# TILINGS OF CONVEX POLYHEDRAL CONES AND TOPOLOGICAL PROPERTIES OF SELF-AFFINE TILES 

YA-MIN YANG AND YUAN ZHANG $\dagger$


#### Abstract

Let $\boldsymbol{a}_{1}, \ldots, \boldsymbol{a}_{r}$ be vectors in a half-space of $\mathbb{R}^{n}$. We call $$
C=\boldsymbol{a}_{1} \mathbb{R}^{+}+\cdots+\boldsymbol{a}_{r} \mathbb{R}^{+}
$$ a convex polyhedral cone, and call $\left\{\boldsymbol{a}_{1}, \ldots, \boldsymbol{a}_{r}\right\}$ a generator set of $C$. A generator set with the minimal cardinality is called a frame. We investigate the translation tilings of convex polyhedral cones.

Let $T \subset \mathbb{R}^{n}$ be a compact set such that $T$ is the closure of its interior, and $\mathcal{J} \subset \mathbb{R}^{n}$ be a discrete set. We say $(T, \mathcal{J})$ is a translation tiling of $C$ if $T+\mathcal{J}=C$ and any two translations of $T$ in $T+\mathcal{J}$ are disjoint in Lebesgue measure.

We show that if the cardinality of a frame of $C$ is larger than $\operatorname{dim} C$, the dimension of $C$, then $C$ does not admit any translation tiling; if the cardinality of a frame of $C$ equals $\operatorname{dim} C$, then the translation tilings of $C$ can be reduced to the translation tilings of $\left(\mathbb{Z}^{+}\right)^{n}$. As an application, we characterize all the self-affine tiles possessing polyhedral corners, which generalizes a result of Odlyzko [A. M. Odlyzko, Non-negative digit sets in positional number systems, Proc. London Math. Soc., 37(1978), 213-229.].


## 1. Introduction

Let $\boldsymbol{a}_{1}, \boldsymbol{a}_{2}, \cdots, \boldsymbol{a}_{m}$ be $m$ non-zero vectors in a half space of $\mathbb{R}^{n}$, that is, there is a nonzero vecter $\beta \in \mathbb{R}^{n}$ such that the inner product $\left\langle\boldsymbol{a}_{j}, \beta\right\rangle>0$ for all $j=1, \ldots, m$. We call the set of all non-negative combinations of these vectors

$$
C=\boldsymbol{a}_{1} \mathbb{R}^{+}+\cdots+\boldsymbol{a}_{m} \mathbb{R}^{+}=\left\{\lambda_{1} \boldsymbol{a}_{1}+\cdots+\lambda_{m} \boldsymbol{a}_{m}: \text { all } \lambda_{i} \geq 0\right\}
$$

a convex polyhedral cone. In this case, we also say $C$ is spanned by $\left\{\boldsymbol{a}_{1}, \ldots, \boldsymbol{a}_{m}\right\}$.

Date: May 18, 2020.
$\dagger$ The correspondence author.
This work is supported by NSFC Nos. 11431007, 11601172, and Fundamental Research Funds for Central Universities no.2662015PY217, and Self-Determined Research Funds of CCNU from the Colleges' Basic Research and Operation of MOE under Grant CCNU17XJ034.

2000 Mathematics Subject Classification: 52C22, 51M20
Key words and phrases: convex polyhedral cone, translation tiling, self-affine tile.

The convex polyhedral cone is an important object in convex analysis, see for instance, Rockafellar [18]. The main purpose of the present paper is to characterize the translation tilings of convex polyhedral cones.

Definition 1.1. Let $X \subset \mathbb{R}^{n}, T \subset \mathbb{R}^{n}$ be a compact set, and $\mathcal{J} \subset \mathbb{R}^{n}$ be a (finite or infinite) discrete set.

We say that $(T, \mathcal{J})$ is a packing of $X$ if $T+\mathcal{J} \subset X$, and $T+t_{1}$ and $T+t_{2}$ are disjoint in Lebesgue measure for any $t_{1} \neq t_{2} \in \mathcal{J}$.
$(T, \mathcal{J})$ is called a covering of $X$ if $X \subset T+\mathcal{J}$.
$(T, \mathcal{J})$ is called a translation tiling of $X$ if it is a packing as well as a covering of $X$. In this case, we call $T$ a $X$-tile and $(T, \mathcal{J})$ a $X$-tiling. (In literature, usually it is assumed in addition that $T$ is the closure of the interior of $T$.)
$(T, \mathcal{J})$ is called a local tiling of convex polyhedral cone $C$, if it is a packing of $C$ and it covers a neighborhood of $\mathbf{0}$ in $C$.

Remark 1.1. Let $(T, \mathcal{J})$ be a local tiling of a convex polyhedral cone $C$. Let $T+t_{1}$ be the tile containing $\mathbf{0}$. Set $T^{\prime}=T+t_{1}$ and $\mathcal{J}^{\prime}=\mathcal{J}-t_{1}$, then $T^{\prime}+\mathcal{J}^{\prime}$ is a local tiling of $C$. It follows that $\mathbf{0} \in T^{\prime}, \mathbf{0} \in \mathcal{J}^{\prime}$ and consequently $T^{\prime} \subset C, \mathcal{J}^{\prime} \subset C$. Therefore, from now on, without loss of generality, we always assume that

$$
\begin{equation*}
\mathbf{0} \in T, \mathbf{0} \in \mathcal{J}, T \subset C, \text { and } \mathcal{J} \subset C \tag{1.1}
\end{equation*}
$$

1.1. Translation tilings of convex polyhedral cones. Let $C$ be a convex polyhedral cone. The dimension of $C$, denoted by $\operatorname{dim} C$, is the minimum of the dimensions of subspaces of $\mathbb{R}^{n}$ containing $C$. We call $A=\left\{\boldsymbol{a}_{1}, \ldots, \boldsymbol{a}_{m}\right\}$ a frame of $C$, if $A$ spans $C$, and any proper subset of $A$ does not. It is seen that the frame $A$ of a convex polyhedral cone $C$ is unique if we require all members of $A$ to be unit vectors.

Definition 1.2. We say $C$ is regular, if the cardinality of a frame of $C$ equals $\operatorname{dim} C$, and irregular otherwise.

Denote $\mathbb{R}^{+}=\{x \in \mathbb{R} ; x \geq 0\}$ and $\mathbb{Z}^{+}=\{x \in \mathbb{Z} ; x \geq 0\}$. Clearly, an $n$-dimensional convex polyhedral cone $C$ is regular if and only if $C$ is the image of $\left(\mathbb{R}^{+}\right)^{n}$ under an invertible linear transformation.

We show that if $T$ can tile a 'large' ball of $C$ at the origin, then not only $C$ must be regular, but also $T$ must be a union of translations of unit cubes up to a linear transformation. Denote by $B_{n}(x, r)$, or simply $B(x, r)$, the ball in $\mathbb{R}^{n}$ with center $x$ and with radius $r$.

Theorem 1.1. Let $C$ be a convex polyhedral cone. If $(T, \mathcal{J})$ is a local tiling of $C$ which covers $C \cap B(\mathbf{0}, R)$ for some $R>\operatorname{diam}(T)$, then
(i) $C$ is regular.
(ii) if in addition $T=\overline{T^{\circ}}$, then there exist a finite set $E \subset\left(\mathbb{Z}^{+}\right)^{n}$, and a linear transformation $\varphi$ of $\mathbb{R}^{n}$ such that $T=\varphi\left(E+[0,1]^{n}\right)$.

Sometimes we call $(T, \mathcal{J})$ in the above theorem a 'large' local tiling. As a consequence of Theorem 1.1, we have

Corollary 1.1. An irregular convex polyhedral cone admits no translation tiling.
Corollary 1.2. If $(T, \mathcal{J})$ is a tiling of $\left(\mathbb{R}^{+}\right)^{n}$ and $T=\overline{T^{\circ}}$, then there exists $E, \mathcal{J}^{\prime} \subset\left(\mathbb{Z}^{+}\right)^{n}$, and a positive diagonal matrix $U=\operatorname{diag}\left(u_{1}, \ldots, u_{n}\right)$ such that

$$
U T=E+[0,1]^{n}, \quad U \mathcal{J}=\mathcal{J}^{\prime}
$$

and $\left(E, \mathcal{J}^{\prime}\right)$ is a translation tiling of $\left(\mathbb{Z}^{+}\right)^{n}$.
Therefore, to characterize the translation tilings of regular convex polyhedral cones, we need only characterize the translation tilings of the special cone $\left(\mathbb{R}^{+}\right)^{n}$, and this can be further reduced to the problem of characterization of $\left(\mathbb{Z}^{+}\right)^{n}$-tilings.

We call $A+B$ the direct sum of $A$ and $B$, and denoted by $A \oplus B$, if every element $x \in A+B$ has a unique decomposition as $x=a+b$ with $a \in A, b \in B$. For $A, B \subset\left(\mathbb{Z}^{+}\right)^{n}$, we say $(A, B)$ is a $\left(\mathbb{Z}^{+}\right)^{n}$-complementing pair if $A \oplus B=\left(\mathbb{Z}^{+}\right)^{n}$, furthermore, we say $(A, B)$ is a $\left(\mathbb{Z}^{+}\right)^{n}$-tiling if $\# A<\infty$. (We remark that it is possible that both $A, B$ are infinite sets.)

Remark 1.2. Rao, Yang and Zhang [19] characterizes all the $\left(\mathbb{Z}^{+}\right)^{n}$-complementing pairs, which generalizes the results of de Bruijn [3] and Niven [16], which settled the case $n=1$ and $n=2$, respectively.
1.2. Self-affine tiles possessing polyhedral corners. Let $\mathbf{A} \in M_{n}(\mathbb{R})$ be an expanding matrix (i.e., all its eigenvalues have moduli larger than 1) such that $m=|\operatorname{det}(\mathbf{A})|$ is an integer larger than 1 . Let $\mathcal{D}=\left\{\mathbf{d}_{0}, \mathbf{d}_{1}, \cdots, \mathbf{d}_{m-1}\right\}$ be a subset of $\mathbb{R}^{n}$, which we call the digit set. It is well known ( $[7,[13])$ that there exists a unique non-empty compact set $T:=T(\mathbf{A}, \mathcal{D})$ satisfying the set equation

$$
\begin{equation*}
T=\bigcup_{\boldsymbol{d} \in \mathcal{D}} \mathbf{A}^{-1}(T+\boldsymbol{d}) \tag{1.2}
\end{equation*}
$$

We call $T(\mathbf{A}, \mathcal{D})$ a self-affine tile and $\mathcal{D}$ a tile digit set, if $T(\mathbf{A}, \mathcal{D})$ has non-void interior. A self-affine tile can tile $\mathbb{R}^{n}$ by translation ([13]). Self-affine tiles have been studied extensively in literature ( $[1,2,8,5,13,14,15,10,11,12,20]$ ), since it is related to many fields of mathematics, such as number theory, dynamical system, spectral theory and wavelet, etc. As an application of Theorem 1.1 and 1.2, we study the topological properties of $T(\mathbf{A}, \mathcal{D})$.

Definition 1.3. Let $T(\mathbf{A}, \mathcal{D})$ be a self-affine tile of $\mathbb{R}^{n}$. We say $T(\mathbf{A}, \mathcal{D})$ has a polyhedral corner, if there exists a point $x_{0} \in T(\mathbf{A}, \mathcal{D})$, a real $r>0$, and a convex polyhedral cone $C$, such that

$$
B\left(x_{0}, r\right) \cap T(\mathbf{A}, \mathcal{D})=x_{0}+B(\mathbf{0}, r) \cap C .
$$

The following result generalizes a one-dimensional result of Odlyzko [17.
Theorem 1.2. If a self-affine tile $T(\mathbf{A}, \mathcal{D})$ of $\mathbb{R}^{n}$ has a polyhedral corner, then there exists an affine transformation $\varphi$ such that $\varphi(T(\mathbf{A}, \mathcal{D}))$ is a $\left(\mathbb{R}^{+}\right)^{n}$-tile. Consequently, $T(\mathbf{A}, \mathcal{D})$ is a finite union of translations of $n$-dimensional unit cubes up to an affine transformation.

We close this section with some notations. We use $\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{n}\right\}$ to denote the canonical basis of $\mathbb{R}^{n}$. Let $\partial A$ denote the boundary of $A, A^{\circ}$ denote the interior of $A$, and $\bar{G}$ denote the closure of $G$.

The paper is organized as follows. In Sections 2-4, we show that an irregular convex polyhedral cone $C$ has 2-dimensional slices which are corner-cut regions. In Sections 5-6, we show that if $C$ has a 'large' local tiling, then a 2-dimensional corner-cut slice of $C$ also has a 'large' local tiling; however, we show in Section 7 that this is impossible. Theorem 1.1(i) is proved in Section 6. Section 8 is devoted to the translation tilings of $\left(\mathbb{R}^{+}\right)^{n}$; Theorem 1.1(ii) and Corollary 1.2 are proved there. Section 9, the last section, studies the topological properties of self-affine tiles and Theorem 1.2 is proved there.

## 2. Preliminaries on convex polyhedral cones

First, we recall some notions about convex set, see [18, 4]. Let $F$ be a convex subset of the convex set $C$. We say $F$ is a face of $C$, if any closed line segment in $C$ with a relative interior in $F$ has both endpoints in $F$. An extreme point of a convex set is one which is not a proper convex linear combination of any two points of the set.

Let $\boldsymbol{a}_{1}, \boldsymbol{a}_{2}, \cdots, \boldsymbol{a}_{m}$ be non-zero vectors in $\mathbb{R}^{n}$ located in a half space. Recall that

$$
C=\boldsymbol{a}_{1} \mathbb{R}^{+}+\cdots+\boldsymbol{a}_{m} \mathbb{R}^{+}
$$

is called a convex polyhedral cone. Clearly $C$ is a closed set.

For a set $F \subset \mathbb{R}^{n}$, we use span $(F)$ to denote the smallest subspace containing $F$. Then the dimension of $F$, denoted by $\operatorname{dim} F$, is the dimension of the subspace $\operatorname{span}(F)$. Moreover, we call $F$ a $r$-face of $C$, if $F$ is a face of $C$ with dimension $r$.

A convex polyhedral cone $C$ has exactly one extreme point, or 0 -face, the origin. 1faces of $C$ are the half-lines $\boldsymbol{a}_{j} \mathbb{R}^{+}$with $\boldsymbol{a}_{j}$ in the frame of $C$, and we call $\boldsymbol{a}_{j} \mathbb{R}^{+}$an extreme direction.

We list some facts about faces of convex polyhedral cones.
Lemma 2.1. Let $C$ be a convex polyhedral cone with dimension n. Then
(i) (Theorem 21 in [4.) A convex cone in $\partial C$ is contained in an $(n-1)$-face $Q$. Consequently, $\partial C$ is the union of the $(n-1)$-faces of $C$.
(ii) (Theorem 22 in [4]) Let $Q$ be an $(n-1)$-face of $C$, then

$$
Q=C \cap \operatorname{span}(Q)
$$

(iii) (Theorem 27 in [4].) If $G$ and $F$ are faces of $C$ and $F \subset G$, then $F$ is a face of $G$. (iv) If $G$ is a face of $C$, then any face of $G$ is also a face of $C$.

We shall use the following easy facts.
Lemma 2.2. Let $C \subset \mathbb{R}^{n}$ be a convex polyhedral cone. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^{n}$.
(i) If a ray $R=\boldsymbol{a}+\mathbf{b} \mathbb{R}^{+}$belongs to $C$, then $\mathbf{b} \in C$.
(ii) If $\boldsymbol{a} \mathbb{R}^{+}$is an extreme direction of $C$, and $\mathbf{b}$ is a vector in $C$ but not in $\boldsymbol{a} \mathbb{R}^{+}$, then $\boldsymbol{a}-\mathbf{b} \notin C$.

Proof. (i) The sequence $(\boldsymbol{a}+k \mathbf{b})_{k \geq 1}$ belongs to $C$ implies that $\boldsymbol{a} / k+\mathbf{b} \in C$, hence the limit $\mathbf{b}$ belongs to $C$. (ii) follows from the fact that $\boldsymbol{a} \mathbb{R}^{+}$is a 1 -face.

Recall that a convex polyhedral cone $C$ is regular if the cardinality of a frame of $C$ equals $\operatorname{dim} C$, and irregular otherwise.

Lemma 2.3. Let $C$ be a regular convex polyhedral cone and let $A$ be its frame. Then the convex polyhedral cone spanned by a subset of $A$ is a face of $C$.

Proof. This follows from the definition of face.

## 3. Slices of irregular convex polyhedral cones

In this section, we investigate the intersection of a convex polyhedral cone $C$ and a 2-dimensional hyperplane $H$, where $H$ parallel to a 2-face of $C$.


Figure 1. The left side is a corner-cut slice, while the right side is not.
Definition 3.1. Let $C$ be a convex polyhedral cone and $F$ be a 2-face of $C$. We call $F$ a feasible 2-face of $C$, if there exists a point $x_{0} \in C^{\circ}$ such that the intersection

$$
\begin{equation*}
\left(\operatorname{span}(F)+x_{0}\right) \cap C \tag{3.1}
\end{equation*}
$$

is a convex set with at least two extreme points; in this case, we call the set in (3.1) a corner-cut slice of $C$.

The following lemma is obvious, see Figure 1 .
Lemma 3.1. Let $X$ be the set in (3.1), then
(i) $X$ is a corner-cut slice if and only if there exists a line $L$ such that $X \cap L$ is a line segment and $X^{\circ} \cap L=\emptyset$.
(ii) $X$ is a corner-cut slice if and only if there exists a ray $L$ emanating from an extreme point of $X$ such that $X \cap L$ is a line segment.
3.1. Lemmas. We start with several lemmas.

Lemma 3.2. Let $C$ be a convex polyhedral cone, and $H$ be a 2-dimensional hyperplane intersecting the interior of $C$. Then

$$
\partial(H \cap C)=\bigcup_{Q}(Q \cap H)
$$

where $Q$ runs over all the $(n-1)$-faces of $C$.
Proof. Recall that $\partial C$ is the union of all $(n-1)$-faces, so we need only show that

$$
\begin{equation*}
\partial(H \cap C)=H \cap \partial C \tag{3.2}
\end{equation*}
$$

Let $x \in H \cap C$. If $x \in \partial C$, then (at least) a half open ball of $B_{n}(x, r)$ does not belong to $C$, and hence a half open ball of $B_{2}(x, r)$ does not belong to $H \cap C$, so $x \in \partial(H \cap C)$. If $x \in C^{\circ}$, then clearly $x \in(H \cap C)^{\circ}$. Hence (3.2) holds and the lemma follows.

We use $[\boldsymbol{a}, \mathbf{b}]^{+}$to denote the convex polyhedral cone spanned by $\{\boldsymbol{a}, \mathbf{b}\}$.
Lemma 3.3. Let $C$ be a convex polyhedral cone, $Q$ be an $(n-1)$-face of $C,[\boldsymbol{a}, \mathbf{b}]^{+}$be a 2 -face of $C$, and $y \in C^{\circ}$. Denote $\Sigma=\operatorname{span}(\{\boldsymbol{a}, \mathbf{b}\})$. Then
(i) If $\{\boldsymbol{a}, \mathbf{b}\} \subset Q$, then $Q \cap(\Sigma+y)=\emptyset$.
(ii) If $\{\boldsymbol{a}, \mathbf{b}\} \cap Q=\{\boldsymbol{a}\}$ or $\{\mathbf{b}\}$, then $Q \cap(\Sigma+y)$ is either $\emptyset$, or a ray.
(iii) If $\{\mathbf{a}, \mathbf{b}\} \cap Q=\emptyset$, then $Q \cap(\Sigma+y)$ is $\emptyset$, or a singleton, or a line segment.
(iv) $X_{y}:=C \cap(\Sigma+y)$ is a corner-cut slice if and only if there exists an $(n-1)$-face $Q$ of $C$ such that $Q \cap(\Sigma+y)$ is a line segment.

Proof. First, by linear algebra, $Q \cap(\Sigma+y)$ is a (connected) subset of a line since $y \in C^{\circ}$.
(i) The first assertion holds since $(\Sigma+y) \cap \operatorname{span}(Q)=\emptyset$.
(ii) Suppose $\boldsymbol{a} \in Q$ and $x \in Q \cap(\Sigma+y)$. Then clearly $x+\mathbb{R}^{+} \boldsymbol{a}$ also belongs to this intersection. The second assertion is proved.
(iii) To prove the third assertion, we need only show that

$$
I=Q \cap(\Sigma+y)
$$

is not a ray. Let $L^{\prime}$ be the intersection of $\Sigma$ and $\operatorname{span}(Q)$, then $L^{\prime}$ is a subspace of dimension 1 or 2 . Since $\boldsymbol{a}, \mathbf{b} \notin Q$, we have $\operatorname{dim} L^{\prime}=1$. Moveover, we have $L^{\prime}=\mathbb{R}(\boldsymbol{a}-c \mathbf{b})$ for some $c \in \mathbb{R} \backslash\{0\}$.

We claim that $c>0$, or in other words, $L^{\prime}$ locates outside of the cone $C$ (except the origin). Suppose on the contrary $c<0$, then

$$
\boldsymbol{a}-c \mathbf{b} \in C \cap \operatorname{span}(Q)=Q,
$$

where the last equality is due to Lemma 2.1(ii). It follows that $\boldsymbol{a},-c \mathbf{b} \in Q$ since $Q$ is a face, a contradiction. Our claim is proved.

Assume on the contrary that $I$ is a ray, and let $L$ be the line containing $I$, then the direction of $L$ is $\pm(\boldsymbol{a}-c \mathbf{b})$. By Lemma 2.2(ii), $\pm(\boldsymbol{a}-c \mathbf{b}) \notin C$; furthermore, by Lemma 2.2(i), the intersection $L \cap C$ cannot be a ray. It follows that the interval $I$, as a subset of $L \cap C$ is not a ray. This contradiction proves (iii).
(iv) Suppose $X_{y}$ is a corner-cut slice, then there is a line $L$ such that $X_{y} \cap L$ is a line segment and $X_{y}^{\circ} \cap L=\emptyset$ (Lemma 3.1). Clearly,

$$
\left(X_{y} \cap L\right) \subset \bigcup_{Q} Q \cap(\Sigma+y) .
$$

Let $Q$ be an $(n-1)$-face such that $Q \cap(\Sigma+y)$ contains a sub-interval of $X_{y} \cap L$. By Lemma 3.3, $Q \cap(\Sigma+y)$ is a subset of a line, hence $Q \cap(\Sigma+y)$ is a subset of $L$, and thus a subset of $X_{y} \cap L$, and finally must be a line segment.

On the other hand, if $I=Q \cap(\Sigma+y)$ is a line segment, let $L$ be the line containing this segment, then, by (iii), we have $\boldsymbol{a}, \mathbf{b} \notin Q$, and the direction of $L$ is of the form $\boldsymbol{a}-c \mathbf{b}$ for some $c>0$. Hence $L \cap X_{y}$ must be a line segment since it cannot be a ray; moreover, since the sub-interval $I$ of $L \cap X_{y}$ is a subset of $\partial X_{y}, L \cap X_{y}$ itself must be contained in $\partial X_{y}$, since $X_{y}$ is a planar convex set. Therefore $X_{y}$ is a corner-cut slice. The lemma is proved.

### 3.2. Existence of corner-cut slices.

Definition 3.2. We say a convex polyhedral cone $C$ has regular boundary, if all its $(n-1)$ faces are regular cones.

Let $x, y \in \mathbb{R}^{n}$, we will use $\overline{[x, y]}$ to denote the line segment with endpoints $x$ and $y$.
Theorem 3.1. An irregular convex polyhedral cone with regular boundary always has a feasible 2-face.

Proof. We divide the proof into two cases.
Case 1. There exists $\mathbf{b}_{1}, \mathbf{b}_{2}$ in the frame of $C$ such that $\left[\mathbf{b}_{1}, \mathbf{b}_{2}\right]^{+}$is not a 2 -face of $C$.
Denote $G=\left[\mathbf{b}_{1}, \mathbf{b}_{2}\right]^{+}$. We claim that $G \cap C^{\circ} \neq \emptyset$. Suppose on the contrary $G \cap C^{\circ}=\emptyset$, then $G$ is a subset of $\partial C$, so by Lemma 2.1(i), there exists an ( $n-1$ )-face $Q$ of $C$ containing $G$. Since $Q$ is a regular cone, $G$ must be a face of $Q$. By Lemma 2.1(iv), $G$ is also a face of $C$, which is a contradiction. Our claim is proved.

By Lemma[2.1(i), there exists a $(n-1)$-face $Q$ of $C$ containing $\mathbf{b}_{1}$. Using this argument repeatedly, there exists a 2 -face $F$ of $Q$ containing $\mathbf{b}_{1}$. Let $\boldsymbol{a}_{1}$ be the other element in the frame of $F$. Clearly $\boldsymbol{a}_{1}$ is not a multiple of $\mathbf{b}_{2}$, since $F=\left[\mathbf{b}_{1}, \boldsymbol{a}_{1}\right]^{+}$is a 2 -face and $\left[\mathbf{b}_{1}, \mathbf{b}_{2}\right]^{+}$ is not.

Pick any $x_{0} \in\left[\mathbf{b}_{1}, \mathbf{b}_{2}\right]^{+} \cap C^{\circ}$. Then $x_{0}$ can be written as $x_{0}=\lambda \mathbf{b}_{1}+\rho \mathbf{b}_{2}$, where $\lambda, \rho>0$. Denote $\Sigma=\operatorname{span}\left(\left\{\boldsymbol{a}_{1}, \mathbf{b}_{1}\right\}\right)$. We will show that

$$
\begin{equation*}
X=\left(\Sigma+x_{0}\right) \cap C \tag{3.3}
\end{equation*}
$$

is a corner-cut slice.
Clearly $\rho \mathbf{b}_{2}$ is a extreme point of $X$, since it is an extreme direction of $C$.

Choose $\delta>0$ small so that $z=x_{0}-\delta \boldsymbol{a}_{1}$ is still an interior point of $C$. Clearly, both $\rho \mathbf{b}_{2}$ and $z$ belong to $X$, so $\overline{\left[\rho \mathbf{b}_{2}, z\right]}$ also belongs to $X$. Let

$$
\mathbf{c}=z-\rho \mathbf{b}_{2}=\lambda \mathbf{b}_{1}-\delta \boldsymbol{a}_{1} .
$$

By Lemma 2.2, $\mathbf{c} \notin C$ and the intersection $\left(\rho \mathbf{b}_{2}+\mathbf{c} \mathbb{R}^{+}\right) \cap C$ is not ray. Therefore, the intersection of the ray $\rho \mathbf{b}_{2}+\mathbf{c} \mathbb{R}^{+}$and $X$ is a line segment, and by Lemma 3.1 (ii), $X$ is a corner-cut slice. The theorem is proved in this case.

Case 2. For every pair $\mathbf{b}_{1}, \mathbf{b}_{2}$ in the frame of $C,\left[\boldsymbol{b}_{1}, \mathbf{b}_{2}\right]^{+}$is a 2-face of $C$.
Pick any $(n-1)$-face $Q$ of $C$, then the frame of $Q$ has cardinality $n-1$. Since $C$ is irregular, we can find two elements $\mathbf{b}_{1}$ and $\mathbf{b}_{2}$ in the frame of $C$, but not in the frame of $Q$. Let $F=\left[\mathbf{b}_{1}, \mathbf{b}_{2}\right]^{+}$, then $F$ is a 2 -face of $C$ by our assumption. Denote $\Sigma=\operatorname{span}(F)$ and let $L$ be the intersection of $\Sigma$ and $\operatorname{span}(Q)$, then $L$ is a one-dimensional subspace since $\mathbf{b}_{1}, \mathbf{b}_{2} \notin Q$,

Pick $x_{0} \in Q^{\circ}$. Then $y=x_{0}+c \mathbf{b}_{1} \in C^{\circ}$ for small $c>0$. Notice that

$$
(\Sigma+y) \cap Q=\left(\Sigma+x_{0}\right) \cap Q \supset\left(\Sigma+x_{0}\right) \cap\left(Q+x_{0}\right) .
$$

Since $L+x_{0} \subset \Sigma+x_{0}$ and $Q+x_{0}$ contain a small neighborhood of $L+x_{0}$ near $x_{0}$, we deduce that $(\Sigma+y) \cap Q$ contains a line segment; moreover, by Lemma 3.3(iii), this intersection is a line segment since $\mathbf{b}_{1}, \mathbf{b}_{2} \notin Q$. Finally, by Lemma 3.3 (iv), $(\Sigma+y) \cap C$ is a corner-cut slice. The theorem is proved.

The following example shows that Case 2 in the above proof does appear.
Example 3.1. Let $\mathbf{e}_{1}, \ldots, \mathbf{e}_{6}$ be the canonical basis of $\mathbb{R}^{6}$. Let $\boldsymbol{\xi}=(1,1,1,-1,-1,-1)$. Let $C$ be the convex polyhedral cone with the frame $\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{6}, \boldsymbol{\xi}\right\}$. Then any cone spanned by two vectors in this frame is a 2 -face.

## 4. Continuity of corner-cut slices

Let $C$ be an irregular convex polyhedral cone with regular boundary. Now we fix a feasible 2-face of $C$ and denote it by $F$ (The existence of such a face is guaranteed by Theorem 3.1). We will use the notation $X_{y}$ for the slice $C \cap(\operatorname{span}(F)+y)$ for simplicity. The following lemma is a strengthen of Theorem 3.1.

Lemma 4.1. Let $C$ be an irregular convex polyhedral cone with regular boundary, and $F$ be a feasible 2-face of $C$. Then there exists a ball $B\left(x_{0}, r\right) \subset C^{\circ}$ such that for any $y$ in the ball, $X_{y}$ is a corner-cut slice.

Proof. Let $x_{1}$ be a point in $C^{\circ}$ such that $X_{x_{1}}$ is a corner-cut slice. Let $u$ be the intersection of the two rays in the boundary of $X_{x_{1}}$; clearly, $u \notin C$. Choose $r_{1}>0$ small so that $B\left(u, r_{1}\right) \cap C=\emptyset$ and $B\left(x_{1}, r_{1}\right) \subset C^{\circ}$.

We claim that for any $y$ belongs to $B\left(x_{1}, r_{1}\right) \cap\left(C+x_{1}\right)$, a section of $B\left(x_{1}, r_{1}\right), X_{y}$ is a corner-cut slice. Since $y-x_{1} \in C$, we have

$$
\begin{equation*}
X_{x_{1}}+\left(y-x_{1}\right)=(\operatorname{span}(F)+y) \cap\left(C+\left(y-x_{1}\right)\right) \subset X_{y} . \tag{4.1}
\end{equation*}
$$

Suppose on the contrary that $X_{y}$ has only one extreme point, then the relation (4.1) implies that $u+\left(y-x_{1}\right) \in X_{y} \subset C$, which contradicts $B\left(u, r_{1}\right) \cap C=\emptyset$. So our claim is proved. Therefore, any ball $B\left(x_{0}, r\right) \subset B\left(x_{1}, r_{1}\right) \cap\left(C+x_{1}\right)$ meets the requirement of the lemma.

By applying a linear transformation, we may assume that $\boldsymbol{a}$ and $\mathbf{b}$, the generators of $F$, are orthogonal.

For two sets $A, B \subset \mathbb{R}^{n}$, let $d_{H}(A, B)$ be the Hausdorff metric between $A$ and $B$.
We define a mapping

$$
\pi_{a}: C^{\circ} \rightarrow \partial C
$$

as follows: For $x \in C^{\circ}$, by Lemma 2.2 (i), the ray $x-\mathbb{R}^{+} \boldsymbol{a}$ intersects $\partial C$ at a single point, and we denote this point by $\pi_{a}(x)$. Similarly, we define $\pi_{\mathrm{b}}: C^{\circ} \rightarrow \partial C$.

We note that, if $A$ is a subset of $C^{\circ}, Q$ is an $(n-1)$-face of $C$, and $\pi_{a}(A) \subset Q$, then $\left.\pi_{\boldsymbol{a}}\right|_{A}$ is the projection to $Q$ along the direction $-\boldsymbol{a}$. Therefore, $\pi_{\boldsymbol{a}}$ is the pasting of several projections.

Theorem 4.1. Let $C$ be an irregular convex polyhedral cone with regular boundary, and let $F$ be a feasible 2-face of $C$. Then there exists a ball $B^{*} \subset C^{\circ}$, such that for any $y \in B^{*}$, $X_{y}$ is a corner-cut slice, and

$$
d_{H}\left(X_{y}, F\right),\left|a_{1}(y)\right|,\left|b_{1}(y)\right|
$$

are uniformly bounded, where $a_{1}(y)$ and $b_{1}(y)$ are the origins of the rays on $\partial X_{y}$ with direction $\boldsymbol{a}$ and $\mathbf{b}$ respectively.

Proof. Denote $\Sigma=\operatorname{span}(\{\boldsymbol{a}, \mathbf{b}\})$, where $\{\boldsymbol{a}, \mathbf{b}\}$ is the frame of $F$. Let $B\left(x_{0}, r\right)$ be the ball such that $\Sigma_{y} \cap C$ are corner-cut slices for all $y \in B\left(x_{0}, r\right)$ (see Lemma 4.1). From now on, we call a ball with this property a nice ball.

Notice that if $y^{\prime}=y+c_{1} \boldsymbol{a}+c_{2} \mathbf{b}$ with $c_{1}, c_{2} \in \mathbb{R}$, then $X_{y}=X_{y^{\prime}}$. Consequently, If $B$ is a nice ball, then $B+c_{1} \boldsymbol{a}+c_{2} \mathbf{b}$ is also a nice ball. Also, a sub-ball of a nice ball is also nice.


Figure 2.
Let $Q_{1}$ be the $(n-1)$-face of $C$ such that $Q_{1} \cap\left(\Sigma+x_{0}\right)$ is the ray in $\partial X_{x_{0}}$ with direction $\boldsymbol{a}$. Clearly $\pi_{\mathbf{b}}$ maps $X_{x_{0}}$ to $\partial X_{x_{0}}$; indeed, $\pi_{\mathbf{b}}(x)$ is the canonical projection of $x$ if $\pi_{\mathbf{b}}(x)$ belongs to the ray in $\partial X_{x_{0}}$ with direction $\boldsymbol{a}$, or equivalently, belongs to $Q_{1}$.

We choose $c>0$ large so that $\pi_{\mathbf{b}}\left(x_{0}+c \boldsymbol{a}\right) \in Q_{1}$, and denote $x_{1}=x_{0}+c \boldsymbol{a}$.
Let $Q_{1}^{\circ}$ be the set of relative interior points of $Q_{1}$. We choose $x_{2} \in B\left(x_{1}, r\right)$ so that $\pi_{\mathbf{b}}\left(x_{2}\right) \in Q_{1}^{\circ}$. (Indeed, every point in the intersection $Q_{1}^{\circ}+\left(x_{1}-\pi_{\mathbf{b}}\left(x_{1}\right)\right) \cap B\left(x_{1}, r\right)$ fulfills this requirement.) Let $r_{2}>0$ be a real number such that

$$
\pi_{\mathbf{b}}\left(B\left(x_{2}, r_{2}\right)\right) \subset Q_{1} \text { and } B\left(x_{2}, r_{2}\right) \subset B\left(x_{1}, r\right)
$$

Let $Q_{2}$ be the $(n-1)$-face of $C$ such that $Q_{2} \cap\left(\Sigma+x_{2}\right)$ is the ray on $\partial X_{x_{2}}$ with direction $\mathbf{b}$. Similarly as above, we choose $c^{\prime}$ large so that $\pi_{\boldsymbol{a}}\left(x_{2}+c^{\prime} \mathbf{b}\right) \in Q_{2}$, and denote $x_{3}=x_{2}+c^{\prime} \mathbf{b}$. By the same argument as bove, there exists a ball $B\left(x_{4}, r_{4}\right)$ such that

$$
\pi_{\boldsymbol{a}}\left(\overline{B\left(x_{4}, r_{4}\right)}\right) \subset Q_{2} \text { and } \overline{B\left(x_{4}, r_{4}\right)} \subset B\left(x_{3}, r_{2}\right)
$$

Notice that

$$
\begin{equation*}
\pi_{\mathbf{b}}\left(\overline{B\left(x_{4}, r_{4}\right)}\right) \subset \pi_{\mathbf{b}}\left(B\left(x_{3}, r_{2}\right)\right)=\pi_{\mathbf{b}}\left(B\left(x_{3}-c^{\prime} \mathbf{b}, r_{2}\right)\right)=\pi_{\mathbf{b}}\left(B\left(x_{2}, r^{\prime}\right)\right) \subset Q_{1} \tag{4.2}
\end{equation*}
$$

Set $B^{*}=B\left(x_{4}, r_{4}\right)$. Clearly $\overline{B^{*}}$ is a nice ball. For every $y \in \overline{B^{*}}, X_{y}$ is a corner-cut slice. Moreover, by (4.2), $\pi_{\mathbf{b}}(y)$ belongs to $Q_{1} \cap X_{y}$, which is the ray of $\partial X_{y}$ with direction $\boldsymbol{a}$; similarly, $\pi_{\boldsymbol{a}}(y) \in Q_{2}$ and locates on the ray with direction $\mathbf{b}$ on $\partial X_{y}$.

Since $d_{H}(F, F+y) \leq|y|$ and $d_{H}\left(X_{y}, F\right) \leq d_{H}\left(X_{y}, y+F\right)+d_{H}(y+F, F)$, to show $d_{H}\left(F, X_{y}\right)$ is uniformly bounded, we need only show that

$$
\sup _{y \in \overline{B^{*}}} d_{H}\left(X_{y}, F+y\right)<\infty
$$

Clearly

$$
d_{H}\left(X_{y}, F+y\right) \leq \sqrt{\left|y-\pi_{\boldsymbol{a}}(y)\right|^{2}+\left|y-\pi_{\mathbf{b}}(y)\right|^{2}}
$$

(since we assume that $\boldsymbol{a}$ and $\mathbf{b}$ are orthogonal, see Figure 2) so it is uniformly bounded for $y \in \overline{B^{*}}$, since the right hand side of the above formula is a continuous function of $y$.

Finally, it is seen that both $\left|a_{1}(y)\right|$ and $\left|b_{1}(y)\right|$ are less than

$$
|y|+\left|y-\pi_{\boldsymbol{a}}(y)\right|+\left|y-\pi_{\mathbf{b}}(y)\right| .
$$

The theorem is proved.

## 5. Polyhedral bodies

Let $C$ be a convex polyhedral cone, and let $(T, \mathcal{J})$ be a local tiling of $C$. By Remark 1.1. we may assume that $\mathbf{0} \in T, \mathbf{0} \in \mathcal{J}, T \subset C$, and $\mathcal{J} \subset C$.

Lemma 5.1. If $(T, \mathcal{J})$ is a local tiling of a convex polyhedral cone $C$, then the origin $\mathbf{0}$ belongs to only one tile.

Proof. Suppose on the contrary that two tiles $T+a_{1}$ and $T+a_{2}$ both contain $\mathbf{0}$. Then $-a_{1},-a_{2} \in T$. So $a_{1}-a_{2} \in T+a_{1}$ and $a_{2}-a_{1} \in T+a_{2}$. Therefore, both $a_{1}-a_{2}$ and $a_{2}-a_{1}$ belong to $C$, which is a contradiction.

We call a set $\Omega$ a polyhedral corner, if there exist a point $x$, a number $r>0$ and a convex polyhedral cone $D$ such that

$$
\Omega=x+B(\mathbf{0}, r) \cap D,
$$

and we call $x$ the vertex of the polyhedral corner.
Definition 5.1. Let $T \subset \mathbb{R}^{n}$ be a compact set. For a point $x \in \partial T$, if there exists a real $r>0$, such that $B_{n}(x, r) \cap T$ is a non-overlapping union of several polyhedral corners with the same vertex $x$, then we call $x$ a 'nice' point of $T$; otherwise, we call $x$ a 'bad' point of $T$. If all points in $\partial T$ are 'nice', then we call $T$ a polyhedral body.

If $C$ is a convex polyhedral cone and $x \in \partial C$, then $B(x, r) \cap C$ is a finite non-overlapping union of polyhedral corners for $r$ small.

Lemma 5.2. Let $D_{1}, \ldots, D_{k}$ be convex polyhedral cones of dimension $n$ such that their interiors are disjoint, then
(i) $B(\mathbf{0}, r) \backslash\left(D_{1} \cup \cdots \cup D_{k}\right)$ is a finite non-overlapping union of polyhedral corners.
(ii) Let $\Omega=B(\mathbf{0}, r) \cap\left(D_{1} \cup \cdots \cup D_{k}\right)$ and let $\nu$ be the $(n-1)$-dimensional Hausdorff measure, then for $\nu$-almost every point $x \in \partial \Omega$ with $|x|<r$, there exists a real number $\delta>0$ such that $\overline{B_{n}(x, \delta)} \cap \Omega$ is a closed half ball.

Proof. Each $D_{j}$ is bounded by a set of subspaces. Let $S_{1}, \ldots, S_{m}$ be the collection of such spaces for $j=1, \ldots, k$.
(i) Let $S_{m+1}, \ldots, S_{m+n}$ be the subspaces $\left\{\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n} ; x_{j}=0\right\}, j=1, \ldots, n$. Then $S_{1}, \ldots, S_{m+n}$ decompose $\mathbb{R}^{n}$ into at most $2^{m+n}$ non-overlapping convex polyhedral cones, which we denote by $C_{1}, \ldots, C_{h}$. Then

$$
B(\mathbf{0}, r) \backslash\left(D_{1} \cup \cdots \cup D_{k}\right)=\bigcup\left\{B(\mathbf{0}, r) \cap C_{j} ; C_{j} \text { is not a subset of } D_{1} \cup \cdots \cup D_{k}\right\} .
$$

(ii) Let $x \in \partial \Omega$. Then $x \in S_{1} \cup \cdots \cup S_{m}$.

Suppose $x$ belongs to only one element in $\left\{S_{1}, \ldots, S_{m}\right\}$, say, $x \in S_{j^{\prime}}$. Let $\delta$ be a real number such that $d\left(x, S_{i}\right)>\delta$ for all $i \neq j^{\prime}$. The ball $B_{n}(x, \delta)$ is cut into two (closed) half balls by $S_{j^{\prime}}$. For any $D_{j}, j \in\{1, \ldots, k\}$, either $D_{j}$ contains a half ball of $B(x, \delta)$, or disjoint with $B(x, \delta)$. Moreover, only one $D_{j}$ intersects $B(x, \delta)$, for otherwise, the two half balls of $B(x, \delta)$ belong to two different $D_{j}, j=1, \ldots, k$, and so $x \in \Omega^{\circ}$, which is absurd. Therefore, $\overline{B(x, \delta)} \cap \Omega=\overline{B(x, \delta)} \cap D_{j}$ and it is a closed half ball.

Finally, notice that $S_{i} \cap S_{j}$ is a $\nu$-zero set, the lemma is proved.
Theorem 5.1. Let $C$ be a convex polyhedral cone. If $(T, \mathcal{J})$ is a local tiling of $C$ which covers $C \cap B(\mathbf{0}, R)$ with $R>\operatorname{diam}(T)$, then $T$ must be a polyhedral body.

Proof. By Lemma [5.1, an open neighborhood $\mathcal{N}$ of $\mathbf{0}$ is contained in $T$, and $\mathcal{N} \cap \partial T=$ $\mathcal{N} \cap \partial C$, hence every $x \in \mathcal{N} \cap \partial T$ is a 'nice' point in $T$. Also, notice that $x \in \partial(T+t)$ is a 'bad' point of the tile $T+t$ if and only if $x-t \in \partial T$ is a 'bad' point of the tile $T$.

Suppose on the contrary that the set of 'bad' point in $\partial T$, which we denote by $G$, is not empty. Let $\beta$ be a non-zero vector in $\mathbb{R}^{n}$ such that $\langle x, \beta\rangle>0$ for all $x \in C \backslash\{\mathbf{0}\}$. Let $z_{0}$ be a point in $\bar{G}$ such that $\left\langle z_{0}, \beta\right\rangle$ attains the minimal value. Let

$$
\epsilon=\min \{\langle t, \beta\rangle ; t \in \mathcal{J} \backslash\{\mathbf{0}\}\} .
$$

Clearly $\epsilon>0$ by the discreteness of $\mathcal{J}$. Let $z$ be a point in $G$ such that $\left|z-z_{0}\right|<\frac{\epsilon}{|\beta|}$; a simple calculation shows that $\langle z-t, \beta\rangle<\left\langle z_{0}, \beta\right\rangle$ for any $t \in \mathcal{J} \backslash\{\mathbf{0}\}$.

If there is only one tile in $T+\mathcal{J}$ covering $z$, then for a small $r>0, B_{n}(z, r) \cap T=$ $B_{n}(z, r) \cap C$, which implies that $z$ is a 'nice' point, a contradiction.

If there are exactly two tiles in $T+\mathcal{J}$ covering $z$, say, $z \in T \cap\left(T+t_{1}\right)$ where $t_{1} \neq \mathbf{0}$, then

$$
B_{n}(z, r) \cap C=B_{n}(z, r) \cap\left(T \cup\left(T+t_{1}\right)\right)
$$

for $r$ small. So, by Lemma 5.2, $z$ is a 'bad' point of $T+t_{1}$ since $z$ is a 'bad' point of $T$, and $z-t_{1}$ is a 'bad' point of $T$. Hence $z-t_{1} \in G$ and $\left\langle z-t_{1}, \beta\right\rangle<\left\langle z_{0}, \beta\right\rangle$, which contradicts the minimality of $z_{0}$.

If there are exactly $k+1$ number of tiles covering $z$, say, $z \in T \cap\left(T+t_{1}\right) \cap \cdots \cap\left(T+t_{k}\right)$, then by Lemma 5.2, for at least one $1 \leq j \leq k, z$ is a 'bad' of $T+t_{j}$. By the same argument as above, we get a contradiction. The theorem is proved.

Next, we show a local tiling of $C$ induces a local tiling of $Q$ for every $(n-1)$-face $Q$.
Theorem 5.2. Let $C \subset \mathbb{R}^{n}$ be a convex polyhedral cone, and $(T, \mathcal{J})$ be a local tiling of $C$ which covers $C \cap B(\mathbf{0}, R)$ with $R>\operatorname{diam}(T)$. Then for any $(n-1)$-face $Q$ of $C$, $(T \cap Q, \mathcal{J} \cap Q)$ is a local tiling of $Q$ which covers $Q \cap B(\mathbf{0}, R)$.

Proof. By Theorem 5.1, $T$ is a polyhedral body. Let $\nu$ be the $(n-1)$-dimensional Hausdorff measure. We claim that for $\nu$-almost every point $x \in \partial T$, there exists a real number $\delta>0$ such that $\overline{B_{n}(x, \delta)} \cap T$ is a closed half ball. By compactness of $T$, there exists $x_{j} \in \partial T$, $r_{j}>0, j=1, \ldots, k$, such that each $B\left(x_{j}, r_{j}\right) \cap T$ is a finite non-overlapping union of polyhedral corners and $\partial T$ is covered by $\left\{B\left(x_{j}, r_{j}\right) \cap T ; j=1, \ldots, k\right\}$. So our claim holds by Lemma 5.2(ii).

Since

$$
(T+\mathcal{J}) \cap Q=(T \cap Q)+(\mathcal{J} \cap Q)
$$

we have that $(T \cap Q, \mathcal{J} \cap Q)$ is a covering of $Q \cap B(\mathbf{0}, R)$. To prove the theorem, it suffices to show that $(T \cap Q, \mathcal{J} \cap Q)$ is a packing of $Q$, that is, the intersection

$$
\begin{equation*}
\left((T \cap Q)+t_{1}\right) \cap\left((T \cap Q)+t_{2}\right) \tag{5.1}
\end{equation*}
$$

is a $\nu$-zero set for $t_{1}, t_{2} \in \mathcal{J} \cap Q$ and $t_{1} \neq t_{2}$.
Suppose the intersection (5.1) is not a $\nu$-zero set, then by the claim above, there exists a point $x$ in the above intersection such that both $\overline{B_{n}(x, \delta)} \cap\left(T+t_{1}\right)$ and $\overline{B_{n}(x, \delta)} \cap\left(T+t_{2}\right)$ are closed half balls for $\delta$ small enough. It follows that the closed half ball $\overline{B_{n}(x, \delta)} \cap C$ coincide with the above two half balls, and is a subset of both $T+t_{1}$ and $T+t_{2}$, a contradiction. The theorem is proved.

## 6. Proof of Theorem 1.1(i)

To prove Theorem 1.1(i), we need only show that every irregular convex polyhedral cone with regular boundary admits no translation tiling. For if $C$ is an irregular cone whose boundary is not regular and if $C$ admits a 'large' local tiling $(T, \mathcal{J})$, then there is
an ( $n-1$ )-face $Q$ of $C$ is irregular, and by Theorem 5.2, $Q$ also admits a 'large' local tiling. Therefore, Theorem 1.1(i) can be proved by induction.

Assumptions In the rest of this section, we always assume that $C$ is an irregular convex polyhedral cone with regular boundary, and $(T, \mathcal{J})$ is a packing of $C$ as well as a covering of $C \cap B(\mathbf{0}, R)$ with $R>\operatorname{diam}(T)$.

Under the above assumptions, we are going to deduce a contradiction.
Notations Let $F$ be a feasible 2-face of $C$ with frame $\boldsymbol{a}$ and $\mathbf{b}$. We may assume that $\mathbf{a}$ and $\mathbf{b}$ are orthogonal by applying a linear transformation. Let $B^{*} \subset C^{\circ}$ be a ball in Theorem [4.1, that is, for any $y \in B^{*}$,

$$
X_{y}=(\operatorname{span}(F)+y) \cap C
$$

is a corner-cut slice, and

$$
\begin{gather*}
N=\sup _{y \in B^{*}} d_{H}\left(X_{y}, F\right)<\infty,  \tag{6.1}\\
M=\sup _{y \in B^{*}}\left|a_{1}(y)\right|<\infty, \quad M^{\prime}=\sup _{y \in B^{*}}\left|b_{1}(y)\right|<\infty, \tag{6.2}
\end{gather*}
$$

where $a_{1}(y)$ and $b_{1}(y)$ are the origins of the rays on $\partial X_{y}$ with directions $\boldsymbol{a}$ and $\mathbf{b}$, respectively. Denote $\Sigma=\operatorname{span}(F)$.

Lemma 6.1. Let $\delta>0$. For any $y \in \delta B^{*}, X_{y}$ is a corner-cut slice and

$$
\begin{equation*}
\sup _{y \in \delta B^{*}} d_{H}\left(X_{y}, F\right)=\delta N \tag{6.3}
\end{equation*}
$$

where $N$ is defined in (6.1).
Proof. Since

$$
X_{y}=(\Sigma+y) \cap C=\delta((\Sigma+y / \delta) \cap C)=\delta X_{y / \delta}
$$

that is, $X_{y}$ is a dilation of $X_{y / \delta}$, we infer that $X_{y}$ is a corner-cut slice for $y \in \delta B^{*}$. (6.3) follows from the fact that $d_{H}(\delta A, \delta B)=\delta d_{H}(A, B)$.

Now we study the intersection of $\Sigma+y$ and the local tiling $(T, \mathcal{J})$. We define the $x$-section of a set $A$ as

$$
\begin{equation*}
(A)_{x}=A \cap(\Sigma+x) . \tag{6.4}
\end{equation*}
$$

Here are some easy facts.
Lemma 6.2. (i) If $t \in \mathcal{J} \backslash F$, then $(T+t) \cap F=\emptyset$.
(ii) For $t \in \mathcal{J} \cap F$, we have $(T+t)_{x}=(T)_{x}+t$.

Proof. (i) For otherwise, there exists $x \in T$ and $t \in \mathcal{J} \backslash F$ such that $x+t \in F$. So $x / 2+t / 2$, a convex combination of $x$ and $t$ belongs to $F$. By the definition of a face, $x, t \in F$, which is a contradiction.
(ii) Since $(T+t)_{x}=(T+t) \cap(\Sigma+x)=(T \cap(\Sigma-t+x))+t=T \cap(\Sigma+x)+t=(T)_{x}+t$.

Let $\mu_{r}$ be the $r$-dimensional Lebesgue measure.
Lemma 6.3. For any $\delta>0$, there exists $y \in \delta B^{*}$ such that $\left((T)_{y}, \mathcal{J} \cap F\right)$ is a packing of $X_{y}$.

Proof. Fix $t_{1}, t_{2} \in \mathcal{J} \cap F$ with $t_{1} \neq t_{2}$. Notice that $T+t_{1}$ and $T+t_{2}$ are disjoint in measure $\mu_{n}$. Let $M \subset \delta B^{*}$ be a cube of dimension $n-2$ such that $M$ is orthogonal to $\Sigma$. Let $f: C^{\circ} \rightarrow \mathbb{R}$ be the function

$$
f(x)=\mu_{2}\left(\left(T+t_{1}\right)_{x} \cap\left(T+t_{2}\right)_{x}\right) .
$$

By Fubini Theorem,

$$
\int_{x \in M} f(x) d x \leq \mu_{n}\left(\left(T+t_{1}\right) \cap\left(T+t_{2}\right)\right)=0
$$

so $f(x)=0$ a.e. $x \in M$.
Hence, for a pair $t_{1}, t_{2} \in \mathcal{J}$, to insure $\mu_{2}\left(\left(T+t_{1}\right)_{x} \cap\left(T+t_{2}\right)_{x}\right)=0$, we need to eliminate a measure zero set of $M$. After eliminating the measure zero sets for all pairs in $\mathcal{J} \cap F$, a point $y$ in the remaining set fulfills the requirement of the lemma.

Lemma 6.4. There exists $\delta>0$ such that for any $y \in \delta B^{*}, B(\mathbf{0}, R) \cap X_{y}$ is covered by $(T)_{y}+(\mathcal{J} \cap F)$.

Proof. Without loss of generality, we may assume that $\mathcal{J}$ is a finite set. Let

$$
\varepsilon=\min \{d(T+t, F) ; t \in \mathcal{J} \backslash F\} .
$$

By Lemma 6.2, $d(T+t, F)>0$ for all $t \in \mathcal{J} \backslash F$, so $\varepsilon>0$. In other words, we have that

$$
\begin{equation*}
d(T+t, F) \geq \varepsilon \quad \text { for all } t \in \mathcal{J} \backslash F \tag{6.5}
\end{equation*}
$$

We choose $0<\delta<\varepsilon / N$, where $N$ is defined in (6.1). Pick any $y \in \delta B^{*}$, by Lemma6.1,

$$
\begin{equation*}
d_{H}\left(X_{y}, F\right)<\varepsilon \tag{6.6}
\end{equation*}
$$

Equation (6.5) together with (6.6) imply that

$$
(T+t) \cap X_{y}=\emptyset \text { for } t \in \mathcal{J} \backslash F,
$$

so $X_{y} \cap B(\mathbf{0}, R)$ is covered by $(T+t)_{y}$ with $t \in \mathcal{J} \cap F$. Finally, $(T+t)_{y}=(T)_{y}+t$ by Lemma 6.2(ii). The lemma is proved.

Proof of Theorem 1.1(i). We prove assertion (i) of the theorem by induction on the dimension of $C$.

For $\operatorname{dim} C=1$ or 2 , the cone must be regular, and the theorem holds automatically.
Suppose $\operatorname{dim} C=n$. Assume on the contrary that $C$ is irregular, $(T, \mathcal{J})$ is a packing of $C$ as well as a covering of $C \cap B(\mathbf{0}, R)$ with $R>\operatorname{diam}(T)$.

If $C$ has irregular boundary, then an $(n-1)$-face $Q$ of $C$ is an irregular cone. By Theorem 5.2. ( $T \cap Q, \mathcal{J} \cap Q$ ) is a packing of $Q$ and a covering of $Q \cap B(\mathbf{0}, R)$, which is impossible by our induction hypothesis. So $C$ must be an irregular cone with regular boundary. Now we use the notations listed in the beginning of this section.

Let $0<\kappa<R-\operatorname{diam}(T)$. Let $\delta>0$ be the constant in Lemma 6.4 and satisfies the additional requirement

$$
\delta<\min \left\{\frac{\kappa}{M}, \frac{R-\kappa}{M+M^{\prime}}\right\}
$$

where $M$ and $M^{\prime}$ are the constants in formula (6.2).
Let $y$ be a point in $\delta B^{*}$ satisfying the requirements of Lemma 6.3. Then members in the cluster

$$
\begin{equation*}
\left\{(T)_{y}+t ; t \in \mathcal{J} \cap F\right\} \tag{6.7}
\end{equation*}
$$

are disjoint in $\mu_{2}$, and cover the set $B(\mathbf{0}, R) \cap X_{y}$.
Since $\delta<\kappa / M$, by (6.2) and $X_{y}=\delta X_{y / \delta}$, for any $y \in \delta B^{*},\left|a_{1}(y)\right|<\kappa$ where $a_{1}(y)$ is the origin of the ray on $\partial X_{y}$ with direction $\boldsymbol{a}$. Then $B\left(a_{1}(y), R-\kappa\right) \subset B(\mathbf{0}, R)$ and hence the cluster in (6.7) is a packing of $X_{y}$ and covers the set $B\left(a_{1}(y), R-\kappa\right) \cap X_{y}$. Similarly, from $\delta<(R-\kappa) /\left(M+M^{\prime}\right)$ ), we deduce that $R-\kappa>\left|a_{1}(y)-b_{1}(y)\right|$ for any $y \in \delta B^{*}$. However, in next section, we prove that this is impossible (Theorem 7.1), and we get a contradiction. This completes the proof of the theorem.

## 7. Corner-Cut region can not be tiled by translations of one set

Let $X \subset \mathbb{R}^{2}$ be a unbounded convex region determined by a system of liner inequalities:

$$
\begin{equation*}
a_{j} x+b_{j} y+c_{j} \geq 0, \quad 1 \leq j \leq N . \tag{7.1}
\end{equation*}
$$

We call $X$ a corner-cut region, if $X$ has at least two extreme points, and the two rays on $\partial X$ are not parallel.

Let $a_{1}, a_{2}, \cdots, a_{q}$ be the extreme points of $\partial X$ from left to right, and write

$$
\partial X=\ell_{0} \cup \ell_{1} \cup \cdots \cup \ell_{q},
$$

where $\ell_{0}$ and $\ell_{q}$ are two rays, and the other $\ell_{j}=\overline{\left[a_{j}, a_{j+1}\right]}$ are line segments.
Theorem 7.1. Let $X \subset \mathbb{R}^{2}$ be a corner-cut region, $T$ be a compact set, and $\mathcal{J} \subset \mathbb{R}^{2}$ be a finite set. Let

$$
R>\max \left\{\operatorname{diam}(T),\left|a_{1}-a_{2}\right|\right\} .
$$

Then the following two items can not be fulfilled at the same time:
(i) $X \cap B\left(a_{1}, R\right)$ is covered by $T+\mathcal{J}$ and $T+\mathcal{J} \subset X$;
(ii) The members in $\{T+t ; t \in \mathcal{J}\}$ are disjoint in Lebesgue measure.

Note that in the above theorem, we do not assume that $T=\overline{T^{\circ}}$.
Proof. By applying an affine map, without loss of generality, we may assume that $a_{1}=\mathbf{0}$ is the origin, and the two rays on $\partial X$ are $\ell_{0}=\{0\} \times[0,+\infty)$ and $\ell_{q}=a_{q}+[0,+\infty) \times\{0\}$. (We remark that under this assumption, it holds that $\left|a_{1}-a_{2}\right| \leq\left|a_{1}-a_{q}\right|$. So we can use Theorem 7.1 in the previous section.) For simplicity, we identify $\mathbb{R}^{2}$ to the complex plane $\mathbb{C}$.

Suppose on the contrary that $(T, \mathcal{J})$ is a pair satisfying the two items in Theorem 7.1. By Remark 1.1, without loss of generality, we may assume $\mathbf{0} \in T, \mathbf{0} \in \mathcal{J}, \mathcal{J} \subset X$, and $T \subset X$.

Lemma 7.1. For any $b \in\left\{a_{1}, \cdots, a_{q}\right\}, b$ belongs to exact one tile in $\{T+t: t \in \mathcal{J}\}$. Especially, $T$ is the only tile contains the point $a_{1}=\mathbf{0}$.

Proof. Assume that $b \in\left(T+t_{1}\right) \cap\left(T+t_{2}\right)$, where $t_{1}, t_{2} \in \mathcal{J}$ and $t_{1} \neq t_{2}$. Then $b-t_{1}, b-t_{2} \in$ $T$. So

$$
z_{1}=b-t_{1}+t_{2} \in T+t_{2} \subset X \text { and } z_{2}=b-t_{2}+t_{1} \in T+t_{1} \subset X
$$

Hence $b=\left(z_{1}+z_{2}\right) / 2$, which contradicts that $b$ is an extreme point.
The following is a technical lemma we need in the proof of the following Lemma 7.3
Lemma 7.2. Let $A \subset \mathbb{R}^{2}$ be a trapezoid, and let $L=[0,1] \times\{0\}$ be the base line of $A$ with shorter length. Let $T$ be a compact subset of $A$, and $\mathcal{J}=\left\{t_{0}=\mathbf{0}, t_{1}, \ldots, t_{p}\right\}$ be a subset of $L$ with $p \geq 1$. Then $(T, \mathcal{J})$ can not be a tiling of $A$.

The proof of the above lemma is very similar to the proof of Theorem 8.2, but much more simpler. We put the proof in Appendix A. In the following, the topology we use is the relative topology of $X$.

Lemma 7.3. A neighborhood of $\overline{\left[a_{1}, a_{2}\right]} \subset T$.
Proof. By our assumption, $\overline{\left[a_{1}, a_{2}\right]} \subset B(\mathbf{0}, R)$ and is covered by $(T, \mathcal{J})$.
Let $t_{0}=\mathbf{0}, t_{1}, \cdots, t_{p}$ be the elements in $\mathcal{J}$ satisfying

$$
\left(T+t_{j}\right) \cap \overline{\left[a_{1}, a_{2}\right]} \neq \emptyset .
$$

Then $t_{0}, t_{1}, \cdots, t_{p} \in \overline{\left[a_{1}, a_{2}\right]}$, since $\overline{\left[a_{1}, a_{2}\right]}$ is a face of $X$.
To prove the lemma, we need to show that $p=0$. Suppose on the contrary $p \geq 1$.
Assume that $t_{0}, t_{1}, \cdots, t_{p}$ are arranged from left to right on $\overline{\left[a_{1}, a_{2}\right]}$. Let $L$ be the line containing $\overline{\left[a_{1}, a_{2}\right]}$. There exists $\delta>0$ such that no element of $\mathcal{J} \backslash\left\{t_{0}, t_{1}, \ldots, t_{p}\right\}$ locates in strip between $L$ and $L+\delta$ i. Denote

$$
A=X \cap(L+[0, \delta] \cdot \mathbf{i})
$$

then using $\left(T+t_{j}\right) \cap(L+[0, \delta] \cdot \mathbf{i})=\left(T+t_{j}\right) \cap A$, it is easy to show that

$$
A=(T \cap A)+\left\{t_{0}, \ldots, t_{p}\right\}
$$

and the right hand side is a tiling of $A$. We choose $\delta$ small to ensure $A$ is a trapezoid. By Lemma 7.2 this is impossible. The lemma is proved.

Let $z_{0} \in \partial X$ be the first point on the right side of $a_{1}$ such that $z_{0}$ is not a relative interior point of $T$, that is, there exists $t^{*} \in \mathcal{J} \backslash\{\mathbf{0}\}$, such that $z_{0} \in T+t^{*}$. Then $z_{0} \notin\left\{a_{1}, \cdots, a_{q}\right\}$ by Lemma 7.1 and $z_{0}$ is on the right side of $a_{2}$ by Lemma 7.3 ,

Let $\gamma$ be the open broken line from $a_{1}$ to $z_{0}$ on $\partial X$. Since $z_{0}-t^{*} \in T \subset X$ and $t^{*} \in \mathcal{J} \subset X$, their real parts must be non-negative, i.e.,

$$
\begin{equation*}
\operatorname{Re}\left(z_{0}-t^{*}\right) \geq 0 \text { and } \operatorname{Re}\left(t^{*}\right) \geq 0 \tag{7.2}
\end{equation*}
$$

If $\operatorname{Re}\left(z_{0}-t^{*}\right)=0$, then $t^{*}$ and $z_{0}$ are located in the same vertical line, so to guarantee $z_{0} \in T+t^{*}$, we must have $z_{0}=t^{*}$. Since the slope of the line containing $z_{0}$ is larger than the slope of the line containing $\overline{\left[a_{1}, a_{2}\right]}$, we get that $T+t^{*}$ is not a subset of $X$, a contradiction. If $\operatorname{Re}\left(t^{*}\right)=0$, then $T+t^{*}$ can not contain $z_{0}$, which is also a contradiction. So the equalities in (7.2) are strict, and consequently,

$$
\begin{equation*}
0<\operatorname{Re}\left(z_{0}-t^{*}\right)<\operatorname{Re}\left(z_{0}\right), \tag{7.3}
\end{equation*}
$$



Figure 3.

$$
\begin{equation*}
0<\operatorname{Re}\left(t^{*}\right)<\operatorname{Re}\left(z_{0}\right) \tag{7.4}
\end{equation*}
$$

Moreover, since $z_{0}$ is a lowest point in $t^{*}+T$ (w.r.t. the vertical direction), $z_{0}-t^{*}$ must be a lowest point in $T$ (w.r.t. the vertical direction), hence, by formula (7.3), we have

$$
\begin{equation*}
z_{0}-t^{*} \in \gamma^{\circ} . \tag{7.5}
\end{equation*}
$$

Let $b \neq \mathbf{0}$ be the smallest point on $\ell_{0} \cap \mathcal{J}$. Clearly the interval $\overline{\left[b, a_{1}\right]} \backslash\{b\}$ is contained in $T$ and does not intersect any other tiles. Denote by $\kappa$ the open broken line from $b$ to $z_{0}$ on $\partial X$, then there is an open set $U$ such that

$$
\kappa \subset U \subset T
$$

See Figure 3, where we use a polygon to illustrate the open set $U$.
Next we show that

$$
\begin{equation*}
(b+\kappa) \cap\left(t^{*}+\kappa\right) \neq \emptyset . \tag{7.6}
\end{equation*}
$$

We claim that $t^{*}+b$, the initial point of $t^{*}+\kappa$, is above $b+\kappa$, and $b+z_{0}$, the terminal point of $b+\kappa$, is above $t^{*}+\kappa$. The first assertion holds, since by formula (7.4), $t^{*}$ is above the curve $\kappa$, and the second assertion holds since the point $b+\left(z_{0}-t^{*}\right)$ is above $\kappa$. Our claim is proved. If $t^{*}$ is below the curve $b+\gamma$, then apparently (7.6) holds. If $t^{*}$ is above $b+\gamma$, regarding $b+\gamma$ and $t^{*}+\gamma$ as graphs of two functions and using the Intermediate Value Theorem, we conclude $(b+\gamma) \cap\left(t^{*}+\gamma\right) \neq \emptyset$ and (7.6) is confirmed.

Let $x$ be the intersection of the two curves in (7.6), then $x+[0, \delta]^{2}$ belongs to both $b+U$ and $t^{*}+U$ for $\delta$ small. It follows that $(b+T)^{\circ} \cap\left(t^{*}+T\right)^{\circ} \neq \emptyset$, which is a contradiction.

## 8. From tilings of $\left(\mathbb{R}^{+}\right)^{n}$ to tilings of $\left(\mathbb{Z}^{+}\right)^{n}$

Let $T$ be a compact set satisfying $T=\overline{T^{\circ}}$. Let $R>\operatorname{diam}(T)$. Let $(T, \mathcal{J})$ be a packing of $\left(\mathbb{R}^{+}\right)^{n}$ as well as a covering of $\left(\mathbb{R}^{+}\right)^{n} \cap B(\mathbf{0}, R)$; we call such $(T, \mathcal{J})$ a local tiling in this section. As before, we may assume that $\mathbf{0} \in T, \mathbf{0} \in \mathcal{J}, T \subset\left(\mathbb{R}^{+}\right)^{n}$, and $\mathcal{J} \subset\left(\mathbb{R}^{+}\right)^{n}$.

Since $R>\operatorname{diam}(T)$, for each $j \in\{1, \ldots, n\}, \mathcal{J} \cap \mathbf{e}_{j} \mathbb{R}^{+}$contains at least one non-zero element. By applying a dilation matrix $U=\operatorname{diag}\left(u_{1}, \ldots, u_{n}\right)$, we may assume that

$$
\begin{equation*}
\mathbf{e}_{1}, \ldots, \mathbf{e}_{n} \in \mathcal{J}, \text { and } \lambda \mathbf{e}_{j} \notin \mathcal{J} \text { for all } 0<\lambda<1 \text { and all } j=1, \ldots, n . \tag{8.1}
\end{equation*}
$$

Since $(T, \mathcal{J} \cap B(\mathbf{0}, R))$ also covers $\left(\mathbb{R}^{+}\right)^{n} \cap B(\mathbf{0}, R)$, without loss of generality, we assume that

$$
\begin{equation*}
\mathcal{J} \subset B(\mathbf{0}, R) ; \tag{8.2}
\end{equation*}
$$

especially, $\mathcal{J}$ is a finite set.
The tiling of $\mathbb{R}^{+}$has been characterized by Odlyzko implicitly. Some idea of this section comes from Odlyzko [17].

Theorem 8.1. (17]) Let $(T, \mathcal{J})$ be a tiling of $\mathbb{R}^{+}$. Then there exist a real number $c>0$ such that $c T=E+[0,1]$, where $E \subset \mathbb{Z}^{+}$.

For $x=\left(x_{1}, \ldots, x_{n}\right) \in \mathbb{R}^{n}$, we use $\|x\|_{1}=\max \left|x_{j}\right|$ to denote the 1-norm. We say $x>0$ (or $x \geq 0$ ) if $x_{j}>0\left(\right.$ or $\left.x_{j} \geq 0\right)$ for all $j=1, \ldots, n$.

Lemma 8.1. Let $(T, \mathcal{J})$ be a local tiling of $\left(\mathbb{R}^{+}\right)^{n}$ satisfying (8.1) and (8.2). Then

$$
\mathcal{J} \cap[0,1)^{n}=\{\mathbf{0}\} \text { and }[0,1]^{n} \subset T .
$$

Consequently, if $t, t^{\prime} \in \mathcal{J}$, then $\left\|t-t^{\prime}\right\|_{1} \geq 1$.
Proof. Let $Q$ be a $m$-face of $\left(\mathbb{R}^{+}\right)^{n}$ with $1 \leq m \leq n$, denote

$$
D=Q \cap[0,1)^{n} .
$$

Let $Q^{\perp}$ be the $(n-\operatorname{dim} Q)$-face of $\left(\mathbb{R}^{+}\right)^{n}$ complement to $Q$, that is, $Q+Q^{\perp}=\left(\mathbb{R}^{+}\right)^{n}$. Denote $D^{\perp}=Q^{\perp} \cap[0,1)^{n}$, then $D+D^{\perp}=[0,1)^{n}$.

For a $m$-face $Q$ with $1 \leq m \leq n-1$, we define $\delta_{Q}$ as

$$
\delta_{Q}=\left\{\begin{array}{l}
1, \text { if } \mathcal{J} \cap[0,1)^{n} \backslash Q=\emptyset \\
\min \left\{d(t, Q) ; t \in \mathcal{J} \cap[0,1)^{n} \backslash Q\right\}, \text { otherwise }
\end{array}\right.
$$

Set

$$
\begin{equation*}
\delta=\min \left\{\delta_{Q} ; Q \text { is a face of }\left(\mathbb{R}^{+}\right)^{n}\right\} . \tag{8.3}
\end{equation*}
$$

We shall prove by induction on $\operatorname{dim} Q$ that

$$
\begin{equation*}
\mathcal{J} \cap D=\{\mathbf{0}\}, \text { and } D \subset T \tag{8.4}
\end{equation*}
$$

The result holds when $m=1$ by the assumption (8.1). Suppose $m \geq 2$ and (8.4) holds for $m-1$. Without loss of generality, we assume $Q$ is generated by $\mathbf{e}_{1}, \ldots, \mathbf{e}_{m}$.

Denote

$$
Q_{j}=\operatorname{span}\left(\left\{\mathbf{e}_{1}, \ldots, \mathbf{e}_{m}\right\} \backslash\left\{\mathbf{e}_{j}\right\}\right), \quad D_{j}=[0,1)^{n} \cap Q_{j}
$$

By the induction hypothesis, we have

$$
\begin{equation*}
D_{j} \subset T, \text { and } \mathcal{J} \cap D_{j}=\{\mathbf{0}\}, \quad 1 \leq j \leq m \tag{8.5}
\end{equation*}
$$

Denote $\delta^{\prime}=\delta / \sqrt{n}$ and

$$
L_{j}=D_{j}+\left[0, \delta^{\prime}\right) D_{j}^{\perp}
$$

Pick $x \in L_{j}$ and let $T+t$ be the tile containing $x$. Notice that $x \in[0,1)^{n}$ and $d\left(x, D_{j}\right)<\delta$. It follows that $t \in[0,1)^{n}$ and $d\left(t, D_{j}\right)<\delta$ since $x-t \geq 0$. Moreover, since $\operatorname{dim} D_{j} \leq n-1$, by the definition of $\delta$, we have $t \in D_{j}$, which forces $t=\mathbf{0}$ by our induction hypothesis. This proves that

$$
\begin{equation*}
L_{j} \subset T, j=1, \ldots, m \tag{8.6}
\end{equation*}
$$

Suppose on the contrary that there exists $t=\left(t_{1}, \ldots, t_{m}, 0, \ldots, 0\right) \in \mathcal{J} \cap D$ and $t \neq \mathbf{0}$, then by (8.5), $t_{j}>0$ for $j=1, \ldots, m$. It is seen that

$$
L_{1}=\left[0, \delta^{\prime}\right) \times[0,1)^{m-1} \times\left[0, \delta^{\prime}\right)^{n-m} \text { and } L_{m}=[0,1)^{m-1} \times\left[0, \delta^{\prime}\right) \times\left[0, \delta^{\prime}\right)^{n-m}
$$

We have that

$$
\begin{gathered}
L_{1}+t=\left[t_{1}, t_{1}+\delta^{\prime}\right) \times \prod_{j=2}^{m}\left[t_{j}, t_{j}+1\right) \times\left[0, \delta^{\prime}\right)^{n-m} \\
L_{m}+\mathbf{e}_{m}=[0,1)^{m-1} \times\left[1,1+\delta^{\prime}\right) \times\left[0, \delta^{\prime}\right)^{n-m}
\end{gathered}
$$

Hence
$\left(L_{1}+t\right) \cap\left(L_{m}+\mathbf{e}_{m}\right)=\left[t_{1}, \min \left\{1, t_{1}+\delta^{\prime}\right\}\right) \times \prod_{j=2}^{m-1}\left[t_{j}, 1\right) \times\left[1,1+\min \left\{t_{m}, \delta^{\prime}\right\}\right) \times\left[0, \delta^{\prime}\right)^{n-m}$,
which implies that $(T+t) \cap\left(T+\mathbf{e}_{m}\right)$ has positive Lebesgue measure. This contradiction proves that $\mathcal{J} \cap D=\{\mathbf{0}\}$, the first assertion of (8.4).

Now $\mathcal{J} \cap D=\{\mathbf{0}\}$ implies that $T$ is the only tile intersecting $D$. It follows that $D \cap B(\mathbf{0}, R)$ is a subset of $T$. By Lemma 8.2 we list below, $D$ is a subset of $T$, which verifies the second assertion of (8.4).

Finally, set $D=[0,1)^{n}$ in (8.4), we obtain the lemma.

Lemma 8.2. If $U$ is a connected set and $B(\mathbf{0}, R) \cap U \subset T$, then $U \subset T$.
Proof. If $U$ is not a subset of $B(\mathbf{0}, R)$, then there exists $x \in U \cap B(\mathbf{0}, R)$ such that $|x|$ is as closer to $R$ as we want, so $x \notin T$, which is a contradiction. Therefore, we must have $U \subset T$.

Let ' $\prec$ ' be the order on $\left(\mathbb{R}^{+}\right)^{n}$ defined by $\boldsymbol{a} \prec \mathbf{b}$ if $\mathbf{b}-\boldsymbol{a} \geq 0$ and $\boldsymbol{a} \neq \mathbf{b}$.
Theorem 8.2. Let $(T, \mathcal{J})$ be a local tiling of $\left(\mathbb{R}^{+}\right)^{n}$ satisfying (8.1) and (8.2). Then
(i) $\mathcal{J} \subset\left(\mathbb{Z}^{+}\right)^{n}$.
(ii) There exists a subset $E \subset\left(\mathbb{Z}^{+}\right)^{n}$ such that $T=E+[0,1]^{n}$.

Proof. To prove the theorem, we need only prove the following two statements: For each $\boldsymbol{z} \in\left(\mathbb{Z}^{+}\right)^{n} \cap B(\mathbf{0}, R)$,

$$
\begin{equation*}
\{t \in \mathcal{J} ; t \preceq \boldsymbol{z}\} \subset\left(\mathbb{Z}^{+}\right)^{n}, \tag{8.7}
\end{equation*}
$$

$$
\begin{equation*}
\mathbf{u}+(0,1)^{n} \text { belongs to exactly one tile for all } \mathbf{u} \in\left(\mathbb{Z}^{+}\right)^{n} \cap B(\mathbf{0}, R) \text { with } \mathbf{u} \preceq \boldsymbol{z} . \tag{8.8}
\end{equation*}
$$

In the following, we prove the above statements by induction on $\boldsymbol{z}$. If $\boldsymbol{z}=\mathbf{0}$, the statements are valid by Lemma 8.1. Assume $\mathbf{0} \prec \boldsymbol{z}$. Suppose (8.7) and (8.8) hold for all $\boldsymbol{z}^{\prime} \in\left(\mathbb{Z}^{+}\right)^{n}$ with $\boldsymbol{z}^{\prime} \prec \boldsymbol{z}$, and we show they hold for $\boldsymbol{z}$ in the following.

First, we prove (8.7). Write $\boldsymbol{z}=\left(z_{1}, \ldots, z_{n}\right)$. By the induction hypothesis of (8.8), we have

$$
Y=\bigcup\left\{T+t ; t \in \mathcal{J} \cap\left(\mathbb{Z}^{+}\right)^{n} \text { and } t \prec \boldsymbol{z}\right\}
$$

covers the set

$$
\Omega=\left[0, z_{1}+1\right] \times \cdots \times\left[0, z_{n}+1\right] \backslash\left(\boldsymbol{z}+[0,1]^{n}\right),
$$

which is a rectangle missing the 'last cube' $\boldsymbol{z}+[0,1]^{n}$. So if $t \in \mathcal{J}$ and $t \prec \boldsymbol{z}$, then $t+[0,1]^{n}$ and $Y$ overlap, which forces $t \in\left(\mathbb{Z}^{+}\right)^{n}$. Therefore, no matter $\boldsymbol{z} \in \mathcal{J}$ or not, (8.7) holds.

Next, we prove (8.8). Assume on the contrary (8.8) is false for $\boldsymbol{z}$, then the 'last cube' $\boldsymbol{z}+(0,1)^{n}$ intersects at least two tiles. Denote the tiles intersecting $\boldsymbol{z}+(0,1)^{n}$ by $T+t_{1}$, $T+t_{2}, \ldots, T+t_{\ell}$. In particular, $\boldsymbol{z}$ does not belong to $\mathcal{J}$. Clearly

$$
\left\{t_{1}, \ldots, t_{\ell}\right\} \subset\left[0, z_{1}+1\right) \times \cdots \times\left[0, z_{n}+1\right) .
$$

Since $\Omega$ is covered by tiles $T+t$ with integral $t \prec \boldsymbol{z}$, we conclude that $t_{j}$ is either integral and precedes $\boldsymbol{z}$, or $t_{j}$ belongs to the 'last cube' $\boldsymbol{z}+[0,1]^{n}$. In the formal case, we must have $t_{j}=\mathbf{0}$, for otherwise, $T+t_{j}$ contains only a proper subset of $\boldsymbol{z}+(0,1)^{n}$, so $T$ contains only a proper subset of $\left(\boldsymbol{z}-t_{j}\right)+(0,1)^{n}$, which contradicts our induction hypothesis on
(8.8). In the later case, by Lemma 8.1, at most one element of $\mathcal{J}$ belongs to $\boldsymbol{z}+[0,1)^{n}$. It follows that there are exactly two tiles intersecting $\boldsymbol{z}+(0,1)^{n}$, one is $T$, and the other one, which we denote by $T+t_{2}$, satisfies $\boldsymbol{z} \prec t_{2}$. So

$$
\begin{equation*}
\left(\boldsymbol{z}+(0,1)^{n}\right) \cap B(\mathbf{0}, R) \subset T \cup\left(T+t_{2}\right) \tag{8.9}
\end{equation*}
$$

Let $\pi_{j}$ be the projection such that $\pi_{j}\left(x_{1}, \ldots, x_{n}\right)=x_{j}$. Notice that $\pi_{j}\left(t_{2}\right)>\pi_{j}(\boldsymbol{z})$ for at least one $j \in\{1, \ldots, n\}$, without loss of generality, let us assume that $\pi_{1}\left(t_{2}\right)>\pi_{1}(\boldsymbol{z})$. Clearly, $\left(T+t_{2}\right) \cap\left(\boldsymbol{z}+(0,1)^{n}\right)=\left(t_{2}+[0,1]^{n}\right) \cap\left(\boldsymbol{z}+(0,1)^{n}\right)$, so the open rectangle

$$
U=\boldsymbol{z}+\left(0, \pi_{1}\left(t_{2}\right)-\pi_{1}(\boldsymbol{z})\right) \times(0,1)^{n-1}
$$

as a subset of $\boldsymbol{z}+(0,1)^{n}$, is not covered by $T+t_{2}$. Consequently, $U \cap B(\mathbf{0}, R)$ must be covered by $T$. It follows that $U \subset T$ by Lemma 8.2 ,

Recall that $\mathbf{e}_{1} \in \mathcal{J}$. Now $\left(T+\mathbf{e}_{1}\right) \cap\left(T+t_{2}\right)$ contains $\left(\mathbf{e}_{1}+U\right) \cap\left(t_{2}+(0,1)^{n}\right)$ as a subset, and the later one has positive Lebesgue measure, which is a contradiction. This contradiction proves that

$$
\left(\boldsymbol{z}+(0,1)^{n}\right) \cap B(\mathbf{0}, R) \subset T \text { or } T+t_{2}
$$

In the former case, $\boldsymbol{z}+(0,1)^{n} \subset T$ by Lemma 8.2 , in the later case, $t_{2}=\boldsymbol{z}$ and clearly $\boldsymbol{z}+(0,1)^{n} \subset T+\boldsymbol{z}$. This verifies (8.8) and finishes the proof of the theorem.

Proof of Theorem 1.1 (ii). It is the immediate consequence of Theorem 8.2 ,
Proof of Corollary 1.2 , Let $(T, \mathcal{J})$ be a tiling of $\left(\mathbb{R}^{+}\right)^{n}$ with $T=\overline{T^{\circ}}$. Then there is a diagonal matrix $U$ such that $\left(T^{\prime}=U T, \mathcal{J}^{\prime}=U \mathcal{J}\right)$ satisfies the normalization condition (8.1). For any $R>\operatorname{diam}\left(T^{\prime}\right),\left(T^{\prime}, \mathcal{J}^{\prime} \cap B(\mathbf{0}, R)\right)$ is a local tiling of $\left(\mathbb{R}^{+}\right)^{n}$ satisfying the conditions of Theorem 8.2. It follows that $T^{\prime}=E+[0,1]^{n}$ for some $E \subset\left(\mathbb{Z}^{+}\right)^{n}$, and $\mathcal{J}^{\prime} \cap B(\mathbf{0}, R) \subset\left(\mathbb{Z}^{+}\right)^{n}$ for all $R>0$, so $\mathcal{J}^{\prime} \subset\left(\mathbb{Z}^{+}\right)^{n}$.

Finally, notice that $\left(E+[0,1]^{n}, \mathcal{J}^{\prime}\right)$ is a tiling of $\left(\mathbb{R}^{+}\right)^{n}$ if and only if $\left(E, \mathcal{J}^{\prime}\right)$ is a tiling of $\left(\mathbb{Z}^{+}\right)^{n}$.

## 9. Self-affine tiles with polyhedral corners

Let $T=T(\mathbf{A}, \mathcal{D})$ be a $n$-dimensional self-affine tile with expanding matrix $\mathbf{A}$ and digit set $\mathcal{D}$. Denote

$$
\mathcal{D}_{k}=\mathcal{D}+\mathbf{A} \mathcal{D}+\cdots+\mathbf{A}^{k-1} \mathcal{D}
$$

then iterating $\mathbf{A} T(\mathbf{A}, \mathcal{D})=T(\mathbf{A}, \mathcal{D})+\mathcal{D} k$-times, we obtain

$$
\mathbf{A}^{k} T(\mathbf{A}, \mathcal{D})=T(\mathbf{A}, \mathcal{D})+\mathcal{D}_{k}
$$

Recall that $T$ has a polyhedral corner at $x_{0}$ means that there is a convex polyhedral cone $C$ and a number $r>0$ such that

$$
\begin{equation*}
T \cap B_{n}\left(x_{0}, r\right)=x_{0}+B_{n}(\mathbf{0}, r) \cap C \tag{9.1}
\end{equation*}
$$

Proof of Theorem 1.2. Take $k \geq 1$. Let $B\left(\mathbf{0}, \ell_{k}\right)$ be the maximal ball centered at $\mathbf{0}$ and contained in $\mathbf{A}^{k} B(\mathbf{0}, 1)$. Since $\mathbf{A}$ is expanding, it is seen that $\ell_{k} \rightarrow \infty$ when $k \rightarrow \infty$. Applying $\mathbf{A}^{k}$ to both sides of (9.1), we have

$$
\mathbf{A}^{k} T \cap \mathbf{A}^{k} B\left(x_{0}, r\right)=\mathbf{A}^{k} x_{0}+\left(\mathbf{A}^{k} C \cap A^{k} B(\mathbf{0}, r)\right)
$$

Using $\mathbf{A}^{k} T=T+\mathcal{D}_{k}$, we deduce that

$$
\begin{equation*}
\left(T+\mathcal{D}_{k}\right) \cap \mathbf{A}^{k} B\left(x_{0}, r\right)=\mathbf{A}^{k} x_{0}+\left(\mathbf{A}^{k} C \cap A^{k} B(\mathbf{0}, r)\right) . \tag{9.2}
\end{equation*}
$$

Notice that $B\left(\mathbf{A}^{k} x_{0}, r \ell_{k}\right) \subset \mathbf{A}^{k} B\left(x_{0}, r\right)$. Let

$$
\mathcal{J}_{k}=\left\{t \in \mathcal{D}_{k} ; T+t \text { intersects } \mathbf{A}^{k} x_{0}+\mathbf{A}^{k} C \cap B\left(\mathbf{0}, r \ell_{k}-\operatorname{diam}(T)\right)\right\}
$$

Since $T+\mathcal{D}_{k}$ is a covering of $\mathbf{A}^{k} x_{0}+\left(\mathbf{A}^{k} C \cap \mathbf{A}^{k} B(\mathbf{0}, r)\right), T+\mathcal{J}_{k}$ is a covering of

$$
\mathbf{A}^{k} x_{0}+\mathbf{A}^{k} C \cap B\left(\mathbf{0}, r \ell_{k}-\operatorname{diam}(T)\right)
$$

On the other hand, $T+\mathcal{J}_{k} \subset T+\mathcal{D}_{k}=\mathbf{A}^{k} T$, and clearly $T+\mathcal{J}_{k} \subset B\left(\mathbf{A}^{k} x_{0}, r \ell_{k}\right) ;$ these together with (9.2) imply that

$$
T+\mathcal{J}_{k} \subset \mathbf{A}^{k} T \cap B\left(\mathbf{A}^{k} x_{0}, r \ell_{k}\right) \subset \mathbf{A}^{k} x_{0}+\mathbf{A}^{k} C
$$

which proves that $\left(T, \mathcal{J}_{k}-\mathbf{A}^{k} x_{0}\right)$ is a packing of $\mathbf{A}^{k} C$.
Let $k$ be large enough so that $\ell_{k} r>2 \operatorname{diam}(T)$, then $\left(T, \mathcal{J}_{k}-\mathbf{A}^{k} x_{0}\right)$ is a 'large' local tiling of $\mathbf{A}^{k} C$. Hence, by Theorem 1.1, $A^{k} C$, and also $C$, are regular, and $T$ is a finite union of translations of $[0,1]^{n}$ up to a linear transformation.

## References

[1] C. Bandt, Self-similar sets. V. Integer matrices and fractal tilings of $\mathbb{R}^{n}$. Proc. Amer. Math. Soc. 112(1991), 549-562.
[2] C. Davis, Theory of positive linear dependence, Amer. J. Math. 76 (1954), No. 4, 733-746.
[3] N. G. de Bruijn, On number systems, Nieuw Arch. Wisk. (3) 4(1956), 15-17.
[4] M. Gerstenhaber, Theory of convex polyhedral cones. Chp. XVIII of Cowles Commission Monograph No. 13, Activity analysis of production and allocation, ed. T. C. Koopmans, Wiley, New York, 1951.
[5] K. Gröchenig and Haas, Self-similar lattice tilings, J. Fourier Anal. Appl. 1(1994),131-170.
[6] B. Grümbaum and G. C. Shephard, Tilings and Patterns. W.H. Freedman and Company, New York, 1987.
[7] J. E. Hutchinson, Fractals and self-similarity. Indiana Univ. Math. J. 30(1981), 713-747.
[8] R. Kenyon, Self-replicating tilings, in Symbolic dynamics and its applications, Contemporary mathematics series,(P. Walters,ed.), American Mathematical Society, Providence, RI,vol. 135, 1992, pp. 239-263.
[9] R.Kenyon, Projecting the one-dimensional Sierpinski gasket, Israel J. Math, 97(1997), 221-238.
[10] K. S. Lau, and H. Rao, On one-dimensional self-similar tilings and pq-tiles,Trans. Amer. Math. Soc.,355(2003),1401-1414.
[11] C. K. Lai, K. S. Lau, and H. Rao, Spectral structure of digit sets of self-similar tiles on $\mathbb{R}^{1}$,Trans. Amer. Math. Soc., 365(2013),3831-3850.
[12] C. K. Lai, K. S. Lau, and H. Rao, Classification of tile digit sets as product-forms, Trans. Amer. Math. Soc., 369(2017),623-644.
[13] J. C. Lagarias and Y. Wang, Self-affine tiles in $\mathbb{R}^{n}$. Adv. Math. 121(1996), 21-49.
[14] J. C. Lagarias and Y. Wang, Integral self-affine tiles in $\mathbb{R}^{n}$. I. Standard and nonstandard digit sets. J. London Math. Soc.(2) 54(1996), 161-179.
[15] J. C. Lagarias and Y. Wang, Integral self-affine tiles in $\mathbb{R}^{n}$. II. Lattice tilings. J. Fourier Anal. Appl. 3(1997), 83-102.
[16] I. Niven, A characterization of complementing sets of pairs of integers, Duke Math. J. 38 (1971), 193-203.
[17] A. M. Odlyzko, Non-negative digit sets in positional number systems, Proc. London Math. Soc., 37(1978), 213-229.
[18] T. Rockafellar, Convex Analysis, Princeton University Press, 1970.
[19] H. Rao, Y.M. Yang and Y. Zhang, Characterization of complementing pairs of $\left(\mathbb{Z}^{+}\right)^{n}$. Preprint 2019.
[20] A. Vince, Digit tiling of Euclidean space. Directions in Mathematical Quasicrystals. Amer. Math. Soc., Providence, RI, 2000, 329-370.

## Appendix A. Proof of Lemma 7.2,

Proof. For simplicity, we identify $\mathbb{R}^{2}$ to the complex plane $\mathbb{C}$. Let $L_{0}=\{x+h \mathbf{i} ; x \in[a, b]\}$ be the base line with longer line. Assume that $t_{1}, t_{2}, \cdots, t_{p}$ are points in $L$ from left to right. Suppose on the contrary that $\left(T,\left\{t_{0}, t_{1}, \ldots, t_{p}\right\}\right)$ is a tiling of $A$. Let

$$
I=\left\{x+y(a+h \mathbf{i}) ; x \in\left[0, t_{1}\right], y \in[0,1]\right\}
$$

be a parallelogram on the left part of $A$. Clearly $I \subset T$.
Let $M$ be the largest integer such that $M t_{1}<1$. We claim that $T \cap\left(\bigcup_{m=0}^{M-1}\left(I+m t_{1}\right)\right)$ is a union of translations of $I$. To prove this, we need only prove the following two statements: for each integer $m, 0 \leq m \leq M-1$, we have
(i) $\mathcal{J} \cap\left[0, m t_{1}\right] \subset t_{1} \mathbb{Z}^{+}$;
(ii) For every integer $0 \leq u \leq m, I+u t_{1}$ belongs to one tile except a measure zero set.

We prove (i) and (ii) by induction on $m$. Clearly (i) and (ii) holds for $m=0$. Now we assume that (i) and (ii) holds for $m-1$ with $m \geq 1$.

First, we prove (i). If $t \in \mathcal{J} \cap\left[0, m t_{1}\right)$, then $t+I$ and $\bigcup_{j=0}^{m-1}\left(I+j t_{1}\right)$ overlap, we have $t \notin\left((m-1) t_{1}, m t_{1}\right)$ by the induction hypothesis of (ii). Therefore, by the induction hypothesis of (i), no matter $m t_{1} \in \mathcal{J}$ or not, (i) holds for $m$.

Now we prove (ii). Suppose on the contrary that (ii) is false. Then $I+m t_{1}$ does not belong to one tile. This first implies that $m t_{1} \notin \mathcal{J}$. Secondly, if there exists $1 \leq$ $m^{\prime} \leq m-1$, such that $m^{\prime} t_{1} \in \mathcal{J}$ and $0<\mu\left(\left(T+m^{\prime} t_{1}\right) \cap\left(I+m t_{1}\right)\right)<\mu(I)$, then $0<\mu\left(T \cap\left(I+\left(m-m^{\prime}\right) t_{1}\right)\right)<\mu(I)$, which contradicts the assumption (ii). Therefore, if a tile $T+t$ satisfying that $0<\mu\left((T+t) \cap\left(I+m t_{1}\right)\right)<\mu(I)$, then either $t=0$, or $m t_{1}<t<(m+1) t_{1}$. In the latter case, there is only one $t$ satisfying this property, and we denote it by $t^{*}$. Then

$$
I+m t_{1} \subset T \cup\left(T+t^{*}\right)
$$

Denote $U=\left\{x+y(a+h \mathbf{i}) ; x \in\left[m t_{1}, t^{*}\right), y \in[0,1]\right\}$. By $U \cap\left(T+t^{*}\right)=\emptyset$ and the above equation, we have $U \subset T$. Then the intersection of $T+t_{1}$ and $T+t^{*}$ contains $U+t_{1}$ as a subset, which is a contradiction. So (ii) holds for $m$.

Since $t_{p}$ is the rightmost point of $\mathcal{J}, T+t_{p}$ must contains a relative neighborhood $B(1, r) \cap A$ of 1 , for all small enough $r\left(<1-M t_{1}\right)$. Moreover, we have $B(1, r) \cap A=$ $\left(T+t_{p}\right) \cap B(1, r)$, thus

$$
\begin{equation*}
(B(1, r) \cap A)-t_{p}=T \cap B\left(1-t_{p}, r\right) . \tag{A.1}
\end{equation*}
$$

On the other hand, since $0 \leq 1-t_{p} \leq M t_{1}$ and $T \cap\left(\bigcup_{m=0}^{M-1}\left(I+m t_{1}\right)\right)$ is a union of translations of $I$, then for small enough $r, T \cap B\left(1-t_{p}, r\right)$ is a half ball or a translation of $I \cap B(0, r)$, or a translation of $I \cap B\left(t_{1}, r\right)$, which contradicts with the shape of $T \cap B\left(1-t_{p}, r\right)$ in (A.1). The lemma is proved.

Institute of applied mathematics, College of Science, Huazhong Agriculture of UniverSity, Wuhan, 430070, China.

E-mail address: yangym09@mail.hzau.edu.cn

Department of Mathematics and Statistics, Central China Normal University, Wuhan, 430079, China,

E-mail address: yzhang@mail.ccnu.edu.cn

