Moreover, the number n can be chosen smaller than

$$n = \mathcal{N}(C_1, \epsilon) \leq \exp\left(\mathrm{const}(d)\epsilon^{\frac{1-d}{2}}\right). \tag{19}$$

We consider N i.i.d. random points $X_1, \dots, X_N$ on the unit sphere whose distribution is given by the measure μ. For a fixed i, the support function h_{K_i} is bounded by one by Lemma 2.3, and one can apply Hoeffding's inequality to the random variables $(\mathrm{h}_{K_i}(X_k))_{1\leq k\leq N}$. By definition of the empirical measure μ_N, this gives

$$\mathbb{P}\left(\left|\int_{\mathcal{S}^{d-1}} \mathrm{h}_{K_i}(x)\mathrm{d}\mu_N(x) - \int_{\mathcal{S}^{d-1}} \mathrm{h}_{K_i}\mathrm{d}\mu\right| \geq \epsilon\right) \leq 2\exp(-2N\epsilon^2). \tag{20}$$

Taking the union bound, we get

$$\mathbb{P}\left(\max_{1\leq i\leq n}\left|\int_{\mathcal{S}^{d-1}} \mathrm{h}_{K_i}(x)\mathrm{d}(\mu_N - \mu)(x)\right| \geq \epsilon\right) \leq 2n\exp(-2N\epsilon^2) \tag{21}$$

Now, given any convex body M included in the unit ball, there exists i in $\{1, \dots, n\}$ such that the distance $\|\mathrm{h}_M - \mathrm{h}_{K_i}\| = \mathrm{d_H}(M, K_i)$ is at most ϵ. Thus, for any probability measure ν on the sphere,

$$\left|\int_{\mathcal{S}^{d-1}} \mathrm{h}_{K_i}(x)\mathrm{d}\nu(x) - \int_{\mathcal{S}^{d-1}} \mathrm{h}_M\mathrm{d}\nu\right| \leq \left|\int_{\mathcal{S}^{d-1}} \|\mathrm{h}_{K_i} - \mathrm{h}_M\|_\infty \mathrm{d}\nu\right| \leq \epsilon,$$

and as a consequence,

$$\left|\int_{\mathcal{S}^{d-1}} \mathrm{h}_M(x)\mathrm{d}(\mu_N - \mu)(x)\right| \leq \max_{1\leq i\leq n}\left|\int_{\mathcal{S}^{d-1}} \mathrm{h}_{K_i}(x)\mathrm{d}(\mu_N - \mu)(x)\right| + 2\epsilon. \tag{22}$$

The combination of inequalities (21) and (22) imply that

$$\mathbb{P}(\mathrm{d}_C(\mu_N, \mu) \geq 3\epsilon) \leq 2\exp(\log(n) - 2N\epsilon^2) \tag{23}$$

Using the upper bound on n from Eq. (19) concludes the proof. $\square$

4.3. **Proof of Theorem 4.1.** We assume first that $\mathrm{d_C}(\mu_K, \mu_{K,N})$ is small enough, and more precisely that $\mathrm{d_C}(\mu_K, \mu_{K,N}) \leq \frac{1}{3}\epsilon_0$, where ϵ_0 is the constant given by Theorem 3.1. By Proposition 4.2 we can construct a probability measure $\overline{\mu}_{K,N}$ with zero mean such that

$$\mathrm{d_C}(\mu_K, \overline{\mu}_{K,N}) \leq 3\mathrm{d_C}(\mu_K, \mu_{K,N}) \leq \epsilon_0.$$

This allows us to apply Theorem 3.1 to the measure $\overline{\mu}_{K,N}$. There exists a convex body L_N whose surface area measure μ_{L_N} coincides with $\overline{\mu}_{K,N}$ and moreover,

$$\min_{x\in\mathbb{R}^d} \mathrm{d_H}(x + K, L_N) \leq c\mathrm{d_C}(\mu_K, \mu_{L_N})^{\frac{1}{d}} = 3^{\frac{1}{d}} c\mathrm{d_C}(\mu_K, \mu_{K,N})^{\frac{1}{d}}.$$

Therefore, using Proposition 4.4, and assuming $\epsilon \leq \frac{1}{3}\epsilon_0$, we have

$$\begin{aligned}\mathbb{P}\left[\min_{x\in\mathbb{R}^d} \mathrm{d_H}(x + K, L_N) \leq 3^{\frac{1}{d}} c\epsilon^{\frac{1}{d}}\right] &\geq \mathbb{P}\left[\mathrm{d_C}(\mu_K, \mu_{K,N}) \leq \epsilon\right] \\ &\geq 1 - 2\exp\left(\mathrm{const}(d)\epsilon^{\frac{1-d}{2}} - N\epsilon^2/2\right).\end{aligned}$$

Finally, we set $\eta = 3^{\frac{1}{d}} c\epsilon^{\frac{1}{d}}$, we get

$$\mathbb{P}\left[\min_{x\in\mathbb{R}^d} \mathrm{d_H}(x + K, L_N) \leq \eta\right] \geq 1 - 2\exp\left[C \cdot \left(\eta^{\frac{d(1-d)}{2}} - N\eta^{2d}\right)\right].$$

for some constant C that only depends on d and K, thus concluding the proof.

5. Special case: polyhedra

When the underlying convex body is a convex polyhedron, one can get much better probabilistic bounds on the speed of convergence. This model is quite simplistic, however, because of the assumptions that each of the measured normals must coincide with the one of normals of the underlying polyhedron. In particular, one cannot hope to extend this result to handle noise. The proof of this proposition relies on Theorem 2.1 of [12] and on a lemma of Devroye.

Proposition 5.1. *Let K be a convex polyhedron of $\mathbb{R}^d$ with k facets, non-empty interior and whose surface area $\mathcal{H}^{d-1}(\partial K)$ equals one. Then one can construct a convex polyhedron L_N such that*

$$\mathbb{P}(\min_{x\in\mathbb{R}^d} \mathrm{d_H}(x + K, L_N) \leq \eta) \geq 1 - p$$

from N random normal measurements with $N \geq \mathrm{const}(d, \mathrm{rotund}(\mu_K), k) \cdot \eta^{-2(d-1)} \log(1/p)$.

Proof of Proposition 5.1. The surface area measure of K can be written as $\mu_K = \sum_{1\leq i\leq k} a_i\delta_{\mathbf{n}_i}$ where the areas (a_i) sum to one. It is well-known that the empirical measure $\mu_{K,N}$ constructed from a finitely support probability measures such as μ_K converges to the probability measure μ_K in the total variation distance with high probability. For instance, using Lemma 3 in [5] we get

$$\mathbb{P}(\mathrm{d_{TV}}(\mu_K, \mu_{K,N}) \geq \epsilon) \leq 3\exp(-N\epsilon^2/25), \quad \text{assuming } \epsilon \geq \sqrt{20k/N}. \tag{24}$$

Now, let ν be an instance of $\mu_{K,N}$ such that $\mathrm{d_{TV}}(\mu_K, \nu) \leq \epsilon$. The measure ν can be written as $\nu = \sum_{1\leq i\leq k} b_i\delta_{\mathbf{n}_i}$, and the assumption that the total variation distance between μ_K and ν is at most ϵ can be rewritten as $\sum_{1\leq i\leq k} |a_i - b_i| \leq \epsilon$. The measure

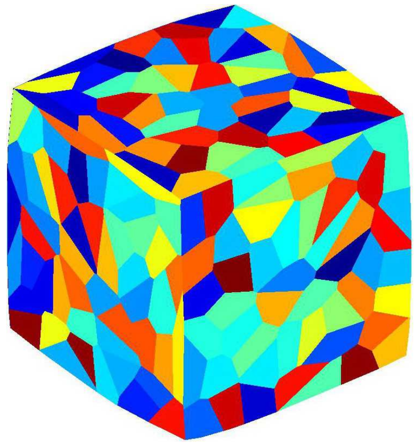

FIGURE 1. Reconstruction of a unit cube from 300 random normal measurements with a uniform noise of radius 0.05. The reconstruction is obtained using the variational approach proposed in [13].

ν does not necessarily have zero mean, but one can search for a perturbed measure $\overline{\nu} = \sum_{1\leq i\leq k} \overline{b}_i \delta_{\mathbf{n}_i}$ with zero mean. More precisely, we let

$$\overline{\nu} = \arg\min\{\mathrm{d}_{\mathrm{TV}}(\nu, \overline{\nu}); \mathrm{mean}(\overline{\nu}) = 0\}.$$

Solving this problem is equivalent to the minimization of a convex functional on a finite-dimensional subspace. Moreover, since $\mathrm{d}_{\mathrm{TV}}(\nu, \mu_K) \leq \epsilon$, we are sure that $\mathrm{d}_{\mathrm{TV}}(\nu, \overline{\nu}) \leq \epsilon$, so that $\mathrm{d}_{\mathrm{TV}}(\overline{\nu}, \mu_K) \leq 2\epsilon$. Finally, assuming ϵ small enough we have

$$\mathrm{d}_{\mathrm{C}}(\overline{\nu}, \mu_K) \leq \mathrm{const}(d)\mathrm{d}_{\mathrm{TV}}(\overline{\nu}, \mu_K) \leq 2\epsilon \leq \frac{1}{2}\mathrm{rotund}(\mu_K).$$

Using Lemma 3.4, this inequality ensures that $\mathrm{rotund}(\overline{\nu}) \geq \mathrm{rotund}(\mu_K)/2 > 0$. By Alexandrov's theorem, there exists a convex set L_N whose surface area measure μ_{L_N} coincides with $\overline{\nu}$, and whose inradius and circumradius can be bounded in term of the weak rotundity $\mathrm{rotund}(\mu_K)$. This allows us to apply Theorem 2.1 of [12] to the sets K and L_N to show that

$$\min_{x\in\mathbb{R}^d} \mathrm{d}_{\mathrm{H}}(K + x, L_N) \leq \mathrm{const}(d, \mathrm{rotund}(\mu_K))\epsilon^{\frac{1}{d-1}} \tag{25}$$

Combining Eqs (24) and (25), we get

$$\begin{aligned}\mathbb{P}\left(\min_{x\in\mathbb{R}^d} \mathrm{d}_{\mathrm{H}}(K + x, L_N) \geq \eta\right) &\leq \mathbb{P}(\mathrm{d}_{\mathrm{TV}}(\mu_K, \mu_{K,N}) \geq c \cdot \eta^{d-1}) \\ &\leq \exp(-c \cdot N\eta^{2(d-1)})\end{aligned}$$

where c depends on d and $\mathrm{rotund}(\mu_K)$). This probability becomes lower than p, and the assumption in Eq (24) is satisfied, as soon as

$$N \geq \mathrm{const}(d, \mathrm{rotund}(\mu_K), k) \cdot \eta^{-2(d-1)} \log(1/p).$$

□

6. Discussion

In this article, we introduced the convex-dual distance between surface area measures. This distance is weaker than the bounded-Lipschitz distance and is yet sufficient to control the Hausdorff distance between convex bodies in term of the distance between their surface area measures. This stability result has then been used to deduce probabilistic reconstruction results (Theorem 4.1). The main open problem consists in improving the exponent in the lower bound on the number of samples in this theorem.

What would happen if we would have used the bounded-Lipschitz distance in the probabilistic part of the proof of Theorem 4.1 instead of the convex-dual distance? The lower bound on the number N of necessary normal measurements to get a Hausdorff error of ϵ in the reconstruction would increase substantially:

$$N \geq \operatorname{const}(d, \operatorname{rotund}(\mu_K)) \cdot \eta^{d(1-d)-2d} \log(1/p). \tag{26}$$

In particular, the exponents would become $N = \Omega(\eta^{-6})$ in dimension two and $N = \Omega(\eta^{-12})$ in dimension three, compared to $N = \Omega(\eta^{-5})$ and $N = \Omega(\eta^{-9})$ with our analysis. This difference is due to the fact that the space BL_1 of functions on $\mathcal{S}^{d-1}$ that are 1-Lipschitz and bounded by one is *much larger* than the space C_1 of support function of convex sets included in the unit ball. More precisely,

$$\mathcal{N}(\mathrm{C}_1, \epsilon) = \Theta\left(\epsilon^{\frac{1-d}{2}}\right) \text{ while } \mathcal{N}(\mathrm{BL}_1, \epsilon) = \Theta\left(\epsilon^{1-d}\right),$$

where the constants in the $\Theta(.)$ notation only depend on the ambient dimension.

It is therefore tempting to pursue in this direction, and to try to consider a weaker dual distance between surface area measures, i.e. defined with an even smaller space of functions. This idea is not hopeless, as if one looks closely at the proofs of Theorem 3.1, Lemma 3.4 and Proposition 4.2, there are only a handful of probing functions that are used to control the Hausdorff distance between two convex bodies K and L as a function of their surface area measures μ_K and μ_L, and more precisely,

$$\mathrm{C}_{K,L} = \{\mathrm{h}_K, \mathrm{h}_L\} \cup \left\{s_u; u \in \mathcal{S}^{d-1}\right\} \cup \left\{\max(s_u, 0); u \in \mathcal{S}^{d-1}\right\} \cup \{\mathrm{h}_{\mathrm{B}(0,1)}\},$$

where $s_u : x \mapsto u \cdot x$. This set of function is *exponentially much smaller* than the set of support functions of convex bodies included in the unit ball:

$$\mathcal{N}(C_{K,L}, \epsilon) \simeq \operatorname{const}(d) \cdot \epsilon^{1-d} \ll \mathcal{N}(\mathrm{C}_1, \epsilon) \simeq \exp\left(\operatorname{const}(d) \cdot \epsilon^{\frac{1-d}{2}}\right).$$

However, turning this remark into an improvement of the probabilistic analysis seems quite challenging, because in the probabilistic setting, the second convex body L_N is reconstructed from random normal measurements and is itself random.

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