On Planar Greedy Drawings of 3-Connected Planar Graphs*

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Abstract

A graph drawing is *greedy* if, for every ordered pair of vertices (x, y), there is a path from x to y such that the Euclidean distance to y decreases monotonically at every vertex of the path. Greedy drawings support a simple geometric routing scheme, in which any node that has to send a packet to a destination "greedily" forwards the packet to any neighbor that is closer to the destination than itself, according to the Euclidean distance in the drawing. In a greedy drawing such a neighbor always exists and hence this routing scheme is guaranteed to succeed.

In 2004 Papadimitriou and Ratajczak stated two conjectures related to greedy drawings. The *greedy embedding conjecture* states that every 3-connected planar graph admits a greedy drawing. The *convex greedy embedding conjecture* asserts that every 3-connected planar graph admits a planar greedy drawing in which the faces are delimited by convex polygons. In 2008 the greedy embedding conjecture was settled in the positive by Leighton and Moitra.

In this paper we prove that every 3-connected planar graph admits a *planar* greedy drawing. Apart from being a strengthening of Leighton and Moitra's result, this theorem constitutes a natural intermediate step towards a proof of the convex greedy embedding conjecture.

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1 Introduction

Geographic routing is a family of routing protocols for *ad-hoc networks*, which are networks with no fixed infrastructure – such as routers or access points – and with dynamic topology [17, 30, 31]. In a geographic routing scheme each node of the network actively sends, forwards, and receives packets; further, it does so by only relying on the knowledge of its own geographic coordinates, of those of its neighbors, and of those of the packet destination. *Greedy routing* – originally called *Cartesian routing* [16] – is the simplest and most renowned geographic routing scheme. In this protocol, a node that has to send a packet simply forwards it to any neighbor that is closer – according to the Euclidean distance – to the destination than itself. The greedy routing scheme might fail to deliver packets because

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of the presence of a *void* in the network; this is a node with no neighbor closer to the destination than itself. For this reason, several variations of the greedy routing scheme have been proposed; see, e.g., [8, 21, 22].

Apart from its failure in the presence of voids, the greedy routing protocol has two disadvantages which limit its applicability. First, in order for the protocol to work, each node of the network has to be equipped with a GPS, which might be expensive and might consume excessive energy. Second, two nodes that are close geographically might be unable to communicate with each other because of the presence of topological obstructions. Rao *et al.* [29] introduced the following brilliant idea for extending the applicability of geographic routing in order to overcome the above issues. Suppose that a network topology is known; then one can assign *virtual coordinates* to the nodes and use these coordinates instead of the geographic locations of the nodes in the greedy routing protocol. The virtual coordinates can then be chosen so that the greedy routing protocol is guaranteed to succeed.

Computing the virtual coordinate assignment for the nodes of a network corresponds to the following graph drawing problem: Given a graph G, construct a *greedy drawing* of G, that is a drawing in the plane such that, for any ordered pair of vertices (x, y), there exists a neighbor of x in G that is closer – in terms of Euclidean distance – to y than x. Equivalently, a greedy drawing of G is such that, for any ordered pair of vertices (x, y), there exists a *distance-decreasing* path from x to y, that is, a path $(u_1, u_2, \dots, u_m)$ in G such that $x = u_1$, $y = u_m$, and the Euclidean distance between u_{i+1} and u_m is smaller than the one between u_i and u_m, for any $i = 1, 2, \dots, m - 2$.

Greedy drawings experienced a dramatical surge of popularity in the theory community in 2004, when Papadimitriou and Ratajczak [27] proposed the following two conjectures about greedy drawings of 3-connected planar graphs.[1]

▶ Conjecture 1. (*Greedy embedding conjecture*) Every 3-connected planar graph admits a greedy drawing.

▶ Conjecture 2. (*Convex greedy embedding conjecture*) Every 3-connected planar graph admits a convex greedy drawing.

Papadimitriou and Ratajczak [27, 28] provided several reasons why 3-connected planar graphs are central to the study of greedy drawings. First, there exist non-3-connected planar graphs and 3-connected non-planar graphs that do not admit any greedy drawing. Thus, the 3-connected planar graphs form the largest class of graphs that might admit a greedy drawing, in a sense. Second, all the graphs with no $K_{3,3}$-minor admit a 3-connected planar spanning graph, hence they admit a greedy drawing, provided the truth of the greedy embedding conjecture. Third, the preliminary study of Papadimitriou and Ratajczak [27, 28] provided evidence for the mathematical depth of their conjectures.

In 2008 Leighton and Moitra [23, 24] settled the greedy embedding conjecture in the affirmative; the same result was established (independently and slightly later) by Angelini et al. [4, 5]. In this paper we show the following result.

▶ **Theorem 1.** *Every 3-connected planar graph admits a planar greedy drawing.*

Given a 3-connected planar graph G, both the algorithm by Leighton and Moitra [23, 24] and the one by Angelini et al. [4, 5] find a certain spanning subgraph S of G and construct a (planar) greedy drawing of S; then they embed the edges of G not in S as straight-line segments obtaining a, in general, *non-planar* greedy drawing of G. Thus, Theorem 1

[1] The convex greedy embedding conjecture has not been stated in the journal version [28] of Papadimitriou and Ratajczak paper [27].

strengthens Leighton and Moitra's and Angelini et al.'s results. Furthermore, *convex* drawings, in which all the faces are delimited by convex polygons, are planar, hence Theorem 1 provides a natural step towards a proof of the convex greedy embedding conjecture.

Our proof employs a structural decomposition for 3-connected planar graphs which finds its origins in a paper by Chen and Yu [9]. This decomposition actually works for a superclass of the 3-connected planar graphs known as *strong circuit graphs*. We construct a planar greedy drawing of a given strong circuit graph G recursively: We apply the structural decomposition to G in order to obtain some smaller strong circuit graphs, we recursively construct planar greedy drawings for them, and then we suitably arrange these drawings together to get a planar greedy drawing of G. For this arrangement to be feasible, we need to ensure that the drawings we construct satisfy some coercive geometric requirements; these are described in the main technical theorem of the paper – Theorem 7.

Related results. Planar greedy drawings always exist for *maximal planar graphs* [12]. Further, every planar graph G with a *Hamiltonian path* $P = (u_1, u_2, \dots, u_n)$ has a planar greedy drawing. Namely, construct a planar straight-line drawing Γ of G such that $y(u_1) < y(u_2) < \cdots < y(u_n)$; such a drawing always exists [13]; scale Γ down horizontally, so that P is "almost vertical". Then, for any $1 \leq i < j \leq n$, the paths $(u_i, u_{i+1} \dots, u_j)$ and $(u_j, u_{j-1} \dots, u_i)$ are distance-decreasing. A characterization of the *trees* that admit a (planar) greedy drawing is known [25]; indeed, a greedy drawing of a tree is always planar [2].

Algorithms have been designed to construct *succinct* greedy drawings, in which the vertex coordinates are represented with a polylogarithmic number of bits [14, 18, 19]; this has been achieved by allowing the embedding space to be different from the Euclidean plane or the metric to be different from the Euclidean distance.

Planar graph drawings have been studied in which paths between pairs of vertices are required to exist satisfying properties different from being distance-decreasing. Consider a path $P = (u_1, u_2, \dots, u_m)$ in a graph drawing. We say that P is *self-approaching* [1, 26] if, for any three points a, b, c in this order along P from u_1 to u_m, the Euclidean distance between a and c is larger than the one between b and c – then a self-approaching path is also distance-decreasing. We say that P is *increasing-chord* [1, 11, 26] if it is self-approaching in both directions. We say that P is *strongly monotone* [3, 15, 20] if the orthogonal projections of the vertices of P on the line ℓ through u_1 and u_m appear in the order $u_1, u_2, \dots, u_m$. We explicitly mention [15] the recent proof that every 3-connected planar graph admits a planar drawing in which every pair of vertices is connected by a strongly monotone path.

2 Preliminaries

In this section we introduce some preliminaries. For a graph G, we denote by $V(G)$ and $E(G)$ its vertex and edge sets, respectively.

Subgraphs and connectivity. Let G be a graph and $U \subseteq V(G)$; we denote by $G - U$ the graph obtained from G by removing the vertices in U and their incident edges. Further, if $e \in E(G)$, we denote by $G - e$ the graph obtained from G by removing the edge e. Let H be a subgraph of G. An *H-bridge* B of G is either an edge of G not in H with both the end-vertices in H (then we say that B is a *trivial H-bridge*), or a connected component of $G - V(H)$ together with the edges from that component to the vertices in $V(H)$ (then we say that B is a *non-trivial H-bridge*); the vertices in $V(H) \cap V(B)$ are the *attachments* of B in H. Further, for a vertex $v \in V(G) - V(H)$, we denote by $H \cup \{v\}$ the subgraph of G composed of H and of the isolated vertex v. Further,

A *vertex k-cut* (in the following simply called *k-cut*) in a connected graph G is a set of

k vertices whose removal disconnects G. For $k \geq 2$, a connected graph is *k-connected* if it has no $(k-1)$-cut. A *k-connected component* of a graph G is a maximal (with respect to both vertices and edges) k-connected subgraph of G. Given a 2-cut $\{a, b\}$ in a 2-connected graph G, an *$\{a, b\}$-component* is either the edge ab (then we say that the $\{a, b\}$-component is *trivial*) or a subgraph of G induced by a, b, and the vertices of a connected component of $G - \{a, b\}$ (then we say that the $\{a, b\}$-component is *non-trivial*).

Plane graphs and embeddings. A drawing of a graph is *planar* if no two edges intersect except at common end-vertices. A *plane graph* is a planar graph together with a plane embedding; a *plane embedding* of a connected planar graph G is an equivalence class of planar drawings of G, where two drawings Γ_1 and Γ_2 are *equivalent* if: (i) for each $v \in V(G)$, the clockwise order of the edges incident to v coincides in Γ_1 and in Γ_2; and (ii) the clockwise order of the edges composing the walks delimiting the outer faces of Γ_1 and Γ_2 is the same. When we talk about a planar drawing of a plane graph G, we always mean that it respects the plane embedding of G. We assume that any subgraph H of G is associated with the plane embedding obtained from the one of G by deleting the vertices and edges not in H. In a plane graph G a vertex is *external* or *internal* depending on whether it is or it is not incident to the outer face of G, respectively.

Refer to Fig. 1. For two external vertices u and v of a 2-connected plane graph G, let $\tau_{uv}(G)$ and $\beta_{uv}(G)$ be the paths composed of the vertices and edges encountered when walking along the boundary of the outer face of G in clockwise and counter-clockwise direction from u to v, respectively. Note that $\tau_{uv}(G)$ and $\beta_{vu}(G)$ have the same vertices and edges, however in reverse linear orders.

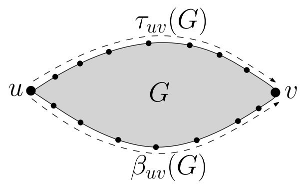


Figure 1 The paths $\tau_{uv}(G)$ and $\beta_{uv}(G)$ in a 2-connected plane graph G.

Geometry. In this paper every angle is measured in radians, even when not explicitly stated. The *slope of a half-line* ℓ is defined as follows. Denote by p the starting point of ℓ and let ℓ' be the vertical half-line starting at p and directed towards decreasing y-coordinates. Then the slope of ℓ is the angle spanned by a counter-clockwise rotation around p bringing ℓ' to coincide with ℓ, minus $\frac{\pi}{2}$. Note that, because of this definition, the slope of any half-line is assumed to be between -$\frac{\pi}{2}$ (included) and $\frac{3\pi}{2}$ (excluded); in the following, there will be very few exceptions to this assumption, which will be however evident from the text. Every angle expressed as $\arctan(\cdot)$ is assumed to be between -$\frac{\pi}{2}$ and $\frac{\pi}{2}$. We define the *slope of an edge uv* in a graph drawing as the slope of the half-line from u through v. Note that the slope of an edge uv is equal to the slope of the edge vu plus or minus π. For a directed line ℓ, we let its slope be equal to the slope of any half-line starting at a point of ℓ and directed as ℓ. We denote by Δpqr a triangle with vertices p, q, r, and we denote by $\measuredangle pqr$ the angle of Δpqr incident to q; note that $\measuredangle pqr$ is between 0 and π.

Let Γ be a drawing of a graph G and let u, v be vertices in $V(G)$. We denote by $d(\Gamma, uv)$ the Euclidean distance between u and v in Γ. We also denote by $d_H(\Gamma, uv)$ the horizontal distance between u and v in Γ, that is, the absolute value of the difference between the x-coordinates of u and v in Γ; the vertical distance $d_V(\Gamma, uv)$ between u and v in Γ is defined analogously. With a slight abuse of notation, we will use $d(\Gamma, pq)$, $d_H(\Gamma, pq)$, and $d_V(\Gamma, pq)$

even if p and q are points in the plane (and not necessarily vertices of G). A drawing of a graph is a *straight-line* drawing if each edge is represented by a straight-line segment.

The following lemma argues that the planarity and the greediness of a drawing are not lost as a consequence of any sufficiently small perturbation of the vertex positions.

▶ **Lemma 2.** *Let Γ be a planar straight-line drawing of a graph G. There exists a value $\varepsilon^*_\Gamma > 0$ such that the following holds true. Let Γ' be any straight-line drawing in which, for every vertex $z \in V(G)$, the Euclidean distance between the positions of z in Γ and Γ' is at most ε^*_Γ; then Γ' is planar and any path which is distance-decreasing in Γ is also distance-decreasing in Γ'.*

Proof. Let δ be the minimum Euclidean distance in Γ between any two vertices, or between any vertex and any non-incident edge, or between any two non-adjacent edges, where the Euclidean distance between a point p and a straight-line segment s is the minimum Euclidean distance between p and any point of s, and the Euclidean distance between two straight-line segments s_1 and s_2 is the minimum Euclidean distance between any point of s_1 and any point of s_2. Note that $\delta > 0$, since Γ is planar. Further, let $\gamma = \min\{d(\Gamma, uz) - d(\Gamma, vz)\}$, where the minimum is taken over all the ordered triples (u, v, z) of distinct vertices of G such that $d(\Gamma, uz) > d(\Gamma, vz)$. Note that $\gamma > 0$. Set $\varepsilon^*_\Gamma = \min\{\frac{\delta}{3}, \frac{\gamma}{5}\}$. Note that $\varepsilon^*_\Gamma > 0$.

Consider any straight-line drawing Γ' of G in which, for each vertex $z \in V(G)$, the Euclidean distance between the positions of z in Γ and Γ' is at most ε^*_Γ.

We prove that Γ' is planar. In order to do that, we exploit the following observation. For any point p' that belongs to the straight-line segment s' representing an edge e in Γ', there exists a point p whose distance from p' is at most ε^*_Γ and that belongs to the straight-line segment s representing e in Γ. This is because s' is contained in the convex hull of the two disks with radius ε^*_Γ centered at the end-points of s or, equivalently, in the region which is the Minkowski sum of s with a disk with radius ε^*_Γ. Now suppose, for a contradiction, that in Γ' two distinct vertices v_1 and v_2 coincide at a point p', or an edge e overlaps a non-incident vertex v at a point p', or two non-adjacent edges e_1 and e_2 cross at a point p'. Then there exist two points p_1 and p_2 in Γ that are at distance at most ε^*_Γ from p' and hence at most $2\varepsilon^*_\Gamma$ from each other and such that v_1 and v_2 are placed at p_1 and p_2 in Γ, or such that v is placed at p_1 and p_2 belongs to the straight-line segment representing e in Γ, or such that p_1 and p_2 belong to the straight-line segments representing e_1 and e_2 in Γ, respectively. However, $2\varepsilon^*_\Gamma \leq \frac{2\delta}{3} < \delta$, which contradicts the definition of δ.

We prove that any path $P = (u_1, u_2, \dots, u_m)$ which is distance-decreasing in Γ is also distance-decreasing in Γ'. Since P is distance-decreasing, we have that $d(\Gamma, u_i u_m) > d(\Gamma, u_{i+1} u_m)$, for every $i = 1, 2 \dots, m-2$. Since the Euclidean distance between the positions of any vertex $z \in V(G)$ in Γ and Γ' is at most ε^*_Γ, for any $i = 1, \dots, m-2$, we have that $d(\Gamma', u_i u_m) \geq d(\Gamma, u_i u_m) - 2\varepsilon^*_\Gamma$ and that $d(\Gamma', u_{i+1} u_m) \leq d(\Gamma, u_{i+1} u_m) + 2\varepsilon^*_\Gamma$. It follows that $d(\Gamma', u_i u_m) - d(\Gamma', u_{i+1} u_m) \geq d(\Gamma, u_i u_m) - d(\Gamma, u_{i+1} u_m) - 4\varepsilon^*_\Gamma \geq \frac{d(\Gamma, u_i u_m) - d(\Gamma, u_{i+1} u_m)}{5} > 0$. Hence, $d(\Gamma', u_i u_m) > d(\Gamma', u_{i+1} u_m)$ for $i = 1, \dots, m-2$. It follows that P is distance-decreasing in Γ'. ◀

We conclude this section with a technical lemma we are going to exploit heavily in the next section. Refer to Fig. 2a.

▶ **Lemma 3.** *Let G be a 2-connected plane graph whose outer face consists of two paths $(u = u_1, u_2, \dots, u_p = v)$ and $(u = v_1, v_2, \dots, v_q = v)$. Let ℓ_1, ℓ_2, and ℓ_3 be three directed lines that pass through a point p_u and that have slopes s_1, s_2, and s_3, respectively, where $0 < s_1 \leq s_2 \leq s_3 < \pi$. Let Γ be a planar drawing of G such that u lies at p_u; let*

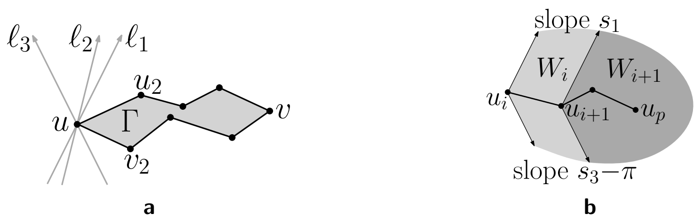


Figure 2 (a) Illustration for the statement of Lemma 3. (b) Illustration for the proof of Lemma 3.

s_m (s_M) be the minimum (maximum, respectively) slope of an edge u_iu_{i+1} or v_jv_{j+1}. If $s_3 - \pi < s_m \leq s_M < s_1$ (if $s_3 < s_m \leq s_M < s_1 + \pi$), then Γ lies entirely to the right (to the left, respectively) of ℓ_2, except for the vertex u.

Proof. We only prove that, if $s_3 - \pi < s_m \leq s_M < s_1$, then Γ lies entirely to the right of ℓ_2, except for the vertex u; the proof that, if $s_3 < s_m \leq s_M < s_1 + \pi$, then Γ lies entirely to the left of ℓ_2, except for the vertex u, is symmetric.

Further, it suffices to prove that the paths $(u = u_1, u_2, \dots, u_p)$ and $(u = v_1, v_2, \dots, v_q)$ lie to the right of ℓ_2, except for the vertex u; indeed, if that is the case, then the planarity of Γ implies that the entire drawing Γ, except for the vertex u, lies to the right of ℓ_2.

We now prove that the path $(u = u_1, u_2, \dots, u_p)$ lies to the right of ℓ_2, except for the vertex u; the proof for the path $(u = v_1, v_2, \dots, v_q)$ is analogous.

For $i = 1, \dots, p$, let W_i be the open wedge delimited by the half-lines starting at u_i with slopes $s_3 - \pi$ and s_1; that is, W_i is the region of the plane that is spanned by a half-line starting at u_i with slope $s_3 - \pi$ while rotating counter-clockwise around u_i until it has slope s_1. We claim that W_i contains the path $(u_i, u_{i+1}, \dots, u_p)$ in its interior, except for the vertex u_i which is on the boundary of W_i. Observe that the claim (with $i = 1$) implies the lemma, since W_i lies to the right of ℓ_2, given that $s_2 - \pi \leq s_3 - \pi$ and $s_1 \leq s_2$.

We now prove the claim by reverse induction on i. The case $i = p$ is trivial. Hence, assume that W_{i+1} contains the path $(u_{i+1}, u_{i+2}, \dots, u_p)$ in its interior, except for the vertex u_{i+1} which is on the boundary of W_{i+1}. See Fig. 2b. Since $s_3 - \pi < s_m \leq s_M < s_1$, the edge u_iu_{i+1} lies in the interior of the wedge W_i, except for the vertex u_i which is on the boundary of W_i. Further, since u_{i+1} lies in the interior of W_i, the entire wedge W_{i+1}, and hence the path $(u_{i+1}, u_{i+2}, \dots, u_p)$, lies in the interior of W_i. This completes the induction and hence concludes the proof of the lemma. ◀

3 Proof of Theorem 1

In this section we prove Theorem 1. Throughout the section, we will work with *plane graphs*. Further, we will deal with a class of graphs that is wider than the one of 3-connected planar graphs. The graphs in this class have been introduced by Chen and Yu [9] with the name of *strong circuit graphs*, as they constitute a subclass of the well-known *circuit graphs*, whose definition is due to Barnette and dates back to 1966 [6]. Here we rephrase the definition of strong circuit graphs as follows.

▶ **Definition 4.** A *strong circuit graph* is a triple (G, u, v) such that either: (i) G is an edge uv or (ii) $|V(G)| \geq 3$ and the following properties are satisfied.

(a) G is a 2-connected plane graph;
(b) u and v are two distinct external vertices of G;

(c) if edge uv exists, then it coincides with the path $\tau_{uv}(G)$; and
(d) for every 2-cut $\{a, b\}$ of G we have that a and b are external vertices of G and at least one of them is an internal vertex of the path $\beta_{uv}(G)$; further, every non-trivial $\{a, b\}$-component of G contains an external vertex of G different from a and b.

Several problems are more easily solved on (strong) circuit graphs than on 3-connected planar graphs. This is because the (strong) circuit graphs can be easily decomposed into smaller (strong) circuit graphs, and hence are suitable for inductive proofs. We now present a structural decomposition for strong circuit graphs whose main ideas can be found in a paper by Chen and Yu [9] (see also a recent paper by Da Lozzo et al. [10] for an application of this decomposition to *cubic* strong circuit graphs).

Consider a strong circuit graph (G, u, v) such that G is neither a single edge nor a simple cycle. The decomposition distinguishes the case in which the path $\tau_{uv}(G)$ coincides with the edge uv (Case A) from the case in which it does not (Case B).

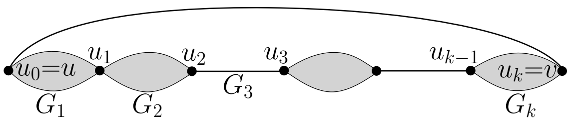


Figure 3 Structure of (G, u, v) in Case A.

▶ **Lemma 5.** *Suppose that we are in Case A (refer to Fig. 3). Then the graph $G' = G - uv$ consists of a sequence of graphs $G_1, \dots, G_k$, with $k \geq 1$, such that:*

- *5a: for $i = 1, \dots, k-1$, the graphs G_i and G_{i+1} share a single vertex u_i; further, G_i is in the outer face of G_{i+1} and vice versa in the plane embedding of G;*
- *5b: for $1 \leq i, j \leq k$ with $j \geq i+2$, the graphs G_i and G_j do not share any vertex; and*
- *5c: for $i = 1, \dots, k$ with $u_0 = u$ and $u_k = v$, (G_i, u_{i-1}, u_i) is a strong circuit graph.*

Proof. Consider the *BC-tree* T' of G', which is the tree that is defined as follows. The tree T' contains a *B-node* for each 2-connected component of G' and a *C-node* for each 1-cut of G'; further, T' contains an edge between a B-node b and a C-node c if the 1-cut corresponding to c is a vertex of the 2-connected component corresponding to b.

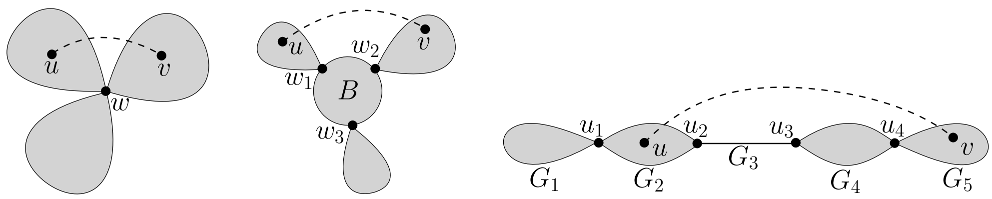


Figure 4 (a) The BC-tree T' of G' contains a node with degree at least 3 corresponding to a 1-cut $\{w\}$ of G'. (b) The BC-tree T' of G' contains a node with degree at least 3 corresponding to a 2-connected component B of G'. (c) The vertices u and v are not in G_1.

First, we have that T' is a path. Namely suppose, for a contradiction, that T' has a node t with degree at least 3. If t corresponds to a 1-cut $\{w\}$ of G', as in Fig. 4a, then w belongs to at least three 2-connected components of G' and the graph $G'' = G' - \{w\}$ consists of at least three connected components. Hence, the graph G'' plus edge uv is disconnected, which implies that $\{w\}$ is a 1-cut of G; this contradicts Property (a) of (G, u, v). Analogously, if

t corresponds to a 2-connected component B of G', as in Fig. 4b, then B contains three distinct 1-cuts $\{w_1\}$, $\{w_2\}$, and $\{w_3\}$ of G'; for each $i \in \{1,2,3\}$, the removal of w_i from G' disconnects G' into at least two connected components, at least one of which, denoted by G_i, does not contain vertices of B. Since G_1, G_2, and G_3 share no vertex, the edge uv connects at most two components G_i and G_j with $i,j \in \{1,2,3\}$, which implies that $\{w_h\}$ is a 1-cut of G, where $h \neq i,j$ and $h \in \{1,2,3\}$; this contradicts Property (a) of (G,u,v). Hence T' is a path $(b_1,c_1,b_2,c_2,\dots,b_{k-1},c_{k-1},b_k)$.

Let G_i be the 2-connected component of G' corresponding to the B-node b_i and let $\{u_i\}$ be the 1-cut of G' corresponding to the C-node c_i. Then, for $i=1,\dots,k-1$, the graphs G_i and G_{i+1} share a single vertex u_i, while for $1 \leq i,j \leq k$ with $j \geq i+2$ the graphs G_i and G_j do not share any vertex. The vertices u and v are one in G_1 and one in G_k; indeed, if say G_1 did not contain any of u and v, as in Fig. 4c, then $\{u_1\}$ would be a 1-cut of G; this would contradict Property (a) of (G,u,v). Assume, w.l.o.g. up to renaming, that u belongs to G_1 and v belongs to G_k. We also have that $u \neq u_1$, as if $u = u_1$ then $\{u_1\}$ would be a 1-cut of G, again contradicting Property (a) of (G,u,v); analogously, $v \neq u_{k-1}$.

We prove that G_{i+1} lies in the outer face of G_i in the plane embedding of G, for every $i=1,\dots,k-1$. Suppose for a contradiction that, for some $i \in \{1,\dots,k-1\}$, the graph G_{i+1} lies inside an internal face f of G_i (except for the vertex u_i, which is on the boundary of f) in the plane embedding of G. Since the graphs $G_{i+2},\dots,G_k$ do not share any vertex with G_i, by planarity they all lie inside f. It follows that the vertex v lies inside f (note that $v \neq u_i$ even if $k=i+1$) and hence it is not incident to the outer face of G, which contradicts Property (b) of (G,u,v). An analogous proof shows that G_i lies in the outer face of G_{i+1} in the plane embedding of G, for every $i=1,\dots,k-1$.

It remains to prove that, for $i=1,\dots,k$, the triple (G_i,u_{i-1},u_i) is a strong circuit graph, where $u_0=u$ and $u_k=v$. We are going to use the fact that $\beta_{uv}(G)$ is composed of the paths $\beta_{uu_1}(G_1),\beta_{u_1u_2}(G_2),\dots,\beta_{u_{k-1}v}(G_k)$. This is because uv coincides with $\tau_{uv}(G)$ by Property (c) of (G,u,v) and because G_{i+1} lies in the outer face of G_i and vice versa in the plane embedding of G.

(a) Graph G_i is 2-connected by assumption and it is associated with a plane embedding, given that it is a subgraph of the plane graph G.

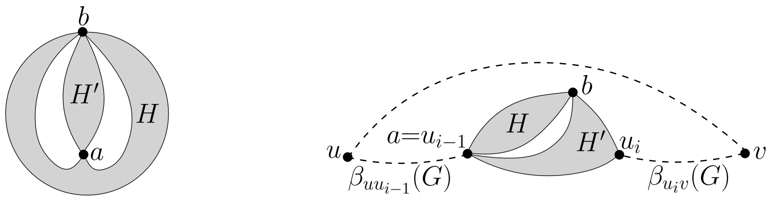
(b) For $i=1,\dots,k-1$, the vertex u_i is external in the plane embedding of G_i, since G_i is in the outer face of G_{i+1} and vice versa; analogously, for $i=2,\dots,k$, the vertex u_{i-1} is external in the plane embedding of G_i. Further, $u_0=u$ and $u_k=v$ are external in the plane embeddings of G_1 and G_k, respectively, since they are external in the plane embedding of G. Finally, for $i=1,\dots,k$, the vertices u_{i-1} and u_i are distinct, as otherwise $\{u_{i-1}=u_i\}$ would be a 1-cut of G, which would contradict Property (a) of (G,u,v).

(c) Suppose, for a contradiction, that the edge $u_{i-1}u_i$ exists and that it does not coincide with $\tau_{u_{i-1}u_i}(G_i)$. This implies that G_i contains vertices different from u_{i-1} and u_i, and hence that $\{u_{i-1},u_i\}$ is a 2-cut of G. The $\{u_{i-1},u_i\}$-component H_i of G that contains $\tau_{u_{i-1}u_i}(G_i)$ is non-trivial, given that $\tau_{u_{i-1}u_i}(G_i)$ does not coincide with $u_{i-1}u_i$. Further, no vertex of H_i other than u_{i-1} and u_i is incident to the outer face of G, given that all the vertices of H_i other than u_{i-1} and u_i lie inside the region delimited by the cycle $\beta_{uu_1}(G_1)\cup\dots\cup\beta_{u_{i-2}u_{i-1}}(G_{i-1})\cup u_{i-1}u_i\cup\beta_{u_iu_{i+1}}(G_{i+1})\cup\dots\cup\beta_{u_{k-1}v}(G_k)\cup vu$. However, this contradicts Property (d) of (G,u,v).

(d) Consider any 2-cut $\{a,b\}$ of G_i; then G_i has at least two non-trivial $\{a,b\}$-components.

We prove that a and b are external vertices of G_i. Suppose, for a contradiction, that a is an internal vertex of G_i (the argument if b is an internal vertex of G_i is analogous), as in Fig. 5a. Then the cycle delimiting the outer face of G_i belongs to a single $\{a,b\}$-component

H of G_i, and there is a non-trivial $\{a,b\}$-component $H' \neq H$ of G_i that does not contain any external vertices of G_i other than b. Since all the edges in $E(G) - E(G_i)$ lie in the outer face of G_i in the plane embedding of G, it follows that $\{a,b\}$ is a 2-cut of G, and that H' is a non-trivial $\{a,b\}$-component of G that does not contain any external vertices of G other than b. However, this contradicts Property (d) of (G,u,v).



a **b**

Figure 5 Illustration for the proof that (G_i, u_{i-1}, u_i) satisfies Property (d) of a strong circuit graph. (a) A vertex a of a 2-cut $\{a,b\}$ of G_i is an internal vertex of G_i. (b) The $\{a,b\}$-component H of G_i containing $\tau_{ab}(G_i)$ contains no internal vertex of $\beta_{ab}(G_i)$.

Next, we prove that at least one of a and b is an internal vertex of $\beta_{u_{i-1}u_i}(G_i)$. Suppose, for a contradiction, that a and b are both in $\tau_{u_{i-1}u_i}(G_i)$. Assume, w.l.o.g. up to renaming of a and b, that u_{i-1}, a, b, and u_i appear in this order in $\tau_{u_{i-1}u_i}(G_i)$, where possibly $u_{i-1} = a$ and/or $b = u_i$. Let H be the $\{a,b\}$-component of G_i containing $\tau_{ab}(G_i)$; let H' be any non-trivial $\{a,b\}$-component of G_i different from H.

- If H contains an internal vertex of $\beta_{ab}(G_i)$, then it contains the entire cycle delimiting the outer face of G_i. The planarity of G_i implies that H' lies inside an internal face of H, except at vertices a and b. This has two consequences. First, since all the edges in $E(G) - E(G_i)$ lie in the outer face of G_i (and of H) in the plane embedding of G, the set $\{a,b\}$ is a 2-cut of G and hence H' is a non-trivial $\{a,b\}$-component of G. Second, no vertex of H' other than a and b is incident to the outer face of G_i or to the outer face of G. These two statements contradict Property (d) for (G,u,v).
- If H contains no internal vertex of $\beta_{ab}(G_i)$, as in Fig. 5b, then $u_{i-1}, u_i \notin V(H) - \{a,b\}$, hence no edge in $E(G) - E(G_i)$ is incident to a vertex of H different from a and b. Since the vertices of $\beta_{u_{i-1}u_i}(G_i)$ are the only external vertices of G in $V(G_i)$, it follows that H is a non-trivial $\{a,b\}$-component of G that contains no external vertex of G other than, possibly, a and b. This contradicts Property (d) for (G,u,v).

Finally, we prove that every non-trivial $\{a,b\}$-component H of G_i contains an external vertex of G_i different from a and b. Namely, if that is not the case for a non-trivial $\{a,b\}$-component H of G_i, then no edge in $E(G) - E(G_i)$ is incident to a vertex of H different from a and b. This implies that the set $\{a,b\}$ is a 2-cut of G and H is a non-trivial $\{a,b\}$-component of G. However, no vertex of H other than, possibly, a and b is incident to the outer face of G. This contradicts Property (d) for (G,u,v). ◀

Given a strong circuit graph (G,u,v) that is not a single edge, the vertex u belongs to one 2-connected component of the graph $G - \{v\}$. Indeed, if it belonged to more than one 2-connected component of $G - \{v\}$, then $\{u\}$ would be a 1-cut of $G - \{v\}$, hence $\{u,v\}$ would be a 2-cut of G, which contradicts Property (d) for (G,u,v). We now present the following.

▶ **Lemma 6.** *Suppose that we are in Case B (refer to Fig. 6). Let H be the 2-connected component of the graph $G - \{v\}$ that contains u; then we have $|V(H)| \geq 3$. Further, let*

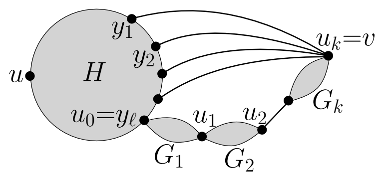


Figure 6 Structure of (G, u, v) in Case B.

H' denote the graph $H \cup \{v\}$. Then G contains ℓ distinct H'-bridges $B_1, \dots, B_\ell$, for some $\ell \geq 2$, such that:

- *6a: each H'-bridge B_i has two attachments, namely v and a vertex $y_i \in V(H)$;*
- *6b: the H'-bridges $B_1, \dots, B_{\ell-1}$ are trivial, while B_ℓ might be trivial or not;*
- *6c: any two among $y_1, \dots, y_\ell$ are distinct except, possibly, for $y_{\ell-1}$ and y_ℓ; also if $\ell = 2$, then y_1 and y_2 are distinct;*
- *6d: y_1 is an internal vertex of $\tau_{uv}(G)$; further, B_1 is an edge that coincides with $\tau_{y_1 v}(G)$;*
- *6e: y_ℓ is an internal vertex of $\beta_{uv}(G)$ and $\beta_{uy_1}(H)$; further, B_ℓ contains the path $\beta_{y_\ell v}(G)$;*
- *6f: $B_1, \dots, B_{\ell-1}$ appear in this counter-clockwise order around v and lie in the outer face of B_ℓ in the plane embedding of G;*
- *6g: the triple (H, u, y_1) is a strong circuit graph; and*
- *6h: B_ℓ consists of a sequence of graphs $G_1, \dots, G_k$, with $k \geq 1$, such that:*
  - *for $i = 1, \dots, k-1$, the graphs G_i and G_{i+1} share a single vertex u_i; further, G_i is in the outer face of G_{i+1} and vice versa in the plane embedding of G;*
  - *for $1 \leq i, j \leq k$ with $j \geq i+2$, the graphs G_i and G_j do not share any vertex; and*
  - *for $i = 1, \dots, k$ with $u_0 = y_\ell$ and $u_k = v$, the triple (G_i, u_{i-1}, u_i) is a strong circuit graph.*

Proof. We first prove that $|V(H)| \geq 3$. Suppose, for a contradiction, that H is a single edge uy_1. If the degree of u in G is one, then $\{y_1\}$ is a 1-cut of G; this contradicts Property (a) for (G, u, v). Otherwise, there is a $(H \cup \{v\})$-bridge B_i of G whose attachment in H is u. If B_i is trivial, then it coincides with the edge uv; however, this contradicts the hypothesis of Case B. Otherwise, B_i is non-trivial; however, this implies that $\{u, v\}$ is a 2-cut of G, as the removal of u and v from G disconnects y_1 from the vertices in $V(B_i) - \{u, v\}$; since neither u nor v is an internal vertex of $\beta_{uv}(G)$, this contradicts Property (d) for (G, u, v).

We now prove the properties of the lemma. First, if G had no H'-bridge, then it would not be connected, while it is 2-connected. Hence, G contains ℓ distinct H'-bridges $B_1, \dots, B_\ell$ with $\ell \geq 1$. Each H'-bridge B_i has at most one attachment $y_i \in V(H)$, as if B_i had at least two attachments in $V(H)$ then it would contain a path (not passing through v) between two vertices of H; however, such a path would be in H, and not in B_i, given that H is a maximal 2-connected subgraph of $G - \{v\}$. It follows that $\ell \geq 2$, as if $\ell = 1$ then y_1 would be a 1-cut of G, whereas G is 2-connected. Further, for $i = 1, 2, \dots, \ell$, the vertex v is an attachment of B_i, as otherwise y_i would be a 1-cut of G, whereas G is 2-connected. Analogously, for $i = 1, 2, \dots, \ell$, the vertex y_i is an attachment of B_i, as otherwise v would be a 1-cut of G, whereas G is 2-connected. This proves Property 6a.

Suppose, for a contradiction, that $y_i = u$, for some $i \in \{1, 2, \dots, \ell\}$. If B_i is a trivial H'-bridge, then it coincides with the edge uv; however, this contradicts the fact that we are in Case B. If B_i is a non-trivial H'-bridge, then $\{u, v\}$ is a 2-cut of G; namely, the removal of u and v from G disconnects the vertices in $V(H) - \{u\}$ from the vertices in $V(B_i) - \{u, v\}$ – the latter set is non-empty given that B_i is non-trivial. However, this contradicts Property (d) for (G, u, v), given that neither u nor v is an internal vertex of $\beta_{uv}(G)$. It follows that $y_i \neq u$, for $i = 1, 2, \dots, \ell$.

We now prove Properties 6b–6f. Since v is incident to the outer face of G, it lies in the outer face of H. It follows that all the H'-bridges $B_1, \dots, B_\ell$ lie in the outer face of H, except at the vertices $y_1, \dots, y_\ell$, respectively. By the planarity of G, there are at most two H'-bridges among $B_1, \dots, B_\ell$ that contain edges incident to the outer face of G. If there were only one H'-bridge B_i containing edges incident to the outer face of G, as in Fig. 7a, then $\{y_i\}$ would be a 1-cut of G, whereas G is 2-connected. Hence, there are exactly two H'-bridges among $B_1, \dots, B_\ell$ containing edges incident to the outer face of G. Denote them by B_1 and B_ℓ, as in Fig. 7b, so that u, y_1, and y_ℓ appear in this clockwise order along the outer face of H. Then $y_1 \neq y_\ell$, as otherwise $\{y_1\}$ would be a 1-cut of G, whereas G is 2-connected; in particular, $y_1 \neq y_2$ if $\ell = 2$. It also follows that y_1 is an internal vertex of $\tau_{uv}(G)$, that y_ℓ is an internal vertex of $\beta_{uv}(G)$ and $\beta_{uy_1}(H)$, that B_1 contains $\tau_{y_1 v}(G)$, that B_ℓ contains $\beta_{y_\ell v}(G)$, and that every vertex $y_i \neq y_1, y_\ell$ is not incident to the outer face of G. Now consider any H'-bridge B_i of G with $y_i \neq y_\ell$. The graph B_i is a $\{y_i, v\}$-component of G, however $\{y_i, v\}$ is a pair of vertices none of which is internal to $\beta_{uv}(G)$, hence if B_i were non-trivial, then Property (d) of (G, u, v) would be violated. It follows that the only H'-bridges which might be non-trivial are those whose attachment in H is y_ℓ; in particular, since $y_1 \neq y_\ell$, we have that B_1 is trivial and coincides with the edge $y_1 v = \tau_{y_1 v}(G)$. If there were at least two non-trivial H'-bridges whose attachment in H is y_ℓ, then at least one of them (in fact all the ones different from B_ℓ) would not contain any vertex incident to the outer face G other than y_ℓ and v; however, this would violate Property (d) of (G, u, v). It follows that B_ℓ is the only H'-bridge of G that is possibly non-trivial. By the planarity of G and the connectivity of $B_\ell - \{y_\ell, v\}$, all the trivial H'-bridges of G lie in the outer face of B_ℓ; denote them by $B_1, \dots, B_{\ell-1}$ in their counter-clockwise order around v. Since $B_1, \dots, B_{\ell-1}$ are trivial and incident to v, then $y_1, \dots, y_{\ell-1}$ are all distinct. By planarity and since $y_1 \neq y_\ell$, it follows that $B_{\ell-1}$ and B_ℓ are the only H'-bridges which might share their attachment in H. This concludes the proof of Properties 6b–6f.

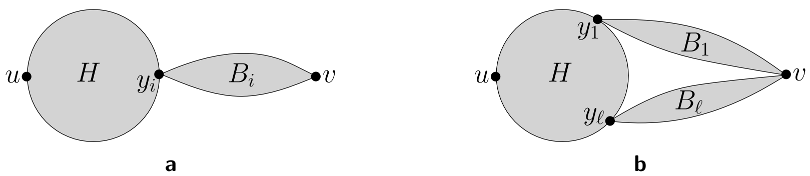


Figure 7 (a) If there were exactly one H'-bridge B_i containing edges incident to the outer face of G, then y_i would be a 1-cut of G. (b) The H'-bridges B_1 and B_ℓ of G.

We now prove that the triple (H, u, y_1) is a strong circuit graph.

(a) Graph H is 2-connected by assumption and it is associated with a plane embedding, given that it is a subgraph of the plane graph G.

(b) The vertex u is incident to the outer face of H since (G, u, v) satisfies Property (b). The vertex y_1 is a vertex of $\tau_{uv}(G)$, as argued above, and hence it is incident to the outer face of G and to the one of H. Finally, u and y_1 are distinct, as otherwise $\tau_{uv}(G)$ would coincide with the edge uv, contradicting the fact that we are in Case B.

(c) Suppose, for a contradiction, that the edge uy_1 exists and does not coincide with $\tau_{uy_1}(H)$. Then $\{u, y_1\}$ is a 2-cut of G, since the removal of u and y_1 disconnects the internal vertices of $\tau_{uy_1}(H)$ (which exist since $\tau_{uy_1}(H)$ is not the edge uy_1) from v. However, none of u and y_1 is an internal vertex of $\beta_{uv}(G)$; this contradicts Property (d) for (G, u, v).

(d) The proof that (H, u, y_1) satisfies Property (d) is very similar to the proof that (G_i, u_{i-1}, u_i) satisfies Property (d) in Lemma 5, hence it is only sketched here.

Consider any 2-cut $\{a, b\}$ of H. If one of a and b is an internal vertex of H, then one non-trivial $\{a, b\}$-component L' of H lies inside an internal face of another non-trivial $\{a, b\}$-component L of H. This implies that $\{a, b\}$ is a 2-cut of G and that L' is an $\{a, b\}$-component of G that does not contain any external vertices of G other than a or b; this contradicts Property (d) for (G, u, v). It follows that a and b are external vertices of H.

If a and b are both in $\tau_{uy_1}(H)$, then assume that u, a, b, and y_1 appear in this order in $\tau_{uy_1}(H)$, where possibly $u = a$ and/or $b = y_1$. Let L be the $\{a, b\}$-component of H containing $\tau_{ab}(H)$ and let L' be any non-trivial $\{a, b\}$-component of H different from L. If L contains an internal vertex of $\beta_{ab}(H)$, then it contains the entire cycle delimiting the outer face of H. It follows that L' lies inside an internal face of L, except at a and b, and hence that L' is an $\{a, b\}$-component of G that does not contain any external vertices of G other than a or b. This contradicts Property (d) for (G, u, v). If L contains no internal vertex of $\beta_{ab}(H)$, then no edge in $E(G) - E(H)$ is incident to a vertex of L different from a and b. It follows that L is a non-trivial $\{a, b\}$-component of G; further, neither a nor b is an internal vertex of $\beta_{uv}(G)$, given that $V(\tau_{ab}(H)) \subseteq V(\tau_{uv}(G))$. This contradicts Property (d) for (G, u, v). It follows that at least one of a and b is an internal vertex of $\beta_{uy_1}(H)$.

Finally, assume that a non-trivial $\{a, b\}$-component L of H contains no external vertex of H other than a and b. Then $\{a, b\}$ is a 2-cut of G and L is a non-trivial $\{a, b\}$-component of G that contains no external vertex of G other than, possibly, a and b. This contradicts Property (d) for (G, u, v). Hence, every non-trivial $\{a, b\}$-component of H contains an external vertex of H other than a and b. This proves Property 6g for (H, u, y_1).

In order to prove Property 6h, assume that B_ℓ does not coincide with the edge $y_\ell v$, as otherwise there is nothing to prove. Let B'_ℓ be the plane graph obtained by adding the edge $y_\ell v$ to B_ℓ, so that y_ℓ immediately precedes v in the clockwise order of the vertices along the outer face of B'_ℓ (both y_ℓ and v are indeed incident to the outer face of B_ℓ); see Fig. 8. We prove that (B'_ℓ, y_ℓ, v) is a strong circuit graph; then Property 6h follows by applying Lemma 5 to (B'_ℓ, y_ℓ, v).

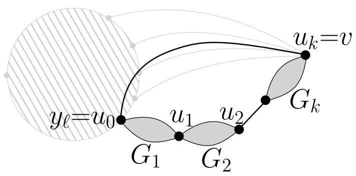


Figure 8 The graph B'_ℓ.

(a) The graph B_ℓ is associated with a plane embedding, given that it is a subgraph of the plane graph G. Further, y_ℓ and v are both incident to the outer face of B_ℓ, hence the plane graph B'_ℓ is well-defined. We prove that B'_ℓ is 2-connected. Since y_ℓ and v are adjacent in B'_ℓ, they belong to the same 2-connected component of B'_ℓ. However, the only vertices of B_ℓ that are incident to edges in $E(G) - E(B_\ell)$ are y_ℓ and v. It follows that any 1-cut of B'_ℓ is also a 1-cut of G. Then B'_ℓ is 2-connected since G is.

(b) The vertices y_ℓ and v are distinct since the first one belongs to H, while the second one does not. Further, both y_ℓ and v are incident to the outer face of B_ℓ, as argued above, and hence are external vertices of B'_ℓ.

(c) The edge $y_\ell v$ exists and coincides with $\tau_{y_\ell v}(B'_\ell)$, by construction.

(d) Consider any 2-cut $\{a, b\}$ of B'_ℓ (possibly $\{a, b\} \cap \{y_\ell, v\} \neq \emptyset$).

Since $y_\ell v \in E(B'_\ell)$, we have that y_ℓ and v are in the same $\{a, b\}$-component L of B'_ℓ. Since y_ℓ and v are the only vertices of B_ℓ incident to edges in $E(G) - E(B_\ell)$, it follows that

$\{a, b\}$ is also a 2-cut of G. Then a and b are external vertices of B'_ℓ since they are external vertices of G.

Next suppose, for a contradiction, that neither a nor b is an internal vertex of $\beta_{y_\ell v}(B'_\ell)$. Since a and b are external vertices of B'_ℓ and since $\tau_{y_\ell v}(B'_\ell)$ coincides with the edge $y_\ell v$, it follows that $a = y_\ell$ and $b = v$ (or vice versa). However, B_ℓ is a $\{y_\ell, v\}$-component of G, hence the removal of y_ℓ and v from B_ℓ (or B'_ℓ) does not disconnect B_ℓ (or B'_ℓ); this contradicts the assumption that $\{a, b\}$ is a 2-cut of B'_ℓ, and implies that one of a and b is an internal vertex of $\beta_{y_\ell v}(B'_\ell)$.

Finally, consider any non-trivial $\{a, b\}$-component L of B'_ℓ. As proved above $\{a, b\}$ is a 2-cut of G and at least one of a and b is not in $\{y_\ell, v\}$. If L contains the edge $y_\ell v$, then it contains an external vertex of B'_ℓ other than a and b, namely whichever vertex of $\{y_\ell, v\}$ that is not in $\{a, b\}$. Otherwise, L is also an $\{a, b\}$-component of G and it contains an external vertex of B'_ℓ other than a and b since it contains an external vertex of G other than a and b.

This concludes the proof that (B'_ℓ, y_ℓ, v) is a strong circuit graph, hence it implies Property 6h via Lemma 5. The lemma follows. ◀

We prove that any strong circuit graph (G, u, v) has a planar greedy drawing by exploiting Lemmata 5 and 6 in a natural way. Indeed, if we are in Case A (in Case B) then Lemma 5 (resp. Lemma 6) is applied in order to construct strong circuit graphs (G_i, u_{i-1}, u_i) with $i = 1, \dots, k$ (resp. strong circuit graphs (H, u, y_1) and (G_i, u_{i-1}, u_i) with $i = 1, \dots, k$) for which planar greedy drawings are inductively constructed and then combined together in order to get a planar greedy drawing of (G, u, v). The base cases of the induction are the ones in which G is an edge or a simple cycle. Then a planar greedy drawing of G is directly constructed.

In order to be able to combine planar greedy drawings for the strong circuit graphs (G_i, u_{i-1}, u_i) (and (H, u, y_1) if we are in Case B) to construct a planar greedy drawing of (G, u, v), we need the inductively constructed drawings to satisfy some restrictive geometric requirements, which are expressed in the following theorem, which is the core of the proof of Theorem 1.

▶ **Theorem 7.** *Let (G, u, v) be a strong circuit graph with at least three vertices and let $0 < \alpha < \frac{\pi}{4}$ be an arbitrary parameter. Let $\beta_{uv}(G) = (u = b_1, b_2, \dots, b_m = v)$. There exists a straight-line drawing Γ of G in the Cartesian plane such that the following holds. For any value $\delta \geq 0$, denote by Γ_δ the straight-line drawing obtained from Γ by moving the position of vertex u by δ units to the left. Then Γ_δ satisfies the following properties (refer to Fig. 9).*

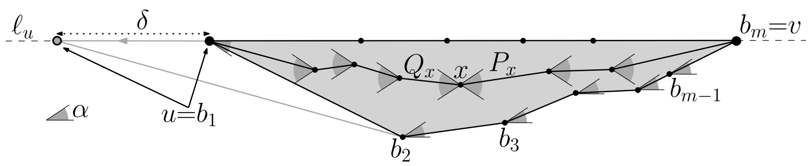


Figure 9 Illustration for the statement of Theorem 7.

1. *Γ_δ is planar;*
2. *$\tau_{uv}(G)$ lies entirely on a horizontal line ℓ_u with u to the left of v;*
3. *the edge $b_1 b_2$ has slope in the interval $(-\alpha; 0)$ and the edge $b_i b_{i+1}$ has slope in the interval $(0; \alpha)$, for each $i = 2, 3, \dots, m-1$;*

4. *for every vertex* $x \in V(G)$ *there is a path* $P_x = (x = v_1, v_2, \dots, v_p = v)$ *from* x *to* v *in* G *such that the edge* $v_i v_{i+1}$ *has slope in the interval* $(-\alpha; \alpha)$ *in* Γ_δ*, for each* $i = 1, 2, \dots, p-1$*; further, if* $x \neq u$*, then* $u \notin V(P_x)$*;*
5. *for every vertex* $x \in V(G)$ *there is a path* $Q_x = (x = w_1, w_2, \dots, w_q = u)$ *from* x *to* u *in* G *such that the edge* $w_i w_{i+1}$ *has slope in the interval* $(\pi - \alpha; \pi + \alpha)$ *in* Γ_δ*, for each* $i = 1, 2, \dots, q-1$*; and*
6. *for every ordered pair of vertices* (x, y) *in* $V(G)$ *there is a path* P_{xy} *from* x *to* y *in* G *such that* P_{xy} *is distance-decreasing in* Γ_δ*; further, if* $x, y \neq u$*, then* $u \notin V(P_{xy})$*.*

Before proceeding with the proof of Theorem 7, we comment on its statement. First, let us set $\delta = 0$ and argue about $\Gamma_0 = \Gamma$. Properties 1 and 6 are those that one would expect, as they state that Γ is planar and greedy, respectively. Properties 2 and 3 state that all the edges incident to the outer face of Γ are "close" to horizontal; indeed, the edges of $\tau_{uv}(G)$ are horizontal, the edge b_1b_2 has a slightly negative slope, and all the other edges of $\beta_{uv}(G)$ have a slightly positive slope. Since Γ is planar, this implies that Γ is contained in a wedge delimited by two half-lines with slopes 0 and $-\alpha$ starting at u. Properties 4 and 5 argue about the existence of certain paths from any vertex to u and v; these two vertices play an important role in the structural decomposition we employ, since distinct subgraphs are joined on those vertices, and the paths incident to them are inductively combined together in order to construct distance-decreasing paths. Finally, all these properties still hold true if u is moved by an arbitrary non-negative amount δ to the left. This is an important feature we exploit in one of our inductive cases.

We now present an inductive proof of Theorem 7. In the base cases G is a single edge (we call this the *Trivial Case*) or a simple cycle (we call this the *Cycle Case*).

We start with the **Trivial Case**, in which G is a single edge. Although Theorem 7 assumes that $|V(G)| \geq 3$, for its proof we need to inductively draw certain subgraphs of G which might be single edges. Whenever we need to draw a strong circuit graph (G, u, v) such that G is a single edge uv, we draw it as a horizontal straight-line segment with positive length, with u to the left of v. We remark that, since Theorem 7 assumes that $|V(G)| \geq 3$, we do not need the constructed drawing to satisfy Properties 1–6.

We next deal with the **Cycle Case**, in which G is a simple cycle with at least 3 vertices. Refer to Fig. 10. By Property (d) of (G, u, v), the set $\{u, v\}$ is not a 2-cut of G, hence u and v appear consecutively along the cycle G. By Property (c) of (G, u, v), the edge uv coincides with the path $\tau_{uv}(G)$. Drawing Γ is constructed as follows. Place b_1, b_2, and b_m at the vertices of an isosceles triangle $\Delta b_1 b_2 b_m$ in which the edge $b_1 b_m$ lies on a horizontal line ℓ_u, with b_1 to the left of b_m, and in which the angles $\measuredangle b_2 b_1 b_m$ and $\measuredangle b_1 b_m b_2$ are $\frac{\alpha}{2}$, with b_2 below ℓ_u. Place the vertices $b_3, \dots, b_{m-1}$ on the straight-line segment $\overline{b_2 b_m}$ in this order from b_2 to b_m. This completes the construction of Γ. We have the following.

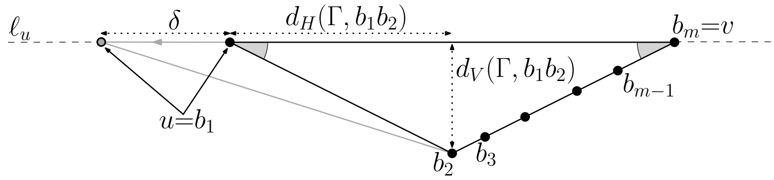


Figure 10 The Cycle Case of the algorithm for the proof of Theorem 7. The gray angles are $\frac{\alpha}{2}$.

▶ **Lemma 8.** *For any* $\delta \geq 0$*, the drawing* Γ_δ *constructed in the Cycle Case satisfies Properties 1–6 of Theorem 7.*

Proof. Properties 1 and 2 are trivially satisfied.

Concerning Property 3, by construction the edge b_ib_{i+1} has slope $\frac{\alpha}{2} \in (0;\alpha)$ for $i = 2,\dots,m-1$. Further, the edge b_1b_2 has slope $-\arctan \frac{d_V(\Gamma,b_1b_2)}{\delta+d_H(\Gamma,b_1b_2)}$, which is smaller than 0, given that $d_V(\Gamma,b_1b_2), d_H(\Gamma,b_1b_2) > 0$ and $\delta \geq 0$, and larger than or equal to $-\arctan \frac{d_V(\Gamma,b_1b_2)}{d_H(\Gamma,b_1b_2)} = -\frac{\alpha}{2}$, hence it is in $(-\alpha;0)$. This implies that Γ_δ satisfies Property 3.

Concerning Property 4, let $x = b_i$ with $i < m$. Then a path P_x satisfying the requirements can be defined as $P_x = (b_1, b_m)$ if $i = 1$ or as $P_x = (b_i, b_{i+1}, \dots, b_m)$ if $i > 1$. In the former case, the only edge of P_x has slope $0 \in (-\alpha;\alpha)$; in the latter case, all the edges of P_x have slope $\frac{\alpha}{2} \in (-\alpha;\alpha)$ and P_x does not pass through u. Hence Γ_δ satisfies Property 4.

Concerning Property 5, let $x = b_i$ with $i > 1$. Then a path Q_x satisfying the requirements can be defined as $Q_x = (b_i, b_{i-1}, \dots, b_1)$. Any edge b_jb_{j-1} with $j \geq 3$ has slope $\pi + \frac{\alpha}{2} \in (\pi - \alpha; \pi+\alpha)$ and edge b_2b_1 has slope $\pi - \arctan \frac{d_V(\Gamma,b_1b_2)}{\delta+d_H(\Gamma,b_1b_2)}$, which is smaller than π, given that $d_V(\Gamma,b_1b_2), d_H(\Gamma,b_1b_2) > 0$ and $\delta \geq 0$, and larger than or equal to $\pi - \arctan \frac{d_V(\Gamma,b_1b_2)}{d_H(\Gamma,b_1b_2)} = \pi - \frac{\alpha}{2}$, hence it is in $(\pi-\alpha;\pi+\alpha)$. This implies that Γ_δ satisfies Property 5.

Finally we deal with Property 6. Let $x = b_i$ and $y = b_j$, for some $1 \leq i, j \leq m$.

- If $2 \leq i, j \leq m$ and $i < j$ (and $j < i$), then the path $P_{xy} = (b_i, b_{i+1}, \dots, b_j)$ (resp. $P_{xy} = (b_i, b_{i-1}, \dots, b_j)$) is distance-decreasing in Γ_δ. Namely, it suffices to observe that the vertex b_{h+1} (resp. b_{h-1}) lies on the open straight-line segment $\overline{b_hb_j}$ for $h = i, i+1, \dots, j-2$ (resp. for $h = i, i-1, \dots, j+2$). Further, P_{xy} does not pass through u.
- If $i = 1$ and $j \geq 3$, then the path $P_{xy} = (b_1, b_2, \dots, b_j)$ is distance-decreasing in Γ_δ. Namely, since the angles $\measuredangle b_2b_1b_m$ and $\measuredangle b_1b_mb_2$ are $\frac{\alpha}{2}$ by construction, the angle $\measuredangle b_mb_2b_1 = \measuredangle b_jb_2b_1$ is equal to $\pi - \alpha$ in Γ, and to at least $\pi - \alpha$ in Γ_δ. Since by assumption $\alpha < \frac{\pi}{4}$, it follows that $\measuredangle b_jb_2b_1$ is the largest angle of the triangle $\Delta b_1b_2b_j$ in Γ_δ, hence $d(\Gamma_\delta, b_1b_j) > d(\Gamma_\delta, b_2b_j)$. That $(b_2, b_3, \dots, b_j)$ is distance-decreasing can be proved as in the previous point.
- If $j = 1$ and $i \geq 3$, then the path $P_{xy} = (b_i, b_{i-1}, \dots, b_1)$ is distance-decreasing in Γ_δ. In order to prove that, it suffices to argue that $d(\Gamma_\delta, b_1b_h) > d(\Gamma_\delta, b_1b_{h-1})$ for any $h = 3, 4, \dots, i$. Since$\measuredangle b_hb_2b_1$ is at least $\pi-\alpha$ in Γ_δ (as from the previous point), the angle $\measuredangle b_hb_{h-1}b_1$ is also at least $\pi - \alpha$. Since by assumption $\alpha < \frac{\pi}{4}$, it follows that $\measuredangle b_hb_{h-1}b_1$ is the largest angle of the triangle $\Delta b_1b_{h-1}b_h$ in Γ_δ, hence $d(\Gamma_\delta, b_1b_h) > d(\Gamma_\delta, b_1b_{h-1})$.

This concludes the proof of the lemma. ◀

We now discuss the inductive cases. In Case A the path $\tau_{uv}(G)$ coincides with the edge uv, while in Case B it does not. We discuss **Case A** first. Let $G' = G - uv$, where G' consists of a sequence of graphs $G_1, \dots, G_k$, with $k \geq 1$, satisfying the properties described in Lemma 5. Our construction is different if $k = 1$ and $k \geq 2$.

Suppose first that $\mathbf{k = 1}$; by Lemma 5 the triple $(G' = G_1, u, v)$ is a strong circuit graph (and G_1 is not a single edge, as otherwise we would be in the Trivial Case). Apply induction in order to construct a straight-line drawing Γ' of G' with $\frac{\alpha}{2}$ as a parameter. Let $\tau_{uv}(G') = (u = a_1, a_2, \dots, a_t = v)$. By Property 2 the path $\tau_{uv}(G')$ lies on a horizontal line ℓ_u in Γ' with u to the left of v. Let $Y > 0$ be the minimum distance in Γ' of any vertex strictly below ℓ_u from ℓ_u. Let

$$\varepsilon = \frac{1}{2} \min\{\varepsilon^*_{\Gamma'}, Y, \tan(\alpha) \cdot d(\Gamma', a_1a_2), \tan(\alpha) \cdot d(\Gamma', a_2a_t)\}.$$

We construct a straight-line drawing Γ of G from Γ' as follows; refer to Fig. 11. Decrease the y-coordinate of the vertex a_2 by ε. Further, decrease the y-coordinate of the vertex a_i, with $i = 3, 4, \ldots, t-1$, so that it ends up on the straight-line segment $\overline{a_2 a_t}$. Draw uv as a straight-line segment. We have the following.

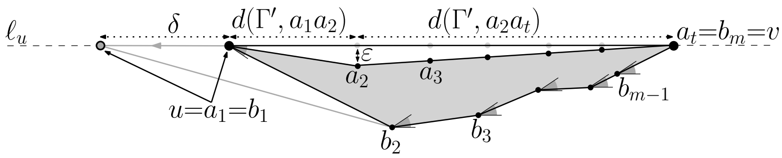


Figure 11 The straight-line drawing Γ of G in Case A if $k = 1$.

▶ **Lemma 9.** *For any $\delta \geq 0$, the drawing Γ_δ constructed in Case A if $k = 1$ satisfies Properties 1–6 of Theorem 7.*

Proof. Concerning Property 1, note first that Γ is planar, given that $\varepsilon < \varepsilon^*_{\Gamma'}$. Since Γ_δ and Γ coincide, except for the position of the vertex u, we only need to prove that no edge incident to u crosses any other edge in Γ_δ. Then consider any two edges uu' and ww' with $u', w, w' \in V(G)$ (possibly $w = u$ or $w = u'$) and suppose, for a contradiction, that they cross or overlap in Γ_δ.

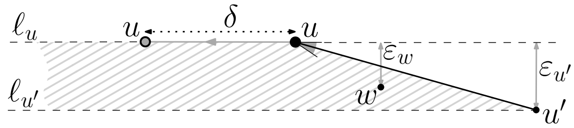


Figure 12 Illustration for the proof that the edges uu' and ww' do not cross in Γ_δ.

Refer to Fig. 12. If $u' = v$, then uu' and ww' do not cross in Γ_δ, given that no vertex other than u and v lies on or above ℓ_u in Γ_δ. We can hence assume that $u' \neq v$ and that $y(u') < y(u)$. By Properties 1–3, we have that Γ' lies in the closed wedge that is delimited by the half-lines starting at u with slopes 0 and $-\frac{\pi}{4}$. It follows that $x(u') > x(u)$ in Γ', Γ, and Γ_δ (given that every vertex has the same x-coordinate in Γ', Γ, and Γ_δ, except for u, whose x-coordinate might be smaller in Γ_δ than in Γ' and Γ). Consider the unbounded region R of the plane that is delimited by ℓ_u from above, by the horizontal line $\ell_{u'}$ through u' from below, and by the representation of the edge uu' in Γ from the right. For any value $\delta > 0$, we have that uu' lies in the interior of R (except at points u and u') in Γ_δ, hence if uu' and ww' cross in Γ_δ then at least one end-vertex of ww', say w, lies in the interior of R, given that $y(w), y(w') \leq y(u)$ and that ww' does not cross uu' in Γ. This implies that $x(u) < x(w) < x(u')$ in Γ', Γ, and Γ_δ. We now distinguish four cases, based on whether u' and/or w belong to $V(\tau_{uv}(G'))$.

- If $u', w \in V(\tau_{uv}(G'))$, then by Property 2 we have that u' and w lie on ℓ_u in Γ'. However, since $x(u) < x(w) < x(u')$, it follows that the edge uu' overlaps the vertex w in Γ', a contradiction to Property 1 of Γ'.
- If $u' \in V(\tau_{uv}(G'))$ and $w \notin V(\tau_{uv}(G'))$, then when transforming Γ' into Γ the y-coordinate of u' has been decreased by a value $\varepsilon_{u'} \leq \varepsilon$ which is larger than the distance $\varepsilon_w \geq Y$ between w and ℓ_u. This contradicts $\varepsilon \leq \frac{Y}{2} < Y$.
- If $u' \notin V(\tau_{uv}(G'))$ and $w \in V(\tau_{uv}(G'))$, then when transforming Γ' into Γ the y-coordinate of w has been decreased by a value $\varepsilon_w \leq \varepsilon$. The point p on the edge uu' with x-coordinate equal to $x(w)$ has y-coordinate larger than $y(w)$, hence the distance from p to ℓ_u is a value $\varepsilon_p < \varepsilon_w$. This implies that the drawing obtained from Γ' by

decreasing the y-coordinate of w by ε_p, while every other vertex stays put, is not planar, given that the edge uu' overlaps the vertex w. However, since $\varepsilon_p < \varepsilon_w$, this contradicts $\varepsilon_w \leq \varepsilon < \varepsilon^*_{\Gamma'}$.

- Finally, if $u', w \notin V(\tau_{uv}(G'))$, then u, u' and w have the same positions in Γ' and Γ. Consider the line through u' and w; let q be its intersection point with ℓ_u and let δ_q be the Euclidean distance between q and u in Γ'. Then the drawing Γ'_{δ_q} is not planar as the edge uu' overlaps the vertex w. This contradicts Property 1 of Γ'.

Concerning Property 2, note that u and v lie on the same horizontal line ℓ_u (with u to the left of v) in Γ since they do in Γ' and since they have not been moved when transforming Γ' into Γ. Since $\tau_{uv}(G)$ coincides with the edge uv, it follows that Γ_δ satisfies Property 2.

Property 3 is satisfied by Γ_δ since it is satisfied by Γ'_δ and since no vertex of $\beta_{uv}(G') = \beta_{uv}(G)$ moves when transforming Γ' into Γ (indeed, $\tau_{uv}(G')$ and $\beta_{uv}(G')$ do not share any vertex other than u and v, given that G' is 2-connected).

We now discuss Property 4. Let $x \in V(G)$. If $x = u$, let $P_x = (u, v)$; then the only edge of P_x has slope $0 \in (-\alpha; \alpha)$. If $x \neq u$, then let $P'_x = (x = v_1, v_2, \dots, v_p = v)$ be a path in G' such that the slope of $v_i v_{i+1}$ in Γ' is in the interval $(-\frac{\alpha}{2}; \frac{\alpha}{2})$, for $i = 1, \dots, p-1$, and such that $u \notin V(P'_x)$. This path exists since Γ' satisfies Property 4, by induction. We distinguish two cases.

- If no vertex of $P'_x - \{v\}$ belongs to $\tau_{uv}(G')$, then $P_x = P'_x$ satisfies the required properties. Indeed, no vertex other than those internal to $\tau_{uv}(G')$ moves when transforming Γ' into Γ and no vertex other than u moves when transforming Γ into Γ_δ; thus, P_x has the same representation (and in particular each edge of P_x has the same slope) in Γ' and Γ_δ.

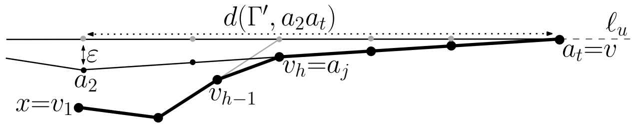


Figure 13 Illustration for the proof that the slope in Γ_δ of every edge in the path $P_x = (x = v_1, v_2, \dots, v_h = a_j, a_{j+1}, \dots, a_t = v)$ is in $(-\alpha; \alpha)$. The path P_x is thick.

- Otherwise, a vertex of $P'_x - \{v\}$ belongs to $\tau_{uv}(G')$; let h be the smallest index such that $v_h = a_j$, for some $a_j \in V(\tau_{uv}(G')) - \{v\}$ and define $P_x = (x = v_1, v_2, \dots, v_h = a_j, a_{j+1}, \dots, a_t = v)$. Refer to Fig. 13. Note that $u \notin V(P_x)$, given that $u \notin V(P'_x)$. Hence, it suffices to argue about the slopes of the edges of P_x in Γ (rather than in Γ_δ). For $i = 1, \dots, h-2$, the slope of the edge $v_i v_{i+1}$ is in $(-\alpha; \alpha)$ in Γ since it is in $(-\frac{\alpha}{2}; \frac{\alpha}{2}) \subset (-\alpha; \alpha)$ in Γ' and since neither v_i nor v_{i+1} moves when transforming Γ' into Γ. Further, for $i = j, \dots, t-1$, the slope of the edge $a_i a_{i+1}$ in Γ is $\arctan\left(\frac{\varepsilon}{d(\Gamma', a_2 a_t)}\right)$, which is in the interval $(0; \alpha) \subset (-\alpha; \alpha)$, given that $\varepsilon, d(\Gamma', a_2 a_t) > 0$ and that $\varepsilon < \tan(\alpha) \cdot d(\Gamma', a_2 a_t)$. Finally, let s' and s be the slopes of the edge $v_{h-1} v_h$ in Γ' and Γ, respectively. Since $v_{h-1} v_h \in E(P'_x)$, we have $s' \in (-\frac{\alpha}{2}; \frac{\alpha}{2})$; since $\alpha \leq \frac{\pi}{4}$, this implies that $x(v_{h-1}) < x(v_h)$ in Γ' and Γ (note that the x-coordinates of the vertices do not change when transforming Γ' into Γ). Further, by Properties 1–4 of Γ', we have that v_{h-1} lies below ℓ_u, which contains v_h; hence, $y(v_{h-1}) < y(v_h)$ in Γ'. Since the vertex v_h moves down (while v_{h-1} stays put) when transforming Γ' into Γ, and since $\varepsilon \leq \frac{Y}{2} < d_V(\Gamma', v_{h-1} v_h)$, it follows that $0 < s < s'$; hence $s \in (0; \frac{\alpha}{2}) \subset (-\alpha; \alpha)$.

We next deal with Property 5. Let $x \in V(G)$ and let $Q'_x = (x = w_1, w_2, \dots, w_q = u)$ be a path in Γ'_δ such that the slope of $w_i w_{i+1}$ is in $(\pi - \frac{\alpha}{2}; \pi + \frac{\alpha}{2})$, for $i = 1, 2, \dots, q-1$.

This path exists since Γ'_δ satisfies Property 5, by induction. Similarly to the proof that Γ_δ satisfies Property 4, we distinguish two cases. If no vertex of $Q'_x - \{u\}$ belongs to $\tau_{uv}(G')$, then let $Q_x = Q'_x$ and observe that Q_x satisfies the required properties in Γ_δ since Q'_x does in Γ'_δ. Otherwise, let h be the smallest index such that $w_h = a_j$, for some $a_j \in V(\tau_{uv}(G')) - \{u\}$ and define $Q_x = (x = w_1, w_2, \ldots, w_h = a_j, a_{j-1}, \ldots, a_1 = u)$. Refer to Fig. 14. For $i = 1, \ldots, h-2$, the slope of the edge $w_i w_{i+1}$ is in $(\pi - \alpha; \pi + \alpha)$ in Γ_δ since it is in $(\pi - \frac{\alpha}{2}; \pi + \frac{\alpha}{2}) \subset (\pi - \alpha; \pi + \alpha)$ in Γ'_δ. Further, similarly to the proof that the edge $v_{h-1}v_h$ has slope in $(-\alpha; \alpha)$ and hence Γ_δ satisfies Property 4, we have that the edge $w_{h-1}w_h$ has slope $s \in (\pi - \alpha; \pi + \alpha)$ in Γ_δ. Indeed, the slope s' of the edge $w_{h-1}w_h$ in Γ' is in the interval $(\pi - \frac{\alpha}{2}; \pi + \frac{\alpha}{2}) \subset (\pi - \alpha; \pi + \alpha)$ in Γ'_δ, given that $w_{h-1}w_h$ belongs to Q'_x. Further, since $x(w_{h-1}) > x(w_h)$ and $y(w_{h-1}) < y(w_h)$, we have that $s' \in (\pi - \alpha; \pi)$. Since the vertex w_h moves down while w_{h-1} stays put when transforming Γ' into Γ, and since $\varepsilon \leq \frac{Y}{2} < d_V(\Gamma', w_{h-1}w_h)$, we have that $s' < s < \pi$, hence $s \in (\pi - \alpha; \pi) \subset (\pi - \alpha; \pi + \alpha)$. For $i = j, j-1 \ldots, 3$, the edge $a_i a_{i-1}$ has slope $\pi + \arctan\left(\frac{\varepsilon}{d(\Gamma', a_2 a_t)}\right)$, which is larger than π, given that $\varepsilon, d(\Gamma', a_2 a_t) > 0$, and smaller than $\pi + \alpha$, given that $\varepsilon < \tan(\alpha) \cdot d(\Gamma', a_2 a_t)$. Finally, the edge $a_2 a_1$ has slope $\pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma', a_1 a_2)}\right)$, which is smaller than π, given that $\varepsilon, d(\Gamma', a_1 a_2) > 0$ and $\delta \geq 0$, and larger than $\pi - \alpha$, given that $\frac{\varepsilon}{\delta + d(\Gamma', a_1 a_2)} \leq \frac{\varepsilon}{d(\Gamma', a_1 a_2)}$ and that $\varepsilon < \tan(\alpha) \cdot d(\Gamma', a_1 a_2)$.

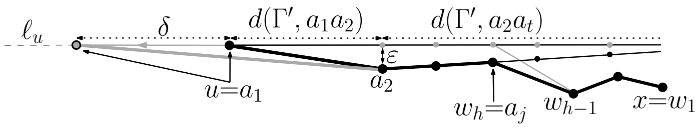


Figure 14 Illustration for the proof that the slope in Γ_δ of every edge in the path $Q_x = (x = w_1, w_2, \ldots, w_h = a_j, a_{j-1}, \ldots, a_1 = u)$ is in $(\pi - \alpha; \pi + \alpha)$. The path Q_x is thick.

Finally, we deal with Property 6. Consider any two vertices $x, y \in V(G)$.

- First, assume that $x, y \neq u$. By induction, there exists a path P_{xy} from x to y in G' that is distance-decreasing in Γ' with $u \notin V(P_{xy})$. By Lemma 2 and since, for every vertex $z \in V(G)$, the Euclidean distance between the positions of z in Γ' and Γ is at most $\varepsilon < \varepsilon^*_{\Gamma'}$, we have that P_{xy} is also distance-decreasing in Γ. Further, since all the vertices other than u have the same position in Γ and Γ_δ, it follows that P_{xy} is a distance-decreasing path from x to y not passing through u in Γ_δ.
- Second, suppose that $y = u$. Consider the path Q_x in G from Property 5, whose every edge has slope in $(\pi - \alpha; \pi + \alpha)$ in Γ_δ. Since $\alpha \leq \frac{\pi}{4}$, it follows that Q_x is a π-path (according to the definition in [11]) or is π-monotone (according to the definition in [7]), where for some angle β a path $(q_1, q_2, \ldots, q_r)$ is a *β-path* or equivalently is *β-monotone* if every edge $q_i q_{i+1}$ has slope in the interval $(\beta - \frac{\pi}{4}; \beta + \frac{\pi}{4})$. In [11, Lemma 3] it is proved that a β-path is distance-decreasing (in fact, it satisfies a much stronger property, namely it is *increasing-chord*); hence, Q_x is distance-decreasing in Γ_δ.
- Finally, suppose that $x = u$ and consider a path P_{xy} from x to y in G' that is distance-decreasing in Γ'. We prove that P_{xy} is distance-decreasing in Γ_δ, as well. Let xx' be the edge of P_{xy} incident to x. Differently from the case in which $x, y \neq u$, we cannot directly apply Lemma 2, given that it is not guaranteed that $\varepsilon < \varepsilon^*_{\Gamma'_\delta}$. However, since for every vertex $z \in V(P_{xy})$ the Euclidean distance between the positions of z in Γ' and Γ is at most $\varepsilon < \varepsilon^*_{\Gamma'}$, by Lemma 2 we have that P_{xy} is distance-decreasing in Γ. Further, the path obtained from P_{xy} by removing the vertex $x = u$ and the edge xx'

has the same representation in Γ_δ and Γ, given that it does not contain u, hence it is distance-decreasing in Γ_δ. Thus, it only remains to show that $d(\Gamma_\delta, xy) > d(\Gamma_\delta, x'y)$. First, since $x = u$, we have that $x', y \neq u$. Hence, $d(\Gamma_\delta, x'y) = d(\Gamma, x'y)$. Second, denote by u_Γ and u_{Γ_δ} the positions of u in Γ and Γ_δ, respectively. By Properties 1–3, the entire drawing Γ, and in particular vertex y, lies in the closed wedge that is delimited by the half-lines starting at u_Γ and with slopes 0 and $-\alpha$. Then the angle incident to u_Γ in the triangle $\Delta y u_\Gamma u_{\Gamma_\delta}$ is at least $\pi - \alpha > \frac{\pi}{2}$, hence the straight-line segment between u_{Γ_δ} and y is the longest side of that triangle. It follows that $d(\Gamma_\delta, xy) \geq d(\Gamma, xy)$. Thus, $d(\Gamma_\delta, xy) \geq d(\Gamma, xy) > d(\Gamma, x'y) = d(\Gamma_\delta, x'y)$, where the second inequality holds true since P_{xy} is distance-decreasing in Γ. Hence, P_{xy} is distance-decreasing in Γ_δ.

This concludes the proof of the lemma. ◀

We now discuss the case in which $\mathbf{k \geq 2}$. Refer to Fig. 15. By Lemma 5, for $i = 1, \dots, k$, the triple (G_i, u_{i-1}, u_i) is a strong circuit graph, where $u_0 = u$, $u_k = v$, and u_i is the only vertex shared by G_i and G_{i+1}, for $i = 1, \dots, k-1$.

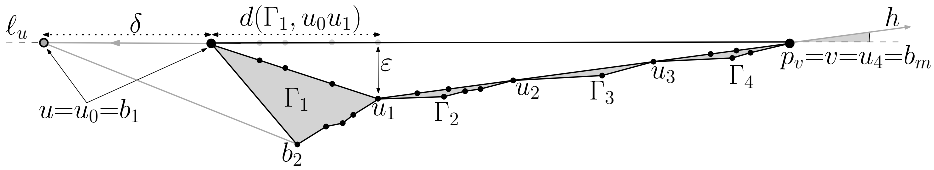


Figure 15 The straight-line drawing Γ of G in Case A if $k \geq 2$. In this example $k = 4$. The gray angle in the drawing is $\frac{\alpha}{2}$.

If G_1 is a single edge, then apply induction in order to construct a straight-line drawing Γ_1 of G_1 and define $\varepsilon = \frac{1}{2}\min\{\varepsilon^*_{\Gamma_1}, \tan(\alpha) \cdot d(\Gamma_1, u_0u_1)\}$.

If G_1 is not a single edge, then apply induction in order to construct a straight-line drawing Γ_1 of G_1 with $\frac{\alpha}{2}$ as a parameter. By Property 2 of Γ_1, the path $\tau_{u_0u_1}(G_1)$ lies on a horizontal line ℓ_u. Let $Y > 0$ be the minimum distance in Γ_1 of any vertex strictly below ℓ_u from ℓ_u. Let $\varepsilon = \frac{1}{2}\min\{\varepsilon^*_{\Gamma_1}, Y, \tan(\alpha) \cdot d(\Gamma_1, u_0u_1)\}$.

In both cases, decrease the y-coordinate of u_1 by ε. Further, decrease the y-coordinate of every internal vertex of the path $\tau_{u_0u_1}(G_1)$, if any, so that it ends up on the straight-line segment $\overline{u_0u_1}$.

Now consider a half-line h with slope $s = \frac{\alpha}{2}$ starting at u_1. Denote by p_v the point at which h intersects the horizontal line ℓ_u through u. For $i = 2, \dots, k$, apply induction in order to construct a straight-line drawing Γ_i of G_i with $\frac{\alpha}{3}$ as a parameter (if G_i is a single edge, then the parameter does not matter). Uniformly scale the drawings $\Gamma_2, \dots, \Gamma_k$ so that the Euclidean distance between u_{i-1} and u_i is equal to $\frac{d(\Gamma_1, u_1p_v)}{k-1}$. For $i = 2, \dots, k$, rotate the scaled drawing Γ_i around u_{i-1} counter-clockwise by s radians. Translate the scaled and rotated drawings $\Gamma_2, \dots, \Gamma_k$ so that the representations of u_i in Γ_i and Γ_{i+1} coincide, for $i = 1, \dots, k-1$. Finally, draw the edge uv as a straight-line segment. This completes the construction of a drawing Γ of G. We have the following.

▶ **Lemma 10.** *For any $\delta \geq 0$, the drawing Γ_δ constructed in Case A if $k \geq 2$ satisfies Properties 1–6 of Theorem 7.*

Proof. Throughout the proof, we denote by $\Gamma_{1,\delta}$ the drawing obtained from Γ_1 by moving the position of the vertex $u_0 = u$ by δ units to the left (where Γ_1 is understood as the drawing of G_1 in which the vertices of $\tau_{u_0u_1}(G_1)$ all lie on ℓ_u).

We first prove Property 2. Because of the uniform scaling which has been applied to $\Gamma_2, \dots, \Gamma_k$, we have that $d(\Gamma, u_{i-1}u_i) = \frac{d(\Gamma, u_1 p_v)}{k-1}$ for $i = 2, \dots, k$. Since the vertices $u_1, \dots, u_k$ all lie on h, we have that $d_V(\Gamma, u_{i-1}u_i) = \frac{d_V(\Gamma, u_1 p_v)}{k-1}$ for $i = 2, \dots, k$. Hence, the y-coordinate of $v = u_k$ is equal to $y(u_1) + d_V(\Gamma, u_1 p_v) = y(u)$, which implies that u and v lie on ℓ_u in Γ and Γ_δ. Further, u is to the left of v in Γ, given that Γ_1 satisfies Property 2 and that $0 < s < \frac{\pi}{2}$. Since $\tau_{uv}(G)$ coincides with the edge uv, it follows that Γ_δ satisfies Property 2.

We next prove that Γ_δ satisfies Property 3. Observe that $\beta_{uv}(G) = \beta_{u_0u_1}(G_1) \cup \beta_{u_1u_2}(G_2) \cup \dots \beta_{u_{k-1}u_k}(G_k)$. We first argue about the slope of the edge b_1b_2.

- If G_1 is a single edge u_0u_1, then we have $u_0 = b_1$ and $u_1 = b_2$. Then the slope of the edge b_1b_2 in Γ_δ is $-\arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0u_1)}\right)$, which is smaller than 0, given that $\varepsilon, d(\Gamma_1, u_0u_1) > 0$ and $\delta \geq 0$, and larger than $-\alpha$, given that $\delta \geq 0$ and that $\varepsilon < \tan\alpha \cdot d(\Gamma_1, u_0u_1)$, hence it is in $(-\alpha; 0)$.
- If G_1 has more than two vertices, then $\beta_{u_0u_1}(G_1)$ is not a single edge u_0u_1, by Property (c) of (G_1, u_0, u_1); hence, $u_0 = b_1$ and $b_2 \neq u_1$. Since $\Gamma_{1,\delta}$ satisfies Property 3 and since u_1 is the only vertex of $\beta_{u_0u_1}(G_1)$ whose positions in $\Gamma_{1,\delta}$ and Γ_δ do not coincide, it follows that the slope of b_1b_2 in Γ_δ is in the interval $(-\alpha; 0)$ since it is in the interval $(-\frac{\alpha}{2}; 0) \subset (-\alpha; 0)$ in $\Gamma_{1,\delta}$.

We now argue about the slope s_j of the edge b_jb_{j+1} in Γ_δ, for any $j = 2, \dots, m-1$.

- If b_jb_{j+1} coincides with a graph G_i, then $s_j = s = \frac{\alpha}{2} \in (0; \alpha)$.
- If b_jb_{j+1} belongs to a graph G_i with $|V(G_i)| \geq 3$, with $i \geq 2$, and with $b_j \neq u_{i-1}$, then s_j is given by the slope b_jb_{j+1} has in Γ_i, which is in $(0; \frac{\alpha}{3})$ by Property 3 of Γ_i, plus s, which results from the rotation of Γ_i. Hence $s_j \in (\frac{\alpha}{2}; \frac{5\alpha}{6}) \subset (0; \alpha)$.
- If b_jb_{j+1} belongs to a graph G_i with $|V(G_i)| \geq 3$, with $i \geq 2$, and with $b_j = u_{i-1}$, then s_j is given by the slope b_jb_{j+1} has in Γ_i, which is in $(-\frac{\alpha}{3}; 0)$ by Property 3 of Γ_i, plus s, which results from the rotation of Γ_i. Hence $s_j \in (\frac{\alpha}{6}; \frac{\alpha}{2}) \subset (0; \alpha)$.
- If b_jb_{j+1} belongs to G_1, if $|V(G_1)| \geq 3$, and if $b_{j+1} \neq u_1$, then since $\Gamma_{1,\delta}$ satisfies Property 3 and since u_1 is the only vertex of $\beta_{u_0u_1}(G_1)$ whose positions in $\Gamma_{1,\delta}$ and Γ_δ do not coincide, it follows that $s_j \in (0; \alpha)$ since the slope of b_jb_{j+1} in $\Gamma_{1,\delta}$ is in $(0; \alpha)$.
- Finally, assume that b_jb_{j+1} belongs to G_1, that $|V(G_1)| \geq 3$, and that $b_{j+1} = u_1$. Note that $b_j \neq u$, given that $j \geq 2$, hence by Property 3 of Γ_1 we have that $x(b_j) < x(b_{j+1})$ and that $y(b_j) < y(b_{j+1})$ in Γ_1. Note that the positions of b_j in Γ_1 and Γ_δ coincide, given that $b_j \notin V(\tau_{u_0u_1}(G_1))$; further, b_{j+1} moves down by ε when transforming Γ_1 into Γ_δ, however its x-coordinate stays unchanged; this implies that s_j is smaller than the slope of b_jb_{j+1} in Γ_1, hence smaller than α. Since $\varepsilon \leq \frac{Y}{2} < d_V(\Gamma_1, b_jb_{j+1})$ – given that u_1 lies on ℓ_u in Γ_1 – it follows that $y(b_j) < y(b_{j+1})$ holds true in Γ_δ, hence $s_j > 0$. Thus, $s_j \in (0; \alpha)$.

We now prove Property 1. First, the edge uv does not cross or overlap any other edge of G, since no vertex other than u and v lies on or above ℓ_u in Γ and Γ_δ. Hence, we only need to argue about crossings among edges in the graphs $G_1, \dots, G_k$.

We first deal with Γ. For $i = 1, \dots, k$, the inductively constructed drawing Γ_i of G_i is planar, by Property 1. Further, for $i = 2, \dots, k$, the drawing of G_i in Γ is congruent to Γ_i, up to affine transformations (a uniform scaling, a rotation, and a translation), which preserve planarity. Moreover, since $\varepsilon < \varepsilon^*_{\Gamma_1}$, by Lemma 2 we have that the drawing of G_1 in Γ is planar, as well. It follows that no two edges in the same graph G_i cross each other in Γ, for each $i = 1, \dots, k$. Since Γ satisfies Property 3, the path $\beta_{uv}(G)$ is represented in

Γ by a curve monotonically increasing in the x-direction from u to v. Further, the path $\tau = \bigcup_{i=1}^{k} \tau_{u_{i-1}u_i}(G_i)$ is also represented in Γ by a curve monotonically increasing in the x-direction from u to v, since it is composed of the straight-line segment $\overline{u_0u_1}$, which has slope $-\arctan\left(\frac{\varepsilon}{d(\Gamma_1,u_0u_1)}\right) \in (-\frac{\pi}{2};0)$, and of the straight-line segment $\overline{u_1u_k}$, which has slope $s = \frac{\alpha}{2} \in (0;\frac{\pi}{2})$. Hence, for $i = 1,2,\dots,k-1$, the vertical line through u_i has the drawings of $G_1,\dots,G_i$ to its left and the drawings of $G_{i+1},\dots,G_k$ to its right in Γ. It follows that no two edges in distinct graphs G_i and G_j cross in Γ. This proves the planarity of Γ.

Since Γ_δ and Γ coincide, except for the position of u, it remains to prove that no edge uu' incident to u with $u' \neq v$ crosses or overlaps any other edge in Γ_δ. Since the vertical line through u_1 has the drawing of G_1 to its left and the drawings of $G_2,\dots,G_k$ to its right in Γ_δ, such a crossing might only occur between uu' and another edge ww' of G_1. The proof that uu' and ww' do not cross or overlap is the same as in the proof of Lemma 9, with G_1 playing the role of G' and Γ_1 playing the role of Γ'.

We now deal with Property 4. Let $x \in V(G)$. If $x = u$, let $P_x = (u,v)$; then the only edge of P_x has slope $0 \in (-\alpha;\alpha)$ in Γ_δ. If $x = u_i$, for some $i \in \{1,\dots,k-1\}$, then let $P_x = \bigcup_{j=i+1}^{k} \tau_{u_{j-1}u_j}(G_j)$ and observe that all the edges of P_x have slope $s = \frac{\alpha}{2} \in (-\alpha;\alpha)$; further P_x does not pass through u. If $x \neq u_i$, for every $i \in \{0,1,\dots,k\}$, then x belongs to a unique graph G_i, for some $i \in \{1,2,\dots,k\}$. We distinguish two cases.

- Assume first that $i \geq 2$. Since Γ_i satisfies Property 4, there exists a path P_x^i from x to u_i in G_i whose every edge has slope in $(-\frac{\alpha}{3};\frac{\alpha}{3})$ in Γ_i; then P_x consists of P_x^i and of the path $\bigcup_{j=i+1}^{k} \tau_{u_{j-1}u_j}(G_j)$. Since the drawing of G_i in Γ_δ is congruent to Γ_i up to a uniform scaling, a counter-clockwise rotation by $s = \frac{\alpha}{2}$ radians, and a translation, it follows that every edge of P_x^i has slope in $(s-\frac{\alpha}{3};s+\frac{\alpha}{3}) = (\frac{\alpha}{6};\frac{5\alpha}{6}) \subset (-\alpha;\alpha)$ in Γ_δ; further, as noted above, all the edges of $\bigcup_{j=i+1}^{k} \tau_{u_{j-1}u_j}(G_j)$ have slope $s = \frac{\alpha}{2} \in (-\alpha;\alpha)$. Hence, all the edges of P_x have slope in $(-\alpha;\alpha)$; further, P_x does not pass through u.

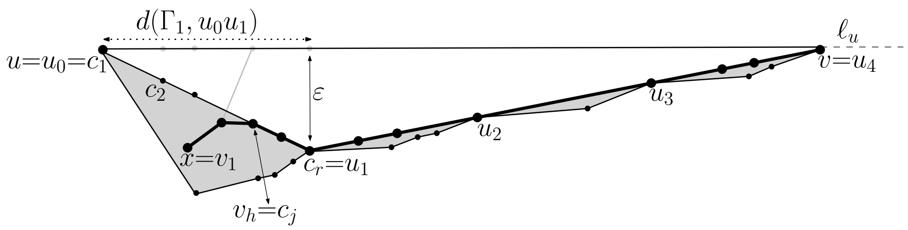


Figure 16 Illustration for the proof that the slope in Γ_δ of every edge in the path P_x is in $(-\alpha;\alpha)$, in the case in which x belongs to G_1. The path P_x is thick.

- Assume next that $i = 1$. Refer to Fig. 16. Let $\tau_{u_0u_1}(G_1) = (u_0 = c_1, c_2, \dots, c_r = u_1)$. Since Γ_1 satisfies Property 4, there exists a path $P_x^1 = (x = v_1, v_2, \dots, v_p = u_1)$ from x to u_1 in G_1, not passing through u, whose every edge has slope in $(-\frac{\alpha}{2};\frac{\alpha}{2})$ in Γ_1; let h be the smallest index such that $v_h = c_j$, for some $j \in \{2,3,\dots,r\}$. Such an index h exists (possibly $h = p$ and $j = r$). Then let P_x consist of the paths $(x = v_1, v_2, \dots, v_h)$, $(v_h = c_j, c_{j+1}, \dots, c_r)$, and $\bigcup_{j=2}^{k} \tau_{u_{j-1}u_j}(G_j)$. Since $u \notin V(P_x^1)$, we have that $u \notin V(P_x)$, hence it suffices to argue about the slopes of the edges of P_x in Γ rather than in Γ_δ.
  For $l = 1,\dots,h-2$, the slope of v_lv_{l+1} in Γ is in the interval $(-\alpha;\alpha)$ since it is in the interval $(-\frac{\alpha}{2};\frac{\alpha}{2}) \subset (-\alpha;\alpha)$ in Γ_1 and since neither v_l nor v_{l+1} moves when transforming Γ_1 into Γ. Further, for $l = j,\dots,r-1$, the slope of the edge c_lc_{l+1} in Γ is $-\arctan\left(\frac{\varepsilon}{d(\Gamma_1,u_0u_1)}\right)$, which is in the interval $(-\alpha;0) \subset (-\alpha;\alpha)$, given that

$\varepsilon, d(\Gamma_1, u_0u_1) > 0$ and that $\varepsilon < \tan(\alpha) \cdot d(\Gamma_1, u_0u_1)$. Moreover, as noted above, the edges of $\bigcup_{j=2}^{k} \tau_{u_{j-1}u_j}(G_j)$ have slope $s = \frac{\alpha}{2} \in (-\alpha; \alpha)$. Finally, let σ_1 and σ be the slopes of the edge $v_{h-1}v_h$ in Γ_1 and Γ, respectively. Since $v_{h-1}v_h \in E(P_x^1)$, we have $\sigma_1 \in (-\frac{\alpha}{2}; \frac{\alpha}{2})$; since $\alpha \leq \frac{\pi}{4}$, we have $x(v_{h-1}) < x(v_h)$ in Γ_1 and Γ (note that the x-coordinates of the vertices do not change when transforming Γ_1 into Γ). Further, by Properties 1–4 of Γ_1, we have that v_{h-1} lies below ℓ_u, which contains v_h; hence, $y(v_{h-1}) < y(v_h)$ in Γ_1. Since the vertex v_h moves down (while v_{h-1} stays put) when transforming Γ_1 into Γ, and since $\varepsilon \leq \frac{Y}{2} < d_V(\Gamma_1, v_{h-1}v_h)$, it follows that $0 < \sigma < \sigma_1$; hence $\sigma \in (0; \frac{\alpha}{2}) \subset (-\alpha; \alpha)$.

We now argue about Property 5. Let $x \in V(G)$. If $x = v$, let $Q_x = (v, u)$; then the only edge of Q_x has slope $\pi \in (\pi - \alpha; \pi + \alpha)$ in Γ_δ. If $x = u_i$, for some $i \in \{1, \dots, k-1\}$, then let $Q_x = \bigcup_{j=1}^{i} \beta_{u_ju_{j-1}}(G_j)$; recall that $\beta_{u_ju_{j-1}}(G_j)$ has the same vertices as $\tau_{u_{j-1}u_j}(G_j)$, however in the reverse linear order. Denote the vertices of Q_x by $(x = w_1, w_2, \dots, w_q = u)$. Consider the edge w_lw_{l+1}, for any $1 \leq l \leq q-1$.

- If w_lw_{l+1} is in G_j, for some $j \geq 2$, then its slope in Γ_δ is $\pi + s = \pi + \frac{\alpha}{2} \in (\pi - \alpha; \pi + \alpha)$.
- If w_lw_{l+1} is in G_1 and $l \leq q-2$, then its slope in Γ_δ is $\pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0u_1)}\right)$, which is in the interval $(\pi - \alpha; \pi) \subset (\pi - \alpha; \pi + \alpha)$, given that $\varepsilon, d(\Gamma_1, u_0u_1) > 0$ and that $\varepsilon < \tan(\alpha) \cdot d(\Gamma_1, u_0u_1)$.
- Finally, the slope of $w_{q-1}w_q$ in Γ_δ is $\pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0u_1)}\right)$, which is in the interval $(\pi - \alpha; \pi) \subset (\pi - \alpha; \pi + \alpha)$, given that $\varepsilon, d(\Gamma_1, u_0u_1) > 0$, that $\delta \geq 0$, and that $\varepsilon < \tan(\alpha) \cdot d(\Gamma_1, u_0u_1)$.

If $x \neq u_i$, for every $i \in \{0, 1, \dots, k\}$, then x belongs to a unique graph G_i, for some $i \in \{1, 2, \dots, k\}$. We distinguish two cases.

- Assume first that $i \geq 2$. Since Γ_i satisfies Property 5, there exists a path Q_x^i from x to u_{i-1} in G_i whose every edge has slope in $(\pi - \frac{\alpha}{3}; \pi + \frac{\alpha}{3})$ in Γ_i; then Q_x consists of Q_x^i and of the path $\bigcup_{j=1}^{i-1} \beta_{u_ju_{j-1}}(G_j)$; denote the vertices of Q_x by $(x = w_1, w_2, \dots, w_q = u)$. Consider the edge w_lw_{l+1}, for any $1 \leq l \leq q-1$.
  - If w_lw_{l+1} is in G_i, then it belongs to the path Q_x^i. Then w_lw_{l+1} has slope in $(\pi - \frac{\alpha}{3}; \pi + \frac{\alpha}{3})$ in Γ_i, hence it has slope $(\pi + s - \frac{\alpha}{3}; \pi + s + \frac{\alpha}{3}) = (\pi + \frac{\alpha}{6}; \pi + \frac{5\alpha}{6}) \subset (\pi - \alpha; \pi + \alpha)$ in Γ_δ.
  - If w_lw_{l+1} is in G_j, for some $2 \leq j \leq i-1$, then its slope in Γ_δ is $\pi + s = \pi + \frac{\alpha}{2} \in (\pi - \alpha; \pi + \alpha)$.
  - If w_lw_{l+1} is in G_1 and $l \leq q-2$, then its slope in Γ_δ is $\pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0u_1)}\right)$, which is in the interval $(\pi - \alpha; \pi + \alpha)$, as proved in the case $x = u_i$.
  - Finally, the slope of $w_{q-1}w_q$ in Γ_δ is $\pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0u_1)}\right)$, which is in the interval $(\pi - \alpha; \pi + \alpha)$, as proved in the case $x = u_i$.
- Assume next that $i = 1$. Let $\tau_{u_0u_1}(G_1) = (u_0 = c_1, c_2, \dots, c_r = u_1)$. Since Γ_1 satisfies Property 5, there exists a path $Q_x^1 = (x = w_1, w_2, \dots, w_q = u)$ from x to u in G_1, whose every edge has slope in $(\pi - \frac{\alpha}{2}; \pi + \frac{\alpha}{2})$ in Γ_1. Similarly to the proof that Γ_δ satisfies Property 5 in Lemma 9, we distinguish two cases. If no vertex of $Q_x^1 - \{u\}$ belongs to $\tau_{u_0u_1}(G_1)$, then let $Q_x = Q_x^1$ and observe that Q_x satisfies the required properties in Γ_δ since Q_x^1 does in $\Gamma_{1,\delta}$. Otherwise, let h be the smallest index such that $w_h = c_j$, for some $j \in \{2, 3, \dots, r\}$ and define $Q_x = (x = w_1, w_2, \dots, w_h = c_j, c_{j-1}, \dots, c_1 = u)$. For $l = 1, \dots, h-2$, the slope of the edge w_lw_{l+1} is in $(\pi - \alpha; \pi + \alpha)$ in Γ_δ since it is in

$(\pi - \alpha; \pi + \alpha)$ in $\Gamma_{1,\delta}$. Further, the edge $w_{h-1}w_h$ has slope in $(\pi - \alpha; \pi + \alpha)$ in Γ_δ since it has slope in that range in Γ_1 (given that it belongs to Q^1_x), since $x(w_{h-1}) > x(w_h)$ and $y(w_{h-1}) < y(w_h)$, since $u \neq w_{h-1}, w_h$, and since $\varepsilon \leq \frac{Y}{2} < d_V(\Gamma_1, w_{h-1}w_h)$. Moreover, for $l = j, j-1 \dots, 3$, the slope of the edge $c_l c_{l-1}$ in Γ_δ is $\pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0u_1)}\right)$, which is in the interval $(\pi - \alpha; \pi + \alpha)$, as proved in the case $x = u_i$. Finally, the slope of the edge c_2c_1 in Γ_δ is $\pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0u_1)}\right)$, which is in the interval $(\pi - \alpha; \pi + \alpha)$, as proved in the case $x = u_i$.

We finally deal with Property 6. Consider any two distinct vertices $x, y \in V(G)$.

If x and y belong to the same graph G_i, for some $i \in \{1, \dots, k\}$, then there exists a distance-decreasing path P_{xy} from x to y in Γ_i, given that Γ_i satisfies Property 6. If $i \in \{2, \dots, k\}$, the drawing of G_i in Γ_δ is congruent to Γ_i, up to three affine transformations (a uniform scaling, a rotation, and a translation) that preserve the property of a path to be distance-decreasing; hence P_{xy} is distance-decreasing in Γ_δ as well. If $i = 1$, then the proof that P_{xy} is distance-decreasing in Γ_δ is the same as the proof that Γ_δ satisfies Property 6 in Lemma 9, with Γ_1 playing the role of Γ'.

We can hence assume that x and y belong to two distinct graphs G_i and G_j, respectively.

- Suppose first that $2 \leq i < j \leq k$. Then let P_{xy} be the path composed of:
  - a path P^i_x in G_i from x to u_i whose every edge has slope in $(-\frac{\alpha}{3}; \frac{\alpha}{3})$ in Γ_i;
  - the path $\bigcup_{l=i+1}^{j-1} \tau_{u_{l-1}u_l}(G_l)$; and
  - a path $P_{u_{j-1}y}$ in G_j that is distance-decreasing in Γ_j.

  By induction, the paths P^i_x and $P_{u_{j-1}y}$ exist since Γ_i satisfies Property 4 and Γ_j satisfies Property 6, respectively. We prove that P_{xy} is distance-decreasing in Γ_δ; note that $u \notin V(P_{xy})$. Let $P_{xy} = (z_1, z_2, \dots, z_s)$; then we need to prove that $d(\Gamma_\delta, z_hz_s) > d(\Gamma_\delta, z_{h+1}z_s)$, for $h = 1, 2, \dots, s-2$. We distinguish three cases.
  - If z_hz_{h+1} is in G_j, then $(z_h, z_{h+1}, \dots, z_s)$ is a sub-path of $P_{u_{j-1}y}$, hence it is distance-decreasing in Γ_δ since it is distance-decreasing in Γ_j and since the drawing of G_j in Γ_δ is congruent to Γ_j, up to three affine transformations (a uniform scaling, a rotation, and a translation) that preserve the property of a path to be distance-decreasing.
  - If z_hz_{h+1} is in $\tau_{u_{l-1}u_l}(G_l)$, for some $l \in \{i+1, i+2, \dots, j-1\}$, as in Fig. 17, then it has slope $s = \frac{\alpha}{2}$. Consider the line ℓ_h with slope $\frac{\pi+\alpha}{2}$ through u_l, oriented towards increasing y-coordinates. By Lemma 3, this line has the drawings of $G_{l+1}, G_{l+2}, \dots, G_k$ to its right; this is because by Property 3 of Γ_δ every edge in $\beta_{u_lv}(G)$ has slope in the interval $(0; \alpha)$, where $\frac{-\pi+\alpha}{2} < 0 < \alpha < \frac{\pi+\alpha}{2}$, and because the path $\bigcup_{m=l+1}^{k} \tau_{u_{m-1}u_m}(G_m)$ has slope $s = \frac{\alpha}{2}$, where $\frac{-\pi+\alpha}{2} < \frac{\alpha}{2} < \frac{\pi+\alpha}{2}$. Further, by Lemma 3, the line ℓ_h has the drawing of the path $\beta_{u_lu_{l-1}}(G_l)$ to its left; this is because every edge in $\beta_{u_lu_{l-1}}(G_l)$ has slope $s = \pi + \frac{\alpha}{2}$, where $\frac{\pi+\alpha}{2} < \pi + \frac{\alpha}{2} < \frac{3\pi+\alpha}{2}$. Then the line ℓ'_h parallel to ℓ_h, passing through the midpoint of the edge z_hz_{h+1}, and oriented towards increasing y-coordinates has ℓ_h to its right, given that the path $\beta_{u_lu_{l-1}}(G_l)$ (and in particular the midpoint of the edge z_hz_{h+1}) is to the left of ℓ_h, hence ℓ'_h has the drawings of $G_{l+1}, G_{l+2}, \dots, G_k$ (and in particular the vertex z_s) to its right. Since the half-plane to the right of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_hz_s) > d(\Gamma_\delta, z_{h+1}z_s)$.
  - If z_hz_{h+1} is in P^i_x, as in Fig. 18, then by Property 4 it has slope in $(-\frac{\alpha}{3}; \frac{\alpha}{3})$ in Γ_i. Since Γ_i is counter-clockwise rotated by s radians in Γ_δ, it follows that z_hz_{h+1} has slope in $(s - \frac{\alpha}{3}; s + \frac{\alpha}{3}) = (\frac{\alpha}{6}; \frac{5\alpha}{6})$ in Γ_δ. Consider the line ℓ_h that passes through u_i, that is directed towards increasing y-coordinates and that is orthogonal to the line through z_h and z_{h+1}. Denote by s_h the slope of ℓ_h. Then $s_h \in (\frac{\pi}{2} + \frac{\alpha}{6}; \frac{\pi}{2} + \frac{5\alpha}{6})$. By

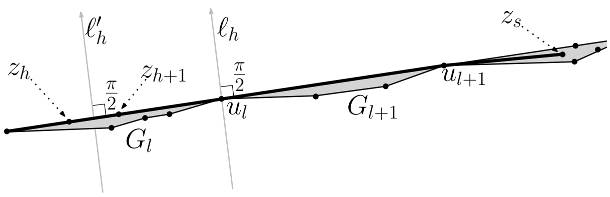


Figure 17 Illustration for the proof that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$ if $z_h z_{h+1}$ is in $\tau_{u_{l-1}u_l}(G_l)$.

Lemma 3, the line ℓ_h has the drawings of $G_{i+1}, \dots, G_k$ to its right; this is because by Property 3 of Γ_δ every edge in $\beta_{u_i v}(G)$ has slope in $(0; \alpha)$ with $s_h - \pi < -\frac{\pi}{2} + \frac{5\alpha}{6} < 0 < \alpha < \frac{\pi}{2} + \frac{\alpha}{6} < s_h$ and because the path $\bigcup_{m=i+1}^{k} \tau_{u_{m-1}u_m}(G_m)$ has slope $s = \frac{\alpha}{2}$, where $s_h - \pi < -\frac{\pi}{2} + \frac{5\alpha}{6} < \frac{\alpha}{2} < \frac{\pi}{2} + \frac{\alpha}{6} < s_h$. Further, by Lemma 3, the line ℓ_h has the drawings of $G_2, \dots, G_i$ to its left; this is because by Property 3 of Γ_δ every edge in $\tau_{u_i u_1}(G)$ has slope in $(\pi; \pi + \alpha)$ with $s_h < \frac{\pi}{2} + \frac{5\alpha}{6} < \pi < \pi + \alpha < \frac{3\pi}{2} + \frac{\alpha}{6} < \pi + s_h$ and because the path $\bigcup_{m=2}^{i} \beta_{u_m u_{m-1}}(G_m)$ has slope $s = \pi + \frac{\alpha}{2}$, where $s_h < \frac{\pi}{2} + \frac{5\alpha}{6} < \pi + \frac{\alpha}{2} < \frac{3\pi}{2} + \frac{\alpha}{6} < \pi + s_h$. Now consider the line ℓ'_h parallel to ℓ_h, passing through the midpoint of the edge $z_h z_{h+1}$, and oriented towards increasing y-coordinates. This line has ℓ_h to its right, given that the drawing of G_i (and in particular the midpoint of $z_h z_{h+1}$) is to the left of ℓ_h in Γ_δ. Thus, ℓ'_h has the drawings of $G_{i+1}, G_{i+2}, \dots, G_k$ (and in particular the vertex z_s) to its right. Since the half-plane to the right of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$.

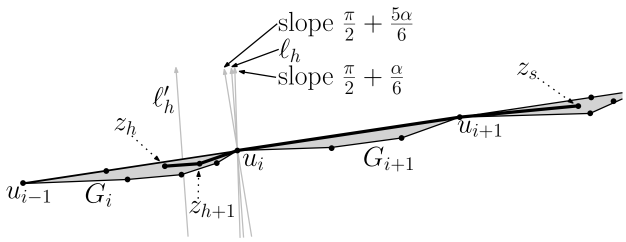


Figure 18 Illustration for the proof that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$ if $z_h z_{h+1}$ is in P_x^i.

- The case in which $2 \le j < i \le k$ is symmetric to the case in which $2 \le i < j \le k$.
- Suppose next that $i = 1$ and $j > 1$. Then let P_{xy} be the path composed of:
  - a path P_x^1 in G_1 from x to u_1 whose every edge has slope in $(-\alpha; \alpha)$ in Γ_δ, where P_x^1 does not pass through u, unless $x = u$;
  - the path $\bigcup_{l=2}^{j-1} \tau_{u_{l-1}u_l}(G_l)$; and
  - a path $P_{u_{j-1}y}$ in G_j that is distance-decreasing in Γ_j.

  The path $P_{u_{j-1}y}$ exists by induction since Γ_j satisfies Property 6.

  We prove that a path P_x^1 satisfying the above properties exists in Γ_δ. Let $\tau_{u_0 u_1}(G_1) = (u_0 = c_1, c_2, \dots, c_r = u_1)$. If $x = u$, then let $P_x^1 = \tau_{u_0 u_1}(G_1)$. Every edge of P_x^1 other than $c_1 c_2$ has slope $-\arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0 u_1)}\right)$ in Γ_δ, while $c_1 c_2$ has slope $-\arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0 u_1)}\right)$. These slopes are smaller than 0, given that $\varepsilon, d(\Gamma_1, u_0 u_1) > 0$ and $\delta \ge 0$, and larger than $-\alpha$, given that $\delta \ge 0$ and $\varepsilon < \tan\alpha \cdot d(\Gamma_1, u_0 u_1)$. Thus, every edge of P_x^1 has slope in $(-\alpha; \alpha)$ in Γ_δ. If $x \ne u$, then P_x^1 can be shown to exist as in the proof that Γ_δ satisfies Property 4, by considering a path $(x = v_1, v_2, \dots, v_p = u_1)$ in G_1 that does not pass through u and whose every edge has slope in $(-\frac{\alpha}{2}; \frac{\alpha}{2})$ in Γ_1 and by defining

$P_x^1 = (v_1, v_2, \dots, v_h = c_j, c_{j+1}, \dots, c_r)$, where h is the smallest index such that $v_h \in V(\tau_{u_0u_1}(G_1))$. This concludes the proof that a path P_x^1 satisfying the required properties exists in Γ_δ.

Note that $u \notin V(P_{xy})$, unless $x = u$, given that $u \notin V(P_x^1)$, unless $x = u$. Let $P_{xy} = (z_1, z_2, \dots, z_s)$; we prove that, for any $h = 1, 2, \dots, s-2$, it holds true that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$. This can be proved exactly as in the case $2 \leq i < j \leq k$ if $z_h z_{h+1}$ is in $P_{u_{j-1}y}$ or if $z_h z_{h+1}$ is in $\bigcup_{l=2}^{j-1} \tau_{u_{l-1}u_l}(G_l)$. Assume hence that $z_h z_{h+1}$ is in P_x^1 and recall that the slope of every edge of P_x^1 in Γ_δ is in $(-\alpha; \alpha)$. Similarly to the case $2 \leq i < j \leq k$, consider the line ℓ_h that passes through u_1, that is directed towards increasing y-coordinates and that is orthogonal to the line through z_h and z_{h+1}. Denote by s_h the slope of ℓ_h. Then $s_h \in (\frac{\pi}{2} - \alpha; \frac{\pi}{2} + \alpha)$. By Lemma 3, the line ℓ_h has the drawings of $G_2, \dots, G_k$ to its right; this is because by Property 3 of Γ_δ every edge in $\beta_{u_1v}(G)$ has slope in $(0; \alpha)$ with $s_h - \pi < -\frac{\pi}{2} + \alpha < 0 < \alpha < \frac{\pi}{2} - \alpha < s_h$ and because the path $\bigcup_{m=2}^{k} \tau_{u_{m-1}u_m}(G_m)$ has slope $s = \frac{\alpha}{2}$, where $s_h - \pi < -\frac{\pi}{2} + \alpha < \frac{\alpha}{2} < \frac{\pi}{2} - \alpha < s_h$. Further, by Lemma 3, the line ℓ_h has the drawing of G_1 to its left; this is because by Property 3 of Γ_1 every edge in $\tau_{u_1u_0}(G)$ has slope in $(\pi - \frac{\alpha}{2}; \pi + \frac{\alpha}{2})$, where $s_h < \frac{\pi}{2} + \alpha < \pi - \frac{\alpha}{2} < \pi + \frac{\alpha}{2} < \frac{3\pi}{2} - \alpha < \pi + s_h$ and because every edge of the path $\beta_{u_1u_0}(G_1)$ has slope either $\pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0u_1)}\right)$ or $\pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0u_1)}\right)$, where $s_h < \frac{\pi}{2} + \alpha < \pi - \alpha < \pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0u_1)}\right) \leq \pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0u_1)}\right) < \pi < \frac{3\pi}{2} - \alpha < \pi + s_h$ – these inequalities exploit $\varepsilon, d(\Gamma_1, u_0u_1) > 0$, $\delta \geq 0$, and $\varepsilon < \tan\alpha \cdot d(\Gamma_1, u_0u_1)$. Now consider the line ℓ'_h parallel to ℓ_h, passing through the midpoint of the edge $z_h z_{h+1}$, and oriented towards increasing y-coordinates. This line has ℓ_h to its right, given that the drawing of G_1 (and in particular the midpoint of $z_h z_{h+1}$) is to the left of ℓ_h in Γ_δ. Thus, ℓ'_h has the drawings of $G_2, G_3, \dots, G_k$ (and in particular the vertex z_s) to its right. Since the half-plane to the right of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_h z_s) > d(\Gamma_\delta, z_{h+1} z_s)$.

- Suppose finally that $i > 1$ and $j = 1$. Then P_{xy} consists of three paths, one contained in G_i, one coinciding with $\bigcup_{l=2}^{i-1} \beta_{u_l u_{l-1}}(G_l)$, and one contained in G_1.

  - The first path in P_{xy} is a path Q_x^i in G_i from x to u_{i-1} whose every slope in Γ_i is in $(\pi - \frac{\alpha}{3}; \pi + \frac{\alpha}{3})$; this path exists since Γ_i satisfies Property 5. Since Γ_i is counter-clockwise rotated by s radians in Γ_δ, it follows that every edge of Q_x^i has slope in $(\pi + s - \frac{\alpha}{3}; \pi + s + \frac{\alpha}{3}) = (\pi + \frac{\alpha}{6}; \pi + \frac{5\alpha}{6})$ in Γ_δ. We prove that, for every edge $z_h z_{h+1}$ of Q_x^i, it holds true that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$. Consider the line ℓ_h that passes through u_{i-1}, that is directed towards increasing y-coordinates and that is orthogonal to the line through z_h and z_{h+1}. Denote by s_h the slope of ℓ_h. Then $s_h \in (\frac{\pi}{2} + \frac{\alpha}{6}; \frac{\pi}{2} + \frac{5\alpha}{6})$. By Lemma 3, the line ℓ_h has the drawing of G_i to its right; this is because by Property 3 of Γ_δ every edge in $\beta_{u_{i-1}u_i}(G)$ has slope in $(-\alpha; \alpha)$ with $s_h - \pi < -\frac{\pi}{2} + \frac{5\alpha}{6} < -\alpha < \alpha < \frac{\pi}{2} + \frac{\alpha}{6} < s_h$ and because the path $\tau_{u_{i-1}u_i}(G_i)$ has slope $s = \frac{\alpha}{2}$, where $s_h - \pi < -\frac{\pi}{2} + \frac{5\alpha}{6} < \frac{\alpha}{2} < \frac{\pi}{2} + \frac{\alpha}{6} < s_h$. Further, by Lemma 3, the line ℓ_h has the drawings of $G_1, \dots, G_{i-1}$ (and in particular y) to its left; this is because by Property 3 of Γ_δ every edge in $\tau_{u_{i-1}u_0}(G)$ has slope in $(\pi - \alpha; \pi + \alpha)$ where $s_h < \frac{\pi}{2} + \frac{5\alpha}{6} < \pi - \alpha < \pi + \alpha < \frac{3\pi}{2} + \frac{\alpha}{6} < \pi + s_h$ and because every edge of the path $\bigcup_{m=1}^{i-1} \beta_{u_m u_{m-1}}(G_m)$ has slope either $s = \pi + \frac{\alpha}{2}$, or $\pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0u_1)}\right)$, or $\pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0u_1)}\right)$, where $s_h < \frac{\pi}{2} + \frac{5\alpha}{6} < \pi - \alpha < \pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0u_1)}\right) \leq \pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0u_1)}\right) < \pi < \pi + \frac{\alpha}{2} < \frac{3\pi}{2} + \frac{\alpha}{6} < \pi + s_h$. Now consider the line ℓ'_h

parallel to ℓ_h, passing through the midpoint of the edge $z_h z_{h+1}$, and oriented towards increasing y-coordinates. This line has ℓ_h to its left, given that the drawing of G_i (and in particular the midpoint of $z_h z_{h+1}$) is to the right of ℓ_h in Γ_δ. Thus, ℓ'_h has the drawings of $G_{i-1}, G_{i-2}, \dots, G_1$ (and in particular the vertex y) to its left. Since the half-plane to the left of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$.

- The second path in P_{xy} is $\bigcup_{l=2}^{i-1} \beta_{u_l u_{l-1}}(G_l)$. Consider an edge $z_h z_{h+1}$ of this path in a graph G_l, for some $l \in \{2, \dots, i-1\}$. Then $z_h z_{h+1}$ has slope $\pi + s = \pi + \frac{\alpha}{2}$. Consider the line ℓ_h that has slope $s_h = \frac{\pi+\alpha}{2}$, that passes through u_{l-1}, and that is oriented towards increasing y-coordinates. By Lemma 3, the line ℓ_h has the drawing of G_l to its right; this is because by Property 3 of Γ_δ every edge in $\beta_{u_{l-1} u_l}(G)$ has slope in $(0;\alpha)$ with $s_h - \pi = \frac{-\pi+\alpha}{2} < 0 < \alpha < \frac{\pi+\alpha}{2} = s_h$ and because the path $\tau_{u_{l-1} u_l}(G_l)$ has slope $s = \frac{\alpha}{2}$, where $s_h - \pi = \frac{-\pi+\alpha}{2} < \frac{\alpha}{2} < \frac{\pi+\alpha}{2} = s_h$. Further, by Lemma 3, the line ℓ_h has the drawings of $G_{l-1}, G_{l-2}, \dots, G_1$ to its left; this is because by Property 3 of Γ_δ every edge in $\tau_{u_{l-1} u_0}(G)$ has slope in $(\pi - \alpha; \pi + \alpha)$, where $s_h = \frac{\pi+\alpha}{2} < \pi - \alpha < \pi + \alpha < \frac{3\pi+\alpha}{2} = s_h + \pi$ and because every edge of the path $\bigcup_{m=1}^{i-1} \beta_{u_m u_{m-1}}(G_m)$ has slope either $\pi + \frac{\alpha}{2}$, or $\pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0 u_1)}\right)$, or $\pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0 u_1)}\right)$, where $s_h = \frac{\pi+\alpha}{2} < \pi - \alpha < \pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0 u_1)}\right) \leq \pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0 u_1)}\right) < \pi < \pi + \frac{\alpha}{2} < \frac{3\pi+\alpha}{2} = s_h + \pi$. Now consider the line ℓ'_h parallel to ℓ_h, passing through the midpoint of the edge $z_h z_{h+1}$, and oriented towards increasing y-coordinates. This line has ℓ_h to its left, given that the drawing of G_l (and in particular the midpoint of $z_h z_{h+1}$) is to the right of ℓ_h in Γ_δ. Thus, ℓ'_h has the drawings of $G_{l-1}, G_{l-2}, \dots, G_1$ (and in particular vertex y) to its left. Since the half-plane to the left of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$.
- The third path $P_{u_1 y}$ in P_{xy} is defined as follows. If $y = u$, let $P_{u_1 y} = \beta_{u_1 u_0}(G_1)$. Then every edge of $P_{u_1 y}$ has slope either $\pi - \arctan\left(\frac{\varepsilon}{d(\Gamma_1, u_0 u_1)}\right)$ or $\pi - \arctan\left(\frac{\varepsilon}{\delta + d(\Gamma_1, u_0 u_1)}\right)$. Both these slopes are smaller than π, given that $\varepsilon, d(\Gamma_1, u_0 u_1) > 0$ and $\delta \geq 0$, and larger than $\pi - \alpha$, given that $\delta \geq 0$ and $\varepsilon < \tan(\alpha) \cdot d(\Gamma_1, u_0 u_1)$. Thus, $P_{u_1 y}$ is a π-path, and hence it is distance-decreasing (see [11] and the proof of Property 6 in Lemma 9). If $y \neq u$, then let $P_{u_1 y}$ be a distance-decreasing path in Γ_1 not passing through u. This path exists by induction, given that Γ_1 satisfies Property 6. Since $P_{u_1 y}$ does not pass through u, it has the same representation in Γ_δ and Γ. Since the Euclidean distance between the positions of any vertex of G_1 in Γ_1 and Γ is at most $\varepsilon < \varepsilon^*_{\Gamma_1}$, by Lemma 2 we have that $P_{u_1 y}$ is distance-decreasing in Γ and hence in Γ_δ.

Hence Γ_δ satisfies Property 6. This concludes the proof of the lemma. ◀

We now discuss **Case B**, in which (G, u, v) is decomposed according to Lemma 6. Refer to Figs. 19 and 20. First, the triple (H, u, y_1) is a strong circuit graph; further, $|V(H)| \geq 3$, hence H is not a single edge. Apply induction in order to construct a straight-line drawing Γ_H of H with $\frac{\alpha}{2}$ as a parameter.

Let $\beta_{uy_1}(H) = (u = b_1, b_2, \dots, b_m = y_1)$. Let ϕ_i be the slope of the edge $b_i b_{i+1}$ in Γ_H and let $\phi = \min_{i=2,\dots,m-1}\{\phi_i\}$. By Property (c) of (H, u, y_1) if edge uy_1 belongs to H then it coincides with the path $\tau_{uy_1}(H)$. Hence, $m \geq 3$ and ϕ is well-defined. Further, ϕ is in the interval $(0; \frac{\alpha}{2})$ by Property 3 of Γ_H.

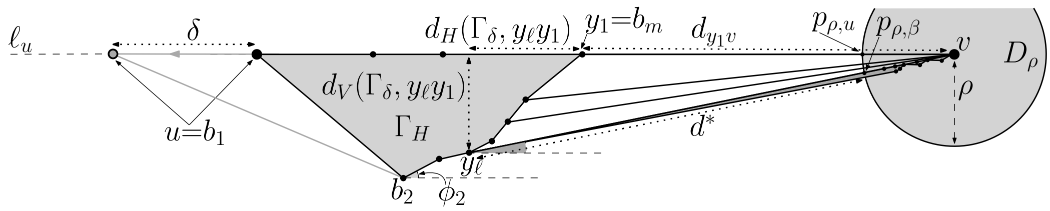


Figure 19 The straight-line drawing Γ of G in Case B. For the sake of readability, ϕ and ρ are larger than they should be. The dark gray angle is equal to β.

Let $\beta = \frac{1}{2}\min\left\{\phi, \arctan\left(\frac{d_V(\Gamma_H, y_\ell y_1)}{3d_V(\Gamma_H, y_\ell y_1) + 3d_H(\Gamma_H, y_\ell y_1)}\right)\right\}$. Note that $\beta > 0$, given that $\phi, d_V(\Gamma_H, y_\ell y_1) > 0$ and $d_H(\Gamma_H, y_\ell y_1) \geq 0$. In particular, $d_V(\Gamma_H, y_\ell y_1) > 0$ because y_1 is an internal vertex of $\tau_{uy_1}(H)$ and y_ℓ is an internal vertex of $\beta_{uy_1}(H)$ by Lemma 6, and because of Properties 1–3 of Γ_H. Also note that $\beta < \frac{\alpha}{4}$, given that $\phi < \frac{\alpha}{2}$.

Consider a half-line h_β with slope β starting at y_ℓ. Place the vertex v at the intersection point between h_β and the horizontal line ℓ_u through u. Draw all the trivial $(H \cup \{v\})$-bridges of G as straight-line segments. This concludes the construction if every $(H \cup \{v\})$-bridge of G is trivial. Otherwise, B_ℓ is the only non-trivial $(H \cup \{v\})$-bridge of G. Then B_ℓ consists of k strong circuit graphs (G_i, u_{i-1}, u_i), where $u_0 = y_\ell$ and $u_k = v$. With a slight change of notation, in the remainder of the section we assume that, if the edge $y_\ell v$ exists, then it is an edge of B_ℓ (rather than an individual trivial $(H \cup \{v\})$-bridge $B_{\ell-1}$ of G); in this case (B_ℓ, u_0, u_k) is a strong circuit graph (this comes from the proof of Lemma 6, where the graph B_ℓ together with the edge $y_\ell v$ was denoted by B'_ℓ).

We claim that v lies to the right of y_1. The polygonal line representing $\beta_{y_\ell y_1}(H)$ in Γ_H and the straight-line segment $\overline{y_\ell v}$ are both incident to y_ℓ. By definition of ϕ and since Γ_H satisfies Property 3, $\beta_{y_\ell y_1}(H)$ is composed of straight-line segments with slopes in the range $[\phi; \frac{\alpha}{2})$, while $\overline{y_\ell v}$ has slope β. The claim then follows from $0 < \beta < \phi < \frac{\pi}{2}$.

Denote by $d_{y_1 v}$ the distance between y_1 and v. Let $Y > 0$ be the minimum distance in Γ_H of any vertex strictly below ℓ_u from ℓ_u.

Let $\rho = \min\{\frac{d_{y_1 v}}{3}, \frac{Y}{2}\}$. Let D_ρ be the disk with radius ρ centered at v. Let $p_{\rho,\beta}$ ($p_{\rho,u}$) be the intersection point closer to y_ℓ (resp. to y_1) of the boundary of D_ρ with h_β (resp. with ℓ_u). Let d^* be the Euclidean distance between y_ℓ and $p_{\rho,\beta}$.

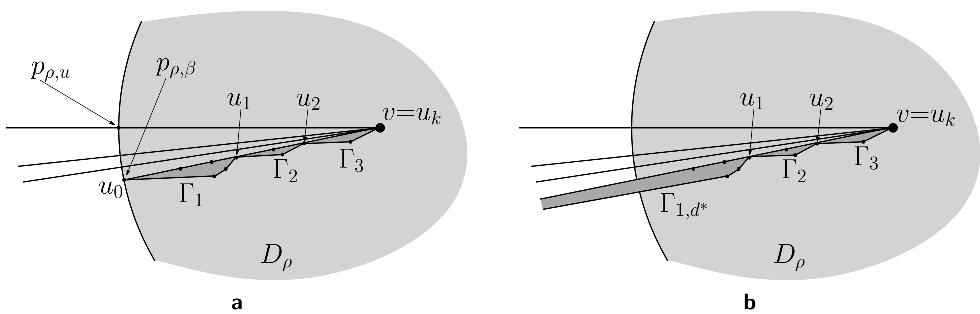


Figure 20 A closer look at D_ρ. Figure (a) represents the drawings $\Gamma_1, \ldots, \Gamma_k$, once they have been uniformly scaled, rotated, and translated, while (b) also has the vertex u_0 moved by d^* units (this movement actually happens before the rotation and translation of Γ_1).

Let $\alpha' = \frac{\beta}{2}$. Since $\beta > 0$, we have $\alpha' > 0$; further, $\alpha' < \frac{\alpha}{8}$, given that $\beta < \frac{\alpha}{4}$. For $i = 1, \dots, k$, apply induction in order to construct a straight-line drawing Γ_i of G_i with α' as a parameter (if G_i is a single edge, then the parameter does not matter). Uniformly scale the drawings $\Gamma_1, \dots, \Gamma_k$ so that the Euclidean distance between u_{i-1} and u_i is equal to $\frac{\rho}{k}$. Move the vertex u_0 in Γ_1 by d^* units to the left, obtaining a drawing Γ_{1,d^*}. Rotate the drawings $\Gamma_{1,d^*}, \Gamma_2, \dots, \Gamma_k$ counter-clockwise by β radians. Translate $\Gamma_{1,d^*}, \Gamma_2, \dots, \Gamma_k$ so that, for $i = 1, \dots, k-1$, the representations of u_i in Γ_i and Γ_{i+1} (in Γ_{1,d^*} and Γ_2 if $i = 1$) coincide and so that the representation of u_0 in the scaled and rotated drawing Γ_{1,d^*} coincides with the one of y_ℓ in Γ_H. This completes the construction of a straight-line drawing Γ of G. We have the following.

▶ **Lemma 11.** *For any $\delta \geq 0$, the drawing Γ_δ constructed in Case B satisfies Properties 1–6 of Theorem 7.*

Proof. Let $\Gamma_{H,\delta}$ be the drawing obtained from Γ_H by moving u by δ units to the left.

We prove Property 2. By Lemma 6, we have that $\tau_{uv}(G) = \tau_{uy_1}(H) \cup y_1 v$. By Property 2 of $\Gamma_{H,\delta}$, we have that $\tau_{uy_1}(H)$ lies entirely on ℓ_u with y_1 to the right of u. By construction v also lies on ℓ_u. As proved before the lemma's statement, v lies to the right of y_1. This implies Property 2 for Γ_δ.

We next prove that Γ_δ satisfies Property 3. By Lemma 6, we have that $\beta_{uv}(G) = \beta_{uy_\ell}(H) \cup \beta_{u_0u_1}(G_1) \cup \beta_{u_1u_2}(G_2) \cup \dots \cup \beta_{u_{k-1}u_k}(G_k)$. Denote $\beta_{uv}(G) = (u = b'_1, b'_2, \dots, b'_m = v)$. The slope of the edge $b'_1b'_2$ in Γ_δ is equal to its slope in $\Gamma_{H,\delta}$; this is because the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$ and because b'_2 is a vertex of H, given that $y_\ell \neq u$ since y_ℓ is an internal vertex of $\beta_{uv}(G)$. Hence the slope of $b'_1b'_2$ is in $(-\frac{\alpha}{2}; 0) \subset (-\alpha; 0)$ in Γ_δ since $\Gamma_{H,\delta}$ satisfies Property 3. We now argue about the slope s_j of the edge $b'_jb'_{j+1}$ in Γ_δ, for any $j = 2, \dots, m-1$.

- If $b'_jb'_{j+1}$ is an edge of $\beta_{uy_\ell}(H)$, then its slope in Γ_δ is equal to its slope in $\Gamma_{H,\delta}$, since the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$. Thus, $s_j \in (0; \frac{\alpha}{2}) \subset (0; \alpha)$, since $\Gamma_{H,\delta}$ satisfies Property 3.
- If $b'_jb'_{j+1}$ coincides with a graph G_i, then $s_j = \beta$. Since $0 < \beta \leq \frac{\alpha}{4}$, we have $s_j \in (0; \alpha)$.
- If $b'_jb'_{j+1}$ belongs to a graph G_i, for some $i \in \{1, \dots, k\}$, with $|V(G_i)| \geq 3$, and with $b'_j \neq u_{i-1}$, then s_j is given by the slope $b'_jb'_{j+1}$ has in Γ_i, which is in $(0; \alpha')$ by Property 3 of Γ_i, plus β, which results from the rotation of Γ_i. Hence $s_j \in (\beta; \beta + \alpha')$; since $\beta > 0$, $\beta < \frac{\alpha}{4}$, and $\alpha' < \frac{\alpha}{8}$, we have that $s_j \in (0; \alpha)$.
- If $b'_jb'_{j+1}$ belongs to a graph G_i, for some $i \in \{2, \dots, k\}$, with $|V(G_i)| \geq 3$, and with $b'_j = u_{i-1}$, then s_j is given by the slope $b'_jb'_{j+1}$ has in Γ_i, which is in $(-\alpha'; 0)$ by Property 3 of Γ_i, plus β, which results from the rotation of Γ_i. Hence $s_j \in (\beta - \alpha'; \beta)$. Since $\alpha' \leq \frac{\beta}{2} < \beta < \frac{\alpha}{4} < \alpha$, we have that $s_j \in (0; \alpha)$.
- Finally, assume that $b'_jb'_{j+1}$ belongs to G_1, that $|V(G_1)| \geq 3$, and that $b'_j = u_0$. Then s_j is given by the slope $b'_jb'_{j+1}$ has in Γ_{1,d^*}, which is in $(-\alpha'; 0)$ by Property 3 of Γ_{1,d^*}, plus β, which results from the rotation of Γ_{1,d^*}. Hence $s_j \in (\beta - \alpha'; \beta) \subset (0; \alpha)$.

We now prove Property 1. Before doing so, we prove the following useful statement: Every vertex $z \neq u_0$ that belongs to a graph G_i, for any $i \in \{1, \dots, k\}$, lies inside the disk D_ρ in Γ_δ. Note that this statement shows a sharp geometric separation between the vertices that are in H and those that are not. Refer to Fig. 21. Consider the drawing Γ_j, for any $j \in \{1, \dots, k\}$ (note that Γ_1 is considered before moving u_0 by d^* units to the left) and consider the disk D_j centered at u_j with radius $d(\Gamma_j, u_{j-1}u_j)$. By Properties 1 and 2 of Γ_j, the path $\tau_{u_{j-1}u_j}(G_j)$ lies on the straight-line segment $\overline{u_{j-1}u_j}$ in Γ_j, hence it lies inside

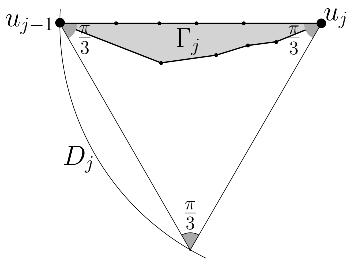


Figure 21 The drawing Γ_j and the disk D_j centered at u_j with radius $d(\Gamma_j, u_{j-1}u_j)$.

D_j. Further, all the edges of $\beta_{u_{j-1}u_j}(G_j)$ have slope in $(-\alpha'; \alpha') \subset (-\frac{\alpha}{8}; \frac{\alpha}{8}) \subset (-\frac{\pi}{32}; \frac{\pi}{32}) \subset (-\frac{\pi}{3}; \frac{\pi}{3})$; hence $\beta_{u_{j-1}u_j}(G_j)$ also lies inside D_j. By Property 1 of Γ_j, the entire drawing Γ_j lies inside D_j. Hence, u_{j-1} is the farthest vertex of G_j from u_j in Γ_j. This property holds true also after the drawings $\Gamma_1, \dots, \Gamma_k$ are uniformly scaled; further, after the scaling, the distance between u_{j-1} and u_j is $\frac{\rho}{k}$, by construction. By the triangular inequality, we have that $d(\Gamma_\delta, vz) \leq \sum_{j=i+1}^{k} d(\Gamma_\delta, u_{j-1}u_j) + d(\Gamma_\delta, u_i z)$. Since $d(\Gamma_\delta, u_{j-1}u_j) = \frac{\rho}{k}$ for any $j \in \{2, \dots, k\}$, and since $d(\Gamma_\delta, u_i z) \leq \frac{\rho}{k}$ (this exploits $z \neq u_0$ and hence $d(\Gamma_\delta, u_i z) = d(\Gamma_i, u_i z)$, where Γ_i is understood as already scaled), we have that $d(\Gamma_\delta, vz) \leq \frac{(k-i+1)\rho}{k} \leq \rho$. Thus z lies inside D_ρ.

We now discuss the possible crossings that might occur in Γ_δ.

- The drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$, hence it is planar since $\Gamma_{H,\delta}$ satisfies Property 1 by induction.
- Analogously, the drawings of $G_1, G_2, \dots, G_k$ in Γ_δ are planar since they coincide with $\Gamma_{1,d^*}, \Gamma_2, \dots, \Gamma_k$, which satisfy Property 1 by induction.
- Since Γ_δ satisfies Property 3, the path $\beta_{y_\ell v}(G)$ is represented in Γ_δ by a curve monotonically increasing in the x-direction from y_ℓ to v. Further, the path $\tau = \bigcup_{i=1}^{k} \tau_{u_{i-1}u_i}(G_i)$ is represented in Γ_δ by a straight-line segment with slope $\beta \in (0; \frac{\alpha}{4}) \subset (0; \frac{\pi}{16})$. Hence, for $i = 1, \dots, k-1$, the vertical line through u_i has the drawings of $G_1, \dots, G_i$ to its left and those of $G_{i+1}, \dots, G_k$ to its right in Γ_δ. It follows that no two edges in distinct graphs G_i and G_j cross in Γ_δ.
- Recall that $\beta_{uy_1}(H) = (u = b_1, b_2, \dots, b_m = y_1)$. Let $y_\ell = b_j$, for some $j \in \{2, 3, \dots, m-1\}$. We prove that the straight-line segments $\overline{b_j v}, \overline{b_{j+1} v}, \dots, \overline{b_m v}$ appear in this clockwise order around v and have slopes in $[0; \beta]$ in Γ_δ (note that these straight-line segments do not necessarily correspond to edges of G). Refer to Fig. 22. Note that the slope of $\overline{b_j v}$

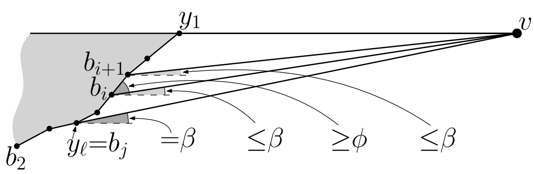


Figure 22 illustration for the proof that the straight-line segments $\overline{b_j v}, \overline{b_{j+1} v}, \dots, \overline{b_m v}$ appear in this clockwise order around v and have slopes in $[0; \beta]$ in Γ_δ.

is β, by construction; now assume that $\overline{b_j v}, \overline{b_{j+1} v}, \dots, \overline{b_i v}$ appear in this clockwise order around v and have slopes in $[0; \beta]$ in Γ_δ, for some $i \in \{j, j+1, \dots, m-1\}$. The edge

$b_i b_{i+1}$ has slope in $[\phi; \frac{\alpha}{2})$, by definition of ϕ and since $\Gamma_{H,\delta}$ satisfies Property 3. Since $\beta < \phi$, the edge $b_i b_{i+1}$ lies above the line through b_i and v. Hence, $b_{i+1}v$ immediately follows $b_i v$ in the clockwise order of the edges incident to v and it has slope smaller than the one of $b_i v$, hence smaller than β. The repetition of this argument concludes the proof that $\overline{b_j v}, \overline{b_{j+1} v}, \dots, \overline{b_m v}$ appear in this clockwise order around v and have slopes in $[0; \beta]$ in Γ_δ.
Since the straight-line segments $\overline{b_j v}, \overline{b_{j+1} v}, \dots, \overline{b_m v}$ appear in this clockwise order around v, then no two $(H \cup \{v\})$-bridges of G cross one another. Further, since the straight-line segments $\overline{b_j v}, \overline{b_{j+1} v}, \dots, \overline{b_m v}$ have slopes in $[0; \beta]$ and since $\beta < \phi$, they all lie to the right of the path $\beta_{b_2 b_m}(H)$, whose edges have slopes in $[\phi; \frac{\alpha}{2})$. Then no trivial $(H \cup \{v\})$-bridge of G crosses H in Γ_δ.

- Consider the vertical line ℓ_1 through y_1. By Properties 1–3 of $\Gamma_{H,\delta}$, the line ℓ_1 has $\Gamma_{H,\delta}$ to its left. Further, since $\rho < d_{y_1 v}$, the disk D_ρ lies to the right of ℓ_1. Since all the vertices different from u_0 of the graphs $G_1, \dots, G_k$ lie inside D_ρ, it follows that no edge in a graph $G_1, \dots, G_k$ crosses an edge of H, unless the former is incident to u_0. However, all the edges in $G_1, \dots, G_k$ that are incident to u_0 (in fact only G_1 contains such edges) have slope at most β, as they lie on or below h_β. Hence they all lie to the right of the path $\beta_{b_2 b_m}(H)$ and do not cross edges of H in Γ_δ.

We now discuss Property 4. Let $x \in V(G)$. Assume first that $x \in V(H)$. Since the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$, there exists a path P'_x in H from x to y_1, not passing through u unless $x = u$, and whose every edge has slope in $(-\frac{\alpha}{2}; \frac{\alpha}{2}) \subset (-\alpha; \alpha)$ in Γ_δ. Further, the edge $y_1 v$ has slope 0. Hence, the path $P_x = P'_x \cup y_1 v$ satisfies the required properties. If $x \notin V(H)$, then $x \in V(G_i)$, for some $i \in \{1, \dots, k\}$. Assume that $i \geq 2$ (that $i = 1$, resp.). Then the path P_x consists of a path P'_x from x to u_i in G_i whose every edge has slope in $(-\alpha'; \alpha')$ in Γ_i (in Γ_{1,d^*}, resp.) – this path exists since Γ_i (Γ_{1,d^*}, resp.) satisfies Property 4 – and of the path $\bigcup_{j=i+1}^{k} \tau_{u_{j-1} u_j}(G_j)$. Since Γ_i (Γ_{1,d^*}, resp.) is rotated by β radians in Γ_δ, its edges have slope in the range $(\beta - \alpha'; \beta + \alpha')$. Since $\alpha' = \frac{\beta}{2}$ and $0 < \beta < \frac{\alpha}{4}$, we have that $(\beta - \alpha'; \beta + \alpha') \subset (0; \frac{3\alpha}{8}) \subset (-\alpha; \alpha)$. Further, every edge in $\tau_{u_{j-1} u_j}(G_j)$ has slope 0 in Γ_j and hence β in Γ_δ. Since $0 < \beta < \frac{\alpha}{4}$, we have that every edge in $\bigcup_{j=i+1}^{k} \tau_{u_{j-1} u_j}(G_j)$ has slope in $(-\alpha; \alpha)$. Note that P_x does not pass through u, since u does not belong to any graph among $G_1, \dots, G_k$. Thus, P_x satisfies the required properties.

We now deal with Property 5. Let $x \in V(G)$. Assume first that $x \in V(H)$. Since the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$ and since $\Gamma_{H,\delta}$ satisfies Property 5, there exists a path Q'_x from x to u whose every edge has slope in $(\pi - \frac{\alpha}{2}; \pi + \frac{\alpha}{2}) \subset (\pi - \alpha; \pi + \alpha)$. Thus, the path $Q_x = Q'_x$ satisfies the required properties. If $x \notin V(H)$, then $x \in V(G_i)$, for some $i \in \{1, \dots, k\}$. Then the path Q_x consists of three paths. First, Q_x contains a path Q'_x from x to u_{i-1} in G_i whose every edge has slope in $(\pi - \alpha'; \pi + \alpha')$ in Γ_i (in Γ_{1,d^*}, if $x \in V(G_1)$). This path exists since Γ_i (Γ_{1,d^*}, resp.) satisfies Property 5. Since Γ_i (Γ_{1,d^*}, resp.) is rotated by β radians in Γ_δ, its edges have slopes in the range $(\pi + \beta - \alpha'; \pi + \beta + \alpha')$. Since $\alpha' = \frac{\beta}{2}$ and $0 < \beta < \frac{\alpha}{4}$, we have that $(\pi + \beta - \alpha'; \pi + \beta + \alpha') \subset (\pi; \pi + \frac{3\alpha}{8}) \subset (\pi - \alpha; \pi + \alpha)$. Second, Q_x contains the path $\bigcup_{j=1}^{i-1} \beta_{u_j u_{j-1}}(G_j)$; by Properties 1 and 2, every edge in $\beta_{u_j u_{j-1}}(G_j)$ has slope π in Γ_i (in Γ_{1,d^*} if $j = 1$), hence it has slope $\pi + \beta$ in Γ_δ. Since $0 < \beta < \frac{\alpha}{4}$, we have that every edge in the path $\bigcup_{j=1}^{i-1} \beta_{u_j u_{j-1}}(G_j)$ has slope in $(\pi - \alpha; \pi + \alpha)$. Third, Q_x contains a path Q'_{y_ℓ} from y_ℓ to u in H whose every edge has slope in $(\pi - \frac{\alpha}{2}; \pi + \frac{\alpha}{2}) \subset (\pi - \alpha; \pi + \alpha)$; this path exists since the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$ and since $\Gamma_{H,\delta}$ satisfies Property 5. Thus, the path Q_x satisfies the required properties.

Finally, we deal with Property 6. Consider any two vertices $x, y \in V(G)$. We prove the existence of a path P_{xy} from x to y in G which does not pass through u, unless $x = u$ or $y = u$, and which is distance-decreasing in Γ_δ. We distinguish several cases, based on which graphs among $H, G_1, \dots, G_k$ the vertices x and y belong to.

- Suppose first that x and y belong to H. Since $\Gamma_{H,\delta}$ satisfies Property 6, there exists a path P_{xy} from x to y in H which does not pass through u, unless $x = u$ or $y = u$, and which is distance-decreasing in $\Gamma_{H,\delta}$. Since the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$, it follows that P_{xy} is distance-decreasing in Γ_δ.
- Suppose next that x and y belong to the same graph G_i, for some $i \in \{1, 2, \dots, k\}$. Since the drawing Γ_i (or Γ_{1,d^*} if $i = 1$) satisfies Property 6, there exists a path P_{xy} from x to y in G_i that is distance-decreasing in Γ_i (in Γ_{1,d^*} if $i = 1$). Since the drawing of G_i in Γ_δ is congruent to Γ_i (to Γ_{1,d^*} if $i = 1$) up to three affine transformations, namely a uniform scaling, a rotation, and a translation, that preserve the property of a path to be distance-decreasing, it follows that P_{xy} is distance-decreasing in Γ_δ. Note that $u \notin V(P_{xy})$.
- Suppose now that x belongs to a graph G_i and y belongs to a graph G_j for some $1 \le i < j \le k$. Then let P_{xy} be the path composed of a path P_x in G_i from x to u_i whose every slope in Γ_i is in $(-\alpha'; \alpha')$, of the path $\bigcup_{l=i+1}^{j-1} \tau_{u_{l-1}u_l}(G_l)$, and of a path $P_{u_{j-1}y}$ in G_j that is distance-decreasing in Γ_j. The path P_x exists since Γ_i (Γ_{1,d^*} if $i = 1$) satisfies Property 4; the path $P_{u_{j-1}y}$ exists since Γ_j satisfies Property 6.
  The proof that P_{xy} is distance-decreasing in Γ_δ is the same as the one that P_{xy} is distance-decreasing in Γ_δ when $x \in V(G_i)$, $y \in V(G_j)$, and $2 \le i < j \le k$ in Lemma 10, with β in place of s and $(-\alpha'; \alpha') \subset (-\frac{\alpha}{8}; \frac{\alpha}{8})$ in place of $(-\frac{\alpha}{3}; \frac{\alpha}{3})$ as the interval of possible slopes for the edges of P_x.
- The case in which $1 \le j < i \le k$ is symmetric to the previous one.
- Suppose now that x belongs to H and y belongs to G_i, for some $i \in \{1, \dots, k\}$. If $i = 1$ and $y = u_0$, then $y \in V(H)$ and P_{xy} is defined as above. Assume hence that $y \ne u_0$. Then the path P_{xy} consists of three sub-paths.
  - The first sub-path of P_{xy} is a path P_x in H from x to y_1.
    Suppose first that $x = u$. Then let $P_x = \tau_{uy_1}(H)$. Let $P_x = (x = z_1, z_2, \dots, z_s = y_1)$; we prove that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$ holds true for any $h = 1, \dots, s-1$. Consider the vertical line ℓ_1 through y_1, oriented towards increasing y-coordinates; as argued above, the disk D_ρ is to the right of ℓ_1 and y lies inside D_ρ. By Properties 1 and 2 of Γ_δ, the edge $z_h z_{h+1}$ is horizontal, with z_h to the left of z_{h+1}. Hence, the line ℓ'_h orthogonal to $z_h z_{h+1}$ and passing through its midpoint is also vertical and has ℓ_1 to its right. It follows that y is to the right of ℓ'_h. Since the half-plane to the right of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, we have $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$.
    Suppose next that $x \ne u$. By Property 4 of $\Gamma_{H,\delta}$, there exists a path $P_x = (x = z_1, z_2, \dots, z_s = y_1)$ in H that connects x to y_1, that does not pass through u, and whose every edge has slope in $(-\frac{\alpha}{2}; \frac{\alpha}{2})$ in $\Gamma_{H,\delta}$. We prove that, for any $h = 1, 2, \dots, s-1$, $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$; refer to Fig. 23. Since the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$, the edge $z_h z_{h+1}$ has slope in $(-\frac{\alpha}{2}; \frac{\alpha}{2})$ in Γ_δ. Consider the line ℓ_h that passes through y_1, that is directed towards increasing y-coordinates and that is orthogonal to the line through z_h and z_{h+1}. Denote by s_h the slope of ℓ_h. Then $s_h \in (\frac{\pi-\alpha}{2}; \frac{\pi+\alpha}{2})$. We prove that ℓ_h has the disk D_ρ to its right. In order to do that, consider the point p_T on the half-line with slope $\frac{\pi-\alpha}{2}$ starting at y_1 and such that $d_V(\Gamma_\delta, y_1 p_T) = \rho$. Further, consider the point p_B on the half-line with slope $\frac{-\pi+\alpha}{2}$ starting at y_1 and such that

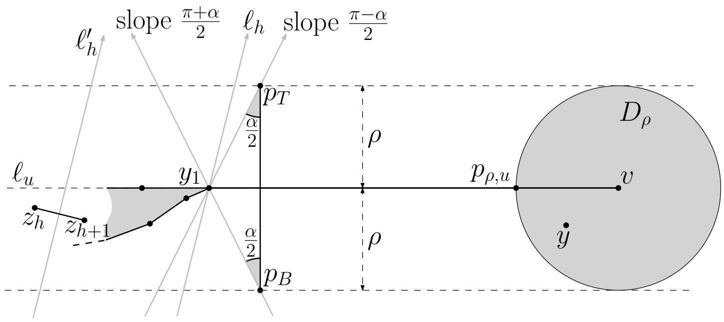


Figure 23 Illustration for the proof that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$ if $z_h z_{h+1}$ is in P_x. For the sake of readability, D_ρ is larger than it should be.

$d_V(\Gamma_\delta, y_1 p_B) = \rho$. Note that $\overline{p_T p_B}$ is a vertical straight-line segment with length 2ρ. Consider the infinite closed strip S with height 2ρ that is delimited by the horizontal lines through p_T and p_B. Since D_ρ has its center on ℓ_u and has radius ρ, it lies inside S. The part of ℓ_h inside S is to the left of $\overline{p_T p_B}$, given that $s_h \in (\frac{\pi-\alpha}{2}; \frac{\pi+\alpha}{2})$. Hence, we only need to show that $p_{\rho,u}$, which is the point of D_ρ with smallest x-coordinate, lies to the right of $\overline{p_T p_B}$. We have that $d(\Gamma_\delta, y_1 p_{\rho,u}) = d_{y_1 v} - \rho$. Further, $d_H(\Gamma_\delta, y_1 p_T) = \rho \cdot \tan(\frac{\alpha}{2})$. Hence, it suffices to prove $\rho \cdot \tan(\frac{\alpha}{2}) < d_{y_1 v} - \rho$, that is $\rho < \frac{d_{y_1 v}}{1+\tan(\frac{\alpha}{2})}$; this holds true since $\rho < \frac{d_{y_1 v}}{3}$ and $\tan(\frac{\alpha}{2}) < 1$, given that $0 < \alpha < \frac{\pi}{4}$.
By Lemma 3, the line ℓ_h has the drawing of H (and in particular the midpoint of the edge $z_h z_{h+1}$) to its left; this is because by Property 2 of $\Gamma_{H,\delta}$ every edge in $\beta_{y_1 u}(H)$ has slope π, where $s_h < \frac{\pi+\alpha}{2} < \pi < \frac{3\pi-\alpha}{2} < \pi + s_h$, and because by Property 3 of $\Gamma_{H,\delta}$ every edge in $\tau_{y_1 u}(H)$ has slope in $(\pi - \frac{\alpha}{2}; \pi + \frac{\alpha}{2})$, where $s_h < \frac{\pi+\alpha}{2} < \pi - \frac{\alpha}{2} < \pi + \frac{\alpha}{2} < \frac{3\pi-\alpha}{2} < \pi + s_h$. Now consider the line ℓ'_h parallel to ℓ_h, passing through the midpoint of the edge $z_h z_{h+1}$, and oriented towards increasing y-coordinates; ℓ'_h has ℓ_h to its right, given that the midpoint of $z_h z_{h+1}$ is to the left of ℓ_h in Γ_δ. Thus, ℓ'_h has D_ρ, and in particular y, to its right. Since the half-plane to the right of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$.

- The second sub-path is the edge $y_1 v$. Since y lies in D_ρ, we have that $d(\Gamma_\delta, vy) \leq \rho \leq \frac{d_{y_1 v}}{3}$. By the triangular inequality, we have that $d(\Gamma_\delta, y_1 y) > d(\Gamma_\delta, y_1 v) - d(\Gamma_\delta, vy) \geq d_{y_1 v} - \rho \geq \frac{2 d_{y_1 v}}{3}$. Hence, $d(\Gamma_\delta, y_1 y) > d(\Gamma_\delta, vy)$.
- The third sub-path is a path P_{vy} that connects v to y, that belongs to $\bigcup_{l=i}^{k} G_l$, and that is distance-decreasing in Γ_δ. This path exists, as from the case in which x and y belong to the same graph G_i or from the case in which x belongs to a graph G_i and y belongs to a graph G_j for some $1 \leq j < i \leq k$.

- Suppose finally that x belongs to G_i, for some $i \in \{1, \dots, k\}$, and y belongs to H. If $i = 1$ and $x = u_0$, then $x \in V(H)$ and P_{xy} is defined as above. Assume hence that $x \neq u_0$. We now describe the path P_{xy}, which consists of three sub-paths.

  - The first sub-path of P_{xy} is a path Q_x in G_i from x to u_{i-1} whose every edge has slope in $(\pi - \alpha'; \pi + \alpha')$ in Γ_i (in Γ_{1,d^*} if $i = 1$). This path exists since Γ_i (Γ_{1,d^*} if $i = 1$) satisfies Property 5. The second sub-path of P_{xy} is $\bigcup_{j=1}^{i-1} \beta_{u_j u_{j-1}}(G_j)$. Since Γ_j (Γ_{1,d^*} when $j = 1$) satisfies Properties 1 and 2, every edge in $\bigcup_{j=1}^{i-1} \beta_{u_j u_{j-1}}(G_j)$ has slope π in Γ_j (in Γ_{1,d^*} when $j = 1$). Let $(x = z_1, z_2, \dots, z_{s-1}, z_s = y_\ell)$ be the union of these two sub-paths of P_{xy}. We prove that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$, for any $h \in \{1, 2, \dots, s-1\}$. Since the drawings $\Gamma_{1,d^*}, \Gamma_2, \dots, \Gamma_k$ are counter-clockwise rotated by β radians in Γ_δ, it follows that $z_h z_{h+1}$ has slope in the interval $(\pi+\beta-\alpha'; \pi+\beta+\alpha')$ in Γ_δ.

We first present a proof that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$ for any $h \in \{1, 2, \dots, s-2\}$; we will later argue that $d(\Gamma_\delta, z_{s-1} y) > d(\Gamma_\delta, z_s y)$. Recall that z_h and z_{h+1} lie in D_ρ in Γ_δ, given that $z_h, z_{h+1} \neq y_\ell$. Consider the line ℓ_h that passes through y_1, that is directed towards increasing y-coordinates and that is orthogonal to the line through z_h and z_{h+1}. Denote by s_h the slope of ℓ_h. Then $s_h \in (\frac{\pi}{2} + \beta - \alpha'; \frac{\pi}{2} + \beta + \alpha')$.

Similarly to the case in which $x \in V(H)$ and $y \in V(G_i)$, we have that ℓ_h has the disk D_ρ to its right and the drawing of H to its left (the main difference is that the gray angles in Fig. 23 are now $\beta + \alpha'$ rather than $\frac{\alpha}{2}$). We now present proofs for these statements.

- ∗ We prove that ℓ_h has D_ρ to its right. Let p_T (p_B) be the point on the half-line with slope $\frac{\pi}{2} - \beta - \alpha'$ (resp. $-\frac{\pi}{2} + \beta + \alpha'$) starting at y_1 and such that $d_V(\Gamma_\delta, y_1 p_T) = \rho$ (resp. $d_V(\Gamma_\delta, y_1 p_B) = \rho$). Then $\overline{p_T p_B}$ is a vertical straight-line segment with length 2ρ and D_ρ lies inside the infinite closed strip S with height 2ρ that is delimited by the horizontal lines through p_T and p_B. The part of ℓ_h inside S is to the left of $\overline{p_T p_B}$, since $s_h \in (\frac{\pi}{2}+\beta-\alpha'; \frac{\pi}{2}+\beta+\alpha')$. Hence, we only need to show that $p_{\rho,u}$ lies to the right of $\overline{p_T p_B}$. We have $d(\Gamma_\delta, y_1 p_{\rho,u}) = d_{y_1 v} - \rho$, while $d_H(\Gamma_\delta, y_1 p_T) = \rho \cdot \tan(\beta+\alpha')$. Hence, it suffices to prove that $\rho < \frac{d_{y_1 v}}{1+\tan(\beta+\alpha')}$; this holds true since $\rho < \frac{d_{y_1 v}}{3}$ and $\tan(\beta + \alpha') < 1$, given that $0 < \beta < \frac{\alpha}{4} < \frac{\pi}{16}$ and $0 < \alpha' < \frac{\alpha}{8} < \frac{\pi}{32}$.
- ∗ By Lemma 3, the line ℓ_h has $\Gamma_{H,\delta}$ (and in particular y) to its left; this is because by Property 2 of $\Gamma_{H,\delta}$ every edge in $\beta_{y_1 u}(H)$ has slope π, where $s_h < \frac{\pi}{2}+\beta+\alpha' < \pi < \frac{3\pi}{2}+\beta-\alpha' < \pi+s_h$, and because by Property 3 of $\Gamma_{H,\delta}$ every edge in $\tau_{y_1 u}(H)$ has slope in $(\pi-\frac{\alpha}{2}; \pi+\frac{\alpha}{2})$, where $s_h < \frac{\pi}{2}+\beta+\alpha' < \pi-\frac{\alpha}{2} < \pi+\frac{\alpha}{2} < \frac{3\pi}{2}+\beta-\alpha' < \pi+s_h$.

Now consider the line ℓ'_h parallel to ℓ_h, passing through the midpoint of the edge $z_h z_{h+1}$, and oriented towards increasing y-coordinates; ℓ'_h has ℓ_h to its left, given that the midpoint of $z_h z_{h+1}$ is in D_ρ, hence to the right of ℓ_h in Γ_δ. Thus, ℓ'_h has $\Gamma_{H,\delta}$ (and in particular y) to its left. Since the half-plane to the left of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, it follows that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$.

We now show that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$ if $h = s-1$. Recall that $z_{h+1} = z_s = u_0 = y_\ell$ and refer to Fig. 24. We exploit again the fact that the line ℓ_h through y_1 orthogonal to the line through z_h and z_{h+1} has $\Gamma_{H,\delta}$ (and in particular y) to its left. Consider the line ℓ'_h parallel to ℓ_h, oriented towards increasing y-coordinates, and passing through the midpoint m_h of the edge $z_h z_{h+1}$. Differently from the case in which $h \in \{1, 2, \dots, s-2\}$, the midpoint m_h of $z_h z_{h+1}$ is not guaranteed to be in D_ρ (in fact it is not in D_ρ, although we do not prove this statement formally as we do not need it in the remainder), given that $z_{h+1} = y_\ell$ is in H and hence not in D_ρ. Since the half-plane to the left of ℓ'_h represents the locus of the points of the plane that are closer to z_{h+1} than to z_h, we only need to show that the intersection point p_h of the lines ℓ'_h and ℓ_u lies to the right of y_1 on ℓ_u; in fact, this implies that ℓ'_h has ℓ_h (and hence y) to its left.

Since z_h lies inside D_ρ, we have that $x(z_h) \geq x(p_{\rho,u})$. Further, $x(p_{\rho,u}) = x(y_1) + d_{y_1 v} - \rho$. Moreover, by Property 3 of $\Gamma_{H,\delta}$, we have that $x(y_1) = x(y_\ell) + d_H(\Gamma_\delta, y_\ell y_1)$. Thus, we have that $x(m_h) = \frac{x(y_\ell)+x(z_h)}{2} \geq \frac{x(y_\ell)+x(y_\ell)+d_H(\Gamma_\delta, y_\ell y_1)+d_{y_1 v}-\rho}{2} = x(y_\ell) + \frac{d_H(\Gamma_\delta, y_\ell y_1)+d_{y_1 v}-\rho}{2}$. For the sake of the simplicity of the description, translate the

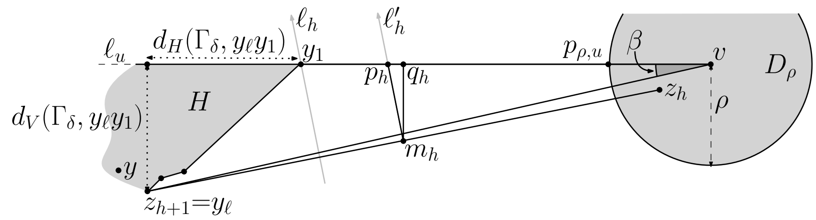


Figure 24 Illustration for the proof that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$ if $h = s-1$.

Cartesian axes so that $x(y_\ell) = 0$. Thus, $x(m_h) \geq \frac{d_H(\Gamma_\delta, y_\ell y_1) + d_{y_1 v} - \rho}{2}$.
By Lemma 6, y_ℓ is an internal vertex of $\beta_{uv}(G)$, hence y_ℓ lies below ℓ_u. Since $\rho \leq \frac{Y}{2}$ and z_h lies in D_ρ, the y-coordinate of y_ℓ is smaller than the one of z_h. It follows that the slope of $z_h z_{h+1}$ is greater than π. Further, z_h and hence m_h lie on or below the line h_β with slope β through y_ℓ. This implies that the slope of $z_h z_{h+1}$ is at most $\pi + \beta$. Thus, the slope s'_h of ℓ'_h is in the interval $(\frac{\pi}{2}; \frac{\pi}{2} + \beta)$.
We now derive a lower bound for the x-coordinate of p_h. Let q_h be the point such that $x(q_h) = x(m_h)$ and $y(q_h) = y(p_h)$. Consider the triangle $\Delta m_h p_h q_h$. Since the y-coordinate of y_ℓ is smaller than the one of z_h, it is also smaller than the one of m_h. Thus, $d(\Gamma_\delta, m_h q_h) \leq d_V(\Gamma_\delta, y_\ell y_1)$. Since $s'_h \in (\frac{\pi}{2}; \frac{\pi}{2} + \beta)$, the angle $\measuredangle p_h m_h q_h$ is at most β. Hence, $d(\Gamma_\delta, p_h q_h) \leq d_V(\Gamma_\delta, y_\ell y_1) \cdot \tan(\beta)$. It follows that $x(p_h) = x(m_h) - d(\Gamma_\delta, p_h q_h) \geq \frac{d_H(\Gamma_\delta, y_\ell y_1) + d_{y_1 v} - \rho}{2} - d_V(\Gamma_\delta, y_\ell y_1) \cdot \tan(\beta)$. It remains to prove that this quantity is larger than $d_H(\Gamma_\delta, y_\ell y_1)$, which is the x-coordinate of y_1.
Since $\beta < \frac{\alpha}{4} < \frac{\pi}{16}$, we have that $\tan(\beta) \leq 1$. It follows that $\frac{d_H(\Gamma_\delta, y_\ell y_1) + d_{y_1 v} - \rho}{2} - d_V(\Gamma_\delta, y_\ell y_1) \cdot \tan(\beta) \geq \frac{d_H(\Gamma_\delta, y_\ell y_1) + d_{y_1 v} - \rho}{2} - d_V(\Gamma_\delta, y_\ell y_1)$. Hence, we want to establish that $\frac{d_H(\Gamma_\delta, y_\ell y_1) + d_{y_1 v} - \rho}{2} - d_V(\Gamma_\delta, y_\ell y_1) > d_H(\Gamma_\delta, y_\ell y_1)$, that is, $d_{y_1 v} > 2 d_V(\Gamma_\delta, y_\ell y_1) + d_H(\Gamma_\delta, y_\ell y_1) + \rho$. Since $\rho \leq \frac{d_{y_1 v}}{3}$, we need to prove that $d_{y_1 v} > \frac{6 d_V(\Gamma_\delta, y_\ell y_1) + 3 d_H(\Gamma_\delta, y_\ell y_1)}{2}$.
We now express $d_{y_1 v}$ as a function of β. This is done by looking at the triangle whose vertices are y_ℓ, v, and the point on ℓ_u with the same x-coordinate as y_ℓ. Since the angle of this triangle at v is β, we get that $d_{y_1 v} = \frac{d_V(\Gamma_\delta, y_\ell y_1)}{\tan(\beta)} - d_H(\Gamma_\delta, y_\ell y_1)$. Substituting this into the previous inequality, we need to have $\frac{d_V(\Gamma_\delta, y_\ell y_1)}{\tan(\beta)} - d_H(\Gamma_\delta, y_\ell y_1) > \frac{6 d_V(\Gamma_\delta, y_\ell y_1) + 3 d_H(\Gamma_\delta, y_\ell y_1)}{2}$, hence $\tan(\beta) < \frac{2 d_V(\Gamma_\delta, y_\ell y_1)}{6 d_V(\Gamma_\delta, y_\ell y_1) + 5 d_H(\Gamma_\delta, y_\ell y_1)}$. This inequality holds true since $\beta < \arctan\left(\frac{d_V(\Gamma_H, y_\ell y_1)}{3 d_V(\Gamma_H, y_\ell y_1) + 3 d_H(\Gamma_H, y_\ell y_1)}\right)$. This concludes the proof that $d(\Gamma_\delta, z_h y) > d(\Gamma_\delta, z_{h+1} y)$ if $h = s-1$.

- The third sub-path of P_{xy} is a path $P_{y_\ell y}$ that connects y_ℓ to y, that belongs to H, and that is distance-decreasing in $\Gamma_{H,\delta}$. This path exists since $\Gamma_{H,\delta}$ satisfies Property 6. Since the drawing of H in Γ_δ coincides with $\Gamma_{H,\delta}$, the path $P_{y_\ell y}$ is also distance-decreasing in Γ_δ.

This concludes the proof of the lemma. ◀

Given a strong circuit graph (G, u, v) such that G is not a single edge or a simple cycle, we are in Case A or Case B depending on whether the edge uv exists or not, respectively. Thus, Lemmata 8–11 prove Theorem 7. It remains to show how to use Theorem 7 in order to prove Theorem 1. This is easily done as follows. Consider any 3-connected planar graph G and associate any plane embedding to it; let u and v be two consecutive vertices in the clockwise order of the vertices along the outer face of G. We have that (G, u, v) is a strong circuit graph. Indeed: (a) by assumption G is 2-connected – in fact 3-connected – and associated with a plane embedding; (b) by construction u and v are two distinct external vertices of G; (c) edge uv exists and coincides with $\tau_{uv}(G)$, given that v immediately follows u in the clockwise order of the vertices along the outer face of G; and (d) G does not have any 2-cut, given that it is 3-connected. Thus, Theorem 7 can be applied in order to construct a planar greedy drawing of G. This concludes the proof of Theorem 1.

4 Conclusions

In this paper we have shown how to construct planar greedy drawings of 3-connected planar graphs. It is tempting to try to use the graph decomposition we employed in this paper for proving that 3-connected planar graphs admit *convex* greedy drawings. However, despite some efforts in this direction, we have not been able to modify the statement of Theorem 7 in order to guarantee the desired convexities of the angles in the drawings. Thus, proving or disproving the convex greedy embedding conjecture remains an elusive goal.

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