# The maximum number of faces of the Minkowski sum of two convex polytopes 

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#### Abstract

We derive tight expressions for the maximum number of $k$-faces, $0 \leq k \leq d-1$, of the Minkowski sum, $P_{1} \oplus P_{2}$, of two $d$-dimensional convex polytopes $P_{1}$ and $P_{2}$, as a function of the number of vertices of the polytopes.

For even dimensions $d \geq 2$, the maximum values are attained when $P_{1}$ and $P_{2}$ are cyclic $d$ polytopes with disjoint vertex sets. For odd dimensions $d \geq 3$, the maximum values are attained when $P_{1}$ and $P_{2}$ are $\left\lfloor\frac{d}{2}\right\rfloor$-neighborly $d$-polytopes, whose vertex sets are chosen appropriately from two distinct $d$-dimensional moment-like curves.


Key words: high-dimensional geometry, discrete geometry, combinatorial geometry, combinatorial complexity, Minkowski sum, convex polytopes
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## 1 Introduction

Given two $d$-dimensional polytopes, or simply $d$-polytopes, $P$ and $Q$, their Minkowski sum, $P \oplus Q$, is defined as the set $\{p+q \mid p \in P, q \in Q\}$. Minkowski sums are fundamental structures in both Mathematics and Computer Science. They appear in a variety of different subjects, including Combinatorial Geometry, Computational Geometry, Computer Algebra, Computer-Aided Design \& Solid Modeling, Motion Planning, Assembly Planning, Robotics (see [18, 4] and the references therein), and, more recently, Game Theory [15], Computational Biology [14] and Operations Research [20].

Despite their apparent importance, little is known about the worst-case complexity of Minkowski sums in dimensions four and higher. In two dimensions, the worst-case complexity of Minkowski sums is well understood. Given two convex polygons $P$ and $Q$ with $n$ and $m$ vertices, respectively, the maximum number of vertices and edges of $P \oplus Q$ is $n+m$ [2]. This result can be immediately generalized (e.g., by induction) to any number of summands. If $P$ is convex and $Q$ is non-convex (or vice versa), the worst-case complexity of $P \oplus Q$ is $\Theta(n m)$, while if both $P$ and $Q$ are non-convex the
complexity of their Minkowski sum can be as high as $\Theta\left(n^{2} m^{2}\right)$ [2]. When $P$ and $Q$ are 3-polytopes (embedded in the 3-dimensional Euclidean space), the worst-case complexity of $P \oplus Q$ is $\Theta(n m)$, if both $P$ and $Q$ are convex, and $\Theta\left(n^{3} m^{3}\right)$, if both $P$ and $Q$ are non-convex (e.g., see 3). For the intermediate cases, i.e., if only one of $P$ and $Q$ is convex, see [17].

Given two convex $d$-polytopes $P_{1}$ and $P_{2}$ in $\mathbb{E}^{d}, d \geq 2$, with $n_{1}$ and $n_{2}$ vertices, respectively, we can easily get a straightforward upper bound of $O\left(\left(n_{1}+n_{2}\right)^{\left\lfloor\frac{d+1}{2}\right\rfloor}\right)$ on the complexity of $P_{1} \oplus P_{2}$ by means of the following reduction: embed $P_{1}$ and $P_{2}$ in the hyperplanes $\left\{x_{d+1}=0\right\}$ and $\left\{x_{d+1}=1\right\}$ of $\mathbb{E}^{d+1}$, respectively; then the weighted Minkowski sum $(1-\lambda) P_{1} \oplus \lambda P_{2}=\left\{(1-\lambda) p_{1}+\lambda p_{2} \mid p_{1} \in\right.$ $\left.P_{1}, p_{2} \in P_{2}\right\}, \lambda \in(0,1)$, of $P_{1}$ and $P_{2}$ is the intersection of the convex hull, $C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$, of $P_{1}$ and $P_{2}$ with the hyperplane $\left\{x_{d+1}=\lambda\right\}$. The embedding and reduction described above are essentially what are known as the Cayley embedding and Cayley trick, respectively [11]. From this reduction it is obvious that the worst-case complexity of $(1-\lambda) P_{1} \oplus \lambda P_{2}$ is bounded from above by the complexity of $C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$, which is $O\left(\left(n_{1}+n_{2}\right)^{\left\lfloor\frac{d+1}{2}\right\rfloor}\right)$. Furthermore, the complexity of the weighted Minkowski sum of $P_{1}$ and $P_{2}$ is independent of $\lambda$, in the sense that for any value of $\lambda \in(0,1)$ the polytopes we get by intersecting $C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$ with $\left\{x_{d+1}=\lambda\right\}$ are combinatorially equivalent. In fact, since $P_{1} \oplus P_{2}$ is nothing but $\frac{1}{2} P_{1} \oplus \frac{1}{2} P_{2}$ scaled by a factor of 2 , the complexity of the weighted Minkowski sum of two convex polytopes is the same as the complexity of their unweighted Minkowski sum. Very recently (cf. [12]), the authors of this paper have considered the problem of computing the asymptotic worst-case complexity of the convex hull of a fixed number $r$ of convex $d$-polytopes lying on $r$ parallel hyperplanes of $\mathbb{E}^{d+1}$. A direct corollary of our results is a tight bound on the worst-case complexity of the Minkowski sum of two convex $d$-polytopes for all odd dimensions $d \geq 3$, which refines the "obvious" upper bound mentioned above. More precisely, we have shown that for $d \geq 3$ odd, the worst-case complexity of $P_{1} \oplus P_{2}$ is in $\Theta\left(n_{1} n_{2}^{\left\lfloor\frac{d}{2}\right\rfloor}+n_{2} n_{1}^{\left\lfloor\frac{d}{2}\right\rfloor}\right)$, which is a refinement of the obvious upper bound when $n$ and $m$ asymptotically differ.

In terms of exact bounds on the number of faces of the Minkowski sum of two polytopes, results are known only when the two summands are convex. Besides the trivial bound for convex polygons (2-polytopes), mentioned in the previous paragraph, the first result of this nature was shown by Gritzmann and Sturmfels [9]: given $r$ polytopes $P_{1}, P_{2}, \ldots, P_{r}$ in $\mathbb{E}^{d}$, with a total of $n$ non-parallel edges, the number of $l$-faces, $f_{l}\left(P_{1} \oplus P_{2} \oplus \cdots \oplus P_{r}\right)$, of $P_{1} \oplus P_{2} \oplus \cdots \oplus P_{r}$ is bounded from above by $2\binom{n}{l} \sum_{j=0}^{d-1-l}\binom{n-l-1}{j}$. This bound is attained when the polytopes $P_{i}$ are zonotopes, and their generating edges are in general position.

Regarding bounds as a function of the number of vertices or facets of the summands, Fukuda and Weibel [5] have shown that, given two 3-polytopes $P_{1}$ and $P_{2}$ in $\mathbb{E}^{3}$, the number of $k$-faces of $P_{1} \oplus P_{2}, 0 \leq k \leq 2$, is bounded from above as follows:

$$
\begin{align*}
& f_{0}\left(P_{1} \oplus P_{2}\right) \leq n_{1} n_{2}, \\
& f_{1}\left(P_{1} \oplus P_{2}\right) \leq 2 n_{1} n_{2}+n_{1}+n_{2}-8,  \tag{1}\\
& f_{2}\left(P_{1} \oplus P_{2}\right) \leq n_{1} n_{2}+n_{1}+n_{2}-6 .
\end{align*}
$$

where $n_{j}$ is the number of vertices of $P_{j}, j=1,2$. Weibel [18] has also derived similar expressions in terms of the number of facets $m_{j}$ of $P_{j}, j=1,2$, namely:

$$
\begin{aligned}
& f_{0}\left(P_{1} \oplus P_{2}\right) \leq 4 m_{1} m_{2}-8 m_{1}-8 m_{2}+16, \\
& f_{1}\left(P_{1} \oplus P_{2}\right) \leq 8 m_{1} m_{2}-17 m_{1}-17 m_{2}+40, \\
& f_{2}\left(P_{1} \oplus P_{2}\right) \leq 4 m_{1} m_{2}-9 m_{1}-9 m_{2}+26 .
\end{aligned}
$$

All these bounds are tight. Fogel, Halperin and Weibel [3] have further generalized some of these bounds in the case of $r$ summands. More precisely, they have shown that given $r$ 3-polytopes
$P_{1}, P_{2}, \ldots, P_{r}$ in $\mathbb{E}^{3}$, where $P_{j}$ has $m_{j} \geq d+1$ facets, the number of facets of the Minkowski sum $P_{1} \oplus P_{2} \oplus \cdots \oplus P_{r}$ is bounded from above by

$$
\sum_{1 \leq i<j \leq r}\left(2 m_{i}-5\right)\left(2 m_{j}-5\right)+\sum_{i=1}^{r} m_{i}+\binom{r}{2},
$$

and this bound is tight.
For dimensions four and higher there are no results that relate the worst-case number of $k$-faces of the Minkowski sum of two of more convex polytopes with the number of facets of the summands. There are, however, bounds on the number of $k$-faces of the Minkowski sum of convex polytopes, as a function of the number of vertices of the summands. More precisely, Fukuda and Weibel [5] have shown that the number of vertices of the Minkowski sum of $r d$-polytopes $P_{1}, \ldots, P_{r}$, where $r \leq d-1$ and $d \geq 2$, is bounded from above by $\prod_{i=1}^{r} n_{i}$, where $n_{i}$ is the number of vertices of $P_{i}$, and this bound is tight. On the other hand, for $r \geq d$ this bound cannot be attained: Sanyal [16] has shown that for $r \geq d, f_{0}\left(P_{1} \oplus \cdots \oplus P_{r}\right)$ is bounded from above by $\left(1-\frac{1}{(d+1)^{d}}\right) \prod_{i=1}^{r} n_{i}$, which is, clearly, strictly smaller than $\prod_{i=1}^{r} n_{i}$. For higher-dimensional faces, i.e., for $k \geq 1$, Fukuda and Weibel [5] have shown that the number of $k$-faces of the Minkowski sum of $r$ polytopes is bounded as follows:

$$
\begin{equation*}
f_{k}\left(P_{1} \oplus P_{2} \oplus \cdots \oplus P_{r}\right) \leq \sum_{\substack{1 \leq s_{i} \leq n_{i} \\ s_{1}+\cdots+s_{r}=k+r}} \prod_{i=1}^{r}\binom{n_{i}}{s_{i}}, \quad 0 \leq k \leq d-1 \tag{2}
\end{equation*}
$$

where $n_{i}$ is the number of vertices of $P_{i}$. These bounds are tight for $d \geq 4, r \leq\left\lfloor\frac{d}{2}\right\rfloor$, and for all $k$ with $0 \leq k \leq\left\lfloor\frac{d}{2}\right\rfloor-r$, i.e., for the cases where both the number of summands and the dimension of the faces considered is small.

We end our discussion of the previous work related to this paper by some results presented in a technical report of Weibel [19]. In this report, Weibel considers the case where the number of summands, $r$, is at least as big as the dimension of the polytopes. In this setting he gives a relation between the number of $k$-faces of the Minkowski sum of $r$ polytopes, $r \geq d \geq 2$, and the number of $k$-faces of the Minkowski sum of subsets of the original set of $r$ polytopes, that are of size at most $d-1$. In more detail, if we have $r d$-polytopes $P_{1}, P_{2}, \ldots, P_{r}$ in $\mathbb{E}^{d}$, where $r \geq d$, that are in general position, then the following relation holds for any $k$ with $0 \leq k \leq d-1$ :

$$
\begin{equation*}
f_{k}\left(P_{1} \oplus P_{2} \oplus \cdots \oplus P_{r}\right)=\alpha+\sum_{j=1}^{d-1}(-1)^{d-1-j}\binom{r-1-j}{d-1-j} \sum_{S \in \mathcal{C}_{j}^{r}}\left(f_{k}\left(P_{S}\right)-\alpha\right) \leq \sum_{S \in \mathcal{C}_{d-1}^{r}} f_{k}\left(P_{S}\right), \tag{3}
\end{equation*}
$$

where $\mathcal{C}_{j}^{r}$ is the family of subsets of $\{1,2, \ldots, r\}$ of cardinality $j, P_{S}$ is the Minkowski sum of the polytopes in $S$, and, finally, $\alpha=2$ if $k=0$ and $d$ is odd, $\alpha=0$, otherwise. Weibel then uses this relation to derive upper bounds on the number of vertices of the Minkowski sum of $r d$-polytopes in $\mathbb{E}^{d}$, when $r \geq d$. An important qualitative consequence of relation (3) is that, when we consider Minkowski sums of $d$-polytopes in $\mathbb{E}^{d}$, essentially it really suffices to consider up to $d-1$ summands. To pose it otherwise, if we know good/tight upper bounds for the worst-case number of $k$-faces of the Minkowski sum of $d-1$ polytopes, then we immediately know upper bounds for the $k$-faces of the Minkowski sum of $r \geq d$ polytopes in $\mathbb{E}^{d}$. If we are to strive for tight exact bounds, however, there is still something to be done in this case, due to the fact that the sum in (3) is an alternating sum: not only do we have to find sets of $r \geq d$ polytopes such that any subset of them of size at most $d-1$ yields the worst possible number of faces, but also prove that such a configuration does indeed maximize the right-hand size of the equality in (3).

In this paper, we extend previous results on the exact maximum number of faces of the Minkowski sum of two convex $d$-polytopes 1 . More precisely, we show that given two $d$-polytopes $P_{1}$ and $P_{2}$ in $\mathbb{E}^{d}$ with $n_{1} \geq d+1$ and $n_{2} \geq d+1$ vertices, respectively, the maximum number of $k$-faces of the Minkowski sum $P_{1} \oplus P_{2}$ is bounded from above as follows:

$$
f_{k-1}\left(P_{1} \oplus P_{2}\right) \leq f_{k}\left(C_{d+1}\left(n_{1}+n_{2}\right)\right)-\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k+1-i}\left(\binom{n_{1}-d-2+i}{i}+\binom{n_{2}-d-2+i}{i}\right)
$$

where $1 \leq k \leq d$, and $C_{d}(n)$ stands for the cyclic $d$-polytope with $n$ vertices. The expressions above are shown to be tight for any $d \geq 2$ and for all $1 \leq k \leq d$, and, clearly, match with the corresponding expressions for two and three dimensions (cf. rel. (11)), as well as the expressions in (2) for $r=2$ and for all $0 \leq k \leq\left\lfloor\frac{d}{2}\right\rfloor-2$.

To prove the upper bounds we use the embedding in one dimension higher already stated above. Given the $d$-polytopes $P_{1}$ and $P_{2}$ in $\mathbb{E}^{d}$, we embed $P_{1}$ and $P_{2}$ in the hyperplanes $\left\{x_{d+1}=0\right\}$ and $\left\{x_{d+1}=1\right\}$ of $\mathbb{E}^{d+1}$. We consider the convex hull $P=C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$ and argue that, for the purposes of the worst-case upper bounds, it suffices to consider the case where $P$ is simplicial, except possibly for its two facets $P_{1}$ and $P_{2}$. We concentrate on the set $\mathcal{F}$ of faces of $P$ that are neither faces of $P_{1}$ nor faces of $P_{2}$. The reason that we focus on $\mathcal{F}$ is that there is a bijection between the $k$-faces of $\mathcal{F}$ and the $(k-1)$-faces of $P_{1} \oplus P_{2}, 1 \leq k \leq d$, and, thus, deriving upper bounds of the number of $(k-1)$-faces of $P_{1} \oplus P_{2}$ reduces to deriving upper bounds for the number of $k$-faces of $\mathcal{F}$. We then proceed in a manner analogous to that used by McMullen [13] to prove the Upper Bound Theorem for polytopes. We consider the $f$-vector $\boldsymbol{f}(\mathcal{F})$ of $\mathcal{F}$, from this we define the $h$-vector $\boldsymbol{h}(\mathcal{F})$ of $\mathcal{F}$, and continue by:
(i) deriving Dehn-Sommerville-like equations for $\mathcal{F}$, expressed in terms of the elements of $\boldsymbol{h}(\mathcal{F})$ and the $g$-vectors of the boundary complexes of $P_{1}$ and $P_{2}$, and,
(ii) establishing a recurrence relation for the elements of $\boldsymbol{h}(\mathcal{F})$.

From the latter, we inductively compute upper bounds on the elements of $\boldsymbol{h}(\mathcal{F})$, which we combine with the Dehn-Sommerville-like equations for $\mathcal{F}$, to get refined upper bounds for the "left-most half" of the elements of $\boldsymbol{h}(\mathcal{F})$, i.e., for the values $h_{k}(\mathcal{F})$ with $k>\left\lfloor\frac{d+1}{2}\right\rfloor$. We then establish our upper bounds by computing $\boldsymbol{f}(\mathcal{F})$ from $\boldsymbol{h}(\mathcal{F})$.

To prove the lower bounds we distinguish between even and odd dimensions. In even dimensions $d \geq 2$, we show that the $k$-faces of the Minkowski sum of any two cyclic $d$-polytopes with $n_{1}$ and $n_{2}$ vertices, respectively, whose vertex sets are distinct, attain the upper bounds we have proved. In odd dimensions $d \geq 3$, the construction that establishes the tightness of our bounds is more intricate. We consider the $(d-1)$-dimensional moment curve $\gamma(t)=\left(t, t^{2}, t^{3}, \ldots, t^{d-1}\right), t>0$, and define two vertex sets $V_{1}$ and $V_{2}$ with $n_{1}$ and $n_{2}$ vertices on $\gamma(t)$, respectively. We then embed $V_{1}$ (resp., $V_{2}$ ) on the hyperplane $\left\{x_{2}=0\right\}$ (resp., $\left\{x_{1}=0\right\}$ ) of $\mathbb{E}^{d}$ and perturb the $x_{2}$-coordinates (resp., $x_{1}$-coordinates) of the vertices in $V_{1}$ (resp., $V_{2}$ ), so that the polytope $P_{1}$ (resp., $P_{2}$ ) defined as the convex hull, in $\mathbb{E}^{d}$, of the vertices in $V_{1}$ (resp., $V_{2}$ ) is full-dimensional. We then argue that by appropriately choosing the vertex sets $V_{1}$ and $V_{2}$, the number of $k$-faces of the Minkowski sum $P_{1} \oplus P_{2}$ attains its maximum possible value. At a very high/qualitative level, the appropriate choice we refer to above amounts to choosing $V_{1}$ and $V_{2}$ so that the parameter values on $\gamma(t)$ of the vertices in $V_{1}$ and $V_{2}$, lie within two disjoint intervals of $\mathbb{R}$ that are far away from each other.

The structure of the rest of the paper is as follows. In Section 2 we formally give various definitions, and recall a version of the Upper Bound Theorem for polytopes that will be useful later

[^0]in the paper. In Section 3 we define what we call bineighborly polytopal complexes and prove some properties associated with them. The reason that we introduce this new notion is the fact that the tightness of our upper bounds is shown to be equivalent to requiring that the $(d+1)$-polytope $P=C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$, defined above, is bineighborly. In Section 4 we prove our upper bounds on the number of faces of the Minkowski sum of two polytopes. In Section 5 we describe our lower bound constructions and show that these constructions attain the upper bounds proved in Section 4. We conclude the paper with Section 6, where we summarize our results, and state open problems and directions for future work.

## 2 Definitions and preliminaries

A convex polytope, or simply polytope, $P$ in $\mathbb{E}^{d}$ is the convex hull of a finite set of points $V$ in $\mathbb{E}^{d}$, called the vertex set of $P$. A polytope $P$ can equivalently be described as the intersection of all the closed halfspaces containing $V$. A face of $P$ is the intersection of $P$ with a hyperplane for which the polytope is contained in one of the two closed halfspaces delimited by the hyperplane. The dimension of a face of $P$ is the dimension of its affine hull. A $k$-face of $P$ is a $k$-dimensional face of $P$. We consider the polytope itself as a trivial $d$-dimensional face; all the other faces are called proper faces. We use the term d-polytope to refer to a polytope the trivial face of which is $d$-dimensional. For a $d$-polytope $P$, the 0 -faces of $P$ are its vertices, the 1 -faces of $P$ are its edges, the $(d-2)$-faces of $P$ are called ridges, while the ( $d-1$ )-faces are called facets. For $0 \leq k \leq d$ we denote by $f_{k}(P)$ the number of $k$-faces of $P$. Note that every $k$-face $F$ of $P$ is also a $k$-polytope whose faces are all the faces of $P$ contained in $F$. A $k$-simplex in $\mathbb{E}^{d}, k \leq d$, is the convex hull of any $k+1$ affinely independent points in $\mathbb{E}^{d}$. A polytope is called simplicial if all its proper faces are simplices. Equivalently, $P$ is simplicial if for every vertex $v$ of $P$ and every face $F \in P, v$ does not belong to the affine hull of the vertices in $F \backslash\{v\}$.

A polytopal complex $\mathcal{C}$ is a finite collection of polytopes in $\mathbb{E}^{d}$ such that (i) $\emptyset \in \mathcal{C}$, (ii) if $P \in \mathcal{C}$ then all the faces of $P$ are also in $\mathcal{C}$ and (iii) the intersection $P \cap Q$ for two polytopes $P$ and $Q$ in $\mathcal{C}$ is a face of both $P$ and $Q$. The dimension $\operatorname{dim}(\mathcal{C})$ of $\mathcal{C}$ is the largest dimension of a polytope in $\mathcal{C}$. A polytopal complex is called pure if all its maximal (with respect to inclusion) faces have the same dimension. In this case the maximal faces are called the facets of $\mathcal{C}$. We use the term $d$-complex to refer to a polytopal complex whose maximal faces are $d$-dimensional (i.e., the dimension of $\mathcal{C}$ is d). A polytopal complex is simplicial if all its faces are simplices. Finally, a polytopal complex $\mathcal{C}^{\prime}$ is called a subcomplex of a polytopal complex $\mathcal{C}$ if all faces of $\mathcal{C}^{\prime}$ are also faces of $\mathcal{C}$.

One important class of polytopal complexes arise from polytopes. More precisely, a $d$-polytope $P$, together with all its faces and the empty set, form a $d$-complex, denoted by $\mathcal{C}(P)$. The only maximal face of $\mathcal{C}(P)$, which is clearly the only facet of $\mathcal{C}(P)$, is the polytope $P$ itself. Moreover, all proper faces of $P$ form a pure ( $d-1$ )-complex, called the boundary complex $\mathcal{C}(\partial P)$, or simply $\partial P$ of $P$. The facets of $\partial P$ are just the facets of $P$, and its dimension is, clearly, $\operatorname{dim}(\partial P)=\operatorname{dim}(P)-1=d-1$.

Given a $d$-polytope $P$ in $\mathbb{E}^{d}$, consider $F$ a facet of $P$, and call $H$ the supporting hyperplane of $F$ (with respect to $P$ ). For an arbitrary point $p$ in $\mathbb{E}^{d}$, we say that $p$ is beyond (resp., beneath) the facet $F$ of $P$, if $p$ lies in the open halfspace of $H$ that does not contain $P$ (resp., contains the interior of $P$ ). Furthermore, we say that an arbitrary point $v^{\prime}$ is beyond the vertex $v$ of $P$ if for every facet $F$ of $P$ containing $v, v^{\prime}$ is beyond $F$, while for every facet $F$ of $P$ not containing $v, v^{\prime}$ is beneath $F$. For a vertex $v$ of $P$, the star of $v$, denoted by $\operatorname{star}(v, P)$, is the polytopal complex of all faces of $P$ that contain $v$, and their faces. The link of $v$, denoted by $\operatorname{link}(v, P)$, is the subcomplex of $\operatorname{star}(v, P)$ consisting of all the faces of $\operatorname{star}(v, P)$ that do not contain $v$.

Definition 1 ([21, Remark 8.3]). Let $\mathcal{C}$ be a pure simplicial polytopal d-complex. A shelling $\mathbb{S}(\mathcal{C})$ of $\mathcal{C}$ is a linear ordering $F_{1}, F_{2}, \ldots, F_{s}$ of the facets of $\mathcal{C}$ such that for all $1<j \leq s$ the intersection, $F_{j} \cap\left(\bigcup_{i=1}^{j-1} F_{i}\right)$, of the facet $F_{j}$ with the previous facets is non-empty and pure $(d-1)$-dimensional. In other words, for every $i<j$ there exists some $\ell<j$ such that the intersection $F_{i} \cap F_{j}$ is contained in $F_{\ell} \cap F_{j}$, and such that $F_{\ell} \cap F_{j}$ is a facet of $F_{j}$.

Every polytopal complex that has a shelling is called shellable. In particular, the boundary complex of a polytope of always shellable. (cf. [1). Consider a pure shellable simplicial polytopal complex $\mathcal{C}$ and let $\mathbb{S}(\mathcal{C})=\left\{F_{1}, \ldots, F_{s}\right\}$ be a shelling order of its facets. The restriction $R\left(F_{j}\right)$ of a facet $F_{j}$ is the set of all vertices $v \in F_{j}$ such that $F_{j} \backslash\{v\}$ is contained in one of the earlier facets ${ }^{2}$ The main observation here is that when we construct $\mathcal{C}$ according to the shelling $\mathbb{S}(\mathcal{C})$, the new faces at the $j$-th step of the shelling are exactly the vertex sets $G$ with $R\left(F_{j}\right) \subseteq G \subseteq F_{j}$ (cf. [21, Section 8.3]). Moreover, notice that $R\left(F_{1}\right)=\emptyset$ and $R\left(F_{i}\right) \neq R\left(F_{j}\right)$ for all $i \neq j$.

The $f$-vector $\boldsymbol{f}(P)=\left(f_{-1}(P), f_{0}(P), \ldots, f_{d-1}(P)\right)$ of a $d$-polytope $P$ (or its boundary complex $\partial P)$ is defined as the $(d+1)$-dimensional vector consisting of the number $f_{k}(P)$ of $k$-faces of $P,-1 \leq$ $k \leq d-1$, where $f_{-1}(P)=1$ refers to the empty set. The $h$-vector $\boldsymbol{h}(P)=\left(h_{0}(P), h_{1}(P), \ldots, h_{d}(P)\right)$ of a $d$-polytope $P$ (or its boundary complex $\partial P$ ) is defined as the $(d+1)$-dimensional vector, where $h_{k}(P):=\sum_{i=0}^{k}(-1)^{k-i}\binom{d-i}{d-k} f_{i-1}(P), 0 \leq k \leq d$. It is easy to verify from the defining equations of the $h_{k}(P)$ 's that the elements of $\boldsymbol{f}(P)$ determine the elements of $\boldsymbol{h}(P)$ and vice versa.

For simplicial polytopes, the number $h_{k}(P)$ counts the number of facets of $P$ in a shelling of $\partial P$, whose restriction has size $k$; this number is independent of the particular shelling chosen (cf. [21, Theorem 8.19]). Moreover, the elements of $\boldsymbol{f}(P)$ (or, equivalently, $\boldsymbol{h}(P)$ ) are not linearly independent; they satisfy the so called Dehn-Sommerville equations, which can be written in a very concise form as: $h_{k}(P)=h_{d-k}(P), 0 \leq k \leq d$. An important implication of the existence of the Dehn-Sommerville equations is that if we know the face numbers $f_{k}(P)$ for all $0 \leq k \leq\left\lfloor\frac{d}{2}\right\rfloor-1$, we can determine the remaining face numbers $f_{k}(P)$ for all $\left\lfloor\frac{d}{2}\right\rfloor \leq k \leq d-1$. Both the $f$-vector and $h$-vector of a simplicial $d$-polytope are related to the so called $g$-vector. For a simplicial $d$ polytope $P$ its $g$-vector is the $\left(\left\lfloor\frac{d}{2}\right\rfloor+1\right)$-dimensional vector $\boldsymbol{g}(P)=\left(g_{0}(P), g_{1}(P), \ldots, g_{\left\lfloor\frac{d}{2}\right\rfloor}(P)\right)$, where $g_{0}(P)=1$, and $g_{k}(P)=h_{k}(P)-h_{k-1}(P), 1 \leq k \leq\left\lfloor\frac{d}{2}\right\rfloor$ (see also [21, Section 8.6]). Using the convention that $h_{d+1}(P)=0$, we can actually extend the definition of $g_{k}(P)$ for all $0 \leq k \leq d+1$, while using the Dehn-Sommerville equations for $P$ yields: $g_{d+1-k}(P)=-g_{k}(P), 0 \leq k \leq d+1$. We can then express $\boldsymbol{f}(P)$ in terms of $\boldsymbol{g}(P)$ as follows:

$$
f_{k-1}(P)=\sum_{j=0}^{\left\lfloor\frac{d}{2}\right\rfloor} g_{j}(P)\left(\binom{d+1-j}{d+1-k}-\binom{j}{d+1-k}\right), \quad 0 \leq k \leq d+1
$$

As a final note for this section, the Upper Bound Theorem for polytopes can be expressed in terms of their $g$-vector:

Corollary 2 ([21, Corollary 8.38]). We consider simplicial d-polytopes $P$ of fixed dimension $d$ and fixed number of vertices $n=g_{1}(P)+d+1 . \boldsymbol{f}(P)$ has its componentwise maximum if and only if all the components of $\boldsymbol{g}(P)$ are maximal, with

$$
\begin{equation*}
g_{k}(P)=\binom{g_{1}(P)+k-1}{k}=\binom{n-d-2+k}{k} . \tag{4}
\end{equation*}
$$

Also, $f_{k-1}(P)$ is maximal if an only if $g_{i}(P)$ is maximal for all $i$ with $i \leq \min \left\{k,\left\lfloor\frac{d}{2}\right\rfloor\right\}$.

[^1]
## 3 Bineighborly polytopal complexes

Let $\mathcal{C}$ be a $d$-complex, and let $V$ be the vertex set of $\mathcal{C}$. Let $\left\{V_{1}, V_{2}\right\}$ be a partition of $V$ and define $\mathcal{C}_{1}$ (resp., $\mathcal{C}_{2}$ ) to be the subcomplex of $\mathcal{C}$ consisting of all the faces of $\mathcal{C}$ whose vertices are vertices in $V_{1}$ (resp., $V_{2}$ ). We start with a useful definition:

Definition 3. Let $\mathcal{C}$ be a d-complex. We say that $\mathcal{C}$ is $\left(k, V_{1}\right)$-bineighborly if we can partition the vertex set $V$ of $\mathcal{C}$ into two non-empty subsets $V_{1}$ and $V_{2}=V \backslash V_{1}$ such that for every $\emptyset \subset S_{j} \subseteq V_{j}$, $j=1,2$, with $\left|S_{1}\right|+\left|S_{2}\right| \leq k$, the vertices of $S_{1} \cup S_{2}$ define a face of $\mathcal{C}$ (of dimension $\left|S_{1}\right|+\left|S_{2}\right|-1$ ).

We introduce the notion of bineighborly polytopal complexes because they play an important role when considering the maximum complexity of the Minkowski sum of two $d$-polytopes $P_{1}$ and $P_{2}$. As we will see in the upcoming section, the number of $(k-1)$-faces of $P_{1} \oplus P_{2}$ is maximal for all $1 \leq k \leq l, l \leq\left\lfloor\frac{d-1}{2}\right\rfloor$, if and only if the convex hull $P$ of $P_{1}$ and $P_{2}$, when embedded in the hyperplanes $\left\{x_{d+1}=0\right\}$ and $\left\{x_{d+1}=1\right\}$ of $\mathbb{E}^{d+1}$, respectively, is $\left(l+1, V_{1}\right)$-bineighborly, where $V_{1}$ stands for the vertex set of $P_{1}$. Even more interestingly, in any odd dimension $d \geq 3$, the number of $k$-faces of $P_{1} \oplus P_{2}$ is maximized for all $0 \leq k \leq d-1$, if and only if $P$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, V_{1}\right)$-bineighborly. In the rest of this section we highlight some properties of bineighborly polytopal complexes that will be useful in the upcoming sections.

A direct consequence of our definition is the following: suppose that $\mathcal{C}$ is a $\left(l, V_{1}\right)$-bineighborly polytopal complex, and let $F$ be a $k$-face $F$ of $\mathcal{C}, 1 \leq k<l$, such that at least one vertex of $F$ is in $V_{1}$ and at least one vertex of $F$ is in $V_{2}$; then $F$ is simplicial (i.e., $F$ is a $k$-simplex). Another immediate consequence of Definition 3 is that a $k$-neighborly $d$-complex is also $\left(k, V^{\prime}\right)$-bineighborly for every non-empty subset $V^{\prime}$ of its vertex set:

Corollary 4. Let $\mathcal{C}$ be a $k$-neighborly d-complex, with vertex set $V$. Then, for every $V^{\prime}$, with $\emptyset \subset V^{\prime} \subset V, \mathcal{C}$ is $\left(k, V^{\prime}\right)$-bineighborly.

It is easy to see that if a $d$-complex $\mathcal{C}$ is $\left(k, V_{1}\right)$-bineighborly, then it is $(k-1)$-neighborly, as the following straightforward lemma suggests.

Lemma 5. Let $\mathcal{C}$ be a ( $k, V_{1}$ )-bineighborly d-complex, $k \geq 2$. Then $\mathcal{C}$ is $(k-1)$-neighborly.
Proof. Let $S$ be a non-empty subset of $V$ of size $k-1$. Consider the following, mutually exclusive cases:
(i) $S$ consists of vertices of both $V_{1}$ and $V_{2}$. In this case choose a vertex $v \in V \backslash\left(V_{1} \cup V_{2}\right)$.
(ii) $S$ consists of vertices of $V_{1}$ only. In this case choose a vertex $v \in V_{2}$.
(iii) $S$ consists of vertices of $V_{2}$ only. In this case choose a vertex $v \in V_{1}$.

Consider the vertex set $S^{\prime}=S \cup\{v\}$, where $v$ is defined as above. $S^{\prime}$ has size $k$, and has at least one vertex from $V_{1}$ and at least one vertex from $V_{2}$. Since $\mathcal{C}$ is $\left(k, V_{1}\right)$-bineighborly, the vertex set $S^{\prime}$ defines a $(k-1)$-face $F_{S^{\prime}}$ of $\mathcal{C}$, which is, in fact, a $(k-1)$-simplex. This implies that $S$ is a $(k-2)$-face of $F_{S^{\prime}}$, and thus a $(k-2)$-face of $\mathcal{C}$. In other words, for every vertex subset $S$ of $\mathcal{C}$ of size $k-1, S$ defines a $(k-2)$-face of $\mathcal{C}$, i.e., $\mathcal{C}$ is $(k-1)$-neighborly.

The following lemma is in some sense the reverse of Lemma 5 .
Lemma 6. Let $\mathcal{C}$ be a $\left(k, V_{1}\right)$-bineighborly d-complex, and let its two subcomplexes $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ be $k$-neighborly. Then $\mathcal{C}$ is also $k$-neighborly.

Proof. Let $S$ be a non-empty subset of $V$ of size $k$. Consider the following, mutually exclusive cases:
(i) $S$ consists of vertices of both $V_{1}$ and $V_{2}$. Then, since $\mathcal{C}$ is $\left(k, V_{1}\right)$-bineighborly, $S$ defines a ( $k-1$ )-face of $\mathcal{C}$.
(ii) $S$ consists of vertices of $V_{j}$ only, $j=1,2$. Since $\mathcal{C}_{j}$ is $k$-neighborly, $S$ defines a $(k-1)$-face of $\mathcal{C}_{j}$. However, $\mathcal{C}_{j}$ is a subcomplex of $\mathcal{C}$, which further implies that $S$ is also a face of $\mathcal{C}$.

Hence, for every vertex subset $S$ of $V$ of size $k, S$ defines a $(k-1)$-face of $\mathcal{C}$, i.e., $\mathcal{C}$ is $k$-neighborly.
Consider again a $d$-complex $\mathcal{C}$ with vertex set $V$. As above, partition $V$ into two subsets $V_{1}$ and $V_{2}$, and let $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ be the corresponding subcomplexes of $\mathcal{C}$. Finally, let $\mathcal{B}$ be the set of faces of $\mathcal{C}$ that are not faces of either $\mathcal{C}_{1}$ or $\mathcal{C}_{2}$. We end this section with the following lemma that gives tight upper bounds for the number of faces in $\mathcal{B}$. In what follows, we denote by $n_{j}$ the cardinality of $V_{j}$, $j=1,2$.

Lemma 7. The number of $(k-1)$-faces of $\mathcal{B}$ is bounded from above as follows:

$$
\begin{equation*}
f_{k-1}(\mathcal{B}) \leq \sum_{j=1}^{k-1}\binom{n_{1}}{j}\binom{n_{2}}{k-j}=\binom{n_{1}+n_{2}}{k}-\binom{n_{1}}{k}-\binom{n_{2}}{k}, \quad 1 \leq k \leq d \tag{5}
\end{equation*}
$$

where equality holds if and only if $\mathcal{C}$ is $\left(k, V_{1}\right)$-bineighborly.
Proof. The case $k=1$ is trivial. We have $f_{0}(\mathcal{B})=0=\binom{n_{1}+n_{2}}{1}-\binom{n_{1}}{1}-\binom{n_{2}}{1}$, since $\mathcal{B}$ does not contain any 0 -faces of $\mathcal{C}$ : the 0 -faces of $\mathcal{C}$, i.e., the vertices of $\mathcal{C}$, are either vertices of $\mathcal{C}_{1}$ or $\mathcal{C}_{2}$.

Let $k \geq 2$, and denote by $V_{F}$ the subset of $V$ defining a face $F \in \mathcal{B}$. Define $\varphi_{k-1}: \mathcal{B} \rightarrow 2^{V}$ to be the mapping that maps a $(k-1)$-face $F \in \mathcal{B}$ to a subset $V_{F}^{\prime}$ of $V_{F}$, of size $k$, such that:
(1) $V_{F}^{\prime}$ is $(k-1)$-dimensional, and
(2) $V_{F}^{\prime} \cap V_{j} \neq \emptyset, j=1,2$.

The mapping $\varphi_{k-1}$ is well defined in the sense that such a subset $V_{F}^{\prime}$ always exists. We are going to show this by induction on $k$. For $k=2$, simply choose $V_{F}^{\prime}=\left\{v_{1}, v_{2}\right\}$, where $v_{1} \in V_{F} \cap V_{1}$ and $v_{2} \in V_{F} \cap V_{2}$. Suppose that our claim holds for $k \geq 2$, i.e., for any ( $k-1$ )-face $F$ of $\mathcal{B}$, there exists a subset $V_{F}^{\prime}$ of $V_{F}$ of size $k$, such that $V_{F}^{\prime}$ is $(k-1)$-dimensional, and $V_{F}^{\prime} \cap V_{j} \neq \emptyset, j=1,2$. We wish to show that this is also true for $k+1$. Indeed, let $F$ be a $k$-face of $\mathcal{B}$. If $F$ is defined by $k+1$ vertices (i.e., $F$ is simplicial), $V_{F}^{\prime}$ is simply $V_{F}$. Clearly, $V_{F}$ is $k$-dimensional, and $V_{F} \cap V_{j} \neq \emptyset$, $j=1,2$, since $F$ is a $k$-face of $\mathcal{B}$. Otherwise, suppose $F$ is defined by more than $k+1$ vertices, i.e., $\left|V_{F}\right|>k+1$. Consider the $(k-1)$-faces of $F$ : at least one of these faces has to be a face in $\mathcal{B}$ (since, otherwise, $F$ would not have been a face of $\mathcal{B}$, but rather a face of either $\mathcal{C}_{1}$ or $\mathcal{C}_{2}$ ), and let $F^{\prime}$ be such a $(k-1)$-face of $F$. By the induction hypothesis there exists a subset $V_{F^{\prime}}^{\prime}$ of $V_{F^{\prime}}$ of size $k$, such that $V_{F^{\prime}}^{\prime}$ is $(k-1)$-dimensional and $V_{F^{\prime}}^{\prime} \cap V_{j} \neq \emptyset, j=1,2$. But then there exists a vertex $v \in V_{F} \backslash V_{F^{\prime}}^{\prime}$, such that the set $V_{F}^{\prime}=V_{F^{\prime}}^{\prime} \cup\{v\}$ is $k$-dimensional (if this is not the case, then $F$ would have been $(k-1)$-dimensional, which contradicts the fact that $F$ is a $k$-face of $\mathcal{B})$. The set $V_{F}^{\prime}$ is the set we were looking for: $V_{F}^{\prime}$ has size $k+1$ (since $\left|V_{F^{\prime}}^{\prime}\right|=k$ ), $V_{F}^{\prime}$ is $k$-dimensional (we just argued that), and $V_{F}^{\prime} \cap V_{j} \neq \emptyset, j=1,2$ (this holds for $V_{F^{\prime}}^{\prime}$, and, thus, it holds for $V_{F}^{\prime}$ as well).

We argue that the mapping $\varphi_{k-1}$ is an injection from the faces of $\mathcal{B}$ to the subsets of size $k$ of $V$ which contain elements from both $V_{1}$ and $V_{2}$. To this end, consider two ( $k-1$ )-faces $F_{1}$ and $F_{2}$ of $\mathcal{B}$, such that $F_{1} \neq F_{2}$, and assume that $\varphi_{k-1}\left(F_{1}\right)=\varphi_{k-1}\left(F_{2}\right)$. Since $\varphi_{k-1}\left(F_{1}\right)=\varphi_{k-1}\left(F_{2}\right)$, we have that $V_{F_{1}}^{\prime}=V_{F_{2}}^{\prime}$ and both $V_{F_{1}}^{\prime}$ and $V_{F_{2}}^{\prime}$ are $(k-1)$-dimensional. Therefore, the intersection
$F_{1} \cap F_{2}$ is not only a face $F$ of both $F_{1}$ and $F_{2}$, but also contains all vertices in $V_{F_{1}}^{\prime}=V_{F_{2}}^{\prime}$. Since $V_{F_{1}}^{\prime}$, or $V_{F_{2}}^{\prime}$, is $(k-1)$-dimensional, $F$ is a $(k-1)$-face of both $F_{1}$ and $F_{2}$. On the other hand, the only $(k-1)$-face of either $F_{1}$, or $F_{2}$, is $F_{1}$, or $F_{2}$, respectively. Hence $F=F_{1}$ and $F=F_{2}$, that is $F_{1}=F_{2}$, which contradicts our assumption that $F_{1} \neq F_{2}$. Summarizing, we have that if $F_{1} \neq F_{2}$, then $\varphi_{k-1}\left(F_{1}\right) \neq \varphi_{k-1}\left(F_{2}\right)$, i.e., the mapping $\varphi_{k-1}$ is an injection.

Having established that $\varphi_{k-1}: \mathcal{B} \rightarrow 2^{V}$ is an injection, we proceed with the upper bound and equality claim of the lemma. The number of the subsets of $V$ of size $k$, that have at least one vertex from both $V_{1}$ and $V_{2}$ is precisely $\sum_{1 \leq j \leq k-1}\binom{n_{1}}{j}\binom{n_{2}}{k-j}$, which is equal to $\binom{n_{1}+n_{2}}{k}-\binom{n_{1}}{k}-\binom{n_{2}}{k}$, according to Vandermonde's convolution identity. This gives the upper bound. Furthermore, notice that the injection $\varphi_{k-1}$ becomes a bijection if and only if for every non-empty subset $S_{1}$ of $V_{1}$ and every non-empty subset $S_{2}$ of $V_{2}$, where $\left|S_{1}\right|+\left|S_{2}\right|=k$, the vertex set $S_{1} \cup S_{2}$ defines a $(k-1)$-face of $\mathcal{C}$. In other words, equality in (5) can only hold if and only if $\mathcal{C}$ is ( $k, V_{1}$ )-bineighborly.

Combining Lemma 5 with Lemma 7 we deduce that, if the inequality in Lemma 7 holds as equality for some $l$, then we also have $f_{k-1}(\mathcal{B})=\binom{n_{1}+n_{2}}{k}-\binom{n_{1}}{k}-\binom{n_{2}}{k}$ for all $k$ with $1 \leq k \leq l-1$.

## 4 Upper bounds

Let $P_{1}$ and $P_{2}$ be two $d$-polytopes in $\mathbb{E}^{d}$, with $n_{1}$ and $n_{2}$ vertices, respectively. The Minkowski sum $P_{1} \oplus P_{2}$ of $P_{1}$ and $P_{2}$ is the $d$-polytope $P_{1} \oplus P_{2}=\left\{p+q \mid p \in P_{1}, q \in P_{2}\right\}$, whereas their weighted Minkowski sum is defined as $(1-\lambda) P_{1} \oplus \lambda P_{2}=\left\{(1-\lambda) p+\lambda q \mid p \in P_{1}, q \in P_{2}\right\}$, where $\lambda \in(0,1)$. Let us embed $P_{1}$ (resp., $P_{2}$ ) in the hyperplane $\Pi_{1}$ (resp., $\Pi_{2}$ ) of $\mathbb{E}^{d+1}$ with equation $\left\{x_{d+1}=0\right\}$ (resp., $\left\{x_{d+1}=1\right\}$ ). Then the weighted Minkowski sum $(1-\lambda) P_{1} \oplus \lambda P_{2}$ is the $d$-polytope we get when intersecting $C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$ with the hyperplane $\left\{x_{d+1}=\lambda\right\}$ (see Fig. (1). From this reduction


Figure 1: The $d$-polytopes $P_{1}$ and $P_{2}$ are embedded in the hyperplanes $\Pi_{1}=\left\{x_{d+1}=0\right\}$ and $\Pi_{2}=\left\{x_{d+1}=0\right\}$ of $\mathbb{E}^{d+1}$. The polytope $\tilde{P}$ is the intersection of $C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$ with the hyperplane $\tilde{\Pi}=\left\{x_{d+1}=\lambda\right\}$.
it is evident that the weighted Minkowski sum $(1-\lambda) P_{1} \oplus \lambda P_{2}, \lambda \in(0,1)$, does not really depend on the specific value of $\lambda$, in the sense that the weighted Minkowski sums of $P_{1}$ and $P_{2}$ for two different $\lambda$ values are combinatorially equivalent. Furthermore, the weighted Minkowski sum of $P_{1}$ and $P_{2}$ is also combinatorially equivalent to the unweighted Minkowski sum $P_{1} \oplus P_{2}$, since $P_{1} \oplus P_{2}$ is nothing but $\frac{1}{2} P_{1} \oplus \frac{1}{2} P_{2}$, scaled by a factor of 2 . In view of these observations, in the rest of the paper we focus on the sum $P_{1} \oplus P_{2}$, with the understanding that our results carry over to the weighted Minkowski sum $(1-\lambda) P_{1} \oplus \lambda P_{2}$, for any $\lambda \in(0,1)$.

As in the previous paragraph, let $\Pi_{1}$ and $\Pi_{2}$ be the hyperplanes $\left\{x_{d+1}=0\right\}$ and $\left\{x_{d+1}=1\right\}$, and let $\tilde{\Pi}$ be a hyperplane in $\mathbb{E}^{d+1}$ parallel and in-between $\Pi_{1}$ and $\Pi_{2}$. Consider two $d$-polytopes $P_{1}$ and $P_{2}$ embedded in $\mathbb{E}^{d+1}$, and in the hyperplanes $\Pi_{1}$ and $\Pi_{2}$, respectively, and call $P$ the convex hull $C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$. Karavelas and Tzanaki [12, Lemma 2] have shown that the vertices of $P_{1}$ and $P_{2}$ can be perturbed in such a way that:
(i) the vertices of $P_{1}^{\prime}$ and $P_{2}^{\prime}$ remain in $\Pi_{1}$ and $\Pi_{2}$, respectively, and both $P_{1}^{\prime}$ and $P_{2}^{\prime}$ are simplicial,
(ii) $P^{\prime}=C H_{d+1}\left(\left\{P_{1}^{\prime}, P_{2}^{\prime}\right\}\right)$ is also simplicial, except possibly the facets $P_{1}^{\prime}$ and $P_{2}^{\prime}$, and
(iii) the number of vertices of $P_{1}^{\prime}$ and $P_{2}^{\prime}$ is the same as the number of vertices of $P_{1}$ and $P_{2}$, respectively, whereas $f_{k}(P) \leq f_{k}\left(P^{\prime}\right)$ for all $k \geq 1$,
where $P_{1}^{\prime}$ and $P_{2}^{\prime}$ are the polytopes in $\Pi_{1}$ and $\Pi_{2}$ we get after perturbing the vertices of $P_{1}$ and $P_{2}$, respectively. In view of this result, it suffices to consider the case where both $P_{1}, P_{2}$ and their convex hull $P=C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$ are simplicial complexes (except possibly the facets $P_{1}$ and $P_{2}$ of $P$ ). In the rest of this section, we consider that this is the case: $P$ is considered simplicial, with the possible exception of its two facets $P_{1}$ and $P_{2}$. Let $\mathcal{F}$ be the set of proper faces of $P$ having non-empty intersection with $\tilde{\Pi}$. Note that $\tilde{P}=P \cap \tilde{\Pi}$ is a $d$-polytope, which is, in general, non-simplicial, and whose proper non-trivial faces are intersections of the form $F \cap \tilde{\Pi}$ where $F \in \mathcal{F}$. As we have already observed above, $\tilde{P}$ is combinatorially equivalent to the Minkowski sum $P_{1} \oplus P_{2}$. Furthermore,

$$
\begin{equation*}
f_{k-1}\left(P_{1} \oplus P_{2}\right)=f_{k-1}(\tilde{P})=f_{k}(\mathcal{F}), \quad 1 \leq k \leq d \tag{6}
\end{equation*}
$$

The rest of this section is devoted to deriving upper bounds for $f_{k}(\mathcal{F})$, which, by relation (6), become upper bounds for $f_{k-1}\left(P_{1} \oplus P_{2}\right)$.

Let $\mathcal{K}$ be the polytopal complex whose faces are all the faces of $\mathcal{F}$, as well as the faces of $P$ that are subfaces of faces in $\mathcal{F}$. It is easy to see that the $d$-faces of $\mathcal{K}$ are exactly the $d$-faces of $\mathcal{F}$, and, thus, $\mathcal{K}$ is a pure simplicial $d$-complex, with the $d$-faces of $\mathcal{F}$ being the facets of $\mathcal{K}$. Moreover, the set of $k$-faces of $\mathcal{K}$ is the disjoint union of the sets of $k$-faces of $\mathcal{F}, \partial P_{1}$ and $\partial P_{2}$. This implies:

$$
\begin{equation*}
f_{k}(\mathcal{K})=f_{k}(\mathcal{F})+f_{k}\left(\partial P_{1}\right)+f_{k}\left(\partial P_{2}\right), \quad-1 \leq k \leq d \tag{7}
\end{equation*}
$$

where $f_{d}\left(\partial P_{j}\right)=0, j=1,2$, and conventionally we set $f_{-1}(\mathcal{F})=-1$.
Let $y_{1}$ (resp., $y_{2}$ ) be a point below $\Pi_{1}$ (resp., above $\Pi_{2}$ ), such that the vertices of $P_{1}$ (resp., $P_{2}$ ) are the only vertices of $P$ visible from $y_{1}\left(\right.$ resp., $\left.y_{2}\right)$ (see Fig. 2). To achieve this, we choose $y_{1}$ (resp., $y_{2}$ ) to be a point beyond the facet $P_{1}$ (resp., $P_{2}$ ) of $P$, and beneath every other facet of $P$. Let $Q$ be the $(d+1)$-polytope that is the convex hull of the vertices of $P_{1}, P_{2}, y_{1}$ and $y_{2}$. Observe that the faces of $\partial P$ (and thus all faces of $\mathcal{F}$ ), except for the facets $P_{1}$ and $P_{2}$ of $\partial P$, are all faces of the boundary complex $\partial Q$. To see that, notice that a supporting hyperplane $H_{F}$ for a facet $F \in P$, with $F \neq P_{1}, P_{2}$, is also a supporting hyperplane for $Q$. Indeed, the vertices of $F$ are vertices of $Q$ different from $y_{1}$ and $y_{2}$ and thus, every vertex of $P$ that is not a vertex of $F$ strictly satisfies all hyperplane inequalities for $P$. Also, by construction, the points $y_{1}$ and $y_{2}$ strictly satisfy


Figure 2: The polytope $Q$ is created by adding two vertices $y_{1}$ and $y_{2}$. The vertex $y_{1}$ (resp., $y_{2}$ ) is below $P_{1}$ (resp., above $P_{2}$ ), and is visible by the vertices of $P_{1}$ (resp., $P_{2}$ ) only.
all hyperplane inequalities apart from those for $\Pi_{1}$ and $\Pi_{2}$, respectively. Since $H_{F}$ is a hyperplane other than $\Pi_{1}$ and $\Pi_{2}$ we deduce that all vertices of $P$, as well as $y_{1}$ and $y_{2}$, lie on the same halfspace defined by $H_{F}$, and therefore $H_{F}$ supports $Q$. The faces of $Q$ that are not faces of $\mathcal{F}$ are the faces in the star $\mathcal{S}_{1}$ of $y_{1}$ and the star $\mathcal{S}_{2}$ of $y_{2}$. To verify this, consider a $k$-face $F$ of $P_{1}$, and let $F^{\prime}$ be a face in $\mathcal{F}$ that contains $F$. Let $H^{\prime}$ be a supporting hyperplane of $F^{\prime}$ with respect to $P$. Tilt $H^{\prime}$ until it hits the point $y_{1}$, while keeping $H^{\prime}$ incident to $F^{\prime}$, and call $H^{\prime \prime}$ this tilted hyperplane. $H^{\prime \prime}$ is a supporting hyperplane for $y_{1}$ and the vertex set of $P_{1}$, and thus is a supporting hyperplane for $Q$. The same argument can be applied for $\operatorname{star}\left(y_{2}, Q\right)$. In fact, the boundary complex $\partial P_{1}$ of $P_{1}$ (resp., $\partial P_{2}$ of $P_{2}$ ) is nothing but the link of $y_{1}$ (resp., $y_{2}$ ) in $Q$.

It is easy to realize that the set of $k$-faces of $\partial Q$ is the disjoint union of the $k$-faces of $\mathcal{F}, \mathcal{S}_{1}$ and $\mathcal{S}_{2}$. This implies that:

$$
\begin{equation*}
f_{k}(\partial Q)=f_{k}(\mathcal{F})+f_{k}\left(\mathcal{S}_{1}\right)+f_{k}\left(\mathcal{S}_{2}\right), \quad 0 \leq k \leq d, \tag{8}
\end{equation*}
$$

where $f_{0}(\mathcal{F})=0$. The $k$-faces of $\partial Q$ in $\mathcal{S}_{j}$ are either $k$-faces of $\partial P_{j}$ or $k$-faces defined by $y_{j}$ and a $(k-1)$-face of $\partial P_{j}$. In fact, there exists a bijection between the $(k-1)$-faces of $\partial P_{j}$ and the $k$-faces of $\mathcal{S}_{j}$ containing $y_{j}$. Hence, we have, for $j=1,2$ :

$$
\begin{equation*}
f_{k}\left(\mathcal{S}_{j}\right)=f_{k}\left(\partial P_{j}\right)+f_{k-1}\left(\partial P_{j}\right), \quad 0 \leq k \leq d \tag{9}
\end{equation*}
$$

where $f_{-1}\left(\partial P_{j}\right)=1$ and $f_{d}\left(\partial P_{j}\right)=0$. Combining relations (8) and (9), we get:

$$
\begin{equation*}
f_{k}(\partial Q)=f_{k}(\mathcal{F})+f_{k}\left(\partial P_{1}\right)+f_{k-1}\left(\partial P_{1}\right)+f_{k}\left(\partial P_{2}\right)+f_{k-1}\left(\partial P_{2}\right), \quad 0 \leq k \leq d \tag{10}
\end{equation*}
$$

We call $\mathcal{K}_{j}, j=1,2$, the subcomplex of $\partial Q$ consisting of either faces of $\mathcal{K}$ or faces of $\mathcal{S}_{j} . \mathcal{K}_{j}$ is a pure simplicial $d$-complex the facets of which are either facets in the star $\mathcal{S}_{j}$ of $y_{j}$ or facets of $\mathcal{K}$. Furthermore, $\mathcal{K}_{j}$ is shellable. To see this first notice that $\partial Q$ is shellable ( $Q$ is a polytope). Consider a line shelling $F_{1}, F_{2}, \ldots, F_{s}$ of $\partial Q$ that shells star $\left(y_{2}, \partial Q\right)$ last, and let $F_{\lambda+1}, F_{\lambda+2}, \ldots, F_{s}$ be the facets of $\partial Q$ that correspond to $\mathcal{S}_{2}$. Trivially, the subcomplex of $\partial Q$, the facets of which are $F_{1}, F_{2}, \ldots, F_{\lambda}$, is shellable; however, this subcomplex is nothing but $\mathcal{K}_{1}$. The argument for $\mathcal{K}_{2}$ is analogous.

Notice that $Q$ is a simplicial $(d+1)$-polytope, while $\mathcal{K}, \mathcal{K}_{1}$ and $\mathcal{K}_{2}$ are simplicial $d$-complexes; hence their $h$-vectors are well defined. More precisely:

$$
\begin{equation*}
h_{k}(\mathcal{Y})=\sum_{i=0}^{k}(-1)^{k-i}\binom{d+1-i}{d+1-k} f_{i-1}(\mathcal{Y}), \quad 0 \leq k \leq d+1, \tag{11}
\end{equation*}
$$

where $\mathcal{Y}$ stands for either $\partial Q, \mathcal{K}, \mathcal{K}_{1}$ or $\mathcal{K}_{2}$. We define the $f$-vector of $\mathcal{F}$ to be the $(d+2)$-vector $\boldsymbol{f}(\mathcal{F})=\left(f_{-1}(\mathcal{F}), f_{0}(\mathcal{F}), \ldots, f_{d}(\mathcal{F})\right)$, where recall that $f_{-1}(\mathcal{F})=-1$, and from this we can also define the $(d+2)$-vector $\boldsymbol{h}(\mathcal{F})=\left(h_{0}(\mathcal{F}), h_{1}(\mathcal{F}), \ldots, h_{d+1}(\mathcal{F})\right)$, where

$$
\begin{equation*}
h_{k}(\mathcal{F})=\sum_{i=0}^{k}(-1)^{k-i}\binom{d+1-i}{d+1-k} f_{i-1}(\mathcal{F}), \quad 0 \leq k \leq d+1 . \tag{12}
\end{equation*}
$$

We call this vector the $h$-vector of $\mathcal{F}$. As for polytopal complexes and polytopes, the $f$-vector of $\mathcal{F}$ defines the $h$-vector of $\mathcal{F}$ and vice versa. In particular, solving the defining equations (12) of the elements of $\boldsymbol{h}(\mathcal{F})$ in terms of the elements of $\boldsymbol{f}(\mathcal{F})$ we get:

$$
\begin{equation*}
f_{k-1}(\mathcal{F})=\sum_{i=0}^{d+1}\binom{d+1-i}{k-i} h_{i}(\mathcal{F}), \quad 0 \leq k \leq d+1 \tag{13}
\end{equation*}
$$

The next lemma associates the elements of $\boldsymbol{h}(\partial Q), \boldsymbol{h}(\mathcal{K}), \boldsymbol{h}\left(\mathcal{K}_{1}\right), \boldsymbol{h}\left(\mathcal{K}_{2}\right), \boldsymbol{h}(\mathcal{F}), \boldsymbol{h}\left(\partial P_{1}\right)$ and $\boldsymbol{h}\left(\partial P_{2}\right)$. The last among the relations in the lemma can be thought of as the analogue of the Dehn-Sommerville equations for $\mathcal{F}$.
Lemma 8. For all $0 \leq k \leq d+1$ we have:

$$
\begin{align*}
h_{k}(\partial Q) & =h_{k}(\mathcal{F})+h_{k}\left(\partial P_{1}\right)+h_{k}\left(\partial P_{2}\right),  \tag{14}\\
h_{k}(\mathcal{K}) & =h_{k}(\mathcal{F})+g_{k}\left(\partial P_{1}\right)+g_{k}\left(\partial P_{2}\right),  \tag{15}\\
h_{k}\left(\mathcal{K}_{j}\right) & =h_{k}(\mathcal{K})+h_{k-1}\left(\partial P_{j}\right),  \tag{16}\\
h_{d+1-k}(\mathcal{F}) & =h_{k}(\mathcal{F})+g_{k}\left(\partial P_{1}\right)+g_{k}\left(\partial P_{2}\right) . \tag{17}
\end{align*}
$$

Proof. Let $\mathcal{Y}$ denote either $\mathcal{F}$ or a pure simplicial subcomplex of $\partial Q$. We define the operator $\mathcal{S}_{k}(\cdot ; \delta, \nu)$ whose action on $\mathcal{Y}$ is as follows:

$$
\begin{equation*}
\mathcal{S}_{k}(\mathcal{Y} ; \delta, \nu)=\sum_{i=1}^{\delta}(-1)^{k-i}\binom{\delta-i}{\delta-k} f_{i-\nu}(\mathcal{Y}) . \tag{18}
\end{equation*}
$$

It is easy to verify $3^{3}$ that if $\mathcal{Y}$ is $\delta$-dimensional (this includes the case $\mathcal{Y} \equiv \mathcal{F}$ ), then

$$
\begin{equation*}
\mathcal{S}_{k}(\mathcal{Y} ; \delta, 1)=h_{k}(\mathcal{Y})-(-1)^{k}\binom{\delta}{\delta-k} f_{-1}(\mathcal{Y}) . \tag{19}
\end{equation*}
$$

[^2]while if $\mathcal{Y}$ is $(\delta-1)$-dimensional, then
\[

$$
\begin{align*}
& \mathcal{S}_{k}(\mathcal{Y} ; \delta, 1)=h_{k}(\mathcal{Y})-h_{k-1}(\mathcal{Y})-(-1)^{k}\binom{\delta}{\delta-k} f_{-1}(\mathcal{Y}), \text { and }  \tag{20}\\
& \mathcal{S}_{k}(\mathcal{Y} ; \delta, 2)=h_{k-1}(\mathcal{Y}) . \tag{21}
\end{align*}
$$
\]

Applying the operator $\mathcal{S}_{k}(\cdot ; d+1,1)$ to $\partial Q$ and using relation (10) we get:

$$
\begin{align*}
\mathcal{S}_{k}(\partial Q ; d+1,1)= & \mathcal{S}_{k}(\mathcal{F} ; d+1,1)+\mathcal{S}_{k}\left(\partial P_{1} ; d+1,1\right)+\mathcal{S}_{k}\left(\partial P_{1} ; d+1,2\right)  \tag{22}\\
& +\mathcal{S}_{k}\left(\partial P_{2} ; d+1,1\right)+\mathcal{S}_{k}\left(\partial P_{2} ; d+1,2\right)
\end{align*}
$$

Substituting in (22), using relations (19)-(21), we get:

$$
\begin{aligned}
h_{k}(\partial Q)-(-1)^{k} & \binom{d+1}{d+1-k} f_{-1}(\partial Q)=\left[h_{k}(\mathcal{F})-(-1)^{k}\binom{d+1}{d+1-k} f_{-1}(\mathcal{F})\right] \\
& +\left[h_{k}\left(\partial P_{1}\right)-h_{k-1}\left(\partial P_{1}\right)-(-1)^{k}\binom{d+1}{d+1-k} f_{-1}\left(\partial P_{1}\right)\right]+h_{k-1}\left(\partial P_{1}\right) \\
& +\left[h_{k}\left(\partial P_{2}\right)-h_{k-1}\left(\partial P_{2}\right)-(-1)^{k}\binom{d+1}{d+1-k} f_{-1}\left(\partial P_{2}\right)\right]+h_{k-1}\left(\partial P_{2}\right) .
\end{aligned}
$$

Given that $f_{-1}(\partial Q)=f_{-1}\left(\partial P_{1}\right)=f_{-1}\left(\partial P_{2}\right)=1$, and $f_{-1}(\mathcal{F})=-1$, the above equality simplifies to relation (14).

Recall that the set of $k$-faces of $\mathcal{K}$ is the disjoint union of the $k$-faces of $\mathcal{F}$, the $k$-faces of $\partial P_{1}$, and the $k$-faces of $\partial P_{2}$. Applying the operator $\mathcal{S}_{k}(\cdot ; d+1,1)$ to $\mathcal{K}$, and using relation (77) we get:

$$
h_{k}(\mathcal{K})=h_{k}(\mathcal{F})+h_{k}\left(\partial P_{1}\right)-h_{k-1}\left(\partial P_{1}\right)+h_{k}\left(\partial P_{2}\right)-h_{k-1}\left(\partial P_{2}\right), \quad 0 \leq k \leq d+1
$$

which reduces to relation (15) if we replace the difference $h_{k}(\cdot)-h_{k-1}(\cdot)$ by the corresponding element of $\boldsymbol{g}\left(P_{j}\right)$.

The $k$-faces of $\mathcal{K}_{j}, j=1,2$, are either $k$-faces of $\mathcal{K}$ or $k$-faces of the star $\mathcal{S}_{j}$ of $y_{j}$ that contain $y_{j}$. The latter faces are in one-to-one correspondence with the $(k-1)$-faces of $\partial P_{j}$, i.e., we get:

$$
\begin{equation*}
f_{k}\left(\mathcal{K}_{j}\right)=f_{k}(\mathcal{K})+f_{k-1}\left(\partial P_{j}\right), \quad 0 \leq k \leq d . \tag{23}
\end{equation*}
$$

Once again, applying the operator $\mathcal{S}_{k}(\cdot ; d+1,1)$ to $\mathcal{K}_{j}$, and using relation (23) we get relation (16).
We end the proof of this lemma by proving relations (17). Since $Q$ is a simplicial ( $d+1$ )-polytope, and $P_{1}, P_{2}$ are simplicial $d$-polytopes, the Dehn-Sommerville equations for these polytopes hold. More precisely:

$$
\begin{array}{rlrl}
h_{d+1-k}(\partial Q) & =h_{k}(\partial Q), & 0 \leq k \leq d+1, &  \tag{24}\\
h_{d-k}\left(\partial P_{j}\right) & =h_{k}\left(\partial P_{j}\right), & 0 \leq k \leq d, \quad j=1,2 .
\end{array}
$$

Combining the above relations with (14) we get, for all $0 \leq k \leq d+1$ :

$$
\begin{equation*}
h_{d+1-k}(\mathcal{F})+h_{d+1-k}\left(\partial P_{1}\right)+h_{d+1-k}\left(\partial P_{2}\right)=h_{k}(\mathcal{F})+h_{k}\left(\partial P_{1}\right)+h_{k}\left(\partial P_{2}\right), \tag{25}
\end{equation*}
$$

or, equivalently:

$$
\begin{equation*}
h_{d+1-k}(\mathcal{F})+h_{k-1}\left(\partial P_{1}\right)+h_{k-1}\left(\partial P_{2}\right)=h_{k}(\mathcal{F})+h_{k}\left(\partial P_{1}\right)+h_{k}\left(\partial P_{2}\right), \tag{26}
\end{equation*}
$$

which finally gives:

$$
h_{d+1-k}(\mathcal{F})=h_{k}(\mathcal{F})+g_{k}\left(\partial P_{1}\right)+g_{k}\left(\partial P_{2}\right)
$$

In the equations above, $g_{0}\left(\partial P_{j}\right)=-g_{d+1}\left(\partial P_{j}\right)=1, j=1,2$.

Recall that the main goal in this section is to derive upper bounds for the elements of $\boldsymbol{h}(\mathcal{F})$. The most critical step toward this goal is the recurrence inequality for the elements of $\boldsymbol{h}(\mathcal{F})$ described in the following lemma.

Lemma 9. For all $0 \leq k \leq d$,

$$
\begin{equation*}
h_{k+1}(\mathcal{F}) \leq \frac{n_{1}+n_{2}-d-1+k}{k+1} h_{k}(\mathcal{F})+\frac{n_{1}}{k+1} g_{k}\left(\partial P_{2}\right)+\frac{n_{2}}{k+1} g_{k}\left(\partial P_{1}\right) \tag{27}
\end{equation*}
$$

Proof. Let us denote by $V$ the vertex set of $\partial Q$, and by $V_{j}$ the vertex set of $\partial P_{j}, j=1,2$. Let $\mathcal{Y} / v$ be a shorthand for $\operatorname{link}(v, \mathcal{Y})$, where $v$ is a vertex of $\mathcal{Y}$, and $\mathcal{Y}$ stands for either $\mathcal{K}_{1}$ or $\mathcal{K}_{2}$, or the boundary complex of a simplicial polytope.

McMullen [13] in his original proof of the Upper Bound Theorem for polytopes proved that for any $d$-polytope $P$ the following relation holds:

$$
\begin{equation*}
(k+1) h_{k+1}(\partial P)+(d-k) h_{k}(\partial P)=\sum_{v \in \operatorname{vert}(\partial P)} h_{k}(\partial P / v), \quad 0 \leq k \leq d-1 . \tag{28}
\end{equation*}
$$

Furthermore, we have $h_{k}(\partial P / v) \leq h_{k}(\partial P)$. To see this consider a shelling of $\partial P$ that shells $\operatorname{star}(v, \partial P)$ first. The contributions to $h_{k}(\partial P)$ coincide with the contributions to $h_{k}(\partial P / v)$ during the shelling of $\operatorname{star}(v, \partial P)$. After the shelling has left $\operatorname{star}(v, \partial P)$ we get no more contributions to $h_{k}(\partial P / v)$, whereas we may get contributions to $h_{k}(\partial P)$. Therefore:

$$
\begin{equation*}
\sum_{v \in \operatorname{vert}(\partial P)} h_{k}(\partial P / v) \leq f_{0}(\partial P) h_{k}(\partial P), \quad 0 \leq k \leq d-1 \tag{29}
\end{equation*}
$$

Applying relation (28) to $Q, P_{1}$ and $P_{2}$ we get the following relations:

$$
\begin{align*}
(k+1) h_{k+1}(\partial Q)+(d+1-k) h_{k}(\partial Q) & =\sum_{v \in V} h_{k}(\partial Q / v), & 0 \leq k \leq d  \tag{30}\\
(k+1) h_{k+1}\left(\partial P_{1}\right)+(d-k) h_{k}\left(\partial P_{1}\right) & =\sum_{v \in V_{1}} h_{k}\left(\partial P_{1} / v\right), & 0 \leq k \leq d-1  \tag{31}\\
(k+1) h_{k+1}\left(\partial P_{2}\right)+(d-k) h_{k}\left(\partial P_{2}\right) & =\sum_{v \in V_{2}} h_{k}\left(\partial P_{2} / v\right), & 0 \leq k \leq d-1 \tag{32}
\end{align*}
$$

Recall that the link of $y_{j}$ in $\partial Q$ is $\partial P_{j}, j=1,2$, and observe that the link of $v \in V_{j}$ in $\partial Q$ coincides with $\mathcal{K}_{j} / v$. Expanding relation (30) by means of relation (14), we deduce:

$$
\begin{align*}
(k+1)[ & \left.h_{k+1}(\mathcal{F})+h_{k+1}\left(\partial P_{1}\right)+h_{k+1}\left(\partial P_{2}\right)\right]+(d+1-k)\left[h_{k}(\mathcal{F})+h_{k}\left(\partial P_{1}\right)+h_{k}\left(\partial P_{2}\right)\right]= \\
= & (k+1) h_{k+1}(\mathcal{F})+(d+1-k) h_{k}(\mathcal{F})+(k+1) h_{k+1}\left(\partial P_{1}\right)+(d-k) h_{k}\left(\partial P_{1}\right) \\
& \quad+(k+1) h_{k+1}\left(\partial P_{2}\right)+(d-k) h_{k}\left(\partial P_{2}\right)+h_{k}\left(\partial P_{1}\right)+h_{k}\left(\partial P_{2}\right) \\
= & \sum_{v \in V} h_{k}(\partial Q / v)=h_{k}\left(\partial Q / y_{1}\right)+h_{k}\left(\partial Q / y_{2}\right)+\sum_{v \in V_{1} \cup V_{2}} h_{k}(\partial Q / v)  \tag{33}\\
= & h_{k}\left(\partial P_{1}\right)+h_{k}\left(\partial P_{2}\right)+\sum_{v \in V_{1}} h_{k}\left(\mathcal{K}_{1} / v\right)+\sum_{v \in V_{2}} h_{k}\left(\mathcal{K}_{2} / v\right) .
\end{align*}
$$

Utilizing relations (31) and (32), the above equation is equivalent to:

$$
\begin{equation*}
(k+1) h_{k+1}(\mathcal{F})+(d+1-k) h_{k}(\mathcal{F})=\sum_{v \in V_{1}}\left[h_{k}\left(\mathcal{K}_{1} / v\right)-h_{k}\left(\partial P_{1} / v\right)\right]+\sum_{v \in V_{2}}\left[h_{k}\left(\mathcal{K}_{2} / v\right)-h_{k}\left(\partial P_{2} / v\right)\right] . \tag{34}
\end{equation*}
$$

Let us now consider a vertex $v \in V_{1}$, and a shelling $S(\partial Q)$ of $\partial Q$ that $\operatorname{shells} \operatorname{star}(v, \partial Q)$ first and $\operatorname{star}\left(y_{2}, \partial Q\right)$ last. Such a shelling does exit: consider a point $v^{\prime}$ (resp., $y_{2}^{\prime}$ ) beyond $v$ (resp., $y_{2}$ ) such that the line $\ell$ defined by $v^{\prime}$ and $y_{2}^{\prime}$ does not pass through $v$ and $y_{2}$. Call $v^{\prime \prime}$ and $y_{2}^{\prime \prime}$ the points of intersection of $\ell$ with $\partial Q$, and notice that, since $v$ and $y_{2}$ are not visible to each other, the only points of intersection of $\ell$ with $\partial Q$ are the points $v^{\prime \prime}$ and $y_{2}^{\prime \prime}$. The shelling $S(\partial Q)$ is the line shelling of $\partial Q$ induced by $\ell$ when we move from $v^{\prime \prime}$ away from $\partial Q$ towards $+\infty$, and then from $-\infty$ to $y_{2}^{\prime \prime}$. Notice that $\mathbb{S}(\partial Q)$ induces a shelling $\mathbb{S}\left(\mathcal{K}_{1}\right)$ for $\mathcal{K}_{1}$ that shells star $\left(v, \mathcal{K}_{1}\right)$ first (any shelling of $\partial Q$, that shells $\operatorname{star}\left(y_{2}, \partial Q\right)$ last, induces a shelling for $\mathcal{K}_{1}$, where the order of the facets of $\mathcal{K}_{1}$ in this shelling is the same as their order in the shelling of $\partial Q)$. On the other hand, $\mathbb{S}\left(\mathcal{K}_{1}\right)$ also induces (cf. [21, Lemma 8.7]):
(i) a shelling $\mathbb{S}\left(\mathcal{K}_{1} / v\right)$ for $\mathcal{K}_{1} / v$, and
(ii) a shelling $\mathbb{S}\left(\partial P_{1}\right)$ for $\partial P_{1}$ that shells star $\left(v, \partial P_{1}\right)$ first (recall that $\left.\partial P_{1} \equiv \partial Q / y_{1} \equiv \mathcal{K}_{1} / y_{1}\right)$,
while $\mathbb{S}\left(\partial P_{1}\right)$ induces a shelling $\mathbb{S}\left(\partial P_{1} / v\right)$ for $\partial P_{1} / v$ (again, cf. [21, Lemma 8.7]). The interested reader may refer to Figs. 33 8, where we show a shelling $\mathbb{S}\left(\mathcal{K}_{1}\right)$ of $\mathcal{K}_{1}$ that shells $\operatorname{star}\left(v, \mathcal{K}_{1}\right)$ first,
 step construction of $\mathcal{K}_{1}$ from $\mathbb{S}\left(\mathcal{K}_{1}\right)$. Fig. 6 shows the step-by-step construction of $\operatorname{star}\left(v, \mathcal{K}_{1}\right)$ from $\mathbb{S}\left(\mathcal{K}_{1}\right)$, as well as the corresponding induced construction of $\mathcal{K}_{1} / v$ from the induced shelling $\mathbb{S}\left(\mathcal{K}_{1} / v\right)$. Finally, Figs. 7 and 8 show the step-by-step construction of $\partial P_{1}$ from the shelling $\mathbb{S}\left(\partial P_{1}\right)$ induced by $\mathbb{S}\left(\mathcal{K}_{1}\right)$, along with the corresponding steps of the construction of $\mathcal{K}_{1}$ from $\mathbb{S}\left(\mathcal{K}_{1}\right)$, i.e., we only depict the steps of $\mathbb{S}\left(\mathcal{K}_{1}\right)$ that induce facets of $\mathbb{S}\left(\partial P_{1}\right)$.

Let $F$ be a facet in $\mathbb{S}\left(\mathcal{K}_{1}\right)$. If $F$ induces a facet for $\mathbb{S}\left(\mathcal{K}_{1} / v\right)$, denote by $F / v$ this facet of $\mathcal{K}_{1} / v$. Similarly, if $F$ induces a facet for $\mathbb{S}\left(\partial P_{1}\right)$, call $F_{1}$ this facet of $\partial P_{1}$. Finally, if $F_{1}$ induces a facet for $\mathbb{S}\left(\partial P_{1} / v\right)$, let $F_{1} / v$ be this facet of $\partial P_{1} / v$. Let $G \subseteq F, G / v \subseteq F / v, G_{1} \subseteq F_{1}$ and $G_{1} / v \subseteq F_{1} / v$ be the minimal new faces associated with $F, F / v, F_{1}$ and $F_{1} / v$ in the corresponding shellings, let $\lambda$ be the cardinality of $G$, and observe that $F_{1}=F \cap \partial P_{1}, F_{1} / v=(F / v) \cap \partial P_{1}$, $G_{1}=G \cap \partial P_{1}$ and $G_{1} / v=(G / v) \cap \partial P_{1}$. As long as we shell $\operatorname{star}\left(v, \mathcal{K}_{1}\right), G$ induces $G / v$, and, in fact, the faces $G$ and $G / v$ coincide (see also Fig. [6): if $F$ is the first facet in $\mathbb{S}\left(\mathcal{K}_{1}\right)$, then $G \equiv G / v \equiv \emptyset$; otherwise, $v$ cannot be a vertex in $G$ or $G / v$ (the minimal new faces are faces of $\left.\mathcal{K}_{1} / v\right)$. Similarly, as long as we shell $\operatorname{star}\left(v, \partial P_{1}\right), G_{1}$ induces $G_{1} / v$, and, in fact, the faces $G_{1}$ and $G_{1} / v$ coincide: if $F_{1}$ is the first facet in $\mathbb{S}\left(\partial P_{1}\right)$, then $G_{1} \equiv G_{1} / v \equiv \emptyset$; otherwise, $v$ cannot be a vertex in $G_{1}$ or $G_{1} / v$ (the minimal new faces are faces of $\left.\partial P_{1} / v\right)$. Hence, as long as we shell $\operatorname{star}\left(v, \mathcal{K}_{1}\right)$ (i.e., as long as $\left.v \in F\right)$, we have $h_{k}\left(\mathcal{K}_{1} / v\right)=h_{k}\left(\mathcal{K}_{1}\right)$ and $h_{k}\left(\partial P_{1} / v\right)=h_{k}\left(\partial P_{1}\right)$, for all $k \geq 0$, and, thus, $h_{k}\left(\mathcal{K}_{1} / v\right)-h_{k}\left(\partial P_{1} / v\right)=h_{k}\left(\mathcal{K}_{1}\right)-h_{k}\left(\partial P_{1}\right)$, for all $k \geq 0$. After the shelling $\mathbb{S}\left(\mathcal{K}_{1}\right)$ has left $\operatorname{star}\left(v, \mathcal{K}_{1}\right)$, there are no more facets in $\mathbb{S}\left(\mathcal{K}_{1} / v\right)$. This implies that, after $\mathbb{S}\left(\mathcal{K}_{1}\right)$ has left $\operatorname{star}\left(v, \mathcal{K}_{1}\right)$ (i.e., $v$ is not a vertex of $F$ anymore), the values of $h_{k}\left(\mathcal{K}_{1} / v\right)$ and $h_{k}\left(\partial P_{1} / v\right)$ remain unchanged for all $k \geq 0$. However, the values of $h_{k}\left(\mathcal{K}_{1}\right)$ and $h_{k}\left(\partial P_{1}\right)$ may increase for some $k$. More precisely, if $F$ does not induce any facet for $\mathbb{S}\left(\partial P_{1}\right)$, then $h_{\lambda}\left(\mathcal{K}_{1}\right)$ is increased by one, $h_{k}\left(\mathcal{K}_{1}\right)$ does not change for $k \neq \lambda$, while $h_{k}\left(\partial P_{1}\right)$ remains unchanged for all $k \geq 0$. Thus, $h_{\lambda}\left(\mathcal{K}_{1} / v\right)-h_{\lambda}\left(\partial P_{1} / v\right)<h_{\lambda}\left(\mathcal{K}_{1}\right)-h_{\lambda}\left(\partial P_{1}\right)$, while $h_{k}\left(\mathcal{K}_{1} / v\right)-h_{k}\left(\partial P_{1} / v\right) \leq h_{k}\left(\mathcal{K}_{1}\right)-h_{k}\left(\partial P_{1}\right)$, for all $k \neq \lambda$. If, however, $F$ induces $F_{1}$, then the minimal new face $G_{1}$ in $S\left(\partial P_{1}\right)$ due to $F_{1}$ coincides with $G$ (see also Figs. 7 and 8). To vefiry this, suppose $G_{1} \subset G$; since $G$ is the minimal new face in $\mathbb{S}\left(\mathcal{K}_{1}\right), G_{1}$ would have been a face already "discovered" at a previous step of $\mathbb{S}\left(\mathcal{K}_{1}\right)$, and thus also at a previous step of $\mathbb{S}\left(\partial P_{1}\right)$, which contradicts the fact that $G_{1}$ is the minimal new face for $\mathbb{S}\left(\partial P_{1}\right)$. Therefore, in this case, both $h_{\lambda}\left(\mathcal{K}_{1}\right)$ and $h_{\lambda}\left(\partial P_{1}\right)$ are increased by one, while $h_{k}\left(\mathcal{K}_{1}\right)$ and $h_{k}\left(\partial P_{1}\right)$ remain unchanged for all $k \neq \lambda$. This implies $h_{k}\left(\mathcal{K}_{1} / v\right)-h_{k}\left(\partial P_{1} / v\right) \leq h_{k}\left(\mathcal{K}_{1}\right)-h_{k}\left(\partial P_{1}\right)$, for all $k \geq 0$. Summarizing the analysis above, we deduce that for all $v \in V_{1}$, and for all $0 \leq k \leq d$, we


Figure 3: Top left: The complex $\mathcal{K}_{1}$ (from Fig. (2) with the vertex $v$ shown in orange. Remaining subfigures (from left to right and top to bottom): the first eight steps of the construction of $\mathcal{K}_{1}$ from a shelling $\mathbb{S}\left(\mathcal{K}_{1}\right)=\left\{F_{1}, F_{2}, \ldots, F_{26}\right\}$ that $\operatorname{shells} \operatorname{star}\left(v, \mathcal{K}_{1}\right)$ first. The facets in green are the facets of $\operatorname{star}\left(v, \mathcal{K}_{1}\right)$. All other facets are shown in either blue or yellow, depending on whether we see their exterior or interior side (w.r.t. the interior of the polytope $Q$ ). The minimal new faces at each step of the shelling are shown in red; recall that the minimal new face corresponding to $F_{1}$ is $\emptyset$. In all subfigures, the faces of $\operatorname{star}\left(y_{2}, \partial Q\right)$ that do not belong to $\partial Q / y_{2} \equiv \partial P_{2}$ are shown in gray.


Figure 4: From left to right and top to bottom: The next twelve steps of the construction of $\mathcal{K}_{1}$ from $\mathbb{S}\left(\mathcal{K}_{1}\right)$. Colors are as in Fig. 3.


Figure 5: From left to right and top to bottom: The final twelve steps of the construction of $\mathcal{K}_{1}$ from $\mathbb{S}\left(\mathcal{K}_{1}\right)$. Colors are, again, as in Fig. 3.


Figure 6: The first six steps of $\mathbb{S}\left(\mathcal{K}_{1}\right)$ and the corresponding steps in the induced shelling $\mathbb{S}\left(\mathcal{K}_{1} / v\right)$ of $\mathcal{K}_{1} / v$ (recall that $\mathbb{S}\left(\mathcal{K}_{1}\right)$ shells $\operatorname{star}\left(v, \mathcal{K}_{1}\right)$ first). Rows $1 \& 3$ : The steps of $\mathbb{S}\left(\mathcal{K}_{1}\right)$. Rows 2 \& 4 : The steps of $\mathbb{S}\left(\mathcal{K}_{1} / v\right)$. $\mathcal{K}_{1} / v$ is shown with green solid segments (the facets of $\mathcal{K}_{1} / v$, that have not been added yet, are highlighted as black solid segments). The minimal new faces at each step of the shellings $\mathbb{S}\left(\mathcal{K}_{1}\right)$ and $\mathbb{S}\left(\mathcal{K}_{1} / v\right)$ are shown in red. As expected, the minimal new faces, at corresponding steps, coincide.


Figure 7: The first six steps of the construction of $\partial P_{1}$ from the shelling $\mathbb{S}\left(\partial P_{1}\right)$ induced by $\mathbb{S}\left(\mathcal{K}_{1}\right)$, along with the corresponding steps of the construction of $\mathcal{K}_{1}$ from $\mathbb{S}\left(\mathcal{K}_{1}\right)$. Rows $1 \& 3$ : the steps of $\mathbb{S}\left(\mathcal{K}_{1}\right)$ that induce facets for $\mathbb{S}\left(\partial P_{1}\right)$. Rows $2 \& 4$ : The corresponding steps of $\mathbb{S}\left(\partial P_{1}\right)$. $\partial P_{1}$ is shown with green solid/dashed segments (the facets of $\partial P_{1}$, that have not been added yet, are highlighted as black solid/dashed segments). The minimal new faces at each step of the shellings $S\left(\mathcal{K}_{1}\right)$ and $S\left(\partial P_{1}\right)$ are shown in red. As expected, the minimal new faces, at corresponding steps, coincide.


Figure 8: The last three steps of the construction of $\partial P_{1}$ from the shelling $\mathbb{S}\left(\partial P_{1}\right)$ induced by $\mathbb{S}\left(\mathcal{K}_{1}\right)$, along with the corresponding steps of the construction of $\mathcal{K}_{1}$ from $\mathbb{S}\left(\mathcal{K}_{1}\right)$. Top row: The steps of $\mathbb{S}\left(\mathcal{K}_{1}\right)$. Bottom row: The steps of $\mathbb{S}\left(\mathcal{K}_{1} / v\right)$. Colors are as in Fig. 7 .
have:

$$
\begin{equation*}
h_{k}\left(\mathcal{K}_{1} / v\right)-h_{k}\left(\partial P_{1} / v\right) \leq h_{k}\left(\mathcal{K}_{1}\right)-h_{k}\left(\partial P_{1}\right) . \tag{35}
\end{equation*}
$$

Using the analogous argument for all vertices of $V_{2}$, we also get that, for all $v \in V_{2}$, and for all $0 \leq k \leq d$ :

$$
\begin{equation*}
h_{k}\left(\mathcal{K}_{2} / v\right)-h_{k}\left(\partial P_{2} / v\right) \leq h_{k}\left(\mathcal{K}_{2}\right)-h_{k}\left(\partial P_{2}\right) . \tag{36}
\end{equation*}
$$

Now, combining relation (15) with relations (16), yields

$$
\begin{align*}
& h_{k}\left(\mathcal{K}_{1}\right)-h_{k}\left(\partial P_{1}\right)=h_{k}(\mathcal{F})+g_{k}\left(\partial P_{2}\right),  \tag{37}\\
& h_{k}\left(\mathcal{K}_{2}\right)-h_{k}\left(\partial P_{2}\right)=h_{k}(\mathcal{F})+g_{k}\left(\partial P_{1}\right), \tag{38}
\end{align*}
$$

Thus, by applying relation (35), and using relation (37), we get for every vertex $v \in V_{1}$ :

$$
\begin{equation*}
\sum_{v \in V_{1}}\left[h_{k}\left(\mathcal{K}_{1} / v\right)-h_{k}\left(\partial P_{1} / v\right)\right] \leq \sum_{v \in V_{1}}\left[h_{k}\left(\mathcal{K}_{1}\right)-h_{k}\left(\partial P_{1}\right)\right]=n_{1}\left[h_{k}(\mathcal{F})+g_{k}\left(\partial P_{2}\right)\right], \tag{39}
\end{equation*}
$$

Similarly, applying relation (36), and using relation (38), we get for every vertex $v \in V_{2}$ :

$$
\begin{equation*}
\sum_{v \in V_{2}}\left[h_{k}\left(\mathcal{K}_{2} / v\right)-h_{k}\left(\partial P_{2} / v\right)\right] \leq \sum_{v \in V_{2}}\left[h_{k}\left(\mathcal{K}_{2}\right)-h_{k}\left(\partial P_{2}\right)\right]=n_{2}\left[h_{k}(\mathcal{F})+g_{k}\left(\partial P_{1}\right)\right] . \tag{40}
\end{equation*}
$$

We thus arrive at the following inequality, for $0 \leq k \leq d$ :

$$
\begin{equation*}
(k+1) h_{k+1}(\mathcal{F})+(d+1-k) h_{k}(\mathcal{F}) \leq\left(n_{1}+n_{2}\right) h_{k}(\mathcal{F})+n_{1} g_{k}\left(\partial P_{2}\right)+n_{2} g_{k}\left(\partial P_{1}\right), \tag{41}
\end{equation*}
$$

which gives the recurrence inequality in the statement of the lemma.

Using the recurrence relation from Lemma 9 we get the following bounds on the elements of $h(\mathcal{F})$.

Lemma 10. For all $0 \leq k \leq d+1$,

$$
\begin{equation*}
h_{k}(\mathcal{F}) \leq\binom{ n_{1}+n_{2}-d-2+k}{k}-\binom{n_{1}-d-2+k}{k}-\binom{n_{2}-d-2+k}{k} . \tag{42}
\end{equation*}
$$

Equality holds for all $k$ with $0 \leq k \leq l$ if and only if $l \leq\left\lfloor\frac{d+1}{2}\right\rfloor$ and $P$ is $\left(l, V_{1}\right)$-bineighborly.
Proof. We show the desired bound by induction on $k$. Clearly, the bound holds (as equality) for $k=0$, since

$$
\begin{equation*}
h_{0}(\mathcal{F})=-1=1-1-1=\binom{n_{1}+n_{2}-d-2+0}{0}-\binom{n_{1}-d-2+0}{0}-\binom{n_{2}-d-2+0}{0} . \tag{43}
\end{equation*}
$$

Suppose now that the bound holds for $h_{k}(\mathcal{F})$, where $k \geq 0$. Using the recurrence relation (27), in conjunction with the upper bounds for the elements of the $g$-vector of a polytope from Corollary 2, and since for $k \geq 0, n_{1}+n_{2}-d-1+k \geq d+1>0$, we have

$$
\begin{align*}
h_{k+1}(\mathcal{F}) \leq & \frac{n_{1}+n_{2}-d-1+k}{k+1} h_{k}(\mathcal{F})+\frac{n_{1}}{k+1} g_{k}\left(\partial P_{2}\right)+\frac{n_{2}}{k+1} g_{k}\left(\partial P_{1}\right) \\
\leq & \frac{n_{1}+n_{2}-d-1+k}{k+1}\left[\binom{n_{1}+n_{2}-d-2+k}{k}-\binom{n_{1}-d-2+k}{k}-\binom{n_{2}-d-2+k}{k}\right] \\
& \quad+\frac{n_{1}}{k+1}\binom{n_{2}-d-2+k}{k}+\frac{n_{2}}{k+1}\binom{n_{1}-d-2+k}{k}  \tag{44}\\
= & \frac{n_{1}+n_{2}-d-1+k}{k+1}\binom{n_{1}+n_{2}-d-2+k}{k}-\frac{n_{1}-d-1+k}{k+1}\binom{n_{1}-d-2+k}{k}-\frac{n_{2}-d-1+k}{k+1}\binom{n_{2}-d-2+k}{k} \\
= & \binom{n_{1}+n_{2}-d-1+k}{k+1}-\binom{n_{1}-d-1+k}{k+1}-\binom{n_{2}-d-1+k}{k+1} .
\end{align*}
$$

Let us now turn to our equality claim. The claim for $l=0$ is obvious (cf. (43)), so we assume below that $l \geq 1$. Suppose first that $P$ is $\left(l, V_{1}\right)$-bineighborly. Then, we have:

$$
\begin{equation*}
f_{i-1}(\mathcal{F})=\binom{n_{1}+n_{2}}{i}-\binom{n_{1}}{i}-\binom{n_{2}}{i}, \quad 0 \leq i \leq l . \tag{45}
\end{equation*}
$$

Substituting $f_{i-1}(\mathcal{F})$ from (45) in the defining equations (12) for $\boldsymbol{h}(\mathcal{F})$, we get, for all $0 \leq k \leq l$ :

$$
\begin{aligned}
h_{k}(\mathcal{F}) & =\sum_{i=0}^{k}(-1)^{k-i}\binom{d+1-i}{d+1-k} f_{i-1}(\mathcal{F}) \\
& =\sum_{i=0}^{k}(-1)^{k-i}\binom{d+1-i}{d+1-k}\left(\binom{n_{1}+n_{2}}{i}-\binom{n_{1}}{i}-\binom{n_{2}}{i}\right) \\
& =\sum_{i=0}^{k}(-1)^{k-i}\binom{d+1-i}{d+1-k}\binom{n_{1}+n_{2}}{i}-\sum_{i=0}^{k}(-1)^{k-i}\binom{d+1-i}{d+1-k}\binom{n_{1}}{i}-\sum_{i=0}^{k}(-1)^{k-i}\binom{d+1-i}{d+1-k}\binom{n_{2}}{i} \\
& =\binom{n_{1}+n_{2}-d-2+k}{k}-\binom{n_{2}-d-2+k}{k}-\binom{n_{2}-d-2+k}{k},
\end{aligned}
$$

where for the last equality we used the fact that $\binom{d+1-i}{d+1-k}=0$ for $i>k$, in conjunction with the following combinatorial identity (cf. [8, eq. (5.25)], [21, Exercise 8.20]):

$$
\sum_{0 \leq k \leq l}\binom{l-k}{m}\binom{s}{k-n}(-1)^{k}=(-1)^{l+m}\binom{s-m-1}{l-m-n} .
$$

In the equation above we set $k \leftarrow i, l \leftarrow d+1, m \leftarrow d+1-k, n \leftarrow 0$, while $s$ stands for either $n_{1}+n_{2}, n_{1}$ or $n_{2}$. We thus conclude that (42) holds as equality for all $0 \leq k \leq l$.

Suppose now that inequality (42) holds as equality for all $0 \leq k \leq l$. Substituting $h_{i}(\mathcal{F})$, $0 \leq i \leq l$, from (42) in (13) we get:

$$
\begin{align*}
f_{l-1}(\mathcal{F}) & =\sum_{i=0}^{d+1}\binom{d+1-i}{l-i} h_{i}(\mathcal{F}) \\
& \left.=\sum_{i=0}^{d+1}\binom{d+1-i}{l-i}\binom{n_{1}+n_{2}-d-2+i}{i}-\binom{n_{1}-d-2+i}{i}-\binom{n_{2}-d-2+i}{i}\right) \\
& =\sum_{i=0}^{d+1}\binom{d+1-i}{l-i}\binom{n_{1}+n_{2}-d-2+i}{i}-\sum_{i=0}^{d+1}\binom{d+1-i}{l-i}\binom{n_{1}-d-2+i}{i}-\sum_{i=0}^{d+1}\binom{d+1-i}{l-i}\binom{n_{2}-d-2+i}{i} \\
& =\sum_{i=0}^{d+1}\binom{d+1-i}{d+1-l}\binom{n_{1}+n_{2}-d-2+i}{n_{1}+n_{2}-d-2}-\sum_{i=0}^{d+1}\binom{d+1-i}{d+1-l}\binom{n_{1}-d-2+i}{n_{1}-d-2}-\sum_{i=0}^{d+1}\binom{d+1-i}{d+1-l}\binom{n_{2}-d-2+i}{n_{2}-d-2}  \tag{46}\\
& =\binom{(d+1)+\left(n_{1}+n_{2}-d-2\right)+1}{(d+1-l)+\left(n_{1}+n_{2}-d-2\right)+1}-\binom{(d+1)+\left(n_{1}-d-2\right)+1}{(d+1-l)+\left(n_{1}-d-2\right)+1}-\binom{(d+1)+\left(n_{2}-d-2\right)+1}{(d+1-l)+\left(n_{2}-d-2\right)+1}  \tag{47}\\
& =\binom{n_{1}+n_{2}}{n_{1}+n_{2}-l}-\binom{n_{1}}{n_{1}-l}-\binom{n_{2}}{n_{2}-l} \\
& =\binom{n_{1}+n_{2}}{l}-\binom{n_{1}}{l}-\binom{n_{2}}{l}
\end{align*}
$$

where, in order to get from (46) to (47), we used the combinatorial identity (cf. [8, eq. (5.26)]):

$$
\sum_{0 \leq k \leq l}\binom{l-k}{m}\binom{q+k}{n}=\binom{l+q+1}{m+n+1}
$$

with $k \leftarrow i, l \leftarrow d+1, m \leftarrow d+1-k, q \leftarrow s-d-2, n \leftarrow s-d-2$, and $s$ stands for either $n_{1}+n_{2}$, $n_{1}$ or $n_{2}$. Hence, $P$ is $\left(l, V_{1}\right)$-bineighborly.

Using the Dehn-Sommerville-like relations (17), in conjunction with the bounds from the previous lemma, we derive alternative bounds for $h_{k}(\mathcal{F})$, which are of interest since they refine the bounds for $h_{k}(\mathcal{F})$ from Lemma 10 for large values of $k$, namely for $k>\left\lfloor\frac{d+1}{2}\right\rfloor$. More precisely:

Lemma 11. For all $0 \leq k \leq d+1$,

$$
\begin{equation*}
h_{d+1-k}(\mathcal{F}) \leq\binom{ n_{1}+n_{2}-d-2+k}{k} \tag{48}
\end{equation*}
$$

Equality holds for all $k$ with $0 \leq k \leq l$ if and only if $l \leq\left\lfloor\frac{d}{2}\right\rfloor$ and $P$ is $l$-neighborly.
Proof. The upper bound claim in (48) is a direct consequence of the Dehn-Sommerville-like relations (17) for $\boldsymbol{h}(\mathcal{F})$, the upper bounds from Lemma 10, and the Upper Bound Theorem for polytopes as stated in Corollary 2.

The rest of the proof deals with the equality claim. Inequality (48) holds as equality for all $0 \leq k \leq l$, where $l \leq\left\lfloor\frac{d}{2}\right\rfloor$, if and only if the following two conditions hold:
(i) Inequalities (42) hold as equalities for all $0 \leq k \leq l \leq\left\lfloor\frac{d}{2}\right\rfloor$.
(ii) For $j=1,2$, and for all $0 \leq k \leq l \leq\left\lfloor\frac{d}{2}\right\rfloor$, we have $g_{k}\left(\partial P_{j}\right)=\binom{n_{j}-d-2+k}{k}$.

The first condition holds true if and only if $P$ is $\left(l, V_{1}\right)$-bineighborly, while the second condition holds true if and only if $P_{j}, j=1,2$, is $l$-neighborly. Therefore, inequality (48) holds as equality for all $0 \leq k \leq l$ if and only if $l \leq\left\lfloor\frac{d}{2}\right\rfloor, P$ is $\left(l, V_{1}\right)$-bineighborly and both $P_{1}, P_{2}$ are $l$-neighborly. In view of Lemma 6. we conclude that equality in (48) holds for all $0 \leq k \leq l$ if and only if $l \leq\left\lfloor\frac{d}{2}\right\rfloor$ and $P$ is $l$-neighborly.

We are now ready to compute upper bounds for the face numbers of $\mathcal{F}$. Using relation (13), in conjunction with the bounds on the elements of $\boldsymbol{h}(\mathcal{F})$ from Lemma 10 and Lemma 11, we get, for $0 \leq k \leq d+1$ :

$$
\begin{align*}
f_{k-1}(\mathcal{F}) & =\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i} h_{i}(\mathcal{F})+\sum_{i=\left\lfloor\frac{d+1}{2}\right\rfloor+1}^{d+1}\binom{d+1-i}{k-i} h_{i}(\mathcal{F}) \\
& =\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i} h_{i}(\mathcal{F})+\sum_{i=0}^{\left\lfloor\frac{d}{2}\right\rfloor}\binom{i}{k-d-1+i} h_{d+1-i}(\mathcal{F}) \\
& \leq \sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i}\left(\binom{n_{1}+n_{2}-d-2+i}{i}-\sum_{j=1}^{2}\binom{n_{j}-d-2+i}{i}\right)+\sum_{i=0}^{\left\lfloor\frac{d}{2}\right\rfloor}\binom{i}{k-d-1+i}\binom{n_{1}+n_{2}-d-2+i}{i} \\
& =\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i}\binom{n_{1}+n_{2}-d-2+i}{i}+\sum_{i=0}^{\left\lfloor\frac{d}{2}\right\rfloor}\binom{i}{k-d-1+i}\binom{n_{1}+n_{2}-d-2+i}{i}-\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i} \sum_{j=1}^{2}\binom{n_{j}-d-2+i}{i}  \tag{49}\\
& \left.=\sum_{i=0}^{\frac{d+1}{2}} *\binom{d+1-i}{k-i}+\binom{i}{k-d-1+i}\right)\binom{n_{1}+n_{2}-d-2+i}{i}-\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i} \sum_{j=1}^{2}\binom{n_{j}-d-2+i}{i}  \tag{50}\\
& =f_{k-1}\left(C_{d+1}\left(n_{1}+n_{2}\right)\right)-\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i} \sum_{j=1}^{2}\binom{n_{j}-d-2+i}{i}
\end{align*}
$$

where $C_{d}(n)$ stands for the cyclic $d$-polytope with $n$ vertices, $\sum_{i=0}^{\frac{\delta}{2}}{ }^{*} T_{i}$ denotes the sum of the elements $T_{0}, T_{1}, \ldots, T_{\left\lfloor\frac{\delta}{2}\right\rfloor}$ where the last term is halved if $\delta$ is even, while in order to get from (49) to (50) we used an identity proved in Section B of the Appendix. The following lemma summarizes our results.

Lemma 12. For all $0 \leq k \leq d+1$ :

$$
f_{k-1}(\mathcal{F}) \leq f_{k-1}\left(C_{d+1}\left(n_{1}+n_{2}\right)\right)-\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i}\left(\binom{n_{1}-d-2+i}{i}+\binom{n_{2}-d-2+i}{i}\right)
$$

where $C_{d}(n)$ stands for the cyclic d-polytope with $n$ vertices. Furthermore:
(i) Equality holds for all $0 \leq k \leq l$ if and only if $l \leq\left\lfloor\frac{d+1}{2}\right\rfloor$ and $P$ is $\left(l, V_{1}\right)$-bineighborly.
(ii) For $d \geq 2$ even, equality holds for all $0 \leq k \leq d+1$ if and only if $P$ is $\left\lfloor\frac{d}{2}\right\rfloor$-neighborly.
(iii) For $d \geq 3$ odd, equality holds for all $0 \leq k \leq d+1$ if and only if $P$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, V_{1}\right)$-bineighborly.

Since for all $1 \leq k \leq d, f_{k-1}\left(P_{1} \oplus P_{2}\right)=f_{k}(\mathcal{F})$, we arrive at the central theorem of this section, stating upper bounds for the face numbers of the Minkowski sum of two $d$-polytopes.
Theorem 13. Let $P_{1}$ and $P_{2}$ be two d-polytopes in $\mathbb{E}^{d}$, $d \geq 2$, with $n_{1} \geq d+1$ and $n_{2} \geq d+1$ vertices, respectively. Let also $P$ be the convex hull in $\mathbb{E}^{d+1}$ of $P_{1}$ and $P_{2}$ embedded in the hyperplanes $\left\{x_{d+1}=0\right\}$ and $\left\{x_{d+1}=1\right\}$ of $\mathbb{E}^{d+1}$, respectively. Then, for $1 \leq k \leq d$, we have:
$f_{k-1}\left(P_{1} \oplus P_{2}\right) \leq f_{k}\left(C_{d+1}\left(n_{1}+n_{2}\right)\right)-\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k+1-i}\left(\binom{n_{1}-d-2+i}{i}+\binom{n_{2}-d-2+i}{i}\right)$,
Furthermore:
(i) Equality holds for all $1 \leq k \leq l$ if an only if $l \leq\left\lfloor\frac{d-1}{2}\right\rfloor$ and $P$ is $\left(l+1, V_{1}\right)$-bineighborly.
(ii) For $d \geq 2$ even, equality holds for all $1 \leq k \leq d$ if an only if $P$ is $\left\lfloor\frac{d}{2}\right\rfloor$-neighborly.
(iii) For $d \geq 3$ odd, equality holds for all $1 \leq k \leq d$ if an only if $P$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, V_{1}\right)$-bineighborly.

## 5 Lower bounds

In the previous section we proved upper bounds on the face numbers of the Minkowski sum $P_{1} \oplus P_{2}$ of two polytopes $P_{1}$ and $P_{2}$, and we provided necessary and sufficient conditions for these bounds to hold. However, there is one remaining important question: Are these bounds tight? In this section give a positive answer to this question.

We recall, from the introductory section, the already known results, and discuss how they are related to the results in this paper. It is already known (e.g., cf. [2]) that the maximum number of vertices/edges of the Minkowski sum of two polygons (i.e., 2-polytopes) is the sum of the vertices/edges of the summands. These match our expressions for $d=2$ in Theorem 13, Fukuda and Weibel [5] have shown tight expressions for the number of $k$-faces, $0 \leq k \leq 2$, of the Minkowski sum of two 3-polytopes $P_{1}$ and $P_{2}$, as a function of the number of vertices of $P_{1}$ and $P_{2}$. These maximal values are given in relations (11), and match our expressions for $d=3$ in Theorem 13, In the same paper, Fukuda and Weibel have shown that given $r d$-polytopes $P_{1}, P_{2}, \ldots, P_{r}$, the number of $k$-faces of $P_{1} \oplus P_{2} \oplus \ldots \oplus P_{r}$ is bounded from above as per relation (2). These bounds have been shown to be tight for $d \geq 4, r \leq\left\lfloor\frac{d}{2}\right\rfloor$, and for all $k$ with $0 \leq k \leq\left\lfloor\frac{d}{2}\right\rfloor-r$. For $r=2$, the upper bounds in (2) reduce to

$$
\begin{equation*}
f_{k}\left(P_{1} \oplus P_{2}\right) \leq \sum_{j=1}^{k+1}\binom{n_{1}}{j}\binom{n_{2}}{k+2-j}, \quad 0 \leq k \leq d-1, \tag{51}
\end{equation*}
$$

and are tight for all $k$, with $0 \leq k \leq\left\lfloor\frac{d}{2}\right\rfloor-2$. According to Fukuda and Weibel [5], these upper bounds are attained when considering two cyclic $d$-polytopes $P_{1}$ and $P_{2}$, with $n_{1}$ and $n_{2}$ vertices, respectively, with disjoint vertex sets. As we show below, this construction gives, in fact, tight bounds on the number of $k$-faces of the Minkowski sum for all $0 \leq k \leq d-1$, when the dimension $d$ is even.

Theorem 14. Let $d \geq 2$ and $d$ is even. Consider two cyclic $d$-polytopes $P_{1}$ and $P_{2}$ with disjoint vertex sets on the $d$-dimensional moment curve, and let $n_{j}$ be the number of vertices of $P_{j}, j=1,2$. Then, for all $1 \leq k \leq d$ :
$f_{k-1}\left(P_{1} \oplus P_{2}\right)=f_{k}\left(C_{d+1}\left(n_{1}+n_{2}\right)\right)-\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k+1-i}\left(\binom{n_{1}-d-2+i}{i}+\binom{n_{2}-d-2+i}{i}\right)$,
where $C_{d}(n)$ stands for the cyclic d-polytope with $n$ vertices.
Proof. Let $V_{1}$ and $V_{2}$ be two disjoint sets of points on the $d$-dimensional moment curve of cardinalities $n_{1}$ and $n_{2}$, respectively. Let $P_{1}$ and $P_{2}$ be the corresponding cyclic $d$-polytopes, and embed them, as in the previous section, in the hyperplanes $\left\{x_{d+1}=0\right\}$ and $\left\{x_{d+1}=1\right\}$ of $\mathbb{E}^{d+1}$. Let $P=$ $C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$ and, again as in the previous section, define the set of faces $\mathcal{F}$ as the set of proper faces of $P$ intersected by the hyperplane $\tilde{\Pi}$ with equation $\left\{x_{d+1}=\lambda\right\}, \lambda \in(0,1)$. We then get:

$$
f_{\left\lfloor\frac{d}{2}\right\rfloor-1}(\mathcal{F})=f_{\left\lfloor\frac{d}{2}\right\rfloor-2}\left(P_{1} \oplus P_{2}\right)=\sum_{j=1}^{\left\lfloor\left\lfloor\frac{d}{2}\right\rfloor-1\right.}\binom{n_{1}}{j}\binom{n_{2}}{\left\lfloor\frac{d}{2}\right\rfloor-j}=\binom{n_{1}+n_{2}}{\left\lfloor\frac{d}{2}\right\rfloor}-\binom{n_{1}}{\left\lfloor\frac{d}{2}\right\rfloor}-\binom{n_{2}}{\left\lfloor\frac{d}{2}\right\rfloor},
$$

which, by Lemma 7 implies that $P$ is $\left(\left\lfloor\frac{d}{2}\right\rfloor, V_{1}\right)$-bineighborly. Using Lemma 6, in conjunction with the fact that both $P_{1}$ and $P_{2}$ are $\left\lfloor\frac{d}{2}\right\rfloor$-neighborly, we further conclude that $P$ is $\left\lfloor\frac{d}{2}\right\rfloor$-neighborly. Hence, by Theorem [13, our upper bounds in Theorem 13 are attained for all face numbers of $P_{1} \oplus P_{2}$.

If $d \geq 5$ and $d$ is odd, however, the construction in [5] gives tight bounds for $f_{k}\left(P_{1} \oplus P_{2}\right)$ for all $0 \leq k \leq\left\lfloor\frac{d}{2}\right\rfloor-2$, which according to Theorem 13 are not sufficient to establish that the bounds are tight for the face numbers of all dimensions. To establish the tightness of the bounds in Theorem 13 for all the face numbers of all dimensions, we need to construct two $d$-polytopes $P_{1}$ and $P_{2}$, with $n_{1}$ and $n_{2}$ vertices, respectively, such that

$$
f_{\left\lfloor\frac{d}{2}\right\rfloor}(\mathcal{F})=f_{\left\lfloor\frac{d}{2}\right\rfloor-1}\left(P_{1} \oplus P_{2}\right)=\binom{n_{1}+n_{2}}{\left\lfloor\frac{d+1}{2}\right\rfloor}-\binom{n_{1}}{\left\lfloor\frac{d+1}{2}\right\rfloor}-\binom{n_{2}}{\left\lfloor\frac{d+1}{2}\right\rfloor},
$$

or, equivalently, construct two $d$-polytopes $P_{1}$ and $P_{2}$, such that $P$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, V_{1}\right)$-bineighborly.
The rest of this section is devoted to this construction. Before getting into the technical details we first outline our approach. In what follows $d \geq 3$ and $d$ is odd. We denote by $\gamma(t), t>0$, the ( $d-1$ )-dimensional moment curve, i.e., $\gamma(t)=\left(t, t^{2}, \ldots, t^{d-1}\right)$, and we define two additional curves $\gamma_{1}(t ; \zeta)$ and $\gamma_{2}(t ; \zeta)$ in $\mathbb{E}^{d+1}$, as follows:

$$
\begin{align*}
& \gamma_{1}(t ; \zeta)=\left(t, \zeta t^{d}, t^{2}, t^{3}, \ldots, t^{d-1}, 0\right), \quad t>0, \quad \zeta \geq 0 \\
& \gamma_{2}(t ; \zeta)=\left(\zeta t^{d}, t, t^{2}, t^{3}, \ldots, t^{d-1}, 1\right), \tag{52}
\end{align*}
$$

Notice that $\gamma_{1}(t ; \zeta)$ and $\gamma_{2}(t ; \zeta)$, with $\zeta>0$, are $d$-dimensional moment-like curves, embedded in the hyperplanes $\left\{x_{d+1}=0\right\}$ and $\left\{x_{d+1}=1\right\}$, respectively. Choose $n_{1}+n_{2}$ real numbers $\alpha_{i}, i=1, \ldots, n_{1}$, and $\beta_{i}, i=1, \ldots, n_{2}$, such that $0<\alpha_{1}<\alpha_{2}<\ldots<\alpha_{n_{1}}$ and $0<\beta_{1}<\beta_{2}<\ldots<\beta_{n_{2}}$. Let $\tau$ be a strictly positive parameter determined below, and let $U_{1}$ and $U_{2}$ be the $(d-1)$-dimensional point sets:

$$
\begin{align*}
& U_{1}=\left\{\gamma_{1}\left(\alpha_{1} \tau\right), \gamma_{1}\left(\alpha_{2} \tau\right), \ldots, \gamma_{1}\left(\alpha_{n_{1}} \tau\right)\right\},  \tag{53}\\
& U_{2}=\left\{\boldsymbol{\gamma}_{2}\left(\beta_{1}\right), \boldsymbol{\gamma}_{2}\left(\beta_{2}\right), \ldots, \gamma_{2}\left(\beta_{n_{2}}\right)\right\}
\end{align*}
$$

where $\gamma_{j}(\cdot)$ is used to denote $\gamma_{j}(\cdot ; 0)$, for simplicity. Notice that $U_{1}$ and $U_{2}$ consist of points on the moment curve $\boldsymbol{\gamma}(t)$, embedded in the $(d-1)$-subspaces $\left\{x_{1}=0, x_{d+1}=0\right\}$ and $\left\{x_{2}=0, x_{d+1}=1\right\}$ of $\mathbb{E}^{d+1}$, respectively. Call $Q_{j}$ the cyclic $(d-1)$-polytope defined as the convex hull of the points in $U_{j}, j=1,2$. We first show that, for sufficiently small $\tau$, any subset $U$ of $\left\lfloor\frac{d+1}{2}\right\rfloor$ vertices of $U_{1} \cup U_{2}$, such that $U \cap U_{j} \neq \emptyset, j=1,2$, defines a $\left\lfloor\frac{d}{2}\right\rfloor$-face of $Q=C H_{d+1}\left(\left\{Q_{1}, Q_{2}\right\}\right)$; in other words, we show that, for sufficiently small $\tau$, the $(d+1)$-polytope $Q$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, U_{1}\right)$-bineighborly. We then
appropriately perturb $U_{1}$ and $U_{2}$ (by considering a positive value for $\zeta$ ) so that they become $d$ dimensional. Let $V_{1}, V_{2}$ be the perturbed vertex sets, and $P_{1}, P_{2}$ be the resulting $d$-polytopes ( $V_{j}$ is the vertex set of $P_{j}$ ). The final step of our construction amounts to considering the $(d+1)$-polytope $P=C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$, and arguing that, if the perturbation parameter $\zeta$ is sufficiently small, then $P$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, V_{1}\right)$-bineighborly. In view of Theorem [13, this establishes the tightness of our bounds for all face numbers of $P_{1} \oplus P_{2}$.

We start off with a technical lemma. Its proof may be found in Section $\mathbb{C}$ of the Appendix.
Lemma 15. Fix two integers $k \geq 2$ and $\ell \geq 2$, such that $k+\ell$ is odd. Let $D_{k, \ell}(\tau)$ be the $(k+\ell) \times(k+\ell)$ determinant:

$$
D_{k, \ell}(\tau)=\left|\begin{array}{cccccccc}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
x_{1} \tau & x_{2} \tau & \cdots & x_{k} \tau & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & y_{1} & y_{2} & \cdots & y_{\ell} \\
x_{1}^{2} \tau^{2} & x_{2}^{2} \tau^{2} & \cdots & x_{k}^{2} \tau^{2} & y_{1}^{2} & y_{2}^{2} & \cdots & y_{\ell}^{2} \\
x_{1}^{3} \tau^{3} & x_{2}^{3} \tau^{3} & \cdots & x_{k}^{3} \tau^{3} & y_{1}^{3} & y_{2}^{3} & \cdots & y_{\ell}^{3} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
x_{1}^{m} \tau^{m} & x_{2}^{m} \tau^{m} & \cdots & x_{k}^{m} \tau^{m} & y_{1}^{m} & y_{2}^{m} & \cdots & y_{\ell}^{m}
\end{array}\right|, \quad m=k+\ell-3, \quad \text {, }
$$

where $0<x_{1}<x_{2}<\ldots<x_{k}, 0<y_{1}<y_{2}<\ldots<y_{\ell}$, and $\tau>0$. Then, there exists some $\tau_{0}>0$ (that depends on the $x_{i}$ 's, the $y_{i}$ 's, $k$, and $\ell$ ) such that for all $\tau \in\left(0, \tau_{0}\right)$, the determinant $D_{k, \ell}(\tau)$ is strictly positive.

We now formally proceed with our construction. As described above, consider the vertex sets $U_{1}$ and $U_{2}$ (cf. (531)), and call $Q_{j}$ the cyclic ( $d-1$ )-polytope with vertex set $U_{j}, j=1,2$. Notice that $Q_{1}$ (resp., $Q_{2}$ ) is embedded in the ( $d-1$ )-subspace $\left\{x_{2}=0, x_{d+1}=0\right\}$ (resp., $\left\{x_{1}=0, x_{d+1}=1\right\}$ ) of $\mathbb{E}^{d+1}$. As in the previous section, call $\tilde{\Pi}$ the hyperplane of $\mathbb{E}^{d+1}$ with equation $\left\{x_{d+1}=\lambda\right\}$, $\lambda \in(0,1)$. Let $Q=C H_{d+1}\left(\left\{Q_{1}, Q_{2}\right\}\right)$, and let $\mathcal{F}_{Q}$ be the set of proper faces of $Q$ with non-empty intersection with $\tilde{\Pi}$, i.e., $\mathcal{F}_{Q}$ consists of all the proper faces of $Q$, the vertex set of which has nonempty intersection with both $U_{1}$ and $U_{2}$. The following lemma establishes the first step towards our construction.

Lemma 16. There exists a sufficiently small positive value $\tau^{\star}$ for $\tau$, such that the $(d+1)$-polytope $Q$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, U_{1}\right)$-bineighborly.

Proof. Let $t_{i}=\alpha_{i} \tau, t_{i}^{\epsilon}=\left(\alpha_{i}+\epsilon\right) \tau, 1 \leq i \leq n_{1}$, and $s_{i}=\beta_{i}, s_{i}^{\epsilon}=\beta_{i}+\epsilon, 1 \leq i \leq n_{2}$, where $\epsilon>0$ is chosen such that $\alpha_{i}+\epsilon<\alpha_{i+1}$, for all $1 \leq i<n_{1}$, and $\beta_{i}+\epsilon<\beta_{i+1}$, for all $1 \leq i<n_{2}$.

Choose a subset $U$ of $U_{1} \cup U_{2}$ of size $\left\lfloor\frac{d+1}{2}\right\rfloor$, such that $U \cap U_{j} \neq \emptyset, j=1,2$. We denote by $\mu$ (resp., $\nu$ ) the cardinality of $U \cap U_{1}$ (resp., $U \cap U_{2}$ ), and, clearly, $\mu+\nu=\left\lfloor\frac{d+1}{2}\right\rfloor$. Let $\gamma_{1}\left(t_{i_{1}}\right), \gamma_{1}\left(t_{i_{2}}\right), \ldots, \gamma_{1}\left(t_{i_{\mu}}\right)$ be the vertices in $U \cap U_{1}$, where $i_{1}<i_{2}<\ldots<i_{\mu}$, and analogously, let $\gamma_{2}\left(s_{j_{1}}\right), \gamma_{2}\left(s_{j_{2}}\right), \ldots, \gamma_{2}\left(s_{j_{\nu}}\right)$ be the vertices in $U \cap U_{2}$, where $j_{1}<j_{2}<\ldots<j_{\nu}$. Let $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{d+1}\right)$ and define the $(d+2) \times(d+2)$ determinant $H_{U}(\boldsymbol{x})$ as follows:

$$
H_{U}(\boldsymbol{x})=\left|\begin{array}{cccccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1  \tag{54}\\
\boldsymbol{x}^{2} & \gamma_{1}\left(t_{i_{1}}\right) & \gamma_{1}\left(t_{i_{1}}^{\epsilon}\right) & \cdots & \gamma_{1}\left(t_{i_{\mu}}\right) & \gamma_{1}\left(t_{i_{\mu}}^{\epsilon}\right) & \gamma_{2}\left(s_{j_{1}}\right) & \gamma_{2}\left(s_{j_{1}}^{\epsilon}\right) & \cdots & \gamma_{2}\left(s_{j_{\nu}}\right) \\
\gamma_{2}\left(s_{j_{\nu}}^{\epsilon}\right)
\end{array}\right| .
$$

The equation $H_{U}(\boldsymbol{x})=0$ is the equation of a hyperplane in $\mathbb{E}^{d+1}$ that passes through the points in $U$. We claim that, for any choice of $U$, and for all vertices $\boldsymbol{u}$ in $\left(U_{1} \cup U_{2}\right) \backslash U$, we have $H_{U}(\boldsymbol{u})>0$ for sufficiently small $\tau$.

Consider first the case $\boldsymbol{u} \in U_{1} \backslash U$. Then, $\boldsymbol{u}=\gamma_{1}(t)=\left(t, 0, t^{2}, t^{3}, \ldots, t^{d-1}, 0\right), t=\alpha \tau$, for some $\alpha \notin\left\{\alpha_{i_{1}}, \alpha_{i_{2}}, \ldots, \alpha_{i_{\mu}}\right\}$, in which case $H_{U}(\boldsymbol{u})$ becomes:

$$
\begin{aligned}
H_{U}(\boldsymbol{u}) & =\left|\begin{array}{ccccccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\gamma_{1}(t) & \gamma_{1}\left(t_{i_{1}}\right) & \gamma_{1}\left(t_{i_{1}}^{\epsilon}\right) & \cdots & \gamma_{1}\left(t_{i_{\mu}}\right) & \gamma_{1}\left(t_{i_{\mu}}^{\epsilon}\right) & \gamma_{2}\left(s_{j_{1}}\right) & \gamma_{2}\left(s_{j_{1}}^{\epsilon}\right) & \cdots & \gamma_{2}\left(s_{j_{\nu}}\right) & \gamma_{2}\left(s_{j_{\nu}}^{\epsilon}\right)
\end{array}\right| \\
& =\left|\begin{array}{ccccccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
t & t_{i_{1}} & t_{i_{1}}^{\epsilon} & \cdots & t_{i_{\mu}} & t_{i_{\mu}}^{\epsilon} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & s_{j_{1}} & s_{j_{1}}^{\epsilon} & \cdots & s_{j_{\nu}} & s_{j_{\nu}}^{\epsilon} \\
t^{2} & t_{i_{1}}^{2} & \left(t_{i_{1}}^{\epsilon}\right)^{2} & \cdots & t_{i_{\mu}}^{2} & \left(t_{i_{\mu}}^{\epsilon}\right)^{2} & s_{j_{1}}^{2} & \left(s_{j_{1}}^{\epsilon}\right)^{2} & \cdots & s_{j_{\nu}}^{2} & \left(s_{j_{\nu}}^{\epsilon}\right)^{2} \\
t^{3} & t_{i_{1}}^{3} & \left(t_{i_{1}}^{\epsilon}\right)^{3} & \cdots & t_{i_{\mu}}^{3} & \left(t_{i_{\mu}}^{\epsilon}\right)^{3} & s_{j_{1}}^{3} & \left(s_{j_{1}}^{\epsilon}\right)^{3} & \cdots & s_{j_{\nu}}^{3} & \left(s_{j_{\nu}}^{\epsilon}\right)^{3} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
t^{d-1} & t_{i_{1}}^{d-1} & \left(t_{i_{1}}^{\epsilon}\right)^{d-1} & \cdots & t_{i_{\mu}}^{d-1} & \left(t_{i_{\mu}}^{\epsilon}\right)^{d-1} & s_{j_{1}}^{d-1} & \left(s_{j_{1}}^{\epsilon}\right)^{d-1} & \cdots & s_{j_{\nu}}^{d-1} & \left(s_{j_{\nu}}^{\epsilon}\right)^{d-1} \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1
\end{array}\right| .
\end{aligned}
$$

Observe now that we can transform $H_{U}(\boldsymbol{u})$ in the form of the determinant $D_{k, \ell}(\tau)$ of Lemma 15 , where $k=2 \mu+1$ and $\ell=2 \nu$, by means of the following determinant transformations:
(i) Subtract the last row of $H_{U}(\boldsymbol{u})$ from the first.
(ii) Shift the first column of $H_{U}(\boldsymbol{u})$ to the right, so that the non-zero values of the second row of $H_{U}(\boldsymbol{u})$ occupy columns 1 through $2 \mu+1$ and are in increasing order. This has to be done by an even number of column swaps, since $t$ cannot be between some $t_{i_{k}}$ and $t_{i_{k}}^{\epsilon}$ (due to the way we have chosen $\epsilon$ ).
(iii) Shift the last row of $H_{U}(\boldsymbol{u})$ up, so as to become the third row of $H_{U}(\boldsymbol{u})$. This can be done by $d-1$ row swaps, which implies that the sign of the determinant does not change (recall that $d$ is odd).

Consider now the case $\boldsymbol{u} \in U_{2} \backslash U$. Then, $\boldsymbol{u}=\gamma_{2}(s)=\left(0, s, s^{2}, s^{3}, \ldots, s^{d-1}, 1\right)$, for some $s \notin\left\{s_{j_{1}}, s_{j_{2}}, \ldots, s_{j_{\nu}}\right\}$, in which case $H_{U}(\boldsymbol{u})$ becomes:

$$
\begin{aligned}
H_{U}(\boldsymbol{u}) & =\left|\begin{array}{ccccccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
\gamma_{2}(s) & \gamma_{1}\left(t_{i_{1}}\right) & \gamma_{1}\left(t_{i_{1}}^{\epsilon}\right) & \cdots & \gamma_{1}\left(t_{i_{\mu}}\right) & \gamma_{1}\left(t_{i_{\mu}}^{\epsilon}\right) & \gamma_{2}\left(s_{j_{1}}\right) & \gamma_{2}\left(s_{j_{1}}^{\epsilon}\right) & \cdots & \gamma_{2}\left(s_{j_{\nu}}\right) & \gamma_{2}\left(s_{j_{\nu}}^{\epsilon}\right)
\end{array}\right| \\
& =\left|\begin{array}{ccccccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\
0 & t_{i_{1}} & t_{i_{1}}^{\epsilon} & \cdots & t_{i_{\mu}} & t_{i_{\mu}}^{\epsilon} & 0 & 0 & \cdots & 0 & 0 \\
s & 0 & 0 & \cdots & 0 & 0 & s_{j_{1}} & s_{j_{1}}^{\epsilon} & \cdots & s_{j_{\nu}} & s_{j_{\nu}}^{\epsilon} \\
s^{2} & t_{i_{1}}^{2} & \left(t_{i_{1}}^{\epsilon}\right)^{2} & \cdots & t_{i_{\mu}}^{2} & \left(t_{i_{\mu}}^{\epsilon}\right)^{2} & s_{j_{1}}^{2} & \left(s_{j_{1}}^{\epsilon}\right)^{2} & \cdots & s_{j_{\nu}}^{2} & \left(s_{j_{\nu}}^{\epsilon}\right)^{2} \\
s^{3} & t_{i_{1}}^{3} & \left(t_{i_{1}}^{\epsilon}\right)^{3} & \cdots & t_{i_{\mu}}^{3} & \left(t_{i_{\mu}}^{\epsilon}\right)^{3} & s_{j_{1}}^{3} & \left(s_{j_{1}}^{\epsilon}\right)^{3} & \cdots & s_{j_{\nu}}^{3} & \left(s_{j_{\nu}}^{\epsilon}\right)^{3} \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
s^{d-1} & t_{i_{1}}^{d-1} & \left(t_{i_{1}}^{\epsilon}\right)^{d-1} & \cdots & t_{i_{\mu}}^{d-1} & \left(t_{i_{\mu}}^{\epsilon}\right)^{d-1} & s_{j_{1}}^{d-1} & \left(s_{j_{1}}^{\epsilon}\right)^{d-1} & \cdots & s_{j_{\nu}}^{d-1} & \left(s_{j_{\nu}}^{\epsilon}\right)^{d-1} \\
1 & 0 & 0 & \cdots & 0 & 0 & 1 & 1 & \cdots & 1 & 1
\end{array}\right| .
\end{aligned}
$$

As for $\boldsymbol{u} \in U_{1} \backslash U$, observe that we can transform $H_{U}(\boldsymbol{u})$ in the form of the determinant $D_{k, \ell}(\tau)$ of Lemma 15, where now $k=2 \mu$ and $\ell=2 \nu+1$, by means of the following determinant transformations:
(i) Subtract the last row of $H_{U}(\boldsymbol{u})$ from the first.
(ii) Shift the first column of $H_{U}(\boldsymbol{u})$ to the right, so that the non-zero values of the third row of $H_{U}(\boldsymbol{u})$ occupy columns $2 \mu+1$ through $d+2$ and are in increasing order. This has to be done by an even number of column swaps, since we have to shift through the first $2 \mu$ columns, and since $s$ cannot be between some $s_{j_{k}}$ and $s_{j_{k}}^{\epsilon}$ (due to the way we have chosen $\epsilon$ ).
(iii) Shift the last row of $H_{U}(\boldsymbol{u})$ up, so as to become the third row of $H_{U}(\boldsymbol{u})$. This can be done by $d-1$ row swaps, which implies that the sign of the determinant does not change (recall that $d$ is odd).

We thus conclude that, for any specific choice of $U$, and for any specific point $\boldsymbol{u} \in\left(U_{1} \cup U_{2}\right) \backslash U$, there exists some $\tau_{0}>0$ (cf. Lemma 15) that depends on $\boldsymbol{u}$ and $U$, such that for all $\tau \in\left(0, \tau_{0}\right)$, $H_{U}(\boldsymbol{u})>0$.

Since the total number of subsets $U$ is $\binom{n_{1}+n_{2}}{\left\lfloor\frac{d+1}{2}\right\rfloor}-\binom{n_{1}}{\left(\frac{d+1}{2}\right\rfloor}-\binom{n_{2}}{\left\lfloor\frac{d+1}{2}\right\rfloor}$, while for each such subset $U$ we need to consider the $\left(n_{1}+n_{2}-\left\lfloor\frac{d+1}{2}\right\rfloor\right)$ vertices in $\left(U_{1} \cup U_{2}\right) \backslash U$, it suffices to consider a value $\tau^{\star}$ for $\tau$ that is small enough, so that all $\left(n_{1}+n_{2}-\left\lfloor\frac{d+1}{2}\right\rfloor\right)\left[\binom{n_{1}+n_{2}}{\left\lfloor\frac{d+1}{2}\right\rfloor}-\binom{n_{1}}{\left\lfloor\frac{d+1}{2}\right\rfloor}-\binom{n_{2}}{\left\lfloor\frac{d+1}{2}\right\rfloor}\right]$ possible determinants $H_{U}(\boldsymbol{u})$ are strictly positive. Call $U_{j}^{\star}, j=1,2$, the vertex sets we get for $\tau=\tau^{\star}, Q_{j}^{\star}$ the corresponding polytopes, and $Q^{\star}$ the resulting convex hulld . Our analysis immediately implies that for each $U^{\star} \subseteq U_{1}^{\star} \cup U_{2}^{\star}$, where $U^{\star} \cap U_{j}^{\star} \neq \emptyset, j=1,2$, the equation $H_{U^{\star}}(\boldsymbol{x})=0, \boldsymbol{x} \in \mathbb{E}^{d+1}$, is the equation of a supporting hyperplane for $Q^{\star}$ passing through the vertices of $U^{\star}$ (and those only). In other words, every subset of $U^{\star}$ of $U_{1}^{\star} \cup U_{2}^{\star}$, where $\left|U^{\star}\right|=\left\lfloor\frac{d+1}{2}\right\rfloor, U^{\star} \cap U_{j}^{\star} \neq \emptyset, j=1,2$, defines a $\left\lfloor\frac{d}{2}\right\rfloor$-face of $Q^{\star}$, which means that $Q^{\star}$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, U_{1}^{\star}\right)$-bineighborly.

We are now ready to perform the last step of our construction. We assume we have chosen $\tau$ to be equal to $\tau^{\star}$, and, as in the proof of Lemma [16, call $U_{j}^{\star}, Q_{j}^{\star}, j=1,2$, the corresponding vertex sets and $(d-1)$-polytopes. Finally, call $Q^{\star}$ the convex hull of $Q_{1}^{\star}$ and $Q_{2}^{\star}$, i.e., $Q^{\star}=C H_{d+1}\left(\left\{Q_{1}^{\star}, Q_{2}^{\star}\right\}\right)$. We perturb the vertex sets $U_{1}^{\star}$ and $U_{2}^{\star}$, to get the vertex sets $V_{1}$ and $V_{2}$ by considering vertices on the curves $\gamma_{1}(t ; \zeta)$ and $\gamma_{2}(t ; \zeta)$, with $\zeta>0$ instead of the curves $\gamma_{1}(t)$ and $\gamma_{2}(t)$ (cf. (52)). More precisely, define the sets $V_{1}$ and $V_{2}$ as:

$$
\begin{align*}
& V_{1}=\left\{\gamma_{1}\left(\alpha_{1} \tau^{\star} ; \zeta\right), \gamma_{1}\left(\alpha_{2} \tau^{\star} ; \zeta\right), \ldots, \gamma_{1}\left(\alpha_{n_{1}} \tau^{\star} ; \zeta\right)\right\}, \quad \text { and } \\
& V_{2}=\left\{\gamma_{2}\left(\beta_{1} ; \zeta\right), \gamma_{2}\left(\beta_{2} ; \zeta\right), \ldots, \gamma_{2}\left(\beta_{n_{2}} ; \zeta\right)\right\}, \tag{55}
\end{align*}
$$

where $\zeta>0$. Let $P_{j}$ be the convex hull of the vertices in $V_{j}, j=1,2$, and notice that $P_{j}$ is a $\left\lfloor\frac{d}{2}\right\rfloor$-neighborly $d$-polytope. Let $P=C H_{d+1}\left(\left\{P_{1}, P_{2}\right\}\right)$, and let $\mathcal{F}_{P}$ be the set of proper faces of $P$ with non-empty intersection with $\tilde{\Pi}$, i.e., $\mathcal{F}_{P}$ consists of all the proper faces of $P$, the vertex set of which has non-empty intersection with both $V_{1}$ and $V_{2}$. The following lemma establishes the final step of our construction. In view of Theorem [13] it also establishes the tightness of our bounds for all face numbers of $P_{1} \oplus P_{2}$.

Lemma 17. There exists a sufficiently small positive value $\zeta^{\star}$ for $\zeta$, such that the $(d+1)$-polytope $P$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, V_{1}\right)$-bineighborly.

Proof. Similarly to what we have done in the proof of Lemma 16, let $t_{i}=\alpha_{i} \tau^{\star}, t_{i}^{\epsilon}=\left(\alpha_{i}+\epsilon\right) \tau^{\star}$, $1 \leq i \leq n_{1}$, and $s_{i}=\beta_{i}, s_{i}^{\epsilon}=\beta_{i}+\epsilon, 1 \leq i \leq n_{2}$, where $\epsilon>0$ is chosen such that $\alpha_{i}+\epsilon<\alpha_{i+1}$, for all $1 \leq i<n_{1}$, and $\beta_{i}+\epsilon<\beta_{i+1}$, for all $1 \leq i<n_{2}$.

Choose $V$ a subset of $V_{1} \cup V_{2}$ of size $\left\lfloor\frac{d+1}{2}\right\rfloor$, such that $V \cap V_{j} \neq \emptyset, j=1,2$. Denote by $\mu$ (resp., $\nu$ ) the cardinality of $V \cap V_{1}$ (resp., $V \cap V_{2}$ ). Considering $\zeta$ as a small positive parameter,

[^3]let $\gamma_{1}\left(t_{i_{1}} ; \zeta\right), \gamma_{1}\left(t_{i_{2}} ; \zeta\right), \ldots, \gamma_{1}\left(t_{i_{\mu}} ; \zeta\right)$ be the vertices in $V \cap V_{1}$, where $i_{1}<i_{2}<\ldots<i_{\mu}$, and analogously, let $\gamma_{2}\left(s_{j_{1}} ; \zeta\right), \gamma_{2}\left(s_{j_{2}} ; \zeta\right), \ldots, \gamma_{2}\left(s_{j_{\nu}} ; \zeta\right)$ be the vertices in $V \cap V_{2}$, where $j_{1}<j_{2}<\ldots<$ $j_{\nu}$. Let $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{d+1}\right)$ and define the $(d+2) \times(d+2)$ determinant $F_{V}(\boldsymbol{x} ; \zeta)$ as:
\[

F_{V}(\boldsymbol{x} ; \zeta)=\left|$$
\begin{array}{cccccccc}
1 & 1 & 1 & \cdots & 1 & 1 & 1 & \cdots  \tag{56}\\
\boldsymbol{x} & \gamma_{1}\left(t_{i_{1}} ; \zeta\right) & \gamma_{1}\left(t_{i_{1}}^{\epsilon} ; \zeta\right) & \cdots & \gamma_{1}\left(t_{i_{\mu}}^{\epsilon} ; \zeta\right) & \gamma_{2}\left(s_{j_{1}} ; \zeta\right) & \gamma_{2}\left(s_{j_{1}}^{\epsilon} ; \zeta\right) & \cdots \\
\gamma_{2}\left(s_{j_{\nu}}^{\epsilon} ; \zeta\right)
\end{array}
$$\right| .
\]

The equation $F_{V}(\boldsymbol{x} ; \zeta)=0$ is the equation of a hyperplane in $\mathbb{E}^{d+1}$ that passes through the points in $V$. We claim that for all vertices $\boldsymbol{v} \in\left(V_{1} \cup V_{2}\right) \backslash V$, we have $F_{V}(\boldsymbol{v} ; \zeta)>0$ for sufficiently small $\zeta$.

Indeed, let $U^{\star}$ denote the set of vertices in $U_{1}^{\star} \cup U_{2}^{\star}$ that correspond to the vertices in $V$, i.e., $U^{\star}$ contains the projections, on the hyperplanes $\left\{x_{2}=0\right\}$ or $\left\{x_{1}=0\right\}$ of $\mathbb{E}^{d+1}$, of the vertices in $V$, depending on whether these vertices belong to $V_{1}$ or $V_{2}$, respectively. Choose some $\boldsymbol{v} \in\left(V_{1} \cup V_{2}\right) \backslash V$. If $\boldsymbol{v} \in V_{1} \backslash V, \boldsymbol{v}$ is of the form $\boldsymbol{v}=\gamma_{1}\left(t_{i} ; \zeta\right), \zeta>0$, for some $i \notin\left\{i_{1}, i_{2}, \ldots, i_{\mu}\right\}$, whereas if $\boldsymbol{v} \in V_{2} \backslash V, \boldsymbol{v}$ is of the form $\boldsymbol{v}=\gamma_{2}\left(s_{j} ; \zeta\right), \zeta>0$, for some $j \notin\left\{j_{1}, j_{2}, \ldots, j_{\nu}\right\}$. In the former case, let $\boldsymbol{u}^{\star}=\gamma_{1}\left(t_{i}\right)=\gamma_{1}\left(t_{i} ; 0\right)$, whereas, in the latter case, let $\boldsymbol{u}^{\star}=\gamma_{2}\left(s_{j}\right)=\gamma_{2}\left(s_{j} ; 0\right)$. In more geometric terms, we define $\boldsymbol{u}^{\star}$ to be the projection of $\boldsymbol{v}$ on the hyperplanes $\left\{x_{2}=0\right\}$ or $\left\{x_{1}=0\right\}$ of $\mathbb{E}^{d+1}$, depending in whether $\boldsymbol{v}$ belongs to $V_{1} \backslash V$ or $V_{2} \backslash V$, respectively, or, equivalently, $\boldsymbol{u}^{\star}$ is the (unperturbed) vertex in $\left(U_{1}^{\star} \cup U_{2}^{\star}\right) \backslash U^{\star}$ that corresponds to $\boldsymbol{v}$. Observe that $F_{V}(\boldsymbol{v} ; \zeta)$ is a polynomial function in $\zeta$, and thus it is continuous with respect to $\zeta$ for any $\zeta \in \mathbb{R}$. This implies that

$$
\begin{equation*}
\lim _{\zeta \rightarrow 0^{+}} F_{V}(\boldsymbol{v} ; \zeta)=F_{U^{\star}}(\boldsymbol{u} ; 0)=H_{U^{\star}}\left(\boldsymbol{u}^{\star}\right), \tag{57}
\end{equation*}
$$

where we used the fact that $\lim _{\zeta \rightarrow 0^{+}} \boldsymbol{v}=\boldsymbol{u}^{\star}$, and observed that $F_{U^{\star}}\left(\boldsymbol{u}^{\star} ; 0\right)=H_{U^{\star}}\left(\boldsymbol{u}^{\star}\right)$, where $H_{U^{\star}}\left(\boldsymbol{u}^{\star}\right)$ is the determinant in relation (54) in the proof of Lemma 16. Since $H_{U^{\star}}\left(\boldsymbol{u}^{\star}\right)>0$ (recall that we have chosen $\tau$ to be equal to $\tau^{\star}$ ), we conclude, from (57), that there exists some $\zeta_{0}>0$ that depends on $\boldsymbol{v}$ and $V$, such that for all $\zeta \in\left(0, \zeta_{0}\right), F_{V}(\boldsymbol{v} ; \zeta)>0$.

Since the total number of subsets $V$ is $\binom{n_{1}+n_{2}}{\left\lfloor\frac{d+1}{2}\right\rfloor}-\binom{n_{1}}{\left(\frac{d+1}{2}\right\rfloor}-\binom{n_{2}}{\left(\frac{d+1}{2}\right\rfloor}$, while for each such subset $V$ we need to consider the $\left(n_{1}+n_{2}-\left\lfloor\frac{d+1}{2}\right\rfloor\right)$ vertices in $\left(V_{1} \cup V_{2}\right) \backslash V$, it suffices to consider a value
 determinants $F_{V}(\boldsymbol{v} ; \zeta)$ are strictly positive. Call $V_{j}^{\star}, j=1,2$, the vertex sets we get for $\zeta=\zeta^{\star}$, $P_{j}^{\star}$ the corresponding polytopes, and $P^{\star}$ the resulting convex hull. Then, for each $V^{\star} \subseteq V_{1}^{\star} \cup V_{2}^{\star}$, where $V^{\star} \cap V_{j}^{\star} \neq \emptyset, j=1,2$, the equation $F_{V^{\star}}\left(\boldsymbol{x} ; \zeta^{\star}\right)=0, \boldsymbol{x} \in \mathbb{E}^{d+1}$, is the equation of a supporting hyperplane for $P^{\star}$ passing through the vertices of $V^{\star}$ (and those only). In other words, every subset of $V^{\star}$ of $V_{1}^{\star} \cup V_{2}^{\star}$, where $\left|V^{\star}\right|=\left\lfloor\frac{d+1}{2}\right\rfloor, V^{\star} \cap V_{j}^{\star} \neq \emptyset, j=1,2$, defines a $\left\lfloor\frac{d}{2}\right\rfloor$-face of $P^{\star}$, which means that $P^{\star}$ is $\left(\left\lfloor\frac{d+1}{2}\right\rfloor, V_{1}^{\star}\right)$-bineighborly.

We are now ready to state the second main theorem of this section, that concerns the tightness of our upper bounds on the number of $k$-faces of the Minkowski sum of two $d$-polytopes for all $0 \leq k \leq d-1$ and for all odd dimensions $d \geq 3$.

Theorem 18. Let $d \geq 3$ and $d$ is odd. There exist two $\left\lfloor\frac{d}{2}\right\rfloor$-neighborly $d$-polytopes $P_{1}$ and $P_{2}$ with $n_{1}$ and $n_{2}$ vertices, respectively, such that, for all $1 \leq k \leq d$ :
$f_{k-1}\left(P_{1} \oplus P_{2}\right)=f_{k}\left(C_{d+1}\left(n_{1}+n_{2}\right)\right)-\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k+1-i}\left(\binom{n_{1}-d-2+i}{i}+\binom{n_{2}-d-2+i}{i}\right)$,
where $C_{d}(n)$ stands for the cyclic d-polytope with $n$ vertices.

## 6 Summary and open problems

In this paper we have computed the maximum number of $k$-faces, $f_{k}\left(P_{1} \oplus P_{2}\right), 0 \leq k \leq d-1$ of the Minkowski sum of two $d$-polytopes $P_{1}$ and $P_{2}$ as a function of the number of vertices $n_{1}$ and $n_{2}$ of these two polytopes. In even dimensions $d \geq 2$, these maximal values are attained if $P_{1}$ and $P_{2}$ are cyclic $d$-polytopes with disjoint vertex sets. In odd dimensions $d \geq 3$, the lower bound construction is more intricate. Denoting by $\gamma_{1}(t ; \zeta)$ and $\gamma_{2}(t ; \zeta)$ the $d$-dimensional momentlike curves $\left(t, \zeta t^{d}, t^{2}, t^{3}, \ldots, t^{d-1}\right)$ and $\left(\zeta t^{d}, t, t^{2}, t^{3}, \ldots, t^{d-1}\right)$, where $t>0$ and $\zeta>0$, we have shown that these maximum values are attained if $P_{1}$ and $P_{2}$ are the $d$-polytopes with vertex sets $V_{1}=\left\{\gamma_{1}\left(\alpha_{i} \tau^{\star} ; \zeta^{\star}\right) \mid i=1, \ldots, n_{1}\right\}$ and $V_{2}=\left\{\gamma_{2}\left(\beta_{j} ; \zeta^{\star}\right) \mid j=1, \ldots, n_{2}\right\}$, respectively, where $0<\alpha_{1}<\alpha_{2}<\ldots<\alpha_{n_{1}}, 0<\beta_{1}<\beta_{2}<\ldots<\beta_{n_{2}}$, and $\tau^{\star}, \zeta^{\star}$ are appropriately chosen, sufficiently small, positive parameters.

The obvious open problem is to extend our results for the Minkowski sum of $r d$-polytopes in $\mathbb{E}^{d}$, for $r \geq 3$ and $d \geq 4$. A related problem is to express the number of $k$-faces of the Minkowski sum of $r d$-polytopes in terms of the number of facets of these polytopes. Results in this direction are known for $d=2$ and $d=3$ only (see the introductory section and [3] for the 3-dimensional case). We would like to derive such expressions for any $d \geq 4$ and any number, $r$, of summands.

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## A The summation operator

Let $\mathcal{Y}$ be either $\mathcal{F}$ or a pure simplicial subcomplex of $\partial Q$. Below we compute the action of the operator $\mathcal{S}_{k}(\cdot ; \delta, \nu)$ on $\mathcal{Y}$, for $\nu \in\{1,2\}$ and when $\mathcal{Y}$ is either $\delta$ - or $(\delta-1)$-dimensional.

Recall the action of the operator $\mathcal{S}_{k}(\cdot ; \delta, \nu)$ on $\mathcal{Y}$ :

$$
\mathcal{S}_{k}(\mathcal{Y} ; \delta, \nu)=\sum_{i=1}^{\delta}(-1)^{k-i}\binom{\delta-i}{\delta-k} f_{i-\nu}(\mathcal{Y}),
$$

and consider first the case where $\mathcal{Y}$ is $\delta$-dimensional and $\nu=1$. In this case we have:

$$
\begin{align*}
\mathcal{S}_{k}(\mathcal{Y} ; \delta, 1) & =\sum_{i=1}^{\delta}(-1)^{k-i}\binom{\delta-i}{\delta-k} f_{i-1}(\mathcal{Y}) \\
& =\sum_{i=0}^{\delta}(-1)^{k-i}\binom{\delta-i}{\delta-k} f_{i-1}(\mathcal{Y})-(-1)^{k}\binom{\delta}{\delta-k} f_{-1}(\mathcal{Y})  \tag{58}\\
& =h_{k}(\mathcal{Y})-(-1)^{k}\binom{\delta}{\delta-k} f_{-1}(\mathcal{Y}) .
\end{align*}
$$

If $\mathcal{Y}$ is $(\delta-1)$-dimensional and $\nu=1$, we have:

$$
\begin{align*}
\mathcal{S}_{k}(\mathcal{Y} ; \delta, 1) & =\sum_{i=1}^{\delta}(-1)^{k-i}\binom{\delta-i}{\delta-k} f_{i-1}(\mathcal{Y}) \\
& =\sum_{i=1}^{\delta}(-1)^{k-i}\left(\binom{\delta-i-1}{\delta-k}+\binom{\delta-i-1}{\delta-k-1}\right) f_{i-1}(\mathcal{Y}) \\
& =-\sum_{i=1}^{\delta}(-1)^{(k-1)-i}\binom{\delta-1-i}{\delta-1-(k-1)} f_{i-1}(\mathcal{Y})+\sum_{i=1}^{\delta}(-1)^{k-i}\binom{\delta-1-i}{\delta-1-k} f_{i-1}(\mathcal{Y})  \tag{59}\\
& =-h_{k-1}(\mathcal{Y})-(-1)^{k}\binom{\delta-1}{\delta-k} f_{-1}(\mathcal{Y})+h_{k}(\mathcal{Y})-(-1)^{k}\binom{\delta-1}{\delta-1-k} f_{-1}(\mathcal{Y}) \\
& =h_{k}(\mathcal{Y})-h_{k-1}(\mathcal{Y})-(-1)^{k}\binom{\delta}{\delta-k} f_{-1}(\mathcal{Y}) .
\end{align*}
$$

Finally, if $\mathcal{Y}$ is $(\delta-1)$-dimensional and $\nu=2$, we have:

$$
\begin{align*}
\mathcal{S}_{k}(\mathcal{Y} ; \delta, 2) & =\sum_{i=1}^{\delta}(-1)^{k-i}\binom{\delta-i}{\delta-k} f_{i-2}(\mathcal{Y}) \\
& =\sum_{i=0}^{\delta-1}(-1)^{(k-1)-i}\binom{\delta-1-i}{\delta-1-(k-1)} f_{i-1}(\mathcal{Y})  \tag{60}\\
& =h_{k-1}(\mathcal{Y})
\end{align*}
$$

## B Proof of an identity

In this section we prove the following identity used in Section 4 to prove the upper bound for $f_{k-1}(\mathcal{F})$ (see relations (49) and (50)).

Lemma 19. For any $d \geq 2$, and any sequence of numbers $\alpha_{i}$, where $0 \leq i \leq\left\lfloor\frac{d+1}{2}\right\rfloor$, we have:

$$
\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i} \alpha_{i}+\sum_{i=0}^{\left\lfloor\frac{d}{2}\right\rfloor}\binom{i}{k-d-1+i} \alpha_{i}=\sum_{i=0}^{\frac{d+1}{2}} *\left(\binom{d+1-i}{k-i}+\binom{i}{k-d-1+i}\right) \alpha_{i} .
$$

Proof. We start by recalling the definition of the symbol $\sum_{i=0}^{\frac{\delta}{2}}{ }^{*} T_{i}$. This symbol denotes the sum of
the elements $T_{0}, T_{1}, \ldots, T_{\left\lfloor\frac{\delta}{2}\right\rfloor}$, where the last term is halved if $\delta$ is even. More precisely:

$$
\sum_{i=0}^{\frac{\delta}{2}}{ }^{*} T_{i}= \begin{cases}T_{0}+T_{1}+\ldots+T_{\left\lfloor\frac{\delta}{2}\right\rfloor-1}+\frac{1}{2} T_{\left\lfloor\frac{\delta}{2}\right\rfloor} & \text { if } \delta \text { is even } \\ T_{0}+T_{1}+\ldots+T_{\left\lfloor\frac{\delta}{2}\right\rfloor-1}+T_{\left\lfloor\frac{\delta}{2}\right\rfloor} & \text { if } \delta \text { is odd }\end{cases}
$$

Let us now first consider the case $d$ odd. In this case $d+1$ is even, and we have:

$$
\begin{aligned}
\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i} \alpha_{i} & +\sum_{i=0}^{\left\lfloor\left\lfloor\frac{d}{2}\right\rfloor\right.}\binom{i}{k-d-1+i} \alpha_{i}=\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i} \alpha_{i}+\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor-1}\binom{i}{k-d-1+i} \alpha_{i} \\
& =\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor-1}\left(\binom{d+1-i}{k-i}+\binom{i}{k-d-1+i}\right) \alpha_{i}+\binom{d+1-\left\lfloor\frac{d+1}{2}\right\rfloor}{ k-\left\lfloor\frac{d+1}{2}\right\rfloor} \alpha_{\left\lfloor\frac{d+1}{2}\right\rfloor} \\
& =\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor-1}\left(\binom{d+1-i}{k-i}+\binom{i}{k-d-1+i}\right) \alpha_{i}+\frac{1}{2}\left(\binom{d+1-\left\lfloor\frac{d+1}{2}\right\rfloor}{ k-\left\lfloor\frac{d+1}{2}\right\rfloor}+\left(\begin{array}{c}
\left.\left.\begin{array}{c}
\left.\frac{d+1}{2}\right\rfloor \\
k-d-1+\left\lfloor\frac{d+1}{2}\right\rfloor
\end{array}\right)\right) \alpha_{\left\lfloor\frac{d+1}{2}\right\rfloor} \\
\end{array} \sum_{i=0}^{\frac{d+1}{2}} *\left(\binom{d+1-i}{k-i}+\binom{i}{k-d-1+i}\right) \alpha_{i}\right.\right.
\end{aligned}
$$

The case $d$ even is even simpler to prove. In this case $d+1$ is odd, hence:

$$
\begin{aligned}
\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i}{k-i} \alpha_{i}+\sum_{i=0}^{\left\lfloor\frac{d}{2}\right\rfloor}\binom{i}{k-d-1+i} \alpha_{i} & =\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{d+1-i-i}{k-i} \alpha_{i}+\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\binom{i}{k-d-1+i} \alpha_{i} \\
& =\sum_{i=0}^{\left\lfloor\frac{d+1}{2}\right\rfloor}\left(\binom{d+1-i}{k-i}+\binom{i}{k-d-1+i}\right) \alpha_{i} \\
& =\sum_{i=0}^{\frac{d+1}{2}}{ }^{*}\left(\binom{d+1-i}{k-i}+\binom{i}{k-d-1+i}\right) \alpha_{i}
\end{aligned}
$$

This completes the proof.

## C Proof of Lemma 15

We start by introducing what is known as Laplace's Expansion Theorem for determinants (see [6, 10] for details and proofs). Consider a $n \times n$ matrix $A$. Let $\boldsymbol{r}=\left(r_{1}, r_{2}, \ldots, r_{k}\right)$, be a vector of $k$ row indices for $A$, where $1 \leq k<n$ and $1 \leq r_{1}<r_{2}<\ldots<r_{k} \leq n$. Let $\boldsymbol{c}=\left(c_{1}, c_{2}, \ldots, c_{k}\right)$ be a vector of $k$ column indices for $A$, where $1 \leq k<n$ and $1 \leq c_{1}<c_{2}<\ldots<c_{k} \leq n$. We denote by $S(A ; \boldsymbol{r}, \boldsymbol{c})$ the $k \times k$ submatrix of $A$ constructed by keeping the entries of $A$ that belong to a row in $\boldsymbol{r}$ and a column in $\boldsymbol{c}$. The complementary submatrix for $S(A ; \boldsymbol{r}, \boldsymbol{c})$, denoted by $\bar{S}(A ; \boldsymbol{r}, \boldsymbol{c})$, is the $(n-k) \times(n-k)$ submatrix of $A$ constructed by removing the rows and columns of $A$ in $\boldsymbol{r}$ and $\boldsymbol{c}$, respectively. Then, the determinant of $A$ can be computed by expanding in terms of the $k$ columns of $A$ in $\boldsymbol{c}$ according to the following theorem.

Theorem 20 (Laplace's Expansion Theorem). Let $A$ be a $n \times n$ matrix. Let $\boldsymbol{c}=\left(c_{1}, c_{2}, \ldots, c_{k}\right)$ be a vector of $k$ column indices for $A$, where $1 \leq k<n$ and $1 \leq c_{1}<c_{2}<\ldots<c_{k} \leq n$. Then:

$$
\begin{equation*}
\operatorname{det}(A)=\sum_{\boldsymbol{r}}(-1)^{|\boldsymbol{r}|+|\boldsymbol{c}|} \operatorname{det}(S(A ; \boldsymbol{r}, \boldsymbol{c})) \operatorname{det}(\bar{S}(A ; \boldsymbol{r}, \boldsymbol{c})) \tag{61}
\end{equation*}
$$

where $|\boldsymbol{r}|=r_{1}+r_{2}+\ldots+r_{k},|\boldsymbol{c}|=c_{1}+c_{2}+\ldots+c_{k}$, and the summation is taken over all row vectors $\boldsymbol{r}=\left(r_{1}, r_{2}, \ldots, r_{k}\right)$ of $k$ row indices for $A$, where $1 \leq r_{1}<r_{2}<\ldots<r_{k} \leq n$.

The next item that will be useful is some notation and discussion about Vandermonde and generalized Vandermonde determinants. Given a vector of $n \geq 2$ real numbers $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$, the Vandermonde determinant $\operatorname{VD}(\boldsymbol{x})$ of $\boldsymbol{x}$ is the $n \times n$ determinant

$$
\mathrm{VD}(\boldsymbol{x})=\left|\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
x_{1} & x_{2} & \cdots & x_{n} \\
x_{1}^{2} & x_{2}^{2} & \cdots & x_{n}^{2} \\
\vdots & \vdots & & \vdots \\
x_{1}^{n-1} & x_{2}^{n-1} & \cdots & x_{n}^{n-1}
\end{array}\right|=\prod_{1 \leq i<j \leq n}\left(x_{j}-x_{i}\right)
$$

From the above expression, it is readily seen that if the elements of $\boldsymbol{x}$ are in strictly increasing order, then $\operatorname{VD}(\boldsymbol{x})>0$. A generalization of the Vandermonde determinant is the generalized Vandermonde determinant: if, in addition to $\boldsymbol{x}$, we specify a vector of exponents $\boldsymbol{\mu}=\left(\mu_{1}, \mu_{2}, \ldots, \mu_{n}\right)$, where we require that $0 \leq \mu_{1}<\mu_{2}<\ldots<\mu_{n}$, we can define the generalized Vandermonde determinant $\operatorname{GVD}(\boldsymbol{x} ; \boldsymbol{\mu})$ as the $n \times n$ determinant:

$$
\operatorname{GVD}(\boldsymbol{x} ; \boldsymbol{\mu})=\left|\begin{array}{cccc}
x_{1}^{\mu_{1}} & x_{2}^{\mu_{1}} & \cdots & x_{n}^{\mu_{1}} \\
x_{1}^{\mu_{2}} & x_{2}^{\mu_{2}} & \cdots & x_{n}^{\mu_{2}} \\
x_{1}^{\mu_{3}} & x_{2}^{\mu_{3}} & \cdots & x_{n}^{\mu_{3}} \\
\vdots & \vdots & & \vdots \\
x_{1}^{\mu_{n}} & x_{2}^{\mu_{n}} & \cdots & x_{n}^{\mu_{n}}
\end{array}\right| .
$$

It is a well-known fact that, if the elements of $\boldsymbol{x}$ are in strictly increasing order, then $\operatorname{GVD}(\boldsymbol{x} ; \boldsymbol{\mu})>0$ (for example, see [7] for a proof of this fact).

Before proceeding with the proof of Lemma 15 we need to introduce some additional notation concerning vectors. We denote by $\boldsymbol{e}_{i}$ the vector whose elements are zero except for the $i$-th element, which is equal to 1 . Given two vectors of size $n \boldsymbol{a}=\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ and $\boldsymbol{b}=\left(b_{1}, b_{2}, \ldots, b_{n}\right)$, we denote by $\boldsymbol{a}-\boldsymbol{b}$ the vector we get by element-wise subtracting the elements of the second vector from the elements of the first, i.e., $\boldsymbol{a}-\boldsymbol{b}=\left(a_{1}-b_{1}, a_{2}-b_{2}, \ldots, a_{n}-b_{n}\right)$. Finally, given some $t \in \mathbb{R}$, and a vector $\boldsymbol{x}=\left(x_{1}, x_{2}, \ldots, x_{n}\right)$, we denote by $t \boldsymbol{x}$ the vector $\left(t x_{1}, t x_{2}, \ldots, t x_{n}\right)$.

We now restate Lemma 15 and prove it.
Lemma 15. Fix two integers $k \geq 2$ and $\ell \geq 2$, such that $k+\ell$ is odd. Let $D_{k, \ell}(\tau)$ be the $(k+\ell) \times(k+\ell)$ determinant:

$$
D_{k, \ell}(\tau)=\left|\begin{array}{cccccccc}
1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
x_{1} \tau & x_{2} \tau & \cdots & x_{k} \tau & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 & y_{1} & y_{2} & \cdots & y_{\ell} \\
x_{1}^{2} \tau^{2} & x_{2}^{2} \tau^{2} & \cdots & x_{k}^{2} \tau^{2} & y_{1}^{2} & y_{2}^{2} & \cdots & y_{\ell}^{2} \\
x_{1}^{3} \tau^{3} & x_{2}^{3} \tau^{3} & \cdots & x_{k}^{3} \tau^{3} & y_{1}^{3} & y_{2}^{3} & \cdots & y_{\ell}^{3} \\
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
x_{1}^{m} \tau^{m} & x_{2}^{m} \tau^{m} & \cdots & x_{k}^{m} \tau^{m} & y_{1}^{m} & y_{2}^{m} & \cdots & y_{\ell}^{m}
\end{array}\right|, \quad m=k+\ell-3,
$$

where $0<x_{1}<x_{2}<\ldots<x_{k}, 0<y_{1}<y_{2}<\ldots<y_{\ell}$, and $\tau>0$. Then, there exists some $\tau_{0}>0$ (that depends on the $x_{i}$ 's, the $y_{i}$ 's, $k$, and $\ell$ ) such that for all $\tau \in\left(0, \tau_{0}\right)$, the determinant $D_{k, \ell}(\tau)$ is strictly positive.
Proof. We denote by $\Delta_{k, \ell}(\tau)$ the matrix corresponding to the determinant $D_{k, \ell}(\tau)$. We are now going to apply Laplace's Expansion Theorem to evaluate $D_{k, \ell}(\tau)$ in terms of the first $k$ columns of $\Delta_{k, \ell}(\tau)$. Note that in this case $\boldsymbol{c}=(1,2, \ldots, k)$, so we get:

$$
\begin{align*}
D_{k, \ell}(\tau) & =\sum_{\boldsymbol{r}}(-1)^{|\boldsymbol{r}|+|\boldsymbol{c}|} \operatorname{det}\left(S\left(\Delta_{k, \ell}(\tau) ; \boldsymbol{r}, \boldsymbol{c}\right)\right) \operatorname{det}\left(\bar{S}\left(\Delta_{k, \ell}(\tau) ; \boldsymbol{r}, \boldsymbol{c}\right)\right) \\
& =(-1)^{\frac{k(k+1)}{2}} \sum_{\boldsymbol{r}}(-1)^{|\boldsymbol{r}|} \operatorname{det}\left(S\left(\Delta_{k, \ell}(\tau) ; \boldsymbol{r}, \boldsymbol{c}\right)\right) \operatorname{det}\left(\bar{S}\left(\Delta_{k, \ell}(\tau) ; \boldsymbol{r}, \boldsymbol{c}\right)\right) . \tag{62}
\end{align*}
$$

It is easy to verify that the above sum consists of $\binom{k+\ell}{k}$ terms. Observe that, among these terms:
(i) all terms for which $\boldsymbol{r}$ contains the third or the fourth row vanish (the corresponding row of $S\left(\Delta_{k, \ell}(\tau) ; \boldsymbol{r}, \boldsymbol{c}\right)$ consists of zeros), and
(ii) all terms for which $\boldsymbol{r}$ does not contain the first or the second row vanish (in this case there exists at least one row of $\bar{S}\left(\Delta_{k, \ell}(\tau) ; \boldsymbol{r}, \boldsymbol{c}\right)$ that consists of zeros).
The remaining terms of the expansion are the $\binom{k+\ell-4}{k-2}$ terms for which $\boldsymbol{r}=\left(1,2, r_{3}, r_{4}, \ldots, r_{k}\right)$, with $5 \leq r_{3}<r_{4}<\ldots<r_{k} \leq k+\ell$. For any given such $\boldsymbol{r}$, we have that:
(i) $\operatorname{det}\left(S\left(\Delta_{k, \ell}(\tau), \boldsymbol{r}, \boldsymbol{c}\right)\right)$ is the $k \times k$ generalized Vandermonde determinant GVD $(\tau \boldsymbol{x} ; \boldsymbol{r}-\boldsymbol{\alpha})$, where $\tau \boldsymbol{x}=\left(\tau x_{1}, \tau x_{2}, \ldots, \tau x_{k}\right), \boldsymbol{\alpha}=(1,1,3,3, \ldots, 3)=\boldsymbol{e}_{1}+\boldsymbol{e}_{2}+3 \sum_{i=3}^{k} \boldsymbol{e}_{i}$, and
(ii) $\operatorname{det}\left(\bar{S}\left(\Delta_{k, \ell}(\tau), \boldsymbol{r}, \boldsymbol{c}\right)\right)$ is the $\ell \times \ell$ generalized Vandermonde determinant $\operatorname{GVD}(\boldsymbol{y} ; \overline{\boldsymbol{r}}-\boldsymbol{\beta})$, where $\overline{\boldsymbol{r}}$ is the vector of the $\ell$, among the $k+\ell$, row indices for $\Delta_{k, \ell}(\tau)$ that do not belong to $\boldsymbol{r}$, and $\boldsymbol{\beta}=(3,3, \ldots, 3)=3 \sum_{i=1}^{\ell} \boldsymbol{e}_{i}$.

We can, thus, simplify the expansion in (62) to get:

$$
\begin{equation*}
D_{k, \ell}(\tau)=(-1)^{\frac{k(k+1)}{2}} \sum_{\substack{r=\left(1,2, r_{3}, \ldots, r_{k}\right) \\ 5 \leq r_{3}<r_{4}<\ldots<r_{k} \leq k+\ell}}(-1)^{|\boldsymbol{r}|} \operatorname{GVD}(\tau \boldsymbol{x} ; \boldsymbol{r}-\boldsymbol{\alpha}) \operatorname{GVD}(\boldsymbol{y} ; \overline{\boldsymbol{r}}-\boldsymbol{\beta}) \tag{63}
\end{equation*}
$$

Notice that $\operatorname{GVD}(\tau \boldsymbol{x} ; \boldsymbol{r}-\boldsymbol{\alpha})=\tau^{|\boldsymbol{r}-\boldsymbol{\alpha}|} \operatorname{GVD}(\boldsymbol{x} ; \boldsymbol{r}-\boldsymbol{\alpha})=\tau^{|\boldsymbol{r}|-|\boldsymbol{\alpha}|} \operatorname{GVD}(\boldsymbol{x} ; \boldsymbol{r}-\boldsymbol{\alpha})$. This means that the minimum exponent for $\tau$ is attained when $|\boldsymbol{r}|$ is minimal, which is the case when $\boldsymbol{r}$ is equal to $\boldsymbol{\rho}=(1,2,5,6, \ldots, k+2)$. For this value for $\boldsymbol{r}$, we also have that $\operatorname{GVD}(\boldsymbol{x} ; \boldsymbol{\rho}-\boldsymbol{\alpha})=\operatorname{VD}(\boldsymbol{x})$, while $\overline{\boldsymbol{r}}$ is equal to $\bar{\rho}=(3,4, k+3, k+4, \ldots, k+\ell)$. Hence we get:

$$
\begin{equation*}
D_{k, \ell}(\tau)=(-1)^{\frac{k(k+1)}{2}+|\boldsymbol{\rho}|} \tau^{|\boldsymbol{\rho}|-|\boldsymbol{\alpha}|} \operatorname{VD}(\boldsymbol{x}) \operatorname{GVD}(\boldsymbol{y} ; \overline{\boldsymbol{\rho}}-\boldsymbol{\beta})+O\left(\tau^{|\boldsymbol{\rho}|-|\boldsymbol{\alpha}|+1}\right) \tag{64}
\end{equation*}
$$

Since $|\boldsymbol{\rho}|=\sum_{i=1}^{k+2} i-(3+4)=\sum_{i=1}^{k} i+(k+1)+(k+2)-7=\frac{k(k+1)}{2}+2 k-4$, while $|\boldsymbol{\alpha}|=$ $1+1+3(k-2)=3 k-4$, relation (64) can be rewritten as:

$$
\begin{equation*}
D_{k, \ell}(\tau)=\tau^{\frac{k(k-1)}{2}} \operatorname{VD}(\boldsymbol{x}) \operatorname{GVD}(\boldsymbol{y} ; \overline{\boldsymbol{\rho}}-\boldsymbol{\beta})+O\left(\tau^{\frac{k(k-1)}{2}+1}\right) \tag{65}
\end{equation*}
$$

where we also used the fact that $(-1)^{\frac{k(k+1)}{2}+|\rho|}=(-1)^{k(k+1)+2 k-4}=1$, since $k(k+1)$ is even for all $k$. From relation (65) we immediately deduce that:

$$
\begin{equation*}
\lim _{\tau \rightarrow 0^{+}} \frac{D_{k, \ell}(\tau)}{\tau^{\frac{k(k-1)}{2}}}=\operatorname{VD}(\boldsymbol{x}) \operatorname{GVD}(\boldsymbol{y} ; \overline{\boldsymbol{\rho}}-\boldsymbol{\beta}), \tag{66}
\end{equation*}
$$

which establishes the claim of the lemma, since both $\operatorname{VD}(\boldsymbol{x})$ and $\mathrm{GVD}(\boldsymbol{y} ; \overline{\boldsymbol{\rho}}-\boldsymbol{\beta})$ are strictly positive.
We end the proof of the lemma by commenting on two special cases: $k=2$ and $\ell=2$. In these two cases there is a single non-vanishing term in the expansion of $D_{k, \ell}(\tau)$, namely, the term corresponding to $\boldsymbol{r}=(1,2)$, if $k=2$, and $\boldsymbol{r}=(1,2,5,6, \ldots, k+2)$, if $\ell=2$. More precisely, if $k=2$, then $\boldsymbol{\alpha}=(1,1)=\boldsymbol{e}_{1}+\boldsymbol{e}_{2}, \overline{\boldsymbol{r}}=(3,4,5,6, \ldots, \ell+2)$, and, thus

$$
\begin{aligned}
& \operatorname{det}\left(S\left(\Delta_{k, \ell}(\tau), \boldsymbol{r}, \boldsymbol{c}\right)\right)=\operatorname{GVD}(\tau \boldsymbol{x} ; \boldsymbol{r}-\boldsymbol{\alpha})=\operatorname{VD}(\tau \boldsymbol{x})=\tau \operatorname{VD}(\boldsymbol{x})=\tau\left(x_{2}-x_{1}\right), \quad \text { and } \\
& \operatorname{det}\left(\bar{S}\left(\Delta_{k, \ell}(\tau), \boldsymbol{r}, \boldsymbol{c}\right)\right)=\operatorname{GVD}(\boldsymbol{y} ; \overline{\boldsymbol{r}}-\boldsymbol{\beta})=\operatorname{VD}(\boldsymbol{y}) .
\end{aligned}
$$

If $\ell=2$, then $\overline{\boldsymbol{r}}=(3,4)$, and, thus,

$$
\begin{aligned}
& \operatorname{det}\left(S\left(\Delta_{k, \ell}(\tau), \boldsymbol{r}, \boldsymbol{c}\right)\right)=\operatorname{GVD}(\tau \boldsymbol{x} ; \boldsymbol{r}-\boldsymbol{\alpha})=\operatorname{VD}(\tau \boldsymbol{x})=\tau^{\frac{k(k-1)}{2}} \mathrm{VD}(\boldsymbol{x}), \quad \text { and } \\
& \operatorname{det}\left(\bar{S}\left(\Delta_{k, \ell}(\tau), \boldsymbol{r}, \boldsymbol{c}\right)\right)=\operatorname{GVD}(\boldsymbol{y} ; \overline{\boldsymbol{r}}-\boldsymbol{\beta})=\operatorname{VD}(\boldsymbol{y})=y_{2}-y_{1} .
\end{aligned}
$$

Hence, in both cases, we have:

$$
D_{k, \ell}(\tau)=(-1)^{\frac{k(k+1)}{2}+|\boldsymbol{r}|} \operatorname{GVD}(\tau \boldsymbol{x} ; \boldsymbol{r}-\boldsymbol{\alpha}) \operatorname{GVD}(\boldsymbol{y} ; \overline{\boldsymbol{r}}-\boldsymbol{\beta})=\tau^{\frac{k(k-1)}{2}} \operatorname{VD}(\boldsymbol{x}) \operatorname{VD}(\boldsymbol{y}),
$$

which is strictly positive for any $\tau>0$.


[^0]:    ${ }^{1}$ In the rest of the paper, all polytopes are considered to be convex.

[^1]:    ${ }^{2}$ For simplicial faces, we identify the face with its defining vertex set.

[^2]:    ${ }^{3}$ See Section A of the Appendix for detailed derivations.

[^3]:    ${ }^{4}$ In fact $U_{2}$ is independent of $\tau$, but we use a unified notation for simplicity.

