

CHARACTERIZING FACE AND FLAG VECTOR PAIRS FOR POLYTOPES

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ABSTRACT. Grünbaum, Barnette, and Reay in 1974 completed the characterization of the pairs (f_i, f_j) of face numbers of 4-dimensional polytopes.

Here we obtain a complete characterization of the pairs of flag numbers (f_0, f_{03}) for 4-polytopes. Furthermore, we describe the pairs of face numbers (f_0, f_{d-1}) for d -polytopes; this description is complete for even $d \geq 6$ except for finitely many exceptional pairs that are “small” in a well-defined sense, while for odd d we show that there are also “large” exceptional pairs.

Our proofs rely on the insight that “small” pairs need to be defined and to be treated separately; in the 4-dimensional case, these may be characterized with the help of the characterizations of the 4-polytopes with at most 8 vertices by Altshuler and Steinberg (1984).

1. INTRODUCTION

For a d -dimensional polytope P , let $f_i = f_i(P)$ denote the number of i -dimensional faces of P , and for $S \subseteq \{0, \dots, d-1\}$, let $f_S = f_S(P)$ denote the number of chains $F_1 \subset \dots \subset F_r$ of faces of P with $\{\dim F_1, \dots, \dim F_r\} = S$. The f -vector of P is then $(f_0, f_1, \dots, f_{d-1})$, and the *flag vector* of P is $(f_S)_{S \subseteq \{0, \dots, d-1\}}$. The set of all f -vectors of d -polytopes is $\mathcal{F}^d \subset \mathbb{Z}^d$. (Due to the Euler equation, this set lies on a hyperplane in \mathbb{R}^d ; it spans this hyperplane by Grünbaum [15, Sect. 8.1].)

While the f -vector set \mathcal{F}^3 of 3-polytopes was characterized (easily) by Steinitz [25] in 1906, a complete characterization of \mathcal{F}^d is out of reach for any $d \geq 4$. For $d = 4$, the projections of the f -vector set $\mathcal{F}^4 \subset \mathbb{Z}^4$ onto two of the four coordinates have been determined in 1967–1974 by Grünbaum [15, Sect. 10.4], Barnette–Reay [5] and Barnette [4]. We will review these results in Section 2.1. (Note that a complete characterization of the larger set of flag f -vectors of regular CW 3-spheres not necessarily satisfying the intersection property—so their face posets need not be lattices, in which case they are not polytopal—was provided by Murai and Nevo [21, Cor. 3.5].)

This paper provides new results about coordinate projections of f -vector and flag vector sets: The first part is an extension to the flag vectors of 4-polytopes. In particular, in Theorem 2.5 we fully characterize the projection of the set of all flag vectors of 4-polytopes to the two coordinates f_0 and f_{03} . Our proof makes use of the classification of all combinatorial types of 4-polytopes with up to eight vertices by Altshuler and Steinberg [2, 3]. We have not used the classification of the 4-polytopes with nine vertices recently provided by Firsching [13].

In the second part we look at the set \mathcal{F}^d of f -vectors of d -dimensional polytopes, for $d \geq 5$. Here even a complete characterization of the projection $\Pi_{0,d-1}(\mathcal{F}^d) \subset \mathbb{Z}^2$ to the coordinates f_0 and f_{d-1} seems impossible. We call (n, m) a *polytopal pair* if $(n, m) \in \Pi_{0,d-1}(\mathcal{F}^d)$, that is, if there is a d -polytope P with $f_0(P) = n$ and $f_{d-1}(P) = m$. These polytopal pairs must satisfy the *UBT inequality* $m \leq f_{d-1}(C_d(n))$ given by the Upper Bound Theorem [20] [26, Sect. 8.4], where $C_d(n)$ denotes a d -dimensional cyclic polytope with n vertices, and also $n \leq f_{d-1}(C_d(m))$, by duality.

Our second main result, Theorem 3.2, states that for even $d \geq 4$, every (n, m) satisfying the UBT inequalities as well as $n + m \geq \binom{3d+1}{\lfloor d/2 \rfloor}$ is a polytopal pair. However, for even $d \geq 6$, there are pairs (n, m) with $n + m < \binom{3d+1}{\lfloor d/2 \rfloor}$ that satisfy the UBT inequalities, but for which there is no polytope: We call these *small exceptional pairs*. Theorem 3.3 states, in contrast, that for every odd $d \geq 5$ there are also *arbitrarily large exceptional pairs*.

2. FACE AND FLAG VECTOR PAIRS FOR 4-POLYTOPES

2.1. Face vector pairs for 4-polytopes. The 2-dimensional coordinate projections $\Pi_{i,j}(\mathcal{F}^4)$ of the set of f -vectors of 4-polytopes to the coordinate planes, as determined by Grünbaum, Barnette and Reay, are given by the following theorems. See also Figure 1.

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In memory of Branko Grünbaum (1929–2018).

Theorem 2.1 (Grünbaum [15, Thm. 10.4.1]). *The set of f -vector pairs (f_0, f_3) of 4-polytopes is equal to*

$$\begin{aligned} \Pi_{0,3}(\mathcal{F}^4) = \{ & (f_0, f_3) \in \mathbb{Z}^2 : 5 \leq f_0 \leq \frac{1}{2}f_3(f_3 - 3), \\ & 5 \leq f_3 \leq \frac{1}{2}f_0(f_0 - 3)\}. \end{aligned}$$

Theorem 2.2 (Grünbaum [15, Thm. 10.4.2]). *The set of f -vector pairs (f_0, f_1) of 4-polytopes is equal to*

$$\begin{aligned} \Pi_{0,1}(\mathcal{F}^4) = \{ & (f_0, f_1) \in \mathbb{Z}^2 : 10 \leq 2f_0 \leq f_1 \leq \frac{1}{2}f_0(f_0 - 1)\} \\ & \setminus \{(6, 12), (7, 14), (8, 17), (10, 20)\}. \end{aligned}$$

The existence parts of Theorems 2.1 and 2.2 are proved by taking neighborly polytopes, which yield the polytopal pairs on the upper bound, as well as dual neighborly polytopes for the polytopal pairs on the lower bound, and by finding some polytopes for examples of small polyhedral pairs. From these polytopes, polytopes with all other possible polytopal pairs are constructed by an inductive method of (generalized) stacking (see Sections 2.3 and 2.4).

Theorem 2.3 (Barnette & Reay [5, Thm. 10]). *The set of f -vector pairs (f_0, f_2) of 4-polytopes is equal to*

$$\begin{aligned} \Pi_{0,2}(\mathcal{F}^4) = \{ & (f_0, f_2) \in \mathbb{Z}^2 : 10 \leq \frac{1}{2}(2f_0 + 3 + \sqrt{8f_0 + 9}) \leq f_2 \leq f_0^2 - 3f_0, \\ & f_2 \neq f_0^2 - 3f_0 - 1\} \\ & \setminus \{(6, 12), (6, 14), (7, 13), (7, 15), (8, 15), \\ & (8, 16), (9, 16), (10, 17), (11, 20), (13, 21)\}. \end{aligned}$$

The existence part of Theorem 2.3 is proved similarly to the proofs of Theorems 2.1 and 2.2, additionally considering all 4-dimensional pyramids, bipyramids and prisms (“cylinders”).

Theorem 2.4 (Barnette [4, Thm. 1], with corrections, cf. [17]). *The set of f -vector pairs (f_1, f_2) of 4-polytopes is equal to*

$$\begin{aligned} \Pi_{1,2}(\mathcal{F}^4) = \{ & (f_1, f_2) \in \mathbb{Z}^2 : 10 \leq \frac{1}{2}f_1 + \left\lceil \sqrt{f_1 + \frac{9}{4}} + \frac{1}{2} \right\rceil + 1 \leq f_2, \\ & 10 \leq \frac{1}{2}f_2 + \left\lceil \sqrt{f_2 + \frac{9}{4}} + \frac{1}{2} \right\rceil + 1 \leq f_1, \\ & f_2 \neq \frac{1}{2}f_1 + \sqrt{f_1 + \frac{13}{4}} + 2, \\ & f_1 \neq \frac{1}{2}f_2 + \sqrt{f_2 + \frac{13}{4}} + 2\} \\ & \setminus \{(12, 12), (13, 14), (14, 13), (14, 14), (15, 15), \\ & (15, 16), (16, 15), (16, 17), (16, 18), (17, 16), \\ & (17, 20), (18, 16), (18, 18), (19, 21), (20, 17), (20, 23), \\ & (20, 24), (21, 19), (21, 26), (23, 20), (24, 20), (26, 21)\}. \end{aligned}$$

For the proof of Theorem 2.4, a finite number of polytopes with few edges was found, and polytopes with all other possible polytopal pairs were constructed using an inductive method based on “facet splitting” (see Section 2.5).

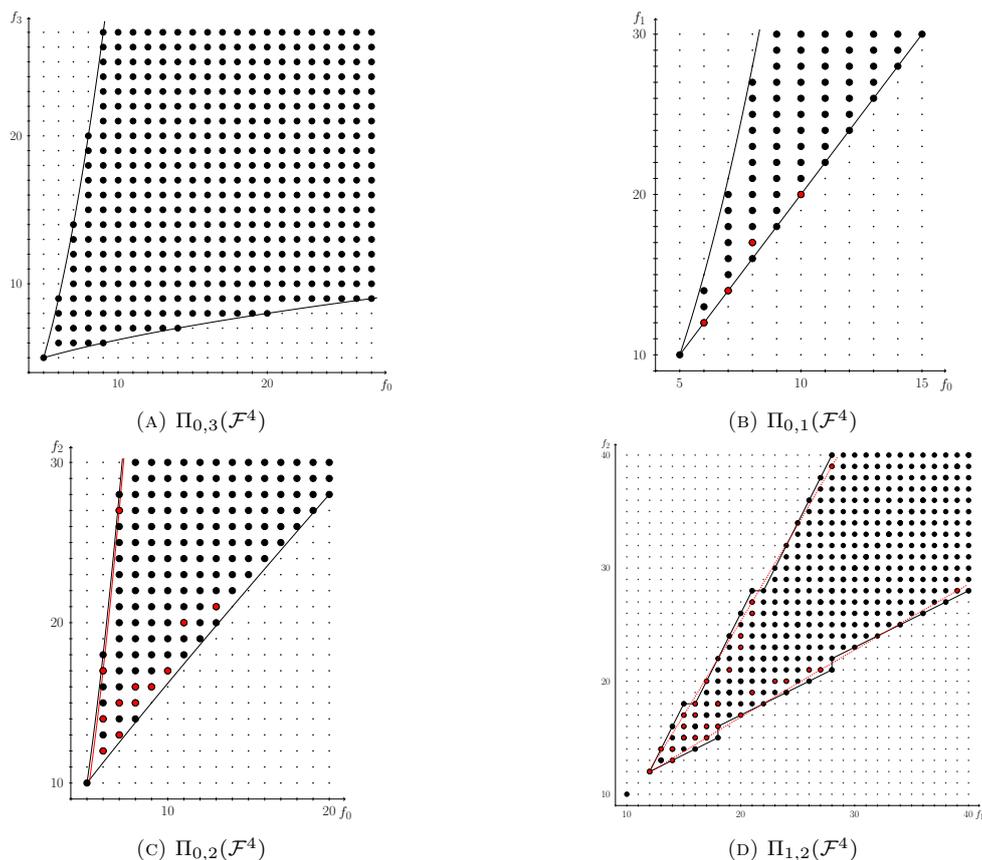
The remaining f -vector projections are given by duality.

2.2. Flag vector pair (f_0, f_{03}) for 4-polytopes. In the following we will characterize the set

$$\Pi_{0,03}(\mathcal{F}^4) = \{(f_0(P), f_{03}(P)) \in \mathbb{Z}^2 \mid P \text{ is a 4-polytope}\},$$

that is, we describe the possible number of vertex-facet incidences of a 4-polytope with a fixed number of vertices. Equivalently, this tells us the possible average number of facets of the vertex figures, $\frac{f_{03}}{f_0}$, for a given number f_0 of vertices.

In 1984 Altshuler and Steinberg classified all combinatorial types of 4-polytopes with up to 8 vertices [2, 3]. This classification makes our proof much easier. We will use the classification to find examples of polytopes for certain small polytopal pairs and also to argue that some pairs cannot be polytopal pairs of any 4-polytope. The following is our first main theorem:

FIGURE 1. f -vector projections, red dots are exceptional pairs

Theorem 2.5. *There exists a 4-polytope P with $f_0(P) = f_0$ and $f_{03}(P) = f_{03}$ if and only if f_0 and f_{03} are integers satisfying*

$$\begin{aligned} 20 \leq 4f_0 \leq f_{03} \leq 2f_0(f_0 - 3), \\ f_{03} \neq 2f_0(f_0 - 3) - k \text{ for } k \in \{1, 2, 3, 5, 6, 9, 13\} \end{aligned}$$

and (f_0, f_{03}) is not one of the 18 exceptional pairs

$$\begin{aligned} (6, 24), (6, 25), (6, 28), \\ (7, 28), (7, 30), (7, 31), \\ (7, 33), (7, 34), (7, 37), (7, 40), \\ (8, 33), (8, 34), (8, 37), (8, 40), \\ (9, 37), (9, 40), (10, 40), (10, 43). \end{aligned}$$

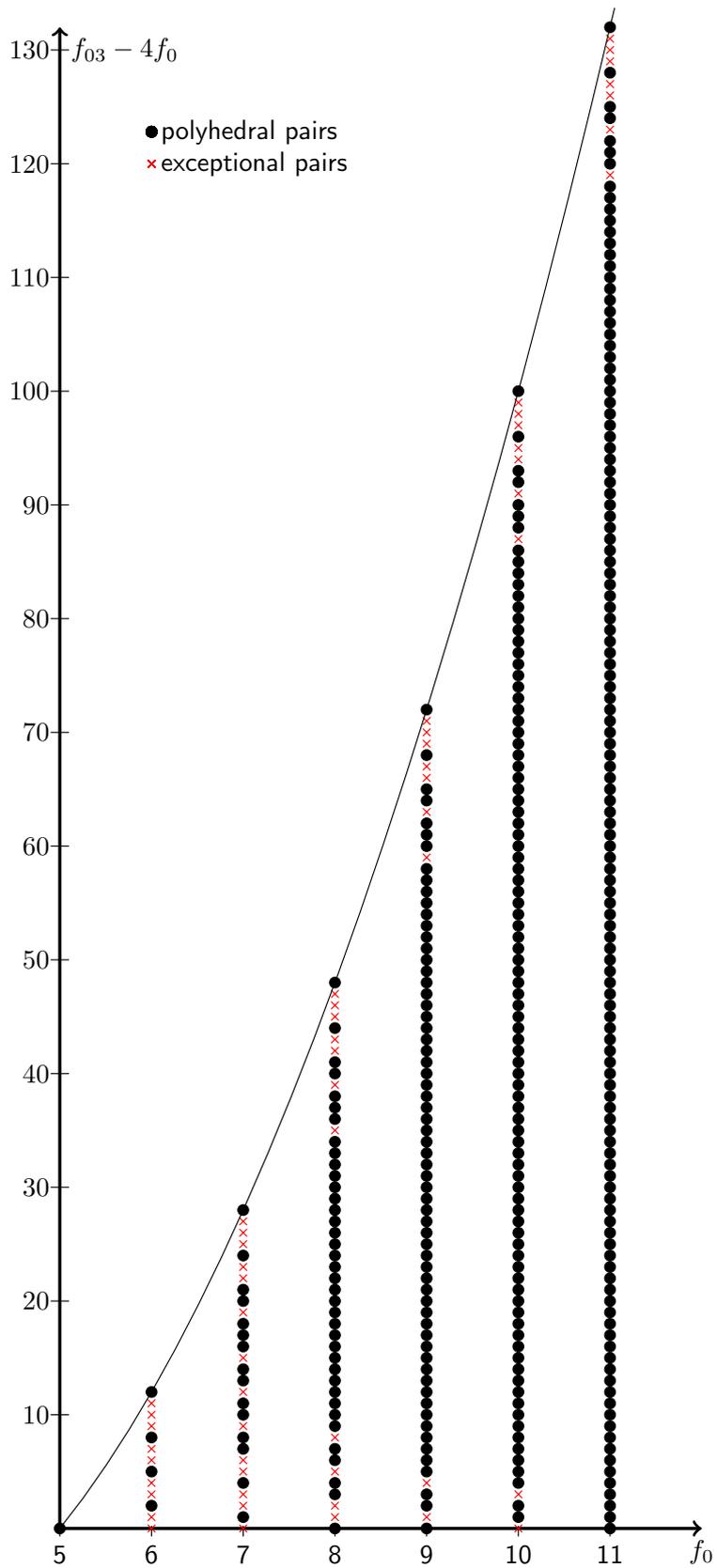
See Figure 2 for a visualization of the projection in the plane $(f_0, f_{03} - 4f_0)$.

The proof of Theorem 2.5 follows the proofs of the projections of the f -vector [4], [5], [15], by taking small polytopal pairs as well as polytopal pairs on the boundaries and constructing new polytopal pairs from the given ones. The inductive methods used for this proof are the stacking and truncating constructions from Theorem 2.1, 2.2 and 2.3 and “facet splitting” methods generalized from the methods used in the proof of Theorem 2.4.

Lemma 2.6. *If P is a 4-dimensional polytope with f_0 vertices and f_{03} vertex-facet incidences, then*

$$4f_0 \leq f_{03} \leq 2f_0(f_0 - 3).$$

Proof. Every vertex of a d -polytope lies in at least d facets, so clearly $4f_0 \leq f_{03}$ holds for all 4-dimensional polytopes, with equality if and only if P is simple.

FIGURE 2. Projection $\Pi_{0,03}(\mathcal{F}^4)$

The second inequality follows from a generalization of the upper bound theorem to flag vectors: For any d -dimensional polytope with n vertices and for any $S \subseteq \{0, \dots, d-1\}$,

$$f_S \leq f_S(C_d(n)),$$

where $C_d(n)$ is the d -dimensional cyclic polytope with n vertices [8, Thm 18.5.9]. In particular, 4-dimensional cyclic polytopes are simplicial, and for any 4-dimensional polytope P ,

$$f_{03}(P) \leq f_{03}(C_4(n)) = 4f_3(C_4(n)) = 2n(n-3)$$

with equality if and only if P is neighborly. \square

Lemma 2.7. *There is no 4-polytope P with $f_0(P) = f_0$ and $f_{03}(P) = f_{03}$ if (f_0, f_{03}) is any of the following pairs:*

$$\begin{aligned} &(6, 24), (6, 25), (6, 28), \\ &(7, 28), (7, 30), (7, 31), \\ &(7, 33), (7, 34), (7, 37), (7, 40), \\ &(8, 33), (8, 34), (8, 37), (8, 40), \\ &(9, 37), (9, 40), (10, 40), (10, 43), \\ &(f_0, 2f_0(f_0 - 3) - k) \text{ for } k \in \{1, 2, 3, 5, 6, 9, 13\} \text{ and for any } f_0 \geq 6. \end{aligned}$$

For the proof of this lemma we need some equations and inequalities which hold for the flag vector of any 4-polytope.

Lemma 2.8 (Generalized Dehn–Sommerville equations, Bayer & Billera [7, Thm. 2.1]). *Let P be a d -polytope and $S \subseteq \{0, 1, \dots, d-1\}$. Let $\{i, k\} \subseteq S \cup \{-1, d\}$ such that $i < k-1$ and such that there is no $j \in S$ for which $i < j < k$. Then*

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S \cup j}(P) = f_S(P)(1 - (-1)^{k-i-1}).$$

For $d = 4$, $S = \{0\}$, $i = 0$, $k = 4$ and with the observation $f_{01} = 2f_1$ we obtain

$$f_{02} = -2f_0 + 2f_1 + f_{03}. \quad (2.1)$$

Lemma 2.9 (Bayer [6, Thm. 1.3, 1.4]). *The flag vector of every 4-polytope satisfies the inequalities*

$$f_{02} - 3f_2 + f_1 - 4f_0 + 10 \geq 0 \quad (2.2)$$

$$\text{and } -6f_0 + 6f_1 - f_{02} \geq 0. \quad (2.3)$$

Inequality (2.3) holds with equality if and only if the 4-polytope is center boolean, that is, if all its facets are simple.

Using Lemma 2.8 and the Euler–Poincaré formula for dimension 4 [15, Thm. 8.1.1] to rewrite Lemma 2.9, we obtain the inequalities

$$-3f_0 - 3f_3 + f_{03} + 10 \geq 0 \quad (2.4)$$

$$\text{and } 4f_0 - 4f_1 + f_{03} \leq 0. \quad (2.5)$$

Proof of Lemma 2.7. We first show that there is no polytope P with

$$(f_0(P), f_{03}(P)) = (f_0, 2f_0(f_0 - 3) - k) \text{ for } k \in \{1, 2, 3, 5, 6, 9, 13\}.$$

For $k = 1, 2, 3$ we prove the non-existence directly. For $k = 5, 6$ we show that if P is a polytope with $2f_0(f_0 - 3) - 5 \leq f_{03} \leq 2f_0(f_0 - 3) - 7$, then necessarily $f_{03} = 2f_0(f_0 - 3) - 7$. The proof for $k = 9$ and $k = 13$ follows similarly.

For $k > 0$ any 4-polytope with polytopal pair $(f_0, 2f_0(f_0 - 3) - k)$ cannot be neighborly, so

$$f_1 < \binom{f_0}{2}.$$

On the other hand, for $(f_0(P), f_{03}(P)) = (f_0, 2f_0(f_0 - 3) - k)$ Inequality (2.5) reads $\frac{1}{2}f_0(f_0 - 1) - \frac{k}{4} \leq f_1$. Both inequalities together give

$$\binom{f_0}{2} - \frac{k}{4} \leq f_1 < \binom{f_0}{2}. \quad (2.6)$$

There is no integer solution for $k = 1, 2, 3$.

For $k = 5, 6, 7$, the only possible integer value for f_1 is $\frac{1}{2}f_0(f_0 - 1) - 1$. Assume that P is a polytope with

$$f_1 = \binom{f_0}{2} - 1$$

and

$$2f_0(f_0 - 3) - 5 \leq f_{03} \leq 2f_0(f_0 - 3) - 7.$$

Since $f_1 = \binom{f_0}{2} - 1$, there is a unique pair v_1, v_2 of vertices of P not forming an edge. We call such a pair of vertices a *non-edge*. Any facet of P which is not a simplex must contain this non-edge, since the only 3-polytope in which every two vertices form an edge is the simplex. Consider a facet F which is not a simplex, and therefore contains the unique non-edge. Such a facet F needs to exist, since if P were simplicial, $f_{03} \equiv 0 \pmod{4}$. Observe that if F would have more than five vertices, then we could find five vertices of F for which every two vertices form an edge. This subpolytope of F could not be d -dimensional, for $d \leq 3$. From this contradiction follows that F has five vertices. The only combinatorial types of 3-polytopes with five vertices are the square pyramid and the bipyramid over a triangle, only the latter has exactly one non-edge. So F is a bipyramid, and the non-edge is between the apices of F . If there were another non-tetrahedral facet of P , it would intersect F in a common face containing the non-edge. Such a face does not exist, and hence P is a polytope with one bipyramidal facet and t tetrahedral facets, for some integer t . This implies that

$$f_{03} = 4t + 5 \equiv 1 \pmod{4}.$$

From the assumption $2f_0(f_0 - 3) - 5 \leq f_{03} \leq 2f_0(f_0 - 3) - 7$ follows now

$$f_{03} = 2f_0(f_0 - 3) - 7.$$

Assume now that there is a polytope P with

$$(f_0(P), f_{03}(P)) = (f_0, 2f_0(f_0 - 3) - 9).$$

Inequality (2.6) implies that

$$f_1 = \binom{f_0}{2} - 2 \text{ or } f_1 = \binom{f_0}{2} - 1.$$

If $f_1 = \binom{f_0}{2} - 1$, then we have just proved that $f_{03} \equiv 1 \pmod{4}$. Since $f_{03}(P) \equiv 3 \pmod{4}$, it follows that P has two non-edges. The inequality $f_1 \leq 3f_0 - 6$ holds for 3-dimensional polytopes and any facet F has at most two non-edges:

$$\binom{f_0(F)}{2} - 2 \leq f_1(F) \leq 3f_0(F) - 6 \Rightarrow f_0(F) < 6.$$

Any non-tetrahedral facet is hence a polytope with five vertices, a bipyramid over a triangle or a square pyramid. Since $f_{03}(P) \equiv 3 \pmod{4}$, there have to be at least three non-tetrahedral facets. Bipyramids have one non-edge, not contained in any other facet. Square pyramids have two non-edges, which are both contained in exactly one other facet. This contradicts the fact that there are only two non-edges in P . In conclusion, there is no polytope with $(f_0, f_{03}) = (f_0, 2f_0(f_0 - 3) - 9)$.

Finally, assume that there exists a polytope P with polytopal pair

$$(f_0(P), f_{03}(P)) = (f_0, 2f_0(f_0 - 3) - 13).$$

From Inequality (2.6) it follows that P has $\binom{f_0}{2} - 3$, $\binom{f_0}{2} - 2$ or $\binom{f_0}{2} - 1$ edges. Each facet F of P has at most three non-edges. For any facet F of P the inequality $f_1(F) \leq 3f_0(F) - 6$ now yields $\binom{f_0(F)}{2} - 3 \leq 3f_0(F) - 6 \Rightarrow f_0(F) < 7$. If F has six vertices, it must have 12 edges and three non-edges. There are only two such combinatorially different 3-polytopes, which both are simplicial.

Assume that P has a facet F with six vertices. Then F contains three non-edges, all of them not in any 2-face of F and hence not in any other facet. So all other facets of P are tetrahedra. This is a contradiction to $f_{03}(P) \equiv 3 \pmod{4}$.

P is not simplicial, so there are non-tetrahedral facets, all of them with five vertices. Observe that since there are at most three non-edges, we cannot have more than three non-tetrahedral facets. Together with $f_{03}(P) \equiv 3 \pmod{4}$, this leaves us with two cases:

- (i) The non-tetrahedral facets of P are three bipyramids over triangles.
- (ii) The non-tetrahedral facets of P are two square pyramids and one bipyramid over a triangle.

In both cases, let t denote the number of tetrahedra in P . Then

$$\begin{aligned} f_{03}(P) &= 2f_0(f_0 - 3) - 13 = 4t + 3 \cdot 5 \\ &\Rightarrow t = \frac{1}{2}f_0(f_0 - 3) - 7 \\ &\Rightarrow f_3(P) = t + 3 = \frac{1}{2}f_0(f_0 - 3) - 4. \end{aligned}$$

We can now calculate $f_2(P)$ in two ways. From the Euler–Poincaré formula,

$$\begin{aligned} f_2 &= f_1 + f_3 - f_0 \\ &= \binom{f_0}{2} - 3 + \frac{1}{2}f_0(f_0 - 3) - 4 - f_0 \\ &= f_0(f_0 - 3) - 7. \end{aligned}$$

Each 2-face lies in exactly two facets. The number of 2-faces of P can therefore also be calculated by counting the number of 2-faces in each facet. In case (i) this gives:

$$f_2 = \frac{1}{2}f_{23} = \frac{1}{2}(4t + 3 \cdot 6) = f_0(f_0 - 3) - 5 \neq f_0(f_0 - 3) - 7.$$

In case (ii) we obtain:

$$f_2 = \frac{1}{2}f_{23} = \frac{1}{2}(4t + 2 \cdot 5 + 6) = f_0(f_0 - 3) - 6 \neq f_0(f_0 - 3) - 7.$$

So there cannot be a polytope with polytopal pair $(f_0, 2f_0(f_0 - 3) - 13)$.

It remains to show the non-existence of 18 pairs (f_0, f_{03}) . All combinatorial types of 4-polytopes with up to 8 vertices have been classified by Altshuler and Steinberg [2, 3]. From this classification it follows that there are no polytopes with polytopal pairs

$$\begin{aligned} &(6, 24), (6, 25), (6, 28), \\ &(7, 28), (7, 30), (7, 31), \\ &(7, 33), (7, 34), (7, 37), (7, 40), \\ &(8, 33), (8, 34), (8, 37) \text{ or } (8, 40). \end{aligned}$$

To see that the four pairs $(9, 37)$, $(9, 40)$, $(10, 40)$ and $(10, 43)$ are exceptional pairs, we make use of the upper bound for the number of facets in terms of the number of vertices and vertex-facet incidences. If there were a polytope P with polytopal pair $(9, 37)$, $(9, 40)$, $(10, 40)$ or $(10, 43)$, due to Inequality (2.4) it would need to have less than 8 facets. By duality, this would give us a polytope P^* with $f_{03}(P^*) = 37, 40$ or 43 and $f_0(P^*) \leq 7$. From the upper bound $f_{03} \leq 2f_0(f_0 - 3)$ it follows that $f_0(P^*) = 7$. As seen above, polytopes with polytopal pair $(7, 37)$ or $(7, 40)$ do not appear in the classification. Pair $(7, 43)$ is of the type $(f_0, 2f_0(f_0 - 3) - 13)$, which is an exceptional pair. \square \square

We will use the classification of 4-dimensional polytopes with up to 8 vertices [2, 3] together with some classes of polytopes, such as cyclic polytopes, pyramids, and some additional polytopes, and from those polytopes and their polytopal pairs construct all other possible polytopal pairs. The methods needed for this construction are described in the following sections.

2.3. Stacking and truncating. The operations *stacking* and *truncating* (see [16, Sect. 16.2.1]) turn out to be essential in finding examples of polytopes for all possible polytopal pairs (f_0, f_{03}) . Let P be a 4-polytope with at least one simplex facet F and v a point beyond F and beneath all other facets of P . Let $Q = \text{conv}(\{v\} \cup P)$. Then

$$f_0(Q) = f_0(P) + 1 \text{ and } f_{03}(Q) = f_{03}(P) + 12.$$

Dually, let Q be a polytope obtained by truncating a simple vertex from a polytope P . Then

$$f_0(Q) = f_0(P) + 3 \text{ and } f_{03}(Q) = f_{03}(P) + 12.$$

The polytopes obtained through these two methods all have both a simple vertex and a simplex facet. This means that we can stack vertices on simplex facets and truncate simple vertices repeatedly. Truncating simple vertices and stacking vertices on simplex facets inductively, starting from a polytope with (f_0, f_{03}) with tetrahedral facet and simple vertex, we obtain new polytopes with

$$(f_0 + 2m + n, f_{03} + 12n) \text{ for } n \geq 0, \quad 0 \leq m \leq n.$$

Given a polytope P with a square pyramidal facet F , let v be a point beyond F and beneath all other facets of P . Let $Q = \text{conv}(\{v\} \cup P)$. Then

$$f_0(Q) = f_0(P) + 1 \text{ and } f_{03}(Q) = f_{03}(P) + 16.$$

The results in this section are simple consequences from [15, Thm. 5.2.1], with corrections from [1].

2.4. Generalized stacking on cyclic polytopes. We need some more methods, especially to create polytopes with polytopal pair (f_0, f_{03}) close to the upper bound $f_{03} = 2f_0(f_0 - 3)$. For our next construction we need the observation that every cyclic 4-polytope with n vertices has edges that lie in exactly $n - 2$ facets. Such edges are called *universal edges*. The following construction was used by Grünbaum [15, Sect. 10.4.1] for the characterization of the sets $\Pi_{0,3}(\mathcal{F}^4)$ and $\Pi_{0,1}(\mathcal{F}^4)$. Starting from a cyclic polytope with n vertices, we can obtain new polytopes by stacking a vertex onto it, such that the vertex lies beyond several facets. Let $R_i(n)$, $i \in \{1, \dots, n - 3\}$, denote a polytope obtained from the cyclic polytope $C_4(n)$ with n vertices as the convex hull of $C_4(n)$ and a point v , where v is beyond i facets of $C_4(n)$ sharing a universal edge. Let F_1, \dots, F_i denote these i facets, such that F_j and F_{j+1} meet in a common 2-face, for all $j = 1, \dots, i - 1$. Then the new polytope $R_i(n)$ has one more vertex than $C_4(n)$ and the following facets:

- (1) All $\frac{1}{2}n(n - 3) - i$ facets of $C_4(n)$ which v lies beneath.
- (2) Facets which are convex hulls of v and 2-faces of $C_4(n)$ that are contained in both a facet which v is beyond and a facet which v is beneath. There are two types of these facets:
 - (2a) Two such facets for each of the $i - 2$ facets F_2, \dots, F_{i-1} which v lies beyond and which share two 2-faces with other facets which v lies beyond.
 - (2b) Three new facets for each of the two facets F_1 and F_i which v lies beyond and which share one 2-face with other facets which v lies beyond.

Note that all these facets are simplices. In conclusion, for $1 \leq i \leq n - 3$,

$$f_0(R_i(n)) = n + 1 \text{ and } f_{03}(R_i(n)) = 2n(n - 3) + 4i + 8.$$

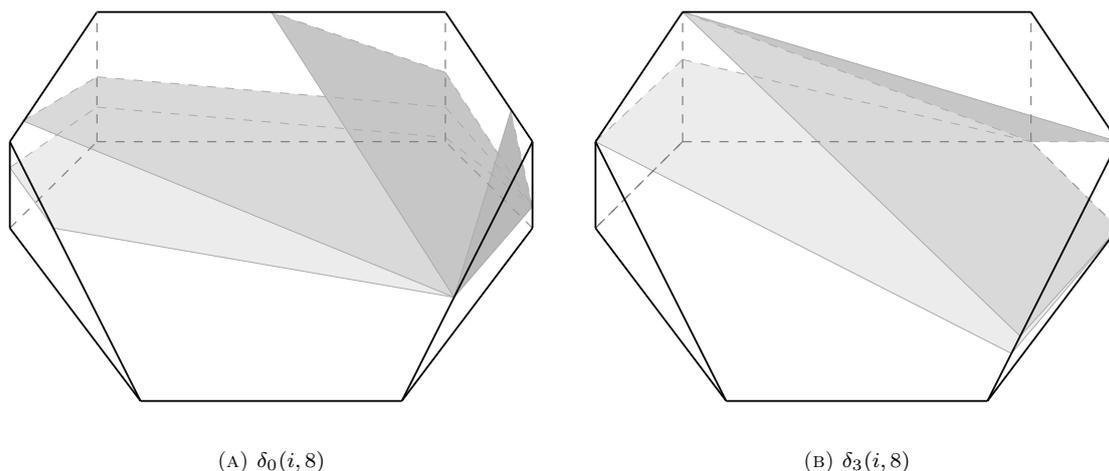
Observe that

$$f_{03}(C_4(n + 1)) = f_{03}(C_4(n)) + 4n - 4,$$

so if $i = n - 3$, we obtain again a neighboring polytope, with $n + 1$ vertices.

2.5. Facet splitting. We need to generalize the stacking method even more to obtain non-simplicial polytopes, Compare to the *A-sewing construction* of Lee and Menzel [19]. For easier visualization, we choose to work in the dual setting. Instead of adding a new vertex to a polytope, we will create a new facet in the dual polytope. This method of *facet splitting* was used by Barnette [4] for the classification of $\Pi_{1,2}(\mathcal{F}^4)$: Consider a facet F of a 4-polytope P and a hyperplane H which intersects the relative interior of F in a polygon X . If on one side of H , the only vertices of P are simple vertices of F , then we can obtain a new polytope P' by separating facet F into two new facets by the polygon X . We say that P' is obtained from P by *facet splitting*.

2.5.1. Dual of a cyclic polytope. We will split a facet of the dual of a cyclic polytope (see Barnette [4]). $C_4^*(n)$, the dual of the cyclic polytope with n vertices, is a simple polytope with n facets, each facet having $2(n - 3)$ vertices. The facets are all *wedges* over $(n - 2)$ -gons, that is, polytopes with two triangular 2-faces, $n - 5$ quadrilateral 2-faces and two $(n - 2)$ -gons meeting in an edge. Let G be a 2-dimensional plane in the affine hull of a facet F of $C_4^*(n)$. Let X be the intersection of F and G . All vertices of $C_4^*(n)$ are simple, so we can obtain a new polytope by facet splitting of $C_4^*(n)$ by choosing a hyperplane H which contains G such that on one side of H the only vertices of $C_4^*(n)$ are vertices of F . Such a hyperplane can be found by taking the facet-defining hyperplane of F and rotating it about G . The combinatorial properties f_1 and f_{03} of the polytope obtained through facet splitting depend on the choice of G : We can choose G not to intersect any vertices of F . Then, for any i such that $3 \leq i \leq n - 2$, $X = G \cap F$ can be chosen to be an i -gon. Let $\delta_0(i, n)$ denote the polytope obtained through facet splitting for this choice of G (see Figure 3a). Now $\delta_0(i, n)$ has one more facet and i more vertices than $C_4^*(n)$. As $C_4^*(n)$ is a simple polytope, all of its edges lie in exactly three facets and each of the i new vertices of $\delta_0(i, n)$ lies in four facets. The new polytope has therefore $4i$

FIGURE 3. Facet of $C_4^*(8)$ split by an i -gon

more vertex-facet incidences than $C_4^*(n)$. If we instead choose G to intersect exactly one vertex of F , X can again be any i -gon for $3 \leq i \leq n - 2$. Call this polytope $\delta_1(i, n)$. It has one more facet and $i - 1$ more vertices than $C_4^*(n)$. The $i - 1$ new vertices are simple, and the one vertex of $C_4^*(n)$ which lies in X is contained in one additional facet. In total, f_{03} increases by $4i - 3$. The polytopes $\delta_0(i, n)$ and $\delta_1(i, n)$ are used in the characterization of $\Pi_{1,2}(\mathcal{F}^4)$ [4].

Similarly, let $\delta_2(i, n)$ denote the polytope obtained when G intersects two vertices of F . As before, i can be chosen to be any integer between 3 and $n - 2$. The new polytope has one more facet, $i - 2$ more vertices and $4i - 6$ more vertex-facet incidences. If we choose G to intersect F in three vertices, as the intersection of G and F we can obtain i -gons for $3 \leq i \leq n - 3$ (see Figure 3b). The new polytope, denoted by $\delta_3(i, n)$, has one more facet, $i - 3$ more vertices and $4i - 9$ more vertex-facet incidences. Let us look at the duals of these polytopes. For $3 \leq i \leq n - 2$ we obtain polytopes $\delta_0^*(i, n)$, $\delta_1^*(i, n)$ and $\delta_2^*(i, n)$ with

$$\begin{aligned} (f_0(\delta_0^*(i, n)), f_{03}(\delta_0^*(i, n))) &= (n + 1, 2n(n - 3) + 4i), \\ (f_0(\delta_1^*(i, n)), f_{03}(\delta_1^*(i, n))) &= (n + 1, 2n(n - 3) + 4i - 3), \\ (f_0(\delta_2^*(i, n)), f_{03}(\delta_2^*(i, n))) &= (n + 1, 2n(n - 3) + 4i - 6). \end{aligned}$$

For $3 \leq i \leq n - 3$ we obtain polytopes $\delta_3^*(i, n)$ with

$$(f_0(\delta_3^*(i, n)), f_{03}(\delta_3^*(i, n))) = (n + 1, 2n(n - 3) + 4i - 9).$$

In particular, the polytopes $\delta_0^*(i, n)$, $\delta_1^*(i, n)$, $\delta_2^*(i, n)$ and $\delta_3^*(i, n)$ have simplex facets.

2.5.2. Polytopes with a bipyramidal facet. Given a polytope P with a facet B which is a bipyramid over a triangle, such that at least one apex v of B is a simple vertex, we can split the bipyramid into two tetrahedra by “moving” v outside the affine hull of B , along the unique edge which contains v and does not belong to B . The new polytope \tilde{P} has the same number of vertices and one more facet than P . The apices of the bipyramid still belong to the same number of facets as before, but the other three vertices now belong to one more facet. In total, the number of vertex-facet incidences increases by 3. Hence,

$$(f_0(\tilde{P}), f_{03}(\tilde{P})) = (f_0(P), f_{03}(P) + 3).$$

Note that \tilde{P} has simplex facets and that any simple vertex of P is a simple vertex of \tilde{P} .

2.6. Construction of polytopal pairs (f_0, f_{03}) . We can now prove Theorem 2.5. First, we list some examples of polytopes with small polytopal pairs (f_0, f_{03}) for $f_{03} \leq 80$ with simplex facet and/or simple vertex, see Table 1. The second column in the table explains how the polytope is found. Polytopes P_i are polytopes with 7 or 8 vertices known from the classification of all polytopes with up to 8 vertices. Facet lists of all polytopes P_i can be found in the appendix. *Dual of (f_0, f_{03})* means that the polytope is the dual of the polytope with polytopal pair (f_0, f_{03}) in the table. A polytope P^* denotes the dual of a polytope P . The methods *stacking on a square pyramidal facet* and *splitting a bipyramidal facet* and the polytopes $R_i(n)$ are explained above.

(f_0, f_{03})	Description	(f_0, f_{03})	Description
Polytopes with Δ_3 -facet and simple vertex		(11, 45)	P_5^*
(5, 20)	4-simplex	(11, 49)	P_{13}^*
(6, 26)	2-fold pyramid over quadrangle	(11, 52)	dual of (9, 52)
(6, 29)	pyramid over triangular bipyramid	(11, 55)	dual of (10, 55)
(7, 29)	pyramid over triangular prism	(12, 52)	P_{14}^*
(7, 32)	2-fold pyramid over pentagon	(13, 55)	P_{15}^*
(7, 35)	P_1	Polytopes with Δ_3 -facet	
(7, 36)	P_2	(6, 36)	cyclic polytope $C_4(6)$
(7, 39)	P_3	(7, 42)	P_4
(7, 45)	P_5	(7, 46)	P_6
(8, 35)	P_1^*	(7, 49)	P_7
(8, 36)	P_2^*	(7, 52)	$R_2(6)$
(8, 38)	2-fold pyramid over hexagon	(7, 56)	cyclic polytope $C_4(7)$
(8, 39)	P_8	(8, 43)	P_{10}
(8, 42)	P_9	(8, 60)	P_{17}
(8, 45)	P_{11}	(8, 63)	P_{19}
(8, 46)	P_{12}	(8, 65)	P_{20}
(8, 49)	P_{13}	(8, 66)	P_{21}
(8, 52)	P_{14}	(8, 68)	P_{22}
(8, 55)	P_{15}	(8, 69)	P_{23}
(8, 59)	P_{16}	(8, 70)	P_{24}
(8, 62)	P_{18}	(8, 72)	P_{25}
(9, 39)	P_3^*	(8, 73)	P_{26}
(9, 42)	P_9^*	(8, 76)	P_{27}
(9, 45)	split bipyramid in (9, 42)	(8, 80)	cyclic polytope $C_4(8)$
(9, 46)	split bipyramid in (9, 43)	(9, 79)	stack onto square pyramid in (8, 63)
(9, 49)	split bipyramid in (9, 46)	Polytopes with simple vertex	
(9, 52)	stack onto square pyramid in (8, 36)	(9, 36)	dual of cyclic polytope $C_4(6)$
(10, 45)	P_{11}^*	(9, 43)	P_{10}^*
(10, 46)	P_{12}^*	(10, 42)	P_4^*
(10, 49)	dual of (9, 49)	(11, 46)	P_6^*
(10, 52)	split bipyramid in (10, 49)	(12, 49)	P_7^*
(10, 55)	stack onto square pyramid in (9, 39)	(13, 52)	$R_2(6)^*$

TABLE 1. Some polytopal pairs

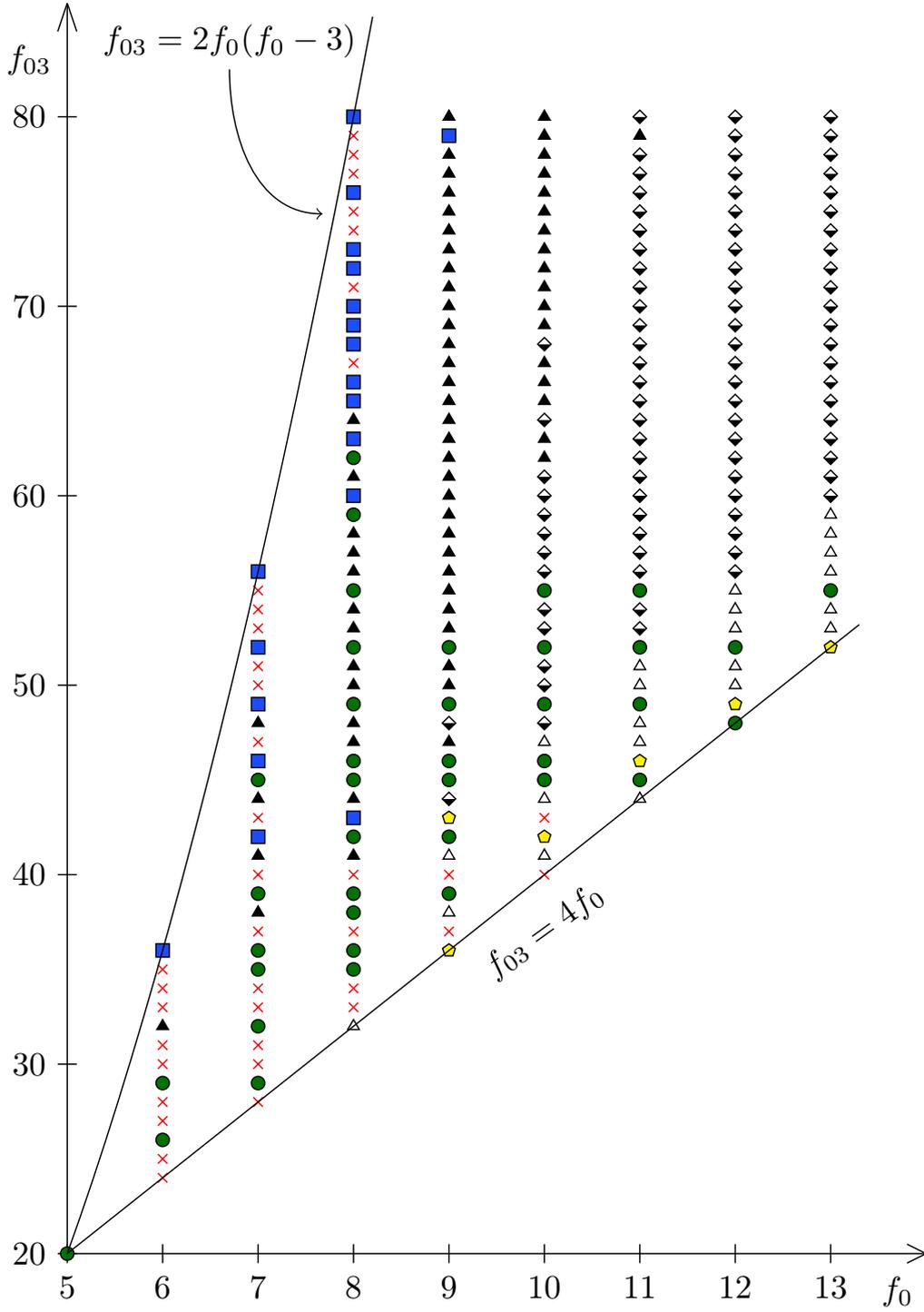
Together with the inductive stacking and truncating methods from Section 2.3, this gives us all possible pairs for $f_{03} \leq 80$ and, in particular, polytopal pairs (f_0, f_{03}) with simple vertex and simplicial facet, for $f_0 \geq 9$, $53 \leq f_{03} \leq 64$ and $4f_0 \leq f_{03}$. See Figure 4. Stacking on simplex facets and truncating simple vertices of these 87 pairs of polytopes inductively will give all polytopal pairs (f_0, f_{03}) bounded by the lower bound $4f_0 \leq f_{03}$, $f_{03} \geq 53$, and a line with slope 12 going through (9, 64). We have hence proved the following.

Lemma 2.10. *There exists a 4-polytope P with $f_0(P) = f_0$ and $f_{03}(P) = f_{03}$ whenever*

$$4f_0 \leq f_{03} \leq 12f_0 - 44 \text{ and } f_{03} \geq 53.$$

In the next step we construct polytopes with $12f_0 - 44 \leq f_{03} \leq 2f_0(f_0 - 3)$. In order to do so, we give examples of polytopes with simplex facet close to the upper bound. The cyclic polytopes have polytopal pairs

$$\begin{aligned} (f_0(C_4(n)), f_{03}(C_4(n))) &= (n, 2n(n-3)), \\ (f_0(C_4(n+1)), f_{03}(C_4(n+1))) &= (n+1, 2n(n-3) + 4n - 4). \end{aligned}$$



- pairs with simplex facet and simple vertex
- pairs with simplex facet
- ⬠ pairs with simple vertex
- ▲ obtained through stacking on simplex facet
- △ obtained through truncating simple vertex
- ◆ obtained through stacking or truncating
- × exceptional pairs

FIGURE 4. Polytopal pairs with $f_{03} \leq 80$

	f_{03}	Example of polytope
$f_{03} \equiv 0 \pmod{4}$	$2n(n-3)$	stack onto Δ_3 -facet of $R_{n-7}(n-1)$
	$2n(n-3) + 4$	stack onto Δ_3 -facet of $R_{n-6}(n-1)$
	$2n(n-3) + 8$	stack onto Δ_3 -facet of $R_{n-5}(n-1)$
	$2n(n-3) + 12$	stack onto Δ_3 -facet of $C_4(n)$
	$2n(n-3) + 16$	$R_2(n)$
	$2n(n-3) + 20$	$R_3(n)$

	$2n(n-3) + 4n - 8$	$R_{n-4}(n)$
	$2n(n-3) + 4n - 4$	$C_4(n+1)$
$f_{03} \equiv 1 \pmod{4}$	$2n(n-3) + 1$	stack onto Δ_3 -facet of $\delta_1^*(n-4, n-1)$
	$2n(n-3) + 5$	stack onto Δ_3 -facet of $\delta_1^*(n-3, n-1)$
	$2n(n-3) + 9$	$\delta_1^*(3, n)$

	$2n(n-3) + 4n - 11$	$\delta_1^*(n-2, n)$
	$2n(n-3) + 4n - 7$	does not exist
$f_{03} \equiv 2 \pmod{4}$	$2n(n-3) + 2$	stack onto Δ_3 -facet of $\delta_2^*(n-3, n-1)$
	$2n(n-3) + 6$	$\delta_2^*(3, n)$

	$2n(n-3) + 4n - 14$	$\delta_2^*(n-2, n)$
	$2n(n-3) + 4n - 10$	does not exist
	$2n(n-3) + 4n - 6$	does not exist
$f_{03} \equiv 3 \pmod{4}$	$2n(n-3) + 3$	$\delta_3^*(3, n)$

	$2n(n-3) + 4n - 21$	$\delta_3^*(n-3, n)$
	$2n(n-3) + 4n - 17$	does not exist
	$2n(n-3) + 4n - 13$	does not exist
	$2n(n-3) + 4n - 9$	does not exist
	$2n(n-3) + 4n - 5$	does not exist

TABLE 2. Polytopal pairs $(n+1, 2n(n-3)+i)$, $n \geq 8$

Our goal is to find polytopes with tetrahedral facets and polytopal pair

$$(n+1, 2n(n-3)+i), \text{ for } i = 0, \dots, 4n-5.$$

If we find such polytopes, combined with the stacking and truncating operations from Section 2.3 this gives us all remaining polytopal pairs. In fact, by Lemma 2.7, there are no polytopes with polytopal pair $(n+1, 2n(n-3)+4n-k)$ for $k \in \{5, 6, 7, 9, 10, 13, 17\}$. In these cases, the “next” polytope in the stacking process, a polytope with polytopal pair $(n+2, 2n(n-3)+4n-k+12)$ for $k \in \{5, 6, 7, 9, 10, 13\}$, is given for $m = n+1$ by the polytope with $(m+1, 2m(m-3)+16-k)$. For $k = 17$, the polytope with polytopal pair $(m+1, 2m(m-3)-1)$ can be obtained through stacking a vertex onto two facets of $\delta_3^*(n-3, n)$ with polytopal pair $(n+1, 2n(n-3)+4n-21) = (m, 2m(m-3)-17)$ (see Section 2.5.1). Stacking a vertex onto $\delta_3^*(n-3, n)$, such that the vertex is beyond two simplex facets which have a common 2-face, yields a new polytope with 16 more vertex-facet incidences and one additional vertex (cf. Section 2.4). So the new polytope has the required polytopal pair $(m+1, 2m(m-3)-1)$.

To find examples of polytopes with $(f_0, f_{03}) = (n+1, 2n(n-3)+i)$, for $f_0 = n+1 \geq 9$, $i = 0, \dots, 4n-4$, $i \neq 4n-j$ for $j \in \{5, 6, 7, 9, 10, 13, 17\}$, we use the constructions from Sections 2.4 and 2.5. Table 2 shows how the polytopes are constructed.

For $f_0 \leq 8$ we use the fact that polytopes with up to 8 vertices have been classified (see Table 1). In particular, we can construct examples of polytopes with simplex facet, simple vertex and polytopal pair

$$(n+2, 2n(n-3)+i), \text{ for all } i = 0, \dots, 4n-5, n \geq 7.$$

If we now inductively stack vertices on simplex facets and truncate simple vertices, we obtain polytopes with polytopal pairs (f_0, f_{03}) with $f_0 \geq 9$ bounded from above by $2f_0(f_0-3)$ and from below by a

line of slope 4, going through (9, 56). So we have found all polytopal pairs with

$$4f_0 + 20 \leq f_{03} \leq 2f_0(f_0 - 3) \quad (2.7)$$

for all $f_0 \geq 9$, with the only exceptions for each value of f_0 being the 7 pairs mentioned above. Lemma 2.10 and Inequality (2.7) together give all pairs (f_0, f_{03}) with $f_0 \geq 9$, $f_{03} \geq 53$ within the bounds, excluding the exceptional pairs. Since we classified all possible polytopal pairs with $f_{03} \leq 80$, and in particular all polytopal pairs with $f_0 \leq 8$, we have now proved Theorem 2.5.

2.7. Other flag vector pairs. The flag vector of a 4-polytope has 16 entries. Besides $f_\emptyset = 1$, the following nine entries depend on only one other entry:

$$\begin{aligned} f_{01} &= 2f_1, & f_{12} &= f_{02}, & f_{13} &= f_{02}, \\ f_{23} &= 2f_2, & f_{012} &= 2f_{02}, & f_{013} &= 2f_{02}, \\ f_{023} &= 2f_{02}, & f_{123} &= 2f_{02}, & f_{0123} &= 4f_{02}. \end{aligned}$$

These equations are some of the Generalized Dehn–Sommerville equations for 4-dimensional polytopes (Lemma 2.8). To obtain all 2-dimensional coordinate projections of the flag vectors of 4-polytopes, we therefore only have to consider the six entries $f_0, f_1, f_2, f_3, f_{02}$ and f_{03} . We still need to determine the projections

$$\Pi_{0,02}(\mathcal{F}^4), \Pi_{1,02}(\mathcal{F}^4), \Pi_{1,03}(\mathcal{F}^4) \text{ and } \Pi_{02,03}(\mathcal{F}^4).$$

All other cases have already been done, or they follow directly, either by duality or by the linear dependence on a single entry.

For the projections $\Pi_{0,02}(\mathcal{F}^4)$ and $\Pi_{1,02}(\mathcal{F}^4)$, the pairs (f_0, f_{02}) in $\Pi_{0,02}(\mathcal{F}^4)$ satisfy the fairly obvious bounds $6f_0 \leq f_{02} \leq 3f_0(f_0 - 3)$. Equality holds for simple and neighborly polytopes, respectively. Similarly the pairs (f_1, f_{02}) in $\Pi_{1,02}(\mathcal{F}^4)$ satisfy $3f_1 \leq f_{02} \leq 6f_1 - 3\sqrt{8f_1 + 1} - 3$, with equality for 2-simple polytopes (each edge is contained in exactly 3 facets) and neighborly polytopes, respectively.

The projection sets $\Pi_{1,03}(\mathcal{F}^4)$ and $\Pi_{02,03}(\mathcal{F}^4)$ are more difficult to describe even approximately. Upper bounds for f_{03} in terms of f_1 are achieved for neighborly polytopes, and in terms of f_{02} for center boolean polytopes. The problem of finding tight lower bounds for f_{03} in terms of f_1 and f_{02} is related to the open problem of finding an upper bound for the *fatness* $F = \frac{f_1 + f_2 - 20}{f_0 + f_3 - 10}$ of a polytope [12].

3. FACE VECTOR PAIR (f_0, f_{d-1}) FOR d -POLYTOPES

Now we work towards analogous results in higher dimensions. In one instance, recently the projection $\Pi_{0,1}(\mathcal{F}^5)$ of the f -vector of 5-polytopes to (f_0, f_1) was determined almost simultaneously by Kusunoki and Murai [18] and by Pineda-Villavicencio, Ugon and Yost [23].

We consider $\Pi_{0,d-1}(\mathcal{F}^d)$, the projection of the set of f -vectors of d -polytopes to (f_0, f_{d-1}) . In the following, for given d , we will consider pairs of integers (n, m) and analyze under which conditions there are d -polytopes with n vertices and m facets.

Definition 3.1. For fixed dimension d , a pair $(n, m) \in \mathbb{N}^n$ is *d-large* if $n + m \geq \binom{3d+1}{\lfloor d/2 \rfloor}$; it is *d-small* otherwise. A pair (n, m) will be called an *exceptional pair* if $m \leq f_{d-1}(C_d(n))$ and $n \leq f_{d-1}(C_d(m))$, and if there is no d -polytope with n vertices and m facets.

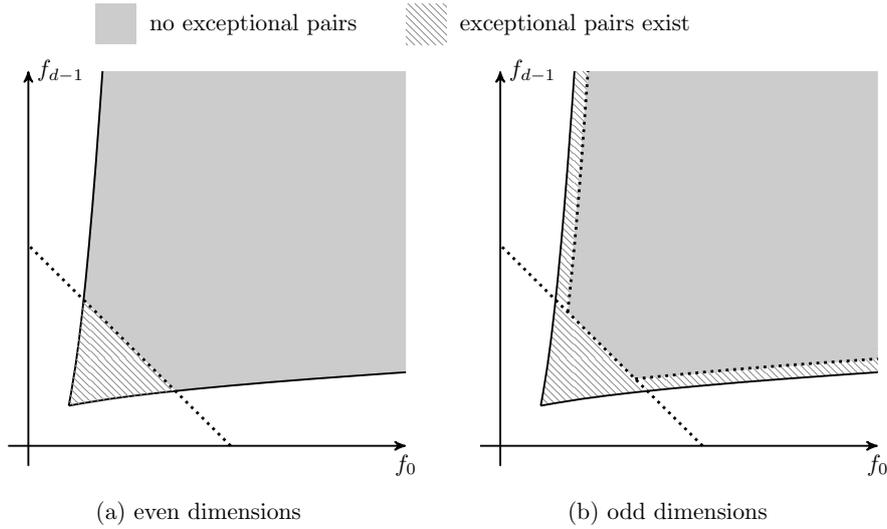
The situation looks as follows:

- (1) If P is a d -polytope with n vertices and m facets, then

$$m \leq f_{d-1}(C_d(n)), \quad n \leq f_{d-1}(C_d(m)).$$

- (2) If (n, m) is a pair of integers, $n, m \geq d + 1$ such that for a given dimension d , $m \leq f_{d-1}(C_d(n))$, $n \leq f_{d-1}(C_d(m))$, then there *usually* exists a d -polytope with n vertices and m facets:

- (2.1) For $d \leq 4$ no exceptional pairs exist.
(2.2) For even $d \geq 6$, only finitely many exceptional pairs exist, all of which are d -small (see Figure 5a).
(2.3) For odd $d \geq 5$, there exist finitely many d -small exceptional pairs and additionally infinitely many d -large exceptional pairs for m odd and $f_{d-1}(C_d(n-1)) < m < f_{d-1}(C_d(n))$ and for n odd and $f_{d-1}(C_d(m-1)) < n < f_{d-1}(C_d(m))$ (see Figure 5b).

FIGURE 5. Projections $\Pi_{0,d-1}(\mathcal{F}^d)$

(1) are the UBT inequalities.

(2.1) holds trivially for $d \leq 2$. For dimension 3, it is given by Steinitz' classification of all 3-dimensional polytopes [25]. For dimension 4, this is Theorem 2.1.

(2.2) is Theorem 3.2 and (2.3) is Theorem 3.3.

Theorem 3.2. *Let $d \geq 2$ be even and (n, m) d -large. Then there exists a d -polytope P with n vertices and m facets if and only if*

$$m \leq f_{d-1}(C_d(n)) \text{ and } n \leq f_{d-1}(C_d(m)).$$

The first inequality holds with equality if and only if P is neighborly, and the second inequality holds with equality if and only if P is dual-neighborly.

However, for $d \geq 6$ d -small exceptional pairs (n, m) exist.

Proof. The necessity of the conditions and the equality cases are direct consequences of the upper bound theorem (McMullen [20]). For the sufficiency, consider the g -vector of simplicial polytopes.

The $\frac{d}{2}$ th entry of the g -vector of a cyclic polytope $C_d(n)$ in even dimension $d = 2k$ with n vertices is

$$g_{d/2}(C_d(n)) = g_k(C_{2k}(n)) = \binom{n-k-2}{k}.$$

A consequence of the sufficiency part of the g -theorem (Billera & Lee [9, 10]) is that there exist simplicial $2k$ -polytopes with n vertices, $g_i = g_i(C_{2k}(n))$ and $g_k = l$ for all $1 \leq i \leq k-1$ and for all $0 \leq l \leq \binom{n-k-2}{k}$.

For all simplicial $2k$ -polytopes,

$$f_{2k-1} = (2k+1) + g_1(2k-1) + g_2(2k-3) + \cdots + g_{k-1} \cdot 3 + g_k.$$

Hence, there exist simplicial $2k$ -polytopes with n vertices and $f_{2k-1}(C_{2k}(n)) - l$ facets, for $0 \leq l \leq \binom{n-k-2}{k}$. Observe that $\binom{n-k-2}{k} > f_{2k-1}(C_{2k}(n)) - f_{2k-1}(C_{2k}(n-1))$ for large n . In particular, this inequality holds for $n \geq 7k+2 = \frac{7}{2}d+2$. This means that for $n \geq \frac{7}{2}d+2$ there are simplicial $2k$ -polytopes with n vertices and m facets, for all integers m such that $f_{2k-1}(C_{2k}(n-1)) \leq m \leq f_{2k-1}(C_{2k}(n))$. Now we can stack a vertex on a facet of each of these polytopes and obtain polytopes with one more vertex and $d-1$ more facets. The new polytope has a simple vertex and simplex facets, so we can repeatedly stack vertices on simplex facets and truncate simple vertices. Truncating simple vertices gives a polytope with $d-1$ more (simple) vertices and one additional (simplex) facet. Consider the pair $(7k+2, m)$: We have just seen that this pair is not an exceptional pair as long as $m \geq f_{2k-1}(C_{2k}(7k+1))$. Stacking a vertex on a facet of a polytope with pair $(7k+2, f_{2k-1}(C_{2k}(7k+1)))$ gives a polytope with simplex facet, simple vertex and pair $(n_0, m_0) := (7k+3, f_{2k-1}(C_{2k}(7k+1)) + d-1)$.

Consider the line ℓ_1 of slope $\frac{1}{d-1}$ through (n_0, m_0) . There are no exceptional pairs with $n \geq n_0$ above ℓ_1 . The line ℓ_1 intersects the line $\ell_2 : m = n$ in a pair (n, n) such that

$$n = \frac{k+1}{12k+2} \binom{6k+1}{k} < \frac{1}{2} \binom{6k+1}{k}.$$

Together with the dual polytope, we have obtained all polytopes with pairs (n, m) within the bounds such that

$$\binom{3d+1}{\frac{d}{2}} \leq n + m.$$

Hence, there are no d -large exceptional pairs.

On the other hand, there are exceptional pairs for d -small (n, m) . As an example consider d -polytopes with $d+2$ vertices. All d -polytopes with $d+2$ vertices are simplicial or (multiple) pyramids over some r -polytope with $r+2$ vertices [26, Sect. 6.5].

There are exactly $\lfloor \frac{d}{2} \rfloor = k$ different combinatorial types of simplicial d -polytopes with $d+2$ vertices ([15, Sect. 6.1]). One of these types is the stacked polytope with $2d$ facets. In particular, for $d \geq 6$, $2d \leq k^2 + k + 1$. Any non-simplicial d -polytope with $d+2$ vertices is a pyramid and has thus at most $f_{d-1}(\text{Pyr}(C_{d-1}(d+1))) = f_{d-2}(C_{d-1}(d+1)) + 1 = k^2 + k + 1$ facets for $d = 2k$. This means that there are at most $k-1$ different combinatorial types of $(2k)$ -polytopes with $2k+2$ vertices and more than $k^2 + k + 1$ facets.

The cyclic polytope with $d+2$ vertices has $k^2 + 2k + 1$ facets for even dimensions $d = 2k$. So there are k pairs (n, m) for given n and $k^2 + k + 1 < m \leq f_{d-1}(C_d(n))$, but at most $k-1$ combinatorially non-equivalent polytopes. Therefore, for $n = d+2$ and even $d \geq 6$ there must be at least one exceptional pair. \square

An example is the pair $(n, m) = (8, 14)$ for dimension 6: There is no 6-polytope with 8 vertices and 14 facets [14], but there are 6-polytopes with 8 vertices and 13 or 15 facets.

Theorem 3.3. *Let $d \geq 3$ be odd. If (n, m) is d -large, then there exist d -polytopes with n vertices and m facets if and only if*

$$m \leq f_{d-1}(C_d(n)) \text{ and } n \leq f_{d-1}(C_d(m))$$

with d -large exceptional pairs occurring only for $d \geq 5$, if m is odd and $f_{d-1}(C_d(n-1)) \leq m$ and if n is odd and $f_{d-1}(C_d(m-1)) \leq n$.

However, for $d \geq 5$ d -small exceptional pairs (n, m) exist.

Proof. The necessity follows again from the upper bound theorem (McMullen [20]). For the sufficiency, we follow the proof of Theorem 3.2.

A cyclic polytope $C_d(n)$ in odd dimension $d = 2k + 1$ with n vertices has $g_{\lfloor \frac{d}{2} \rfloor}$ equal to

$$g_{\lfloor \frac{d}{2} \rfloor}(C_d(n)) = g_k(C_{2k+1}(n)) = \binom{n-k-3}{k}.$$

Again, by the g -theorem [9, 10, 24], there exist simplicial $(2k+1)$ -polytopes with n vertices, $g_i = g_i(C_{2k}(n))$ and $g_k = l$ for all $1 \leq i \leq k-1$ and for all $0 \leq l \leq \binom{n-k-3}{k}$.

For all simplicial $(2k+1)$ -polytopes,

$$f_{d-1} = (d+1) + g_1(d-1) + g_2(d-3) + \cdots + g_{k-1} \cdot 4 + g_k \cdot 2.$$

Hence, there exist simplicial $(2k+1)$ -polytopes with n vertices and $f_{2k}(C_{2k+1}(n)) - 2l$ facets, for $0 \leq l \leq \binom{n-k-3}{k}$.

We have that $2 \binom{n-k-3}{k} > f_{2k}(C_{2k+1}(n)) - f_{2k}(C_{2k+1}(n-1))$ holds for large n , in particular for $d = 5$ if $n \geq 9$ and for general d if $n \geq 5k + 1 = \frac{5}{2}d - \frac{3}{2}$. With the same calculations as before, we obtain polytopes with n vertices and m facets for all pairs (n, m) if n and m are even and if

$$n + m \geq \frac{2}{2k-1} (4k \binom{4k-1}{k} + 4k^2 - 5k - 2).$$

For $d \geq 7$, this implies that

$$n + m \geq \binom{6k+4}{k} = \binom{3d+1}{\lfloor \frac{d}{2} \rfloor}.$$

For $d = 5$, we check that the constructions give us all polytopes with $n + m \geq 58$, where $\binom{3 \cdot 5 + 1}{2} > 58$. We can also construct polytopes with an odd number of facets, as long as $m \leq f_{d-1}(C_d(n-1))$. For

this, we need a generalized stacking construction similar to the one described in Section 2.4. Starting with a simplicial polytope, we place a new vertex beyond one facet, inside the affine hull of a second facet and beneath all other facets. The new polytope has one new (simple) vertex and $d - 2$ new facets. The polytope has one facet which is a bipyramid over a triangle. All other facets are simplices, so we can apply the inductive stacking and truncating method from before.

There are exceptional pairs (n, m) if m is odd and close to $f_{d-1}(C_d(n))$: non-simplicial d -polytopes with n vertices have at most $f_{d-1}(C_d(n)) - \lfloor \frac{d}{2} \rfloor$ facets. This is a direct consequence of the upper bound theorem for *almost simplicial polytopes* by Nevo, Pineda-Villavicencio, Ugon & Yost [22]. These authors give upper bounds for the number of faces of the family $\mathcal{P}(d, n, s)$ of *almost simplicial polytopes*, d -polytopes on n vertices where one facet has $d + s \geq d + 1$ vertices and all other facets are simplices. Such polytopes have at most $f_{d-1}(C_d(n)) - \lfloor \frac{d}{2} \rfloor$ facets. (This follows from [22], Thm. 1.2 and Prop. 4.2.)

For any non-simplicial polytope P on n vertices there exists an almost-simplicial polytope on n vertices (i.e. a polytope with exactly one non-simplicial facet) that has at least as many i -faces as P : Let F be a non-simplicial facet of P . If we successively pull every vertex of $\text{vert } P \setminus \text{vert } F$ (in the sense of [11]) and then pull every vertex $v \in \text{vert } F$ *within* the affine hull of F , then the resulting polytope is almost simplicial, with at least as many i -dimensional faces as P . So the i -faces of non-simplicial d -polytopes on n vertices are maximized among the almost simplicial d -polytopes on n vertices. In particular, for any non-simplicial d -polytope P , $f_{d-1}(P) \leq f_{d-1}(C_d(n)) - \lfloor \frac{d}{2} \rfloor$. Thus, for odd d , odd m , and

$$f_{d-1}(C_d(n)) - \lfloor \frac{d}{2} \rfloor < m < f_{d-1}(C_d(n)),$$

(n, m) is an exceptional pair.

The rest of the theorem for d -large (n, m) follows by duality. For d -small (n, m) , the non-constructive proof for the existence of exceptional pairs in the even-dimensional case works as well in the odd-dimensional case. It can be slightly improved: All d -polytopes with $d + 2$ vertices are simplicial or (multiple) pyramids over some r -polytope with $r + 2$ vertices [26, Sect. 6.5]. In particular, for $d = 2k + 1$ and odd m , any polytope P with $d + 2$ vertices and m facets is a pyramid over some $(d - 1)$ -polytope Q with $d + 1$ vertices and $m - 1$ facets. Hence,

$$m - 1 = f_{d-2}(Q) \leq f_{d-2}(C_{d-1}(d + 1)) = k^2 + 2k + 1.$$

Comparing this to

$$f_{d-1}(C_d(d + 2)) = k^2 + 3k + 2,$$

we see that there are $\lfloor \frac{k}{2} \rfloor$ exceptional pairs (n, m) for which there are no $(2k + 1)$ -polytopes, such that m is odd and

$$k^2 + 2k + 2 < m < k^2 + 3k + 2.$$

□

□

Remark. This implies that for odd d the projection sets $\Pi_{0,d-1}(\mathcal{F}^d)$ have infinitely many exceptional pairs, all of them near the boundary. For a complete characterization of d -large pairs in $\Pi_{0,d-1}(\mathcal{F}^d)$ one would need to analyze closely the possible facet numbers of non-simplicial polytopes with many facets.

For low dimensions, we can improve the bounds for the d -large pairs. We have seen that in dimension 5, a pair can be called d -large if $n + m \geq 58$. Similarly, for dimension 6, the bound for d -large pairs can be reduced to $n + m \geq 132$: It can be seen from the g -theorem that simplicial 6-polytopes with n vertices have $5n - 28$, $5n - 25$, $5n - 24$, or $5n - 22$ to $f_5(C_6(n))$ facets. For $n \geq 11$, it holds that $5n - 22 < f_5(C_6(n - 1))$. From this, the bound $n + m \geq 132$ for d -large pairs can be derived.

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APPENDIX

Table 3 lists all polytopes P_i with 7 and 8 vertices from Table 1 used in the construction of all possible pairs (f_0, f_{03}) . The polytopes are given by their facet list. See Fukuda, Miyata & Moriyama [14] for a complete list of all 31 polytopes with 7 vertices and all 1294 polytopes with 8 vertices. Entry $7.x$ in the last column means that the polytope can be found as the x th polytope listed in the classification of 4-polytopes with 7 vertices.

polytope	facet list	row
P_1	[654321][65430][6520][6420][5310][5210][4310][4210]	7.3
P_2	[65432][65431][65210][64210][5320][5310][4320][4310]	7.21
P_3	[65432][65431][65210][6421][5320][5310][4320][4310][4210]	7.22
P_4	[65432][65410][6531][6431][5420][5321][5210][4320][4310][3210]	7.11
P_5	[65432][6541][6531][6431][5421][5320][5310][5210][4320][4310][4210]	7.16
P_6	[65432][65431][6521][6420][6410][6210][5320][5310][5210][4320][4310]	7.24
P_7	[65432][6541][6531][6430][6410][6310][5421][5320][5310][5210][4320][4210]	7.13
P_8	[765432][765410][76321][75310][64210][5430][4320][3210]	8.186
P_9	[765432][76541][76310][75310][64210][6320][5420][5410][5320]	8.285
P_{10}	[76543][76542][76321][75310][75210][64310][64210][5430][5420]	8.1145
P_{11}	[765432][76541][76310][54310][7531][6421][6320][6210][4320][4210]	8.241
P_{12}	[765432][76541][76320][75310][54310][7610][6421][6210][4320][4210]	8.353
P_{13}	[765432][76541][73210][63210][7631][7520][7510][6420][6410][5420][5410]	8.201
P_{14}	[765432][76541][76310][7531][6430][6410][5420][5410][5321][5210][4320][3210]	8.306
P_{15}	[765432][76510][7641][7541][6530][6421][6321][6310][5420][5410][5320][4210][3210]	8.117
P_{16}	[76543][76521][76420][7542][6531][6431][6410][6210][5432][5320][5310][5210][4320][4310]	8.676
P_{17}	[76543][76542][73210][63210][7632][7531][7520][7510][6431][6420][6410][5431][5420][5410]	8.909
P_{18}	[76543][76521][7642][7542][6530][6510][6432][6320][6210][5430][5421][5410][4321][4310][3210]	8.778
P_{19}	[76543][76542][73210][7631][7621][7530][7520][6431][6420][6410][6210][5431][5420][5410][5310]	8.910
P_{20}	[76543][7652][7642][7531][7521][7431][7421][6530][6521][6510][6430][6420][6210][5310][4310][4210]	8.805
P_{21}	[76543][76542][7632][7531][7521][7320][7310][7210][6431][6420][6410][6320][6310][5431][5421][4210]	8.1227
P_{22}	[7654][7653][7643][7542][7532][7431][7421][7321][6540][6530][6431][6410][6310][5420][5320][4210][3210]	8.1262
P_{23}	[76543][7652][7642][7531][7521][7431][7421][6530][6521][6510][6430][6420][6210][5310][4321][4320][3210]	8.806
P_{24}	[76543][76542][7631][7621][7531][7520][7510][7210][6430][6420][6321][6320][5431][5420][5410][4310][3210]	8.1041
P_{25}	[7654][7653][7643][7542][7532][7431][7421][7321][6542][6530][6520][6430][6420][5321][5310][5210][4310][4210]	8.1263
P_{26}	[76543][7652][7642][7541][7521][7420][7410][7210][6530][6521][6510][6432][6320][6210][5431][5310][4320][4310]	8.815
P_{27}	[7654][7653][7643][7542][7532][7431][7421][7321][6542][6530][6520][6431][6420][6410][6310][5321][5310][5210][4210]	8.1266

TABLE 3. Polytopes P_i with 7 and 8 vertices

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