UNAVOIDABLE MINORS FOR GRAPHS WITH LARGE ℓ_p -DIMENSION

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ABSTRACT. A metric graph is a pair (G, d), where G is a graph and $d : E(G) \to \mathbb{R}_{\geq 0}$ is a distance function. Let $p \in [1, \infty]$ be fixed. An *isometric embedding* of the metric graph (G, d) in $\ell_p^k = (\mathbb{R}^k, d_p)$ is a map $\phi : V(G) \to \mathbb{R}^k$ such that $d_p(\phi(v), \phi(w)) = d(vw)$ for all edges $vw \in E(G)$. The ℓ_p -dimension of G is the least integer k such that there exists an isometric embedding of (G, d) in ℓ_p^k for all distance functions d such that (G, d) has an isometric embedding in ℓ_p^K for some K.

It is easy to show that ℓ_p -dimension is a minor-monotone property. In this paper, we characterize the minor-closed graph classes C with bounded ℓ_p -dimension, for $p \in \{2, \infty\}$. For p = 2, we give a simple proof that C has bounded ℓ_2 -dimension if and only if C has bounded treewidth. In this sense, the ℓ_2 -dimension of a graph is 'tied' to its treewidth.

For $p = \infty$, the situation is completely different. Our main result states that a minorclosed class C has bounded ℓ_{∞} -dimension if and only if C excludes a graph obtained by joining copies of K_4 using the 2-sum operation, or excludes a Möbius ladder with one 'horizontal edge' removed.

1. INTRODUCTION

In this paper, we consider isometric embeddings of metric graphs in metric spaces. Recall that a metric space (X, d) consists of a set of points X and a metric $d : X \times X \to \mathbb{R}_{\geq 0}$. That is, for all $x, y, z \in X$, (i) d(x, y) = d(y, x), (ii) d(x, y) = 0 if and only if x = y, and (iii) $d(x, y) \leq d(x, z) + d(z, y)$. Here, we only consider the metric spaces $\ell_p^k = (\mathbb{R}^k, d_p)$, focusing mainly on the cases $p \in \{2, \infty\}$. We let N denote the set of positive integers, and for $k \in \mathbb{N}$, $[k] = \{1, \ldots, k\}$. Recall that $||x||_p = (\sum_{i=1}^k |x|^p)^{1/p}$ if $p \in [1, \infty)$ and $||x||_{\infty} = \max_{i \in [k]} |x_i|$. We set $d_p(x, y) = ||x - y||_p$ for all $p \in [1, \infty]$.

Comparing different metric spaces is a ubiquitous theme throughout mathematics. One way to do so is by means of *isometric embeddings*, which are functions $\phi : X \to X'$ such that $d(x, y) = d'(\phi(x), \phi(y))$ for all $x, y \in X$. As these are quite restrictive, other approaches have been developed. For instance, Bourgain [5] has shown that every *n*-point metric space can be embedded into an $\ell_p^{O(\log^2 n)}$ space with $O(\log n)$ distortion. (The upper bound on the dimension was subsequently reduced to $O(\log n)$, see [1].)

Another point of view is to require only a subset of distances to be preserved, which is the perspective we take in this paper. Our methods are mostly graph theoretical, although similar problems have been studied using techniques from rigidity theory [15, 21, 22].

All graphs in this paper are finite and do not contain loops or parallel edges, unless otherwise stated. A graph H is a *minor* of a graph G if H can be obtained from a subgraph of G by contracting some edges. When taking minors we remove parallel edges and loops resulting from edge contractions.

A metric graph (G, d) is a pair consisting of a graph G and a function $d : E(G) \to \mathbb{R}_{\geq 0}$ satisfying $d(vw) \leq d(P) = \sum_{i=1}^{r} d(v_{i-1}v_i)$ for all edges $vw \in E(G)$ and all paths $P = v_0v_1 \cdots v_r$

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FIGURE 1. The excluded minors for $f_{\infty}(G) \leq 2$.

with $v_0 = v$ and $v_r = w$. Such a function d is called a *distance function* on G. An *isometric embedding* of a metric graph (G, d) in ℓ_p^k is a map $\phi : V(G) \to \mathbb{R}^k$ such that $d_p(\phi(v), \phi(w)) = d(vw)$ for all edges $vw \in E(G)$.

For each $p \in [1, \infty]$ and graph G, a distance function $d : E(G) \to \mathbb{R}_{\geq 0}$ is ℓ_p -realizable if it has an isometric embedding in ℓ_p^K for some K. If d is ℓ_p -realizable, we define the parameter $f_p(G, d)$ to be the least integer k such that (G, d) can be isometrically embedded in ℓ_p^k . The ℓ_p -dimension of G is defined to be $f_p(G) = \sup_d f_p(G, d)$, where the supremum is over all ℓ_p realizable distance functions d on G. We remark that in the special case $p = \infty$, the supremum is taken over all distance functions on G, since it is well-known that every n-point metric space can be isometrically embedded into ℓ_{∞}^{n-1} . It is known that ℓ_p -dimension is always at most $\binom{|V(G)|}{2}$, see [2] and [8, Proposition 11.2.3]. The ℓ_2 -dimension is also referred to as *Euclidean* dimension.

It is easy to see that every minor H of G satisfies $f_p(H) \leq f_p(G)$ for all $p \in [1, \infty]$. Hence the property $f_p(G) \leq k$ is closed under taking minors. By the Graph Minor Theorem of Robertson and Seymour [19], for each k, there are only a finite number of minor-minimal graphs satisfying $f_p(G) > k$. Formally, an *excluded minor* for $f_p(G) \leq k$ is a graph H such that $f_p(H) > k$ and every proper minor H' of H satisfies $f_p(H') \leq k$.

The complete sets of excluded minors are known in the Euclidean case p = 2 for dimensions k = 1, 2, 3. Belk and Connelly [3, 4] have shown that $\{K_3\}$, $\{K_4\}$, $\{K_5, K_{2,2,2}\}$ are the respective sets of excluded minors. Furthermore, note that $\ell_p^1 = \ell_q^1$ for all $p, q \in [1, \infty]$. Therefore, for all $p \in [1, \infty]$, K_3 is the only excluded minor for $f_p(G) \leq 1$. Fiorini, Huynh, Joret, and Varvitsiotis [13] determined that W_4 , the wheel on 5 vertices, and the graph $K_4 +_e K_4$ (see Figure 1) are the only excluded minors for $f_{\infty}(G) \leq 2$ and for $f_1(G) \leq 2$. As far as we know, the complete set of excluded minors for $f_p(G) \leq k$ is unknown for all other values of p and k.

It is plausible that determining any further set of excluded minors will require significant effort, especially in dimension 3 or higher (see [17]). Therefore, instead of obtaining exact characterizations of the graphs with $f_p(G) \leq k$, we take a different approach and seek collections of *unavoidable minors*. That is, for each $k \in \mathbb{N}$, we look for a finite collection of graphs \mathcal{U}_p^k and an integer $c_p(k)$, such that every graph $H \in \mathcal{U}_p^k$ has $f_p(H) > k$, and every graph Gwith $f_p(G) > c_p(k)$ has a minor in \mathcal{U}_p^k .

For the case p = 2, we show that grids are unavoidable minors, see Theorem 3 in Section 2. Most of the paper is devoted to the case $p = \infty$, which turns out to be much more challenging. Our main result is Theorem 1 that gives unavoidable minors for $p = \infty$.

Now, we introduce the four graphs S_k , P_k , F_k and N_k that form \mathcal{U}_{∞}^k for each $k \in \mathbb{N}$. Examples of all four graphs are given in Figure 2. The first three graphs are obtained by gluing together k copies of K_4 in a certain way, and then deleting each edge that is common to at least two copies. The graph S_k is obtained by gluing the k copies of K_4 along one common edge. The graph P_k is obtained by picking a perfect matching $\{e_i, f_i\}$ in each copy of K_4 , and identifying f_i and e_{i+1} for all $i \in [k-1]$. The graph F_k is constructed in a similar way, except that we



FIGURE 2. The graphs S_5 , P_5 , F_5 and N_5 .

take e_i and f_i to be incident edges. Edges are identified in such a way that the common end of e_i and f_i is identified to the common end of e_{i+1} and f_{i+1} for all $i \in [k-1]$. The notation for these first three families reflect the fact that the corresponding copies of K_4 are arranged as a star, path, and fan, respectively. Notice that $S_2 = P_2 = F_2 = K_4 + K_4$, which is one of the excluded minors for $f_{\infty}(G) \leq 2$. Next, we define our final family of graphs. The graph N_k is the graph with $V(N_k) = \{v_0, \ldots, v_k\} \cup \{w_0, \ldots, w_k\}$ and

$$E(\mathsf{N}_k) = \{v_{i-1}v_i, v_iw_i, v_{i-1}w_i, w_{i-1}w_i \mid i \in [k]\} \cup \{v_0w_0, w_0v_k\}$$

For each $k \in \mathbb{N}$, we let $\mathcal{U}_{\infty}^{k} = \{\mathsf{S}_{k}, \mathsf{P}_{k}, \mathsf{F}_{k}, \mathsf{N}_{k}\}$. We say that a graph *G* contains a \mathcal{U}_{∞}^{k} minor if it contains $\mathsf{S}_{k}, \mathsf{F}_{k}, \mathsf{P}_{k}$ or N_{k} as a minor. Our main theorem shows that if $f_{\infty}(G)$ is large, then *G* necessarily contains a \mathcal{U}_{∞}^{k} minor.

Theorem 1. There exists a computable function $g_1 : \mathbb{N} \to \mathbb{R}$ such that for every $k \in \mathbb{N}$, every graph G with $f_{\infty}(G) > g_1(k)$ contains a \mathcal{U}_{∞}^k minor. Moreover, every graph G that contains a \mathcal{U}_{∞}^k minor has $f_{\infty}(G) > k$.

Let $S = \bigcup_k \{S_k\}, \mathcal{F} = \bigcup_k \{F_k\}, \mathcal{P} = \bigcup_k \{P_k\}, \text{ and } \mathcal{N} = \bigcup_k \{N_k\}$. For a class of graphs C and $p \in [1, \infty]$, we let $f_p(C) = \max\{f_p(G) \mid G \in C\}$, if this number is finite, and $f_p(C) = \infty$, otherwise. As an immediate corollary, our main theorem gives an exact characterization of all minor-closed classes C with $f_\infty(C) = \infty$.

Corollary 2. For all minor-closed classes of graphs C, $f_{\infty}(C) = \infty$ if and only if $S \subseteq C$ or $\mathcal{F} \subseteq C$ or $\mathcal{P} \subseteq C$ or $\mathcal{N} \subseteq C$.

The rest of the paper is organized as follows. In Section 2, we establish that grids are unavoidable minors for large ℓ_2 -dimension. In Section 3, we give a more combinatorial definition of ℓ_{∞} -dimension. In Section 4, we establish some lemmas on ℓ_{∞} -dimension to be used later.

We establish the second part of our main result, Theorem 1, in Section 5, by constructing on each graph $G \in \mathcal{U}_{\infty}^k$ a distance function d that allows us to show $f_{\infty}(G, d) > k$ in a simple, combinatorial way.

In order to prove the first part of Theorem 1, we consider a graph G without a \mathcal{U}_{∞}^k minor and set out to prove that we can upper bound $f_{\infty}(G)$ by some integer $g_1(k)$.

It is straightforward to show that the ℓ_{∞} -dimension of a graph is the maximum ℓ_{∞} -dimension of one of its blocks (see Lemma 12). Therefore, we may assume that G is 2-connected. In Section 6, we prove that we can essentially assume that G is 3-connected. This part relies on SPQR trees.

The 3-connected case is the part of the proof requiring most of the work. The proof techniques here are mostly graph-theoretic, and may be of independent interest. This is done in Section 7 and Section 8.

2. The Euclidean case

The goal of this section is to establish that grids are a collection of unavoidable minors for large Euclidean dimension, which is the analogue of Theorem 1 for ℓ_2 -dimension.

Let $r \in \mathbb{N}$. Recall that the square grid graph \Box_r is the graph with vertex set $[r] \times [r]$, where (i, j) is adjacent to (i', j') if and only if |i - i'| + |j - j'| = 1. The triangular grid graph \triangle_r has vertex set $V(\triangle_r) = \{v_{i,j} \mid i, j \in [r], i \leq j\}$ and edge set $E(\triangle_r) = \{v_{i,j}v_{k,\ell} \mid v_{i,j}, v_{k,\ell} \in V(\triangle_r), (i - k, j - \ell) \in \{\pm(1, 0), \pm(0, 1), \pm(1, 1)\}\}.$

Let G and H be graphs such that H is a minor of G. Then G contains an H-model, that is, a collection $\{X_v \mid v \in V(H)\}$ of disjoint subsets $X_v \subseteq V(G)$ each inducing a connected subgraph of G such that for every edge $vw \in E(H)$ there is an edge of G with one end in X_v and the other in X_w . The sets X_v are called the *vertex images*. The following is the main result of this section.

Theorem 3. There exists a function $g_3(k) = O(k^9 \operatorname{polylog}(k))$ such that every graph G with $f_2(G) > g_3(k)$ contains a \triangle_{k+2} minor. Moreover, every graph G that contains a \triangle_{k+2} minor has $f_2(G) > k$.

In order to prove the first part of Theorem 3, we use the by now standard notion of *treewidth* (see [10] for the definition). We let tw(G) denote the treewidth of a graph G. As observed by Belk and Connelly [4], $f_2(G) \leq tw(G)$ holds for all graphs G. Thus if $f_2(G) > c$, then tw(G) > c.

By the grid theorem [18], there is a function $\gamma(k)$ such that every graph G with $\operatorname{tw}(G) \geq \gamma(k)$ contains \Box_k as a minor. In fact, one can take $\gamma(k) = O(k^9 \operatorname{polylog}(k))$ by very recent results [7] (see [6] for the original polynomial grid theorem). Furthermore, it is easy to check that \Box_{2k+2} has a Δ_{k+2} minor, for all $k \in \mathbb{N}$. Figure 3 illustrates this for k = 4. Therefore, in Theorem 3, we may take $g_3(k) = \gamma(2k+2)$. This proves the first part of the theorem. Notice that for all $r \in \mathbb{N}$, Δ_r has \Box_m as a subgraph, where $m = \lfloor \frac{r-1}{2} \rfloor$. Thus, excluding triangular grids is equivalent to excluding rectangular grids within a factor of 2.



FIGURE 3. On the left is \triangle_6 . On the right is a \triangle_6 -model in \square_{10} . Vertex images are displayed in red, and edges between the vertex images in black or blue.

We now prove the second part of Theorem 3, see Lemma 4 below. We remark that Eisenberg-Nagy, Laurent and Varvitsiotis [11] prove a similar result for a related invariant called *extreme* Gram dimension. This is a variant of the Gram dimension of a graph, that is studied and compared to the Euclidean dimension in Laurent and Varvitsiotis [16]. The idea of considering a triangular grid instead of a rectangular one comes from [11], and our induction-based proof is inspired by their proof. However, to our knowledge, the results of [16] and [11] do not imply our next lemma.

Lemma 4. For all $r \in \mathbb{N}$, $f_2(\Delta_r) \ge r - 1$.

Proof. Let e_1, \ldots, e_r be the r standard basis vectors in \mathbb{R}^r . We recursively define an embedding $\phi: V(\Delta_r) \to \mathbb{R}^r$ by $\phi(v_{1,j}) = e_j$ for all $j \in [r]$ and $\phi(v_{i,j}) = \frac{1}{2}\phi(v_{i-1,j-1}) + \frac{1}{2}\phi(v_{i-1,j})$ for all $2 \leq i \leq j$. We define an ℓ_2 -realizable distance function $d: E(\Delta_r) \to \mathbb{R}_+$ from the embedding ϕ , by letting $d(vv') = ||\phi(v) - \phi(v')||_2$ for each $vv' \in E(\Delta_r)$.

Now consider an arbitrary isometric embedding ψ of (\triangle_r, d) in some Euclidean space \mathbb{E} . By our choice of the distance function, $\psi(v_{i,j})$ is the midpoint of $\psi(v_{i-1,j-1})$ and $\psi(v_{i-1,j})$ for every $i \ge 2$. Hence, the whole embedding ψ is entirely determined by the r points $q_j = \psi(v_{1,j})$, and lies in the affine hull of q_1, \ldots, q_r . By applying an appropriate isometry, we may assume that $\mathbb{E} = \{x \in \mathbb{R}^r \mid \sum_i x_i = 1\}$. We claim that $||q_i - q_j||_2 = \sqrt{2}$ for all distinct $i, j \in [r]$. Hence, these r points are the vertices of a regular simplex, which implies $f_2(G, d) \ge r - 1$.

The proof is by induction on r. Since the statement is clear for r = 2, we may assume that $r \geq 3$. Observe that the induced subgraphs $\Delta_r - \{v_{i,r} \mid i \in [r]\}$ and $\Delta_r - \{v_{i,i} \mid i \in [r]\}$ are both isomorphic to Δ_{r-1} . By the inductive hypothesis, this implies that q_1, \ldots, q_{r-1} are equidistant, and q_2, \ldots, q_r are equidistant. Thus, it remains to show $||q_1 - q_r||_2 = \sqrt{2}$.

Since $||q_i - q_j||_2 = \sqrt{2}$ for all distinct $i, j \in [r-1]$, by applying an appropriate isometry we may assume that $q_k = e_k$ for all $k \in [r-1]$.

Let $x_1, \ldots, x_r \in \mathbb{R}$ denote the coordinates of q_r in \mathbb{R}^r . The following constraints hold:

$$\sum_{i} x_i = 1, \qquad (1)$$

$$\sum_{i} x_{i}^{2} = 1 + 2x_{k} \quad \forall 2 \le k \le r - 1.$$
⁽²⁾

The first constraint is due to the fact that $q_r \in \mathbb{E}$, and the second is equivalent to $||\psi(v_{1,r}) - \psi(v_{1,k})||_2^2 = ||\phi(v_{1,r}) - \phi(v_{1,k})||_2^2$ (for $2 \le k \le r-1$), which holds by induction. Notice that $x_2 = x_3 = \cdots = x_{r-1}$ follows from (2). Since $v_{r-1,r-1}v_{r-1,r}$ is an edge of Δ_r ,

$$||\psi(v_{r-1,r-1}) - \psi(v_{r-1,r})||_2^2 = ||\phi(v_{r-1,r-1}) - \phi(v_{r-1,r})||_2^2.$$
(3)

Since $\psi(v_{1,j}) = \phi(v_{1,j})$ for all $j \in [r-1]$, $\psi(v_{i,j}) = \phi(v_{i,j})$ for all $i \leq j \leq r-1$. Hence, we can rewrite the left-hand side of (3) as

$$\begin{aligned} ||\psi(v_{r-1,r-1}) - \psi(v_{r-1,r})||_2^2 &= ||\phi(v_{r-1,r-1}) - \psi(v_{r-1,r})||_2^2 \\ &= ||(\phi(v_{r-1,r-1}) - \phi(v_{r-1,r})) - (\psi(v_{r-1,r}) - \phi(v_{r-1,r}))||_2^2 \end{aligned}$$

Thus, (3) holds if and only if

$$||\psi(v_{r-1,r}) - \phi(v_{r-1,r})||_2^2 = 2 \langle \phi(v_{r-1,r}) - \phi(v_{r-1,r}), \psi(v_{r-1,r}) - \phi(v_{r-1,r}) \rangle .$$
(4)
By induction, we see that, for all $i \in [r-1]$,

$$\psi(v_{i,r}) - \phi(v_{i,r}) = \frac{1}{2^{i-1}}(\psi(v_{1,r}) - \phi(v_{1,r})) = \frac{1}{2^{i-1}}(q_r - e_r).$$

Using this, we can rewrite the left-hand side of
$$(4)$$
:

$$\begin{aligned} ||\psi(v_{r-1,r}) - \phi(v_{r-1,r})||_2^2 &= \left(\frac{1}{2^{r-2}}\right)^2 ||q_r - e_r||_2^2 \\ &= \frac{1}{2^{2r-4}} (||q_r||_2^2 + ||e_r||_2^2 - 2\langle q_r, e_r \rangle) \\ &= \frac{1}{2^{2r-4}} (1 - 2x_2 + 1 - 2x_r) \,. \end{aligned}$$

Notice that, since $x_2 = x_3 = \ldots = x_{r-1}$,

$$q_r - e_r = x_2 \mathbf{1} + (x_1 - x_2)e_1 + (x_r - x_2 - 1)e_r$$

where $\mathbf{1}$ is the all-ones vector. Also, an easy induction on i shows that

$$\langle \phi(v_{i,i}), e_1 \rangle = \frac{1}{2^{i-1}} = \langle \phi(v_{i,r}), e_r \rangle ,$$

and thus

$$\langle \phi(v_{i,i}) - \phi(v_{i,r}), e_1 \rangle = \frac{1}{2^{i-1}}, \text{ and}$$

 $\langle \phi(v_{i,i}) - \phi(v_{i,r}), e_r \rangle = -\frac{1}{2^{i-1}}.$

Now, we can rewrite the right-hand side of (4) as

$$\frac{1}{2^{r-3}} \langle \phi(v_{r-1,r}) - \phi(v_{r-1,r}), q_r - e_r \rangle
= \frac{1}{2^{r-3}} \langle \phi(v_{r-1,r}) - \phi(v_{r-1,r}), x_2 \mathbf{1} + (x_1 - x_2)e_1 + (x_r - x_2 - 1)e_r \rangle
= \frac{1}{2^{r-3}} \left(0 + \frac{1}{2^{r-2}}(x_1 - x_2) - \frac{1}{2^{r-2}}(x_r - x_2 - 1) \right).$$

Hence, (4) can be rewritten

$$\frac{1}{2^{2r-4}}(1-2x_2+1-2x_r) = \frac{1}{2^{r-3}}\left(\frac{1}{2^{r-2}}(x_1-x_2) - \frac{1}{2^{r-2}}(x_r-x_2-1)\right) \iff x_2 = -x_1$$

Now,

$$||q_r - q_1||_2^2 = ||q_r - e_1||_2^2 = \sum_i x_i^2 + 1 - 2x_1 = (1 - 2x_2) + 1 - 2x_1 = (1 + 2x_1) + 1 - 2x_1 = 2. \quad \Box$$

It is easy to check that $tw(\Delta_r) \leq r-1$ for all $r \geq 3$. Thus, Lemma 4 implies that $f_2(\Delta_r) = r-1$ for all $r \geq 3$. Moreover, since every planar graph is a minor of a sufficiently large triangular grid, Theorem 3 immediately yields the following corollary.

Corollary 5. For all minor-closed classes of graphs C, $f_2(C) = \infty$ if and only if C contains all planar graphs.

3. Alternative view of ℓ_{∞} -dimension

In this section, we provide a more combinatorial definition of ℓ_{∞} -dimension. The equivalence follows by considering potentials on a weighted auxilliary digraph.

Let D be a digraph with edge weights $l : A(D) \to \mathbb{R}$. A potential on (D, l) is a function $p: V(D) \to \mathbb{R}$ such that $p(w) - p(v) \le l(v, w)$ for all arcs $(v, w) \in A(D)$.

Now consider a metric graph (G, d). Let (D, l) be the (edge)-weighted digraph obtained from (G, d) by bidirecting all edges and setting l(v, w) = l(w, v) = d(vw) for all edges $vw \in E(G)$. Note that $p: V(D) \to \mathbb{R}$ is a potential on (D, l) if and only if $|p(w) - p(v)| \leq d(vw)$ for all edges $vw \in E(G)$.

For convenience, we let D(G) and l(d) denote the digraph and edge weights defined above, respectively. Thus the weighted digraph (D, l) we are considering can also be denoted (D(G), l(d)) when more precision is required.

Recall that distances in ℓ_{∞}^k are given by $d_{\infty}(x, y) = \max_{i \in [k]} |x_i - y_i|$. Hence $d_{\infty}(x, y) = \delta$ if and only if $|x_i - y_i| \leq \delta$ for all $i \in [k]$ and there exists some index $j \in [k]$ for which $|x_j - y_j| = \delta$. Therefore, (G, d) has an isometric embedding ϕ in ℓ_{∞}^k if and only if there exist k potentials $p_i : V(G) \to \mathbb{R}$ on (D, l) such that for each edge vw there is at least one index $j \in [k]$ with $|p_j(w) - p_j(v)| = d(vw)$. This can be seen by taking $p_i(v)$ to be the *i*-th coordinate of $\phi(v)$, for all $i \in [k]$ and $v \in V(G)$. We say that a set of arcs $F \subseteq A(D)$ is a *flat set* of (G, d) if there exists a potential $p: V \to \mathbb{R}$ on (D, l) such that $p(w) - p(v) = -d(vw) \iff p(v) - p(w) = d(vw)$ for all arcs $(v, w) \in F$. Given a set $F \subseteq A(D)$, consider the modified edge weights $l_F: A(D) \to \mathbb{R}$ such that

$$l_F(v,w) = \begin{cases} d(vw) & \text{if } (v,w) \notin F \\ -d(vw) & \text{if } (v,w) \in F. \end{cases}$$

When necessary, we denote these edge weights by $l_F(d)$. Then $F \subseteq A(D)$ is a flat set of (G, d) if and only if $(D, l_F) = (D(G), l_F(d))$ admits a potential. By the well-known characterization of the existence of potentials, this is equivalent to the non-existence of a negative weight directed cycle in (D, l_F) . That is, $F \subseteq A(D)$ is a flat set if and only if (D, l_F) does not contain a negative directed cycle. In proofs, we will often use the notation $\langle G, d; F \rangle$ to denote $(D(G), l_F(d))$. Notice that F is a flat set if and only if $F' = \{(w, v) \mid (v, w) \in F\}$ is a flat set.

We say that a flat set $F \subseteq A(D)$ covers an edge $vw \in E(G)$ if F contains (v, w) or (w, v). A flat covering of (G, d) is a collection $\mathcal{F} = \{F_1, \ldots, F_k\}$ of flat sets such that every edge $vw \in E(G)$ is covered by at least one F_i . Then, (G, d) has an isometric embedding into ℓ_{∞}^k if and only if (G, d) has a flat covering of size at most k. To construct an embedding given a flat covering, we pick a potential p_i on $\langle G, d; F_i \rangle$ for each flat set F_i , and use these potentials to define the embedding coordinatewise. That is, each potential p_i associated to F_i gives us the *i*-th coordinate of the vertices in the embedding. Notice that the potentials respect the maximum differences given by the distance function d. Furthermore, because each edge is covered by some potential, the vertices of this edge are at exact distance in the corresponding coordinate. Hence we get an embedding of (G, d). For the other direction, it is sufficient to realize that each coordinate of an embedding defines a potential. Furthermore, for each edge at least one of the potentials defined by the coordinates is such that the distance between the vertices is attained with equality, that is the edge is covered by this potential. Thus, the coordinates define a flat covering of size k.

In our terminology, the ℓ_{∞} -dimension $f_{\infty}(G)$ is the least integer k such that for each distance function d, the metric graph (G, d) has a flat covering of size at most k.

4. Metric tools

In this section, we present several general results related to distance functions and flat coverings.

Given a vertex v of a graph G, we let $N(v) = \{w \in V(G) \mid vw \in E(G)\}$ denote the neighborhood of v in G.

Lemma 6. Let (G, d) be a metric graph and let $v \in V(G)$. The set $F = \{(v, w) \mid w \in N(v)\}$ is a flat set of (G, d).

Proof. Let C be an arbitrary directed cycle in $\langle G, d; F \rangle$. The cycle C uses at most one arc of F. Thus at most one arc of C has negative weight in $\langle G, d; F \rangle$, and all other arcs of C have non-negative weight. Since d is a distance function, it follows that C has non-negative weight in $\langle G, d; F \rangle$. Thus, F is a flat set of (G, d), as required.

A vertex cover of a graph G is a set of vertices $X \subseteq V(G)$ such that every edge of G is incident with some vertex in X. The vertex cover number of G, denoted $\tau(G)$, is the size of a smallest vertex cover of G. By Lemma 6, $f_{\infty}(G)$ is at most the vertex cover number of G.

Lemma 7 ([13], Lemma 9). For every graph G, $f_{\infty}(G) \leq \tau(G)$.

Clearly, if d is a distance function on G, and H is a subgraph of G, then the restriction of d to E(H) is a distance function on H. We denote it by $d|_{H}$. Conversely, sometimes we can define a distance function on a graph from distance functions on certain subgraphs, see Lemma 8 below.

A k-sum is a graph G obtained by gluing two graphs G_1 and G_2 along a common clique K of size k and then possibly deleting some edges of K. We use the following notation for 1-sums and 2-sums. We write $G = G_1 +_v G_2$ if $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v\}$. Now let e = vw be an edge. We write $G = G_1 \oplus_e G_2$ if $G = G_1 \cup G_2$ with $V(G_1) \cap V(G_2) = \{v, w\}$ and $e \in E(G_1) \cap E(G_2)$. Also, we denote by $G_1 +_e G_2$ the graph $G_1 \oplus_e G_2$ minus the edge e.

Lemma 8. Let $G = G_1 \oplus_f G_2$. For $i \in [2]$, let d_i be a distance function on G_i . If $d_1(f) = d_2(f)$, then the function $d : E(G) \to \mathbb{R}_{\geq 0}$ defined by $d(e) = d_i(e)$ if $e \in E(G_i)$ is a distance function on G.

Proof. Let vw be any edge of G. Without loss of generality, we may suppose $vw \in E(G_1)$. Let P be a v-w path in G. If P is contained in G_1 then $d(P) = d_1(P) \ge d_1(vw) = d(vw)$. Otherwise, P uses both ends of f and we may decompose P into a path P_1 from v to an end of f with $E(P_1) \subseteq E(G_1)$, a path P_2 between the two ends of f with $E(P_2) \subseteq E(G_2)$ and a path P'_1 from the other end of f to w with $E(P'_1) \subseteq E(G_1)$. Then we get d(P) = $d(P_1) + d(P_2) + d(P'_1) \ge d(P_1) + d(f) + d(P'_1) \ge d(vw)$, where the first inequality uses that d_2 is a distance function, and the second inequality uses that d_1 is a distance function.

Similarly, every subset of a flat set is flat, and if F is a flat set of (G, d), then F is also a flat set of $(H, d|_H)$, for all subgraphs H of G with $F \subseteq A(D(H))$. The following lemma gives conditions under which a flat set of a subgraph is a flat set of the entire graph.

Lemma 9. Let G be a graph obtained by gluing two graphs G_1 and G_2 along a common clique K. Let d be a distance function on G and $d_i = d|_{G_i}$ its restriction to G_i , where $i \in [2]$. If F is a flat set of (G_j, d_j) for some $j \in [2]$, then F is also a flat set of (G, d). Conversely, if F is a flat set of (G, d) then $F_i = F \cap A(D(G_i))$ is a flat set of (G_i, d_i) for all $i \in [2]$.

Proof. For the first part, it suffices to show that $\langle G, d; F \rangle$ does not contain a negative weight directed cycle. Let C be a minimum weight directed cycle in $\langle G, d; F \rangle$ such that V(C) is inclusion-wise minimal. We may assume that C contains some arc of F, since otherwise C is disjoint from F and has non-negative weight. Thus C intersects $A(D(G_j))$.

We claim that C must be fully contained in $D(G_j)$. Otherwise, C contains a directed path P from v to w, where $v, w \in K$, that is internally disjoint from $D(G_j)$. By replacing P with the arc (v, w) we obtain a new directed cycle C' in $\langle G, d; F \rangle$ whose weight is at most that of C and such that $V(C') \subsetneq V(C)$, a contradiction.

Since C is contained in $D(G_j)$ and F is a flat set of (G_j, d_j) , C has non-negative weight in $\langle G_j, d_j; F \rangle$ and thus in $\langle G, d; F \rangle$.

For the second part, notice that F_i is a flat set of (G, d) because $F_i \subseteq F$ and F is a flat set of (G, d). Since G_i is a subgraph of G, F_i is also clearly a flat set of (G_i, d_i) .

Lemma 10. Let F be a flat set of a metric graph (G, d) and u and v be vertices of G. Let P_1 be a directed path from u to v and let P_2 be a directed path from v to u. Then at least one of P_1 and P_2 has non-negative weight in $\langle G, d; F \rangle$.

Proof. Consider the directed closed walk obtained by concatenating P_1 and P_2 . This directed closed walk decomposes into directed cycles. If P_1 and P_2 both have negative weight in $\langle G, d; F \rangle$, then at least one of these directed cycles has negative weight in $\langle G, d; F \rangle$. But this contradicts the fact that F is a flat set.

In [13], the following result is proved.

Lemma 11 ([13]). For every graph G with $f_{\infty}(G) \geq 2$ and every edge $e \in E(G)$,

 $f_{\infty}(G) = f_{\infty}(G +_e K_3) = f_{\infty}(G \oplus_e K_3).$

Hence, deleting a degree-2 vertex v and adding a new edge between the neighbors of v (if there was none) does not change $f_{\infty}(G)$, provided the resulting graph is not a forest. We

will refer to this operation as suppressing a degree-2 vertex. It follows that for all $k \ge 2$, the excluded minors for $f_{\infty}(G) \le k$ have minimum degree at least 3.

We will use the following bounds on $f_{\infty}(G)$ when G is a k-sum.

Lemma 12. For all graphs G_1 and G_2 (for which the k-sums below exist),

$$f_{\infty}(G_1 +_v G_2) = \max\{f_{\infty}(G_1), f_{\infty}(G_2)\}$$
(5)

and

$$f_{\infty}(G_1 +_{vw} G_2) \le f_{\infty}(G_1 \oplus_{vw} G_2) \le f_{\infty}(G_1) + f_{\infty}(G_2) - 1.$$
(6)

Moreover,

$$f_{\infty}(G) \le f_{\infty}(G_1) + f_{\infty}(G_2) \tag{7}$$

whenever G is a k-sum of G_1 and G_2 .

Proof. Observe that (7) follows from Lemma 9. Next, we prove (5). Let $k = \max\{f_{\infty}(G_1), f_{\infty}(G_2)\}$. Since f_{∞} is minor-monotone, it is clear that $f_{\infty}(G_1 +_v G_2)$ is at least k. The next paragraph proves that it is at most k.

Let d be a distance function on $G_1 +_v G_2$. For $i \in [2]$, let $d_i = d|_{G_i}$. Then d_i is a distance function on G_i . For $i \in [2]$, let ϕ_i be any isometric embedding of (G_i, d_i) into ℓ_{∞}^k . After translating one of the embeddings if necessary, we may assume that $\phi_1(v) = \phi_2(v)$. It is easy to see that the function $\phi : V(G_1 +_v G_2) \to \mathbb{R}^k$ obtained by setting $\phi(w) = \phi_i(w)$ if $w \in V(G_i)$ for $i \in [2]$ is an isometric embedding of $(G_1 +_v G_2, d)$ into ℓ_{∞}^k .

Finally, we prove (6). The first inequality in (6) is trivial since $G_1 +_{vw} G_2$ is a minor of $G_1 \oplus_{vw} G_2$. To prove the second inequality, consider a distance function d on G. For $i \in [2]$, let $d_i = d|_{G_i}$ be the corresponding distance function of G_i .

Let \mathcal{F}_i be a minimum size flat covering of (G_i, d_i) . By Lemma 9, each set in $\mathcal{F}_1 \cup \mathcal{F}_2$ is flat in (G, d). For $i \in [2]$, let F_i be a flat set in \mathcal{F}_i covering vw. By reversing arcs if necessary, we may assume both F_1 and F_2 contain (v, w). We may also assume that neither F_1 nor F_2 contains (w, v), since otherwise we get d(vw) = 0. In this case, we can contract the edge vwand use (5).

We claim that $F_1 \cup F_2$ is a flat set of (G, d). Let C be an arbitrary directed cycle in $\langle G, d; F_1 \cup F_2 \rangle$. For $i \in [2]$, let C_i be the directed cycle obtained by restricting C to $D(G_i)$ and possibly adding (v, w) or (w, v) (possibly $C_i = \emptyset$). Let $l = l_{F_1 \cup F_2}(d)$ be the edge weights on $\langle G, d; F_1 \cup F_2 \rangle$ and $l_i = l_{F_i}(d_i)$ be the edge weights on $\langle G_i, d_i; F_i \rangle$. Notice that l(v, w) = -d(vw) and l(w, v) = d(vw). Then $l(C) = l(C_1) + l(C_2) = l_1(C_1) + l_2(C_2) \ge 0 + 0 = 0$ since l_i is the restriction of l to $A(D(G_i))$ and F_i is flat in (G_i, d_i) . Thus, C has non-negative weight and $F_1 \cup F_2$ is a flat set of (G, d), as claimed.

Now $\mathcal{F} = \{F_1 \cup F_2\} \cup (\mathcal{F}_1 \cup \mathcal{F}_2) \setminus \{F_1, F_2\}$ is a flat covering of (G, d) of size at most $|\mathcal{F}_1| + |\mathcal{F}_2| - 1 \leq f_\infty(G_1) + f_\infty(G_2) - 1$.

Let (G, d) be a metric graph. We say that two edges e and f of G are *incompatible*, if there is no flat set of (G, d) that covers both of them. Note that two such edges are necessarily independent, by Lemma 6. A simple but crucial observation is that if (G, d) contains k pairwise incompatible edges, then $f_{\infty}(G) \geq k$. The following lemma provides sufficient conditions under which two edges are incompatible.

Lemma 13. Let (G, d) be a metric graph and let v_1v_2, w_1w_2 be two independent edges of G. If for all $i, j \in [2]$, there exist paths $P_{i,j}$ between v_i and w_j such that $d(P_{1,1}) + d(P_{2,2}) < d(v_1v_2) + d(w_1w_2)$ and $d(P_{1,2}) + d(P_{2,1}) < d(v_1v_2) + d(w_1w_2)$, then v_1v_2 and w_1w_2 are incompatible.

Proof. Suppose F is a flat set covering v_1v_2 and w_1w_2 . Suppose first $(v_1, v_2), (w_1, w_2) \in F$. Consider the closed directed walk W that starts at v_1 , takes (v_1, v_2) , follows $P_{2,1}$ to w_1 , takes (w_1, w_2) and then follows $P_{1,2}$ back to v_1 . The weight of W in $\langle G, d; F \rangle$ is at most $d(P_{1,2}) + d(P_{2,1}) - d(v_1v_2) - d(w_1w_2) < 0$. Thus, W contains a negative weight directed cycle, which contradicts that F is flat.

By symmetry the remaining case is $(v_1, v_2), (w_2, w_1) \in F$. Again it is easy to find a negative weight directed walk W in $\langle G, d; F \rangle$ using the fact that $d(P_{1,1}) + d(P_{2,2}) < d(v_1v_2) + d(w_1w_2)$. Hence, F cannot simultaneously cover the edges v_1v_2 and w_1w_2 , as claimed.

Finally, we also need the fact that $f_{\infty}(K_4) = 2$.

Lemma 14 ([23], 4.2). $f_{\infty}(K_4) = 2$.

In order to illustrate the concepts introduced in the last two sections, we briefly describe a polynomial reduction from computing the chromatic number of a graph H to computing $f_{\infty}(G,d)$ given a metric graph (G,d). This proves that the latter problem is NP-hard. We remark that there is a different reduction using the PARTITION problem which shows that the problem of deciding if $f_{\infty}(G,d) \leq 1$ given a metric graph (G,d) is NP-complete (see [20]).

Let *H* be a graph. We construct a metric graph (G, d) by replacing each vertex $v \in V(H)$ by two adjacent vertices $v_1, v_2 \in V(G)$, and each edge $vw \in E(H)$ by a $K_{2,2}$ in *G* with edge set $\{v_iw_j \mid i \in [2], j \in [2]\}$. The distance function *d* is defined by $d(v_1v_2) = 2$ for all $v \in V(H)$ and $d(v_iw_j) = 1$ for all $vw \in E(H), i \in [2]$ and $j \in [2]$. We claim that $f_{\infty}(G, d) = \chi(H)$.

To see that $f_{\infty}(G, d) \geq \chi(H)$, notice that edges v_1v_2 and w_1w_2 are incompatible whenever $vw \in E(H)$. Thus every size-k flat covering of (G, d) gives a k-coloring of H.

Finally, $f_{\infty}(G, d) \leq \chi(H)$, since for every stable set S in G, $\{(v_1, v_2) \mid v \in S\} \cup \{(u_i, v_1) \mid i \in [2], uv \in E(H), v \in S\} \cup \{(v_2, w_j) \mid j \in [2], vw \in E(H), v \in S\}$ is a flat set of (G, d). Hence, every k-coloring of H gives a size-k flat covering of (G, d).

5. Certificates of large ℓ_{∞} -dimension

In this section, we show that if $H \in \mathcal{U}_{\infty}^{k} = \{\mathsf{S}_{k},\mathsf{P}_{k},\mathsf{F}_{k},\mathsf{N}_{k}\}$, then $f_{\infty}(H) > k$. It follows that if a graph G contains a \mathcal{U}_{∞}^{k} minor, then $f_{\infty}(G) > k$. Therefore, the existence of one of these four minors is a certificate that $f_{\infty}(G) > k$. Conversely, our main theorem shows that if $f_{\infty}(G) \ge g_{1}(k)$, then G necessarily contains one of these four minors. We also prove that $\mathsf{S}_{k},\mathsf{P}_{k}$, and F_{k} are excluded minors for the property $f_{\infty}(G) \le k$, that is, all their proper minors have ℓ_{∞} -dimension at most k.

We begin by proving that for each $H \in {\{S_k, P_k, F_k\}}, f_{\infty}(H) = k + 1$. We first prove the upper bound.

Lemma 15. For all $k \in \mathbb{N}$ and all $H \in \{S_k, P_k, F_k\}, f_{\infty}(H) \leq k+1$.

Proof. We proceed by induction on k. The base case follows by Lemma 14, since $S_1 = P_1 = F_1 = K_4$. Next note that $S_k = S_{k-1} + K_4$, $P_k = P_{k-1} + K_4$, and $F_k = F_{k-1} + K_4$. Therefore, we are done by induction and Lemmas 12 and 14.

Theorem 16. For all $k \in \mathbb{N}$, $f_{\infty}(S_k) = k + 1$.

Proof. By Lemma 15, it suffices to show $f_{\infty}(S_k) \ge k + 1$. Since $S_1 = K_4$, by Lemma 14, we may assume $k \ge 2$. We now give a distance function d on S_k , which is illustrated in Figure 4, such that there are k + 1 incompatible edges in (S_k, d) .

Let $V(S_k) = \{v, w\} \cup \{v_1, w_1, \dots, v_k, w_k\}$ where v, w, v_i, w_i are the vertices of the *i*th copy of K_4 . We define d as follows:

$$\begin{aligned} d(vv_{1}) &= d(ww_{1}) = 4k ,\\ d(vv_{i}) &= d(ww_{i}) = 2(k+i-1) \\ d(wv_{i}) &= d(vw_{i}) = k+i-1 \\ d(v_{i}w_{i}) &= 3(k+i-1) \end{aligned} \qquad \text{for all } i \in [k] ,\\ \text{for all } i \in [k] . \end{aligned}$$



FIGURE 4. (S_k, d) as in the proof of Theorem 16. The red edges are pairwise incompatible. Vertices with the same label are identified.

First, we show that d is a distance function. For this, let (G, d') be obtained from (S_k, d) by adding the edge vw of length d'(vw) = 3k. Observe that

$$G = K_4 \oplus_{vw} K_4 \oplus_{vw} \cdots \oplus_{vw} K_4,$$

where K_4 appears k times in the righthand side. It is easy to see that the restriction of d' to each K_4 subgraph of G is a distance function. Therefore, by Lemma 8, d' is a distance function on G. Since d is a restriction of d' to S_k it follows that d is a distance function on S_k .

We now show that the k+1 edges $vv_1, ww_1, v_2w_2, v_3w_3, \ldots, v_kw_k$ are pairwise incompatible. For this, we make repeated use of Lemma 13.

First, consider vv_1 and ww_1 . Observe that $d(vv_1) + d(ww_1) = 8k$. However, $d(vw_1) + d(wv_1) = 2k < 8k$ and $d(v_1w_1) + d(vv_2w) = 6k + 3 < 8k$, since $k \ge 2$. By Lemma 13, vv_1 and ww_1 are incompatible.

Next, consider vv_1 and v_iw_i with $i \in \{2, ..., k\}$. Observe that $d(vv_1) + d(v_iw_i) = 7k + 3i - 3$. However, $d(vv_i) + d(w_iwv_1) = 5k + 2i - 2 < 7k + 3i - 3$ and $d(vw_i) + d(v_iwv_1) = 3k + 2i - 2 < 7k + 3i - 3$. Hence, by Lemma 13, vv_1 and v_iw_i are incompatible.

By symmetry, ww_1 and v_iw_i are also incompatible for each $i \in \{2, \ldots, k\}$.

Finally, consider $v_i w_i$ and $v_j w_j$ for $2 \le i < j \le k$. Observe that $d(v_i w_i) + d(v_j w_j) = 6k + 3i + 3j - 6$. However, $d(v_i w v_j) + d(w_i v w_j) = 4k + 2i + 2j - 4 < 6k + 3i + 3j - 6$, and $d(v_i v w_j) + d(w_i w v_j) = 6k + 4i + 2j - 6 < 6k + 3i + 3j - 6$ since i < j. Hence, by Lemma 13, $v_i w_i$ and $v_j w_j$ are incompatible, which completes the proof.

Theorem 17. For all $k \in \mathbb{N}$, $f_{\infty}(\mathsf{P}_k) = k + 1$.

Proof. Again, $f_{\infty}(\mathsf{P}_k) \leq k+1$ follows from Lemma 15. We label the vertices of the topmost path of P_k as v_0, v_1, \ldots, v_k and the vertices of the bottommost path of P_k as w_0, w_1, \ldots, w_k . Thus $V(\mathsf{P}_k) = \{v_0, v_1, \ldots, v_k\} \cup \{w_0, w_1, \ldots, w_k\}$ and $E(\mathsf{P}_k) = \{v_0w_0, v_kw_k\} \cup \{v_{i-1}v_i, v_{i-1}w_i, w_{i-1}v_i, w_{i-1}w_i \mid i \in [k]\}$. For the lower bound, consider the following distance

function d, which is illustrated in Figure 5 (we take $i \in [k]$):

$$\begin{aligned} d(v_0w_0) &= d(v_kw_k) = 2^k \,, \\ d(v_{i-1}v_i) &= d(w_{i-1}w_i) = 2^k + 1 & \text{if } i \equiv 1 \pmod{2} \,, \\ d(v_{i-1}v_i) &= d(w_{i-1}w_i) = 2^k - 1 & \text{if } i \equiv 2 \pmod{4} \,, \\ d(v_{i-1}v_i) &= d(w_{i-1}w_i) = 2^k - 2^{1+i/2} & \text{if } i \equiv 0 \pmod{4} \,, \\ d(v_{i-1}w_i) &= d(w_{i-1}v_i) = 2^{1+i/2} & \text{if } i \equiv 0 \pmod{4} \,, \\ d(v_{i-1}w_i) &= d(w_{i-1}v_i) = 1 & \text{if } i \not\equiv 0 \pmod{4} \,. \end{aligned}$$



FIGURE 5. The top half of the figure depicts the distance function on P_k used in the proof of Theorem 17. The thick double crosses with a circle are each to be replaced with the metric graph shown in the bottom half of the figure.

Let (G, d') be obtained from (P_k, d) by adding edges $v_i w_i$ with $d'(v_i w_i) = 2^k$ for all $i \in [k-1]$. Notice that for all i, the length of a shortest path between v_i and w_i in (P_k, d) is 2^k . Therefore, (P_k, d) is a metric graph if and only if (G, d') is a metric graph. Observe that the restriction of d' to every K_4 subgraph of G is a distance function. Therefore, (G, d') and hence also (P_k, d) is a metric graph by Lemma 8.

Consider the matching $M = \{v_{i-1}v_i, w_{i-1}w_i \mid i \equiv 1 \pmod{2}\}$. If k is even, then we also add the edge $v_k w_k$ to M. Thus |M| = k + 1 always. We claim that the edges of M are pairwise incompatible. To see this, let e = xx' and f = yy' be distinct edges of M. Let P be a shortest x-y path, and P' be a shortest x'-y' path. We claim that $d(P) + d(P') \leq 2 \cdot 2^k$ (see next paragraph for a proof). However, $d(e) + d(f) > 2 \cdot 2^k$ because $e, f \in M$. Therefore, by Lemma 13, e and f are incompatible. Since |M| = k + 1, $f_{\infty}(\mathsf{P}_k) \geq k + 1$, as required.

To prove the claim, we split the discussion into two cases. A segment in P_k is any subgraph induced by $\{v_i, w_i \mid i = 4q + r, r \in \{0, 1, 2, 3\}, i \leq k\}$ for some q. If e and f belong to the same segment, then it is easy to see that $d(P) + d(P') \leq 2 \cdot 2^k$. (Notice that sometimes $d(P) = 2^k + 1$ and $d(P') = 2^k - 1$.) Now if a and b are any two vertices in distinct segments (indexed by q and s, with q < s), then there is a a-b path Q such that

$$\begin{split} d(Q) &\leq 1+1+1+2^{2q+3}+1+1+1+1+\cdots+2^{2s-1}+1+1+1+(2^k-2^{2s+1})+1+1+1\\ &\leq \underbrace{(3s+3)}_{\leq 1+2+4+2^{2s}}+2^3+2^5+\cdots+2^{2s-1}-2^{2s+1}+2^k \leq \sum_{i=0}^{2s}2^i-2^{2s+1}+2^k \leq 2^k\,. \end{split}$$

It follows that $d(P) + d(P') \leq 2 \cdot 2^k$ in this case too.

Theorem 18. For all $k \in \mathbb{N}$, $f_{\infty}(\mathsf{F}_k) = k + 1$.

Proof. For all $i \in [k]$, we label the vertices of the *i*th copy of K_4 in F_k as $v_0, v_{2i-1}, v_{2i}, v_{2i+1}$. Remember that in order to obtain F_k we form the 2-sum of these k copies of K_4 and delete every edge that is in two consecutive copies. Thus $V(\mathsf{F}_k) = \{v_j \mid j \in \{0, \ldots, 2k+1\}\}$ and $E(\mathsf{F}_k) = \{v_0v_1, v_0v_{2k+1}\} \cup \{v_0v_{2i}, v_{2i-1}v_{2i}, v_{2i-1}v_{2i+1}, v_{2i}v_{2i+1}\}.$

By Lemma 15, it suffices to show $f_{\infty}(\mathsf{F}_k) \ge k+1$. Consider the following distance function d on F_k :

$$d(v_0v_1) = 1,$$

$$d(v_0v_{2i}) = 1 \qquad \text{for } i \in [k],$$

$$d(v_{2i-1}v_{2i+1}) = 1 \qquad \text{for } i \in [k],$$

$$d(v_{2i}v_{2i+1}) = i \qquad \text{for } i \in [k],$$

$$d(v_{2i}v_{2i-1}) = i + 1 \qquad \text{for } i \in [k],$$

$$d(v_0v_{2k+1}) = k + 1.$$

As before, by Lemma 8, we can prove that d is a distance function. Notice that v_0 is at distance i + 1 from v_{2i+1} for each $i \in [k-1]$.

Consider the matching $M = \{v_0v_{2k+1}\} \cup \{v_{2i}v_{2i-1} \mid i \in [k]\}$ in (F_k, d) . See Figure 6 for an illustration of the distance function d and the matching M in F_5 . We let the reader verify, with the help of Lemma 13, that all edges of M are pairwise incompatible. Since $|M| = k + 1, f_{\infty}(\mathsf{F}_k) \ge k + 1$ as required. \Box



FIGURE 6. (F_k, d) as in the proof of Theorem 18 and (F_5, d) . The red edges are pairwise incompatible.

Theorem 19. For all $k \ge 2$, $\mathsf{S}_k, \mathsf{P}_k, \mathsf{F}_k$ are excluded minors for the property $f_{\infty}(G) \le k$.

Proof. Let H be one of S_k , P_k , F_k . By Theorems 16, 17, and 18, we know $f_{\infty}(H) > k$.

When deleting or contracting an edge in H, we get a minor H' which can be expressed as a 2-sum of two graphs H_1 , H_2 with the following properties. First, $H_1 \in \{S_\ell, \mathsf{P}_\ell, \mathsf{F}_\ell\}$ for some $\ell < k$ (and H_1 is of the same type as H). Second, H_2 has a degree-2 vertex and recursively suppressing the degree-2 vertices from H_2 results in a graph H'_2 such that $H'_2 \in \{\mathsf{S}_m, \mathsf{P}_m, \mathsf{F}_m\}$ for some $m \leq k - l - 1$ (again H'_2 is of the same type as H), or H'_2 is a single edge (this corresponds to the case m = 0). By Lemma 12 and Lemma 15,

 $f_{\infty}(H') \le f_{\infty}(H_1) + f_{\infty}(H_2) - 1 = f_{\infty}(H_1) + f_{\infty}(H'_2) - 1 \le (l+1) + (m+1) - 1 \le k.$ Thus, *H* is an excluded minor for $f_{\infty}(G) \le k.$

Theorem 20. For all $k \in \mathbb{N}$, $f_{\infty}(N_k) \ge k+1$.



FIGURE 7. (N_k, d) as in the proof of Theorem 20.

Proof. Let
$$V(N_k) = \{v_0, ..., v_k\} \cup \{w_0, ..., w_k\}$$
 and

$$E(\mathsf{N}_k) = \{v_{i-1}v_i, v_iw_i, v_{i-1}w_i, w_{i-1}w_i \mid i \in [k]\} \cup \{v_0w_0, w_0v_k\}.$$

Consider the distance function d such that $d(w_0v_k) = d(v_{i-1}v_i) = d(w_{i-1}w_i) = 1$, $d(v_{i-1}w_i) = k$ for all $i \in [k]$ and $d(v_iw_i) = k + 1$ for all $i = 0, \ldots, k$. It is easy to check that d is indeed a distance function. Let $M = \{v_iw_i \mid i = 0, \ldots, k\}$. See Figure 7 for an illustration of (N_k, d) and M, where $v_0 \cdots v_k$ and $w_0 \cdots w_k$ are the topmost and bottommost paths, respectively.

We claim that the edges in M are pairwise incompatible. To see this, first observe that the shortest $v_i - v_j$ and $w_i - w_j$ paths both have weight $|j - i| \le k$ since all edges in these paths have weight 1, hence the cumulative weight of these paths is at most 2k. If i > j, then

$$d(v_i v_{i+1} \cdots v_k w_0 w_1 \cdots w_j) + d(v_j v_{j+1} \cdots v_{i-1} w_i) = (k - i + j + 1) + (i - j - 1 + k) = 2k.$$

This shows that there exist a $v_i - w_j$ path and a $v_j - w_i$ path of cumulative weight 2k. Since $d(v_i w_i) + d(v_j w_j) = 2k + 2$, the conditions of Lemma 13 are satisfied and we get that $v_i w_i$ and $v_j w_j$ are incompatible for all $i \neq j$. Hence, $f_{\infty}(\mathsf{N}_k) \geq k + 1$.

Since N_k is 3-connected, it is difficult to adapt the proof of Theorem 19 to show that N_k is also an excluded minor for the property $f_{\infty}(G) \leq k$. However, we conjecture that this is true.

6. 2-CONNECTED GRAPHS

In this section, we show that it is enough to prove our main theorem, Theorem 1, for 3connected graphs. To do so, we introduce a variant of SPQR trees in Section 6.1. In section 6.2, we show that in a graph $G_1 +_e G_2$ obtained as a 2-sum of two graphs G_1 and G_2 , we can merge flat sets from G_1 and G_2 under some conditions. In Section 6.3, we present several lemmas that show how to bound $f_{\infty}(H)$, where H is obtained by gluing several 2-connected graphs on a given graph. At the end of this section, we also show how to complete the proof of Theorem 1 under some additional assumptions.

6.1. Contracted SPQR trees. In this context we need to consider *multigraphs* that are minors of a simple 2-connected graph, that is, parallel edges resulting from edge contractions are kept. (Loops on the other hand are not important for our purposes and thus can safely be discarded.) SPQR trees were introduced in [9] as a way to decompose a 2-connected graph across its 2-separations. They are defined as follows.

Let G be a (simple) 2-connected graph. The SPQR tree T_G of G is a tree each of whose node $a \in V(T_G)$ is associated with a multigraph H_a which is a minor of G. Each vertex $x \in V(H_a)$ is a vertex of G, that is, $V(H_a) \subseteq V(G)$. Each edge $e \in E(H_a)$ is classified either as a *real* or *virtual* edge. By the construction of an SPQR tree each edge $e \in E(G)$ appears in exactly one minor H_a as a real edge, and each edge $e \in H_a$ which is classified real is an edge of G. The SPQR tree T_G is defined recursively as follows.

- (1) If G is 3-connected, then T_G consists of a single R-node a for which we have $H_a = G$. All edges of H_a are real in this case.
- (2) If G is a cycle, then T_G consists of a single S-node for which $H_a = G$. Again, all edges of H_a are real in this case.
- (3) Otherwise G has a cutset $\{x, y\}$ such that the vertices x and y have degree at least 3. In this case we construct T_G inductively. First we add a P-node a to T_G , for which H_a is the graph consisting of the single edge xy. The edge xy of H_a is real if xy is an edge of G, and virtual otherwise. Next we consider the connected components C_1, \ldots, C_r $(r \geq 2)$ of $G - \{x, y\}$. Let G_i be the graph $G[V(C_i) \cup \{x, y\}]$ with the additional edge xy if it is not already there. Since we include the edge xy, each G_i is 2-connected and we can construct the corresponding SPQR tree T_{G_i} by induction. Let a_i be the (unique) node in T_{G_i} for which xy is a real edge in H_{a_i} . In order to construct T_G , we make xy a virtual edge in the node a_i , and connect a_i to a in T_G . Finally, we add parallel virtual edges xy to H_a so that it has exactly r virtual edges xy.

Notice that minors corresponding to S-nodes and R-nodes are simple graphs, whereas those corresponding to P-nodes are multigraphs consisting of two vertices linked by at least two virtual edges and possibly a real one. To each edge ab of the SPQR tree T_G corresponds a unique virtual edge $e \in E(H_a) \cap E(H_b)$ with ends $x, y \in V(G)$. Thus we can define a corresponding multigraph $H_{a,b}$ which is the minor of G obtained by taking the 2-sum of H_a and H_b in which the edge e is deleted. (To be precise, one virtual edge xy from each of H_a and H_b is deleted in the operation, other copies of xy, if any, are kept in the resulting graph.) Similarly, we can define a unique minor of G for each subtree of T_G by performing one 2-sum operation as described above for each edge of the subtree.

Let G be a 2-connected graph, and let T_G be the SPQR tree of G. We define the contracted SPQR tree T'_G as the tree obtained from T_G by contracting every maximal connected subtree of T_G each of whose nodes is either a S-node or a P-node, see Figure 8 for an example. We call the new nodes resulting from the contraction O-nodes. Each node a of T'_G has a unique corresponding minor H_a of G. If a is an R-node, then we keep the same minor as in T_G . Otherwise, a is an O-node and H_a is the minor of G corresponding to the subtree of T_G that was contracted to node a of T'_G .

We quickly give some standard terminology before stating our first result of the section. The *length* of a path in G is its number of edges. The *diameter* of a graph G is the maximum length of a shortest path between any two vertices.

Lemma 21. Let G be a 2-connected graph with minimum degree at least 3.

- (1) Every O-node in T'_G corresponds to a 2-connected treewidth-2 graph.
- (2) All leaves of T'_G are R-nodes.
- (3) If the diameter of T'_G is at least 6k, then G contains P_k or F_k as a minor.

Proof. (1) Let o be an O-node of T'_G . Its corresponding minor H_o is obtained by 2-sums from cycles corresponding to S-nodes, and parallel edges corresponding to P-nodes. Hence H_o is 2-connected and has treewidth 2.

(2) Suppose for a contradiction that some leaf o of T'_G is an O-node. Since a P-node cannot be a leaf in T_G , the subtree corresponding to o in T_G has at least one leaf s which is an S-node. Because s is a leaf, H_s contains exactly one virtual edge. Since H_s is a cycle of length at least 3, there is at least one degree-2 vertex in G, a contradiction.



FIGURE 8. An example of a 2-connected graph G, its SPQR tree T_G , and the contracted SPQR tree T'_G .

(3) Let $P = a_0 \cdots a_m$ be a maximum length path in T'_G . By maximality, P is a leaf-to-leaf path in T'_G , a_i is an R-node for even i and an O-node for odd i, and m is even.

For $i \in [m-1]$, we let x_i and y_i be the ends of the virtual edge in $E(H_{a_i}) \cap E(H_{a_{i+1}})$. Since H_{a_i} is 2-connected, exchanging x_i and y_i if necessary we may assume that for each $i \in [m-1]$, H_{a_i} contains an $x_{i-1}-x_i$ path P_i and a $y_{i-1}-y_i$ path Q_i such that P_i and Q_i are vertex-disjoint.

Let $i \in [m-1]$ with *i* even. Let us emphasize that the vertices $x_{i-1}, x_i, y_{i-1}, y_i$ are not necessarily all distinct. We call a K_4 -model in H_{a_i} good if the intersections of the four vertex images with these vertices fall in one of the following cases:

- $\{x_{i-1}\}, \{x_i\}, \{y_{i-1}\}, \{y_i\}, \text{ or }$
- $\{x_{i-1}, x_i\}, \{y_{i-1}\}, \{y_i\}, \emptyset$ with $x_{i-1} \neq x_i$, or
- $\{x_i\}, \{y_{i-1}\}, \{y_i\}, \emptyset$ with $x_{i-1} = x_i$, or
- $\{x_{i-1}\}, \{x_i\}, \{y_{i-1}, y_i\}, \emptyset$ with $y_{i-1} \neq y_i$, or
- $\{x_{i-1}\}, \{x_i\}, \{y_i\}, \emptyset$ with $y_{i-1} = y_i$.

We claim that H_{a_i} has a good K_4 -model for each even $i \in [m-1]$. To see this, let $C_i = P_i + Q_i + x_{i-1}y_{i-1} + x_iy_i$. First suppose $V(C_i) = V(H_{a_i})$. Since H_{a_i} is 3-connected, there is an edge $e \in E(H_{a_i})$ distinct from $x_{i-1}y_{i-1}$ and x_iy_i between $V(P_i)$ and $V(Q_i)$, and another edge f such that $C_i \cup \{e, f\}$ is a subdivision of K_4 . Then $C_i + e + f$ contains a good K_4 -model. Assume now that $V(C_i) \subsetneq V(H_{a_i})$. It follows that there is a component of $H_{a_i} - V(C_i)$ that sends edges to three vertices of C_i which are neither all in $V(P_i)$ nor all in $V(Q_i)$; otherwise $H_{a_i} - \{x_{i-1}, x_i\}$ or $H_{a_i} - \{y_{i-1}, y_i\}$ would be disconnected. Thus, H_{a_i} has a good K_4 -model whose vertex images are a single component of $H_{a_i} - V(C_i)$ and three disjoint connected subgraphs of C_i .

We say that a good K_4 -model in H_{a_i} is type-0 if x_{i-1}, x_i, y_{i-1} , and y_i are in distinct vertex images, type-1 if x_{i-1} and x_i are in the same vertex image, and type-2 if y_{i-1} and y_i are in the same vertex image. We pick a good K_4 -model in each even $i \in [m-1]$. Since $m \ge 6k$, at least k of these good K_4 -models are of the same type, say type-t for some $t \in \{0, 1, 2\}$.

We obtain the required minor of G as follows. First, for each even $i \in [m-1]$ such that H_{a_i} contains a type-t good K_4 -model, we contract the vertex images of the K_4 -model and delete the vertices not belonging to any vertex image. Second, for each index $i \in [m-1]$ not yet

considered, we contract the edges in $E(P_i) \cup E(Q_i)$ and delete all other vertices of H_{a_i} . Note that this second step has the effect of 2-summing the type-t good K_4 -models. Therefore, we obtain a P_k minor in G, if t = 0, and a F_k minor in G in the other two cases.

6.2. Extending flat sets in 2-connected graphs. We now develop some more tools to handle 2-separations in graphs. Assume that $G = G_1 \oplus_e G_2$ with e = vw. The goal is to improve the bounds for $f_{\infty}(G)$ given in Lemma 12. Recall that the proof of Lemma 12 relies on the fact that it is possible to merge a flat set F_1 of (G_1, d_1) and a flat set F_2 of (G_2, d_2) into one flat set $F_1 \cup F_2$ of (G, d) whenever $(v, w) \in F_1 \cap F_2$.

Here is another proof of this fact. Let (D, l), (D_1, l_1) and (D_2, l_2) denote the weighted digraphs obtained by bidirecting (G, d), (G_1, d_1) and (G_2, d_2) respectively. For $i \in [2]$, consider a potential p_i on (D_i, l_i) such that $p_i(x) - p_i(y) = d(xy)$ for all $(x, y) \in F_i$. Since $(v, w) \in$ $F_1 \cap F_2$, we have $p_1(v) - p_1(w) = p_2(v) - p_2(w) = d(vw)$. Hence, it is possible to shift one of the potentials in order to satisfy $p_1(v) = p_2(v)$ and $p_1(w) = p_2(w)$. The potential $p_1 \cup p_2 : V(G) \to \mathbb{R}$ on (D, l) such that $(p_1 \cup p_2)(u) = p_i(u)$ if $u \in V(G_i)$ for $i \in [2]$ witnesses that $F_1 \cup F_2$ is a flat set.

Suppose now that the flat sets F_1 , F_2 of (G_1, d_1) and (G_2, d_2) are such that $(v, w) \in F_1$ but $(v, w), (w, v) \notin F_2$. The previous idea does not work anymore since we could have $|p_2(v) - p_2(w)| < d(vw)$. Hence, we can no longer combine the potentials p_1 and p_2 . However, there possibly exists a potential p'_1 for $F_1 \setminus \{(v, w)\}$ such that $p'_1(v) - p'_1(w) = p_2(v) - p_2(w)$. In that case, $p'_1 \cup p_2$ is a potential for $(F_1 \cup F_2) \setminus \{(v, w)\}$ on (D, l). It follows that in this case $(F_1 \cup F_2) \setminus \{(v, w)\}$ is a flat set.

We now introduce the notion of *compressible edges*, which are edges for which we can apply the idea of the previous paragraph. In this context, it is helpful to switch from directed notions to undirected notions. We call a set F of edges of G flattenable (in (G, d)) if some orientation of F is a flat set in (G, d), that is, if there exists a potential p on (D, l) such that |p(v) - p(w)| = d(vw) for all $vw \in F$. Let $F \subseteq E(G)$ be flattenable in (G, d). An edge subset $\Gamma \subseteq F$ is said to be *compressible* in F if for all $\lambda \in [0, 1]^{\Gamma}$ there exists a potential pon (D, l) such that $|p(v) - p(w)| = \lambda(vw) \cdot d(vw)$ for all $vw \in \Gamma$ and |p(v) - p(w)| = d(vw)for all $vw \in F \setminus \Gamma$. We define a frame in (G, d) as a pair (Γ, F) where $\Gamma \subseteq F \subseteq E(G)$, F is flattenable in (G, d) and Γ is compressible in F.

Notice that subsets of flattenable sets are flattenable, and that $f_{\infty}(G)$ is the least integer k such that for every distance function d the edges of the metric graph (G, d) can be partitioned into k flattenable sets.

The next lemma follows directly from the formal definition of compressible edges.

Lemma 22. Let $G = G_1 \oplus_{vw} G_2$, and let d be a distance function on G. For $i \in [2]$, let d_i be the restriction of d to G_i and let (Γ_i, F_i) be a frame in (G_i, d_i) .

- (i) If $vw \in (F_1 \setminus \Gamma_1) \cap (F_2 \setminus \Gamma_2)$ then $(\Gamma_1 \cup \Gamma_2, F_1 \cup F_2)$ is a frame in (G, d).
- (ii) If $vw \in \Gamma_1 \cup \Gamma_2$ then $((\Gamma_1 \cup \Gamma_2) \setminus \{vw\}, (F_1 \cup F_2) \setminus \{vw\})$ is a frame in (G, d).

We will now use this lemma to improve some bounds given by Lemma 12. For simplicity, we call gluing the 2-sum operation where the edge involved in the 2-sum is kept. Let H be a graph obtained by gluing graphs G_1, \ldots, G_m on distinct edges of a graph G. That is, there are distinct edges e_1, \ldots, e_m such that $H = G \oplus_{e_1} G_1 \cdots \oplus_{e_m} G_m$. The bound obtained by applying Lemma 12 is $f_{\infty}(H) \leq f_{\infty}(G) + \sum_{i \in [m]} (f_{\infty}(G_i) - 1)$. We provide better bounds in the following cases. First, when G is a 2-connected outerplanar graph and all G_i are glued on edges of its outer cycle. Second, when G is a 2-connected treewidth-2 graph and H has no S_k minor.

Lemma 23. Let G be a 2-connected outerplanar graph drawn in the plane with outer cycle C. Let H be obtained from G by gluing graphs G_1, \ldots, G_m on distinct edges of C. Let $M = \max_{i \in [m]} f_{\infty}(G_i)$. Then $f_{\infty}(H) \leq 3M$.



FIGURE 9. Illustration of the proof of Lemma 23: G is a 2-sum of $G' = K_4 - e$ and K_3 . Each color defines a frame (Γ, F) in the corresponding graph. Edges of $F \setminus \Gamma$ are straight and edges of Γ are wavy. The distance function is defined by taking the corresponding Euclidean distance in the figure.

Proof. We will show that G satisfies the following property:

(*) For every distance function d on G, there exist three frames (Γ_j, F_j) , $j \in [3]$, in (G, d) such that each edge of G is in at least one flattenable set F_j , and each edge of its outer cycle C is in exactly two flattenable sets F_j and in exactly one compressible set Γ_j .

For $i \in [m]$, let $\{v_i, w_i\} = V(G_i) \cap V(G)$. Thus, $v_i w_i$ is an edge of C. Without loss of generality, we may assume that $v_i w_i$ is an edge of H.

Now let d be some distance function on H. We will slightly abuse notation and let d also denote the restriction of this distance function to G. For $i \in [m]$, let d_i denote the restriction of d to G_i .

Assuming (\star) , we can find three frames $(\Gamma_j, F_j), j \in [3]$, in (G, d) as above. For each $i \in [m]$, let F_1^i, \ldots, F_M^i be a partition of the edges of (G_i, d_i) into flattenable set. By Lemma 22, for every $j \in [3]$ and $k \in [M]$,

$$\left(F_j \cup \bigcup_{i \in I_j} F_k^i\right) \setminus \{v_i w_i \mid i \in I_j\}$$

is a flattenable set in (H, d), where $I_j = \{i \in [m] \mid v_i w_i \in \Gamma_j\}$. These 3M flattenable sets cover the edges of (H, d), which implies $f_{\infty}(G) \leq 3M$.

To prove the lemma, it remains to show that the claimed frames $(F_j, \Gamma_j), j \in [3]$ exist in (G, d). We can assume that all inner faces of the drawing of G are triangular faces (if not, add extra edges). We show the result by induction on the number of vertices.

The base case is given by $G = K_3$. Let $V(K_3) = \{v_1, v_2, v_3\}$. Without loss of generality, we can assume $d(v_1v_2) \leq d(v_1v_3) \leq d(v_2v_3)$. It is easy to show that $(\Gamma_1, F_1) = (\{v_1v_2, v_1v_3\}, \{v_1v_2, v_1v_3\}), (\Gamma_2, F_2) = (\{v_2v_3\}, \{v_2v_1, v_2v_3\}), \text{ and } (\Gamma_3, F_3) = (\emptyset, \{v_3v_1, v_3v_2\})$ are frames in (G, d). For instance, one can use Lemma 6 to see that each F_j is flattenable, and a direct verification to see that each Γ_j is compressible in F_j . Thus K_3 satisfies (\star) .

Now for the inductive case, suppose that G has at least four vertices. Let v be a degree-2 vertex of G (which exists since G is outerplanar and 2-connected), and consider the graph G' = G - v. Let v_1, v_2 be the two neighbors of v in G, with $d(vv_1) \ge d(vv_2)$. Let C' be the cycle obtained from the outer cycle C in G by shortcutting the path v_1vv_2 to v_1v_2 .

By induction, (*) holds for G'. Let (Γ'_j, F'_j) , $j \in [3]$ denote the corresponding frames. Consider three frames (Γ''_j, F''_j) , $j \in [3]$ for the triangle vv_1v_2v , as described in the base case of the induction.

By permuting the indices if necessary, we may assume that v_1v_2 is in $(F'_1 \setminus \Gamma'_1) \cap (F''_1 \setminus \Gamma''_1)$, Γ'_2 and Γ''_3 . By Lemma 22, $(\Gamma_1, F_1) = (\Gamma'_1 \cup \Gamma''_1, F'_1 \cup F''_1)$ and, for $j \in \{2, 3\}$, $(\Gamma_j, F_j) = ((\Gamma'_j \cup \Gamma''_j) \setminus \{v_1v_2\}, (F'_j \cup F''_j) \setminus \{v_1v_2\})$ are all frames in (G, d). See Figure 9 for an illustration. It is straightforward to check that these frames satisfy the required condition for G. \Box

6.3. Handling several 2-cutsets simultaneously. Before proceeding, we require the following easy lemma. Let $K_4 - e$ be the graph obtained from K_4 by deleting an edge.

Lemma 24 ([13]). Let G be a 2-connected graph with distinct vertices u and v such that $\deg_G(w) \geq 3$ for all $w \in V(G) \setminus \{u, v\}$. Then G has a $K_4 - e$ minor where u and v are contracted to the ends of e.

Let G be a graph together with a subset of E(G) called *glued edges*. We say that G has a k-glumpkin minor if G contains k glued edges in parallel as a minor, that is, if there is a way of choosing a connected subgraph H of G containing at least k glued edges, and of contracting all but k edges of H in such a way that the resulting minor consists of k parallel glued edges. A k-glumpkin minor is rooted at a glued edge r if it contains r. If H is obtained by gluing graphs G_1, \ldots, G_m on distinct edges of G, an edge $e \in E(G)$ is a glued edge if $e \in E(G) \cap E(G_i)$ for some $i \in [m]$. The parameter we are really interested in is the largest S_k minor in H. However, the next lemma relates S_k minors in H to k-glumpkin minors in G.

Lemma 25. Let H be obtained by gluing 2-connected graphs G_1, \ldots, G_m on distinct edges of a graph G such that H has minimum degree at least 3. If G has a k-glumpkin minor, then H has an S_k -minor.

Proof. Let $u_i v_i$ be the glued edge of G_i . Since H has minimum degree at least 3, $\deg_{G_i}(w) \geq 3$ for all $w \in V(G_i) \setminus \{u_i, v_i\}$. By Lemma 24, G_i has a K_4 minor containing the glued edge $u_i v_i$, for all $i \in [m]$. Therefore, since G has a k-glumpkin minor, H has an S_k -minor.

Lemma 26. For all $k, M \in \mathbb{N}$, let $g_{26}(k, M) = 3^k M$. Let H be a graph obtained from a 2-connected outerplanar graph G by gluing 2-connected graphs G_1, \ldots, G_m on distinct edges of G. Let C be the outercycle of G and let $M = \max_{i \in [m]} f_{\infty}(G_i)$. If there exists a glued edge $r \in E(C)$ such that G does not contain a k-glumpkin minor rooted at r, then $f_{\infty}(H) \leq g_{26}(k, M)$.

Proof. We proceed by induction on k. The case k = 1 is vacuous. If k = 2, then by 2connectivity, r is the only glued edge of G. Since G is outerplanar, $f_{\infty}(G) \leq 2$ and so by Lemma 12, $f_{\infty}(H) \leq M + 1 \leq g_{26}(2, M)$. Therefore, we may assume $k \geq 3$. A subpath of C - r is good if its ends are connected by a glued edge. Let $P_1, \ldots P_p$ be the maximal (under inclusion) good subpaths of C - r. Since G is outerplanar, P_i and P_j are internally-disjoint for $i \neq j$. By maximality, every glued edge has both of its ends on some P_i .

Let G'_i be the subgraph of G induced by $V(P_i)$. Let e_i be the glued edge connecting the ends of P_i . Since G does not contain a k-glumpkin minor rooted at r, G'_i does not contain a (k-1)-glumpkin minor rooted at e_i . Let H_i be the subgraph of H induced by G'_i and all the graphs G_j that are glued to some edge of G'_i . By induction, $f_{\infty}(H_i) \leq 3^{k-1}M$ for all $i \in [p]$. Let C' be the cycle obtained from C by replacing P_i with e_i for each $i \in [p]$. Let G' be the subgraph of G induced by the vertices of C'. Notice that G' is a 2-connected outerplanar graph with outer cycle C', and H can be obtained from G' by gluing the graphs H_i on edges of C'. By Lemma 23,

$$f_{\infty}(H) \le 3 \cdot \max_{i \in [p]} f_{\infty}(H_i) \le 3 \cdot 3^{k-1}M = g_{26}(k, M).$$

We now generalize Lemma 26 to 2-connected treewidth-2 graphs.

Lemma 27. For all $k, M \in \mathbb{N}$, let $g_{27}(k, M) = 3^{k^2}M$. Let G be a 2-connected treewidth-2 graph and let H be obtained by gluing 2-connected graphs G_1, \ldots, G_m on distinct edges of G. Let $M = \max_{i \in [m]} f_{\infty}(G_i)$. If for some glued edge r, G does not contain a k-glumpkin minor rooted at r, then $f_{\infty}(H) \leq g_{27}(k, M)$.

Proof. We proceed by lexicographic induction on (k, |V(H)|). Let r be a glued edge such that G does not contain a k-glumpkin minor rooted at r.

The case k = 1 is vacuous. Suppose k = 2. Since G is 2-connected and does not have a 2-glumpkin minor rooted at r, edge r must be the only glued edge of G. Since G is 2-connected and has treewidth 2, $f_{\infty}(G) \leq 2$. By Lemma 12, $f_{\infty}(H) \leq M + 1 \leq g_{27}(2, M)$. Therefore, we may assume $k \geq 3$. If $\deg_H(w) = 2$ for some vertex $w \in V(H)$, then we can suppress w by Lemma 11 and apply induction. Therefore, we may assume H has minimum degree at least 3.

Since G is 2-connected, there is a cycle in G containing r. Let C be a longest cycle in G such that $r \in E(C)$. Let \mathcal{E} be an ear decomposition of G beginning with C. (See for instance [10] for background about ear decompositions.) The *ear-decomposition tree* $T(\mathcal{E})$ of \mathcal{E} is the rooted tree, whose vertices are the ears in \mathcal{E} , defined recursively as follows. The root of $T(\mathcal{E})$ is C. The parent of an ear P is the closest ear Q to C (in $T(\mathcal{E})$) such that both ends of P are on Q. (Such an ear Q is guaranteed to exist since G has treewidth 2 and is 2-connected.)

Let P_1, \ldots, P_ℓ be the set of *C*-ears of \mathcal{E} . Let T_1, \ldots, T_ℓ be the subtrees of $T(\mathcal{E})$ rooted at P_1, \ldots, P_ℓ , respectively. For each $i \in [\ell]$, let x_i and y_i be the ends of P_i on *C*. Let R_i be the $x_i - y_i$ path in *C* containing *r* and let S_i be the other $x_i - y_i$ path in *C*. Notice that $|E(S_i)| \ge |E(P_i)|$, by maximality of *C*. If P_i is an edge, then since *G* is simple, $|E(S_i)| \ge 2$. Otherwise, $|E(S_i)| \ge |E(P_i)| \ge 2$. Therefore, for all $i \in [\ell]$, $|E(S_i)| \ge 2$.

We claim that for all $i \in [\ell]$, $V(S_i)$ contains the ends of a glued edge. Suppose not. Among all S_i such that $V(S_i)$ does not contain the ends of a glued edge, choose S_j so that S_j is inclusion-wise minimal. Since G has treewidth 2 and is 2-connected, for all $i \neq j$, $S_i \subseteq S_j$, $S_j \subseteq S_i$, or S_i and S_j are internally-disjoint. By the minimality of S_j , each internal vertex of S_j has degree 2 in H. However, this contradicts that H has minimum degree at least 3.

For each $i \in [\ell]$, let G'_i be the union of all ears in T_i together with the edge $e_i = x_i y_i$, which we declare to be glued. Since $V(S_i)$ contains the ends of a glued edge and R_i contains r, the graph G'_i does not contain a (k-1)-glumpkin minor rooted at e_i ; otherwise, G contains a k-glumpkin minor rooted at r. Note that each G'_i contains at least one glued edge other than e_i since H has minimum degree at least 3. Let H_i be the graph obtained from G'_i by gluing all G_j such that the glued edge of G_j belongs to G'_i . By induction, $f_{\infty}(H_i) \leq g_{27}(k-1, M)$, for all $i \in [\ell]$. Let e_{i+1}, \ldots, e_L be the glued edges in E(C).

Observe that H is obtained by gluing graphs H_1, \ldots, H_L onto edges of an outerplanar graph G' with outercycle C, where $M' = \max_{i \in [L]} f_{\infty}(H_i) = \max\{M, g_{27}(k-1, M)\} = g_{27}(k-1, M)$. Since G does not contain a k-glumpkin minor rooted at r, neither does G'. Applying Lemma 26 to G' gives

$$f_{\infty}(H) \le g_{26}(k, g_{27}(k-1, M)) = 3^k (3^{(k-1)^2} M) \le g(k, M).$$

Lemma 27 yields the following corollary.

Lemma 28. For all $k, M \in \mathbb{N}$, let $g_{28}(k, M) = 3^{k^2}M$. Let G be a 2-connected treewidth-2 graph and let H be obtained by gluing 2-connected graphs G_1, \ldots, G_m on distinct edges of G. If H does not contain an S_k minor and $M = \max_{i \in [m]} f_{\infty}(G_i)$, then $f_{\infty}(H) \leq g_{28}(k, M)$.

Proof. We proceed by induction on |V(H)|. If $\deg_H(w) = 2$ for some $w \in V(H)$, then by Lemma 11, we can suppress w and apply induction. Since H does not contain an S_k minor, G does not contain a k-glumpkin minor, by Lemma 25. In particular, for each glued edge r, G does not contain a k-glumpkin minor rooted at r. By Lemma 27, $f_{\infty}(H) \leq g_{27}(k, M) = g_{28}(k, M)$.

The following is the main result of this section.

Lemma 29. Suppose there exist computable functions $g_{45} : \mathbb{N} \to \mathbb{R}$ and $g_{46} : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ satisfying the two following conditions.

- (1) $f_{\infty}(G) \leq g_{45}(k)$ for every 3-connected graph G not containing a \mathcal{U}_{∞}^{k} minor. (2) $f_{\infty}(H) \leq g_{46}(k, M)$ for every graph H containing no \mathcal{U}_{∞}^{k} minor, obtained by gluing 2-connected graphs G_1, \ldots, G_m on distinct edges of a 3-connected graph G_0 , where $M = \max_{i \in [m]} f_{\infty}(G_i).$

Then there exists a computable function $g_1 : \mathbb{N} \to \mathbb{R}$ such that $f_{\infty}(G) \leq g_1(k)$ for all graphs G without a \mathcal{U}_{∞}^k minor.

Proof. We define $g_1(k)$ as follows. For all $k, M \in \mathbb{N}$, let $\alpha(k, M)$ be the maximum of $g_{28}(k, M)$ and $g_{46}(k, M)$. Define $\gamma_0(k) = g_{45}(k)$. For all $i, k \in \mathbb{N}$ recursively define $\gamma_i(k) = \alpha(k, \gamma_{i-1}(k))$. Finally, let $g_1(k) = \gamma_{6k}(k)$.

Let G be a graph without a \mathcal{U}_{∞}^k minor. By Lemma 12, we may assume that G is 2-connected. By Lemma 11, we can assume that G has no degree-2 vertices. Let T_G be the SPQR tree of G and let $T = T'_G$ be the contracted SPQR tree, see Lemma 21.

Pick an arbitrary root node r in T. For each node b of T, we denote by T_b the subtree of T rooted at b and by H_b the minor of G corresponding to that subtree. Note that $G = H_r$. By Lemma 21, every leaf of T is an R-node. Hence, each leaf u of T corresponds to a 3-connected minor H_u of G. By our first assumption, $f_{\infty}(H_u) \leq g_{45}(k) = \gamma_0(k)$. Let a be some inner node of T and let a_1, \ldots, a_ℓ denote its children. Let $M_a = \max_{i \in [\ell]} f_\infty(H_{a_i})$. If a is an O-node, then by Lemma 28, $f_{\infty}(H_a) \leq g_{28}(k, M_a)$. If a is a R-node, then $f_{\infty}(H_a) \leq g_{46}(k, M_a)$ by our second assumption. In either case, $f_{\infty}(H_a) \leq \alpha(k, M_a)$. It follows that if i is the maximum length of an a to leaf path of T, then $f_{\infty}(H_a) \leq \gamma_i(k)$. By Lemma 21, the height of T is at most 6k. Therefore, $f_{\infty}(G) = f_{\infty}(H_r) \leq \gamma_{6k}(k) = g_1(k)$.

We will establish the existence of g_{45} and g_{46} in Lemmas 45 and 46, respectively. Lemmas 29, 45, and 46 and the results from Section 5 together establish Theorem 1, which we now restate:

Theorem 1. There exists a computable function $g_1 : \mathbb{N} \to \mathbb{R}$ such that for every $k \in \mathbb{N}$, every graph G with $f_{\infty}(G) > g_1(k)$ contains a \mathcal{U}_{∞}^k minor. Moreover, every graph G that contains a \mathcal{U}_{∞}^k minor has $f_{\infty}(G) > k$.

Proof. For the first part of the theorem, by Lemmas 29, 45, and 46, there exists a computable function $g_1 : \mathbb{N} \to \mathbb{R}$ such that $f_{\infty}(G) \leq g_1(k)$ for all graphs G without a \mathcal{U}_{∞}^k minor. Thus, every graph G satisfying $f_{\infty}(G) > g_1(k)$ contains a \mathcal{U}_{∞}^k minor.

For the second part of the theorem, it is shown in Section 5 that each of the four graphs Gin \mathcal{U}_{∞}^k satisfies $f_{\infty}(G) > k$. Since $f_{\infty}(G)$ is monotone w.r.t. minors, it follows that $f_{\infty}(G) > k$ for every graph G containing a \mathcal{U}_{∞}^k minor.

7. 3-CONNECTED GRAPHS

The results in this section are purely graph theoretical and may be of independent interest. In particular, we prove several lemmas which give sufficient conditions under which a graph contains some specific graphs as minors. We also introduce a reduction operation, called fanreduction. The main result of the section is that if G is a 3-connected, fan-reduced graph having no \mathcal{U}_{∞}^k minor, then the vertex cover number of G, $\tau(G)$, is bounded by a function of k.

Before proceeding, we quickly review some graph theoretical terminology. Let A, B be subsets of vertices of a graph G. An A-B path is a path P in G such that the ends of P are in A and B respectively, and no internal vertex of P is in $A \cup B$. If H is a subgraph of G then an H-path is a path P in G such that the ends of P are in H but no other vertex nor edge of P is in H.



FIGURE 10. The ladder L_5 .

The *n*-ladder L_n is the graph on 2*n* vertices with vertex set $V = \{v_i \mid i \in [n]\} \cup \{w_i \mid i \in [n]\}$ and edge set $E = \{v_i w_i \mid i \in [n]\} \cup \{v_i v_{i+1}, w_i w_{i+1} \mid i \in [n-1]\}$ (see Figure 10). By repeatedly suppressing degree-2 vertices, we can reduce L_n to the graph K_3 . This implies that $f_{\infty}(L_n) = 2$ for all $n \geq 2$ by Lemma 11.

Lemma 30. For all $k \in \mathbb{N}$, let $g_{30}(k) = 12k^2 + 7k$. If G is a 3-connected graph containing a $g_{30}(k)$ -ladder as a minor, then G contains N_k , P_k , or F_k as a minor.

Proof. Since L_n has maximum degree 3, every graph with an L_n minor also contains an L_n subdivision. Let S be a subgraph of G isomorphic to a subdivision of L_n with $n = g_{30}(k)$. We say that the vertices of S that do not correspond to internal vertices of a subdivided edge are branch vertices. We name these branch vertices $\{v_i \mid i \in [n]\} \cup \{w_i \mid i \in [n]\}$ as in the definition of L_n given above. A rung is a path in S corresponding to an edge of L_n of the form $v_i w_i$, for some $i \in [n]$. We say that an S-path P crosses a rung R, if the ends of P are in different components of S - V(R). A rung is crossed if it is crossed by some S-path, and is uncrossed otherwise.

If there exists an S-path in G that crosses at least 2k + 1 rungs, then G contains an N_k minor, and we are done. Hence, we may assume that each S-path crosses at most 2k rungs of S.

We say that the path in S from v_1 to v_n avoiding all w_i for $i \in [n]$ is the upper path of S. Similarly the lower path is the path in S from w_1 to w_n avoiding all vertices v_i for $i \in [n]$. For each $i \in \{2, \ldots, n-1\}$, let S_{ℓ}^i and S_r^i be the components of $S - \{v_i, w_i\}$ that contain v_1 and v_n , respectively.

Suppose there are 8k + 1 uncrossed rungs R_1, \ldots, R_{8k+1} . For each $i \in [8k + 1]$, let $v_{i'}$ and $w_{i'}$ be the ends of R_i . We may assume that i' < j' for all i < j. Since G is 3-connected, $G - \{v_{i'}, w_{i'}\}$ is connected. Therefore, there is a path P in $G - \{v_{i'}, w_{i'}\}$ from $V(S_{\ell}^{i'})$ to $V(S_r^{i'})$. Since R_i is uncrossed, P must use an internal vertex of R_i . Thus, there exists a vertex $y_i \in V(R_i) \setminus \{v_{i'}, w_{i'}\}$ that is connected by an S-path P_i to some vertex $z_i \notin V(R_i)$.

By symmetry and pigeonhole, there is a subset I of size k of $\{2, 4, \ldots, 8k\}$ such that $z_i \in V(S_r^{i'})$ and z_i is not on the lower path of S, for all $i \in I$. Since R_i is uncrossed for all $i \in [8k+1]$ it follows that $z_i \in V(S_{\ell}^{(i+1)'}) \cup V(R_{i+1})$. For the same reason, P_i and P_j are vertex-disjoint for all distinct $i, j \in I$. Therefore, $S \cup \bigcup_{i \in I} P_i$ contains an F_k minor.

We may hence assume that S contains at most 8k uncrossed rungs. Thus, S contains at least $n - 8k = 12k^2 - k$ crossed rungs. Since $12k^2 - k = 1 + (4k+1)(3k-1)$, there is a subset J of [n] of size 3k such that for all distinct $i, j \in J$, $|i - j| \ge 4k + 1$ and R_i is crossed. For each $i \in J$, let P_i be an S-path crossing R_i . Let ℓ_i and r_i be the ends of P_i in S^i_{ℓ} and S^i_r , respectively.

We say that P_i is of type v if ℓ_i and r_i are both on the upper path, type w if ℓ_i and r_i are both on the lower path, and type p otherwise. Since |J| = 3k, there is a subset J' of J of size k such that P_i is of the same type T for all $i \in J'$. Recall that each S-path crosses at most 2krungs and $|i - j| \ge 4k + 1$ for all distinct $i, j \in J'$. Therefore, if $i, j \in J'$ and i < j, then r_i is to the left of ℓ_j . Moreover, for the same reason, P_i and P_j are vertex-disjoint for all distinct $i, j \in J'$. Therefore, $S \cup \bigcup_{i \in J'} P_i$ contains an F_k minor if $\mathsf{T} \in \{v, w\}$ and $S \cup \bigcup_{i \in J'} P_i$ contains a P_k minor if $\mathsf{T} = p$. For each $k \in \mathbb{N}$, the *k*-fan is the graph consisting of a *k*-vertex path called its *outer path*, plus a universal vertex called its *center*. The edges connecting the center to the ends of the *k*-vertex path are called the *boundary edges* of the *k*-fan. A fan is a graph isomorphic to a *k*-fan for some *k*.

Let H be a fan, and assume that G has an H-model. We say that the H-model is rooted at x, y if x and y are contained in the vertex images of vertices a and b of H, respectively, and ab is a boundary edge of the fan.

Lemma 31. For all $k, q \in \mathbb{N}$, let $g_{31}(k, q) = 3(8k^3)^q$. Let G be a graph and let $P = p_1 \cdots p_r$ be a path in G of length at least $g_{31}(k, q)$ such that $V(G) \setminus V(P)$ is a stable set. Then at least one of the following holds:

- (1) G has a k-fan minor;
- (2) there is a model of the q-fan in G rooted at p_2, p_{r-1} and avoiding p_1, p_r ;
- (3) there are non-consecutive indices s,t with 1 < s < t < r such that $\{p_s, p_t\}$ separates in G the p_s-p_t subpath of P from the other vertices of P.

Proof. The proof is by induction on q. For the base case q = 1, observe $g_{31}(k, 1) \ge 24$, for all $k \in \mathbb{N}$. Thus, it suffices to take p_2 and the p_3-p_{r-1} subpath of P as the two vertex images to obtain a model of the 1-fan rooted at p_2, p_{r-1} and avoiding p_1, p_r .

For the inductive step, assume q > 1. Let $S = V(G) \setminus V(P)$. We may assume that every vertex in S has degree at most k - 1 in G, since otherwise there is a k-fan minor in G. Note that $g_{31}(k,q) = 8k^3 \cdot g_{31}(k,q-1)$. A jump is a pair (a,b) of indices $a, b \in [r]$ with $b \ge a + 2$ such that either $p_a p_b \in E(G)$ (type 1) or p_a and p_b have a common neighbor in S (type 2). For definiteness, if both conditions are satisfied then (a,b) is considered to be of type 1. To each jump (a,b) of type 2 we associate a corresponding middle vertex $w \in S$ adjacent to both a and b, that is chosen arbitrarily. A jump (a,b) is called an *outer jump* if a = 1 or b = r; otherwise, (a, b) is an *inner jump*. In what follows we will be mostly interested in inner jumps.

Case 1: There exists an inner jump (a, b) with $b - a \ge k \cdot g_{31}(k, q - 1)$. Let (a, b) be such a jump. If (a, b) is of type 2, we first modify it as follows. Let w be the middle vertex of (a, b). Since w has degree at most k - 1, it follows that there exists a jump (a', b') with $b' - a' \ge k \cdot g_{31}(k, q - 1)/(k - 2) \ge g_{31}(k, q - 1)$ such that w is adjacent to $p_{a'}$ and $p_{b'}$ but to no vertex lying strictly in between them on P. We rename (a', b') to (a, b).

Let G' be the minor of G obtained by contracting the p_1-p_a subpath of P into p_a and the p_b-p_r subpath of P into p_b . Let P' be the path obtained from P by performing these contractions. We regard p_a and p_b as the ends of P'. Note that $V(G') \setminus V(P')$ is a stable set in G'. Since P' has length $b - a \ge g_{31}(k, q - 1)$, by induction at least one of the following holds:

- (1) G' has a k-fan minor;
- (2) there is a model \mathcal{M}' of the (q-1)-fan in G' rooted at p_{a+1}, p_{b-1} and avoiding p_a, p_b ;
- (3) there are non-consecutive indices s, t with a < s < t < b such that $\{p_s, p_t\}$ separates in G' the p_s-p_t subpath of P' from the other vertices of P'.

In the first case, we are done since G' is a minor of G. In the second case, \mathcal{M}' is also such a model in G since the two subpaths that were contracted in the definition of G' resulted in vertices p_a, p_b . By symmetry, we may assume that the vertex image V_0 corresponding to the center of the fan contains p_{a+1} .

Recall that $2 \le a < b \le r - 1$, since (a, b) is an inner jump. Let L and R be the p_2-p_a and p_b-p_{r-1} subpaths of P, respectively. Let w be the middle vertex of (a, b) if (a, b) is type 2. Let R' = R if R is type 1, and $R' = R \cup \{w\}$ if (a, b) is type 2. In either case, observe that L and R' are connected by an edge. By construction, $V(L) \cup V(R)$ is disjoint from all vertex images



FIGURE 11. Illustration of a k-fan-model obtained from a jump sequence $(a_1, b_1), \ldots, (a_{2k}, b_{2k})$ for k = 4. The blue path is the vertex image for the center of the fan, and the red path corresponds to the outer path. Edges incident to the center of the fan map to the first edge of the subpath of P from a_{2i} to b_{2i-1} .

of \mathcal{M}' . Since w is not adjacent to any internal vertex of P', $\{w\}$ is also disjoint from all vertex images of \mathcal{M}' . Finally, the edges $p_a p_{a+1}$ and $p_{b-1} p_b$ connect V(L) and V(R) to the vertex images of \mathcal{M}' containing p_{a+1} and p_{b-1} , respectively. Therefore, $(\mathcal{M}' \setminus \{V_0\}) \cup \{V_0 \cup L, R'\}$ is a model of the q-fan in G rooted at p_2, p_{r-1} and avoiding p_1, p_r , as desired.

It remains to consider the third case. Suppose s, t are non-consecutive indices with a < s < t < b such that $\{p_s, p_t\}$ separates in G' the $p_s - p_t$ subpath of P' from the other vertices of P'. Given how G' was obtained from G, this is also true in G. That is, $\{p_s, p_t\}$ separates in G the $p_s - p_t$ subpath of P from the other vertices of P, as desired.

Case 2: $b - a < k \cdot g_{31}(k, q - 1)$ for all inner jumps (a, b). Let us introduce one more definition. A *jump sequence* is a sequence $(a_1, b_1), \ldots, (a_\ell, b_\ell)$ of inner jumps with $\ell \ge 1$ satisfying $a_i < a_{i+1} < b_i < b_{i+1}$ for each $i \in [\ell - 1]$, and $b_i \le a_{i+2}$ for each $i \in [\ell - 2]$. Its *length* is ℓ and its *spread* is $b_\ell - a_1$.

Case 2.1: There exists a jump sequence of spread at least $2k^2 \cdot g_{31}(k, q-1)$. Let $(a_1, b_1), \ldots, (a_\ell, b_\ell)$ be a jump sequence of spread at least $2k^2 \cdot g_{31}(k, q-1)$ and with ℓ minimum. For each $i \in [\ell]$, if (a_i, b_i) is of type 2 let $w_i \in S$ be the middle vertex of (a_i, b_i) .

We claim that all middle vertices w_i defined above are distinct. Indeed, assume $w_i = w_j$ for some $i, j \in [\ell]$ with i < j. Then (a_i, b_j) is also an inner jump, and $(a_1, b_1), \ldots, (a_{i-1}, b_{i-1}), (a_i, b_j), (a_{j+1}, b_{j+1}), \ldots, (a_{\ell}, b_{\ell})$ is a jump sequence, as the reader can easily check. But the latter jump sequence has length at most $\ell - 1$ and yet its spread is also $b_{\ell} - a_1$, contradicting our choice of the original jump sequence.

Since $b_i - a_i \leq k \cdot g_{31}(k, q-1)$ for each $i \in [\ell]$, we have

$$2k^2 \cdot g_{31}(k, q-1) \le b_{\ell} - a_1 \le \sum_{i \in [\ell]} (b_i - a_i) \le \ell k \cdot g_{31}(k, q-1),$$

implying $\ell \geq 2k$. Now, one can obtain a k-fan-model using the jump sequence $(a_1, b_1), \ldots, (a_{2k}, b_{2k})$ as illustrated in Figure 11.

Case 2.2: All jump sequences have spread less than $2k^2 \cdot g_{31}(k, q-1)$. Let

 $M = \{2, r-1\} \cup \{i \in [r] \mid (1, i) \text{ is an outer jump}\} \cup \{i \in [r] \mid (i, r) \text{ is an outer jump}\}.$

If there are k outer jumps of the form (1, i) then G has a k-fan minor, and the same is true for those of the form (i, r). Thus we may assume that $|M| \leq 2k$. By the pigeonhole principle, there are two indices $i, j \in M$ with i < j and $M \cap [i + 1, j - 1] = \emptyset$ such that

$$j-i \ge \frac{r-1}{|M|-1} \ge \frac{g_{31}(k,q)}{2k} = 4k^2 \cdot g_{31}(k,q-1).$$

If there exists an inner jump (a, b) with a < i < b, let $(a_1, b_1), \ldots, (a_\ell, b_\ell)$ be a jump sequence such that $a_1 < i < b_1$ and maximizing its spread, and let $s = b_\ell$. If no such jump exists, simply let s = i. We claim that there is no inner jump (a, b) with a < s < b. This is obviously true if s = i, so assume $s \neq i$, and consider the corresponding jump sequence $(a_1, b_1), \ldots, (a_\ell, b_\ell)$ defined above. Arguing by contradiction, suppose that there is an inner jump (a, b) with a < s < b. If $a \leq a_1$ then (a, b) is a jump sequence with a < i < b and spread $b - a > b_\ell - a_1$, contradicting our choice of the jump sequence. If $a_1 < a$ then letting $\ell' \in [\ell]$ be the smallest index such that $a < b_{\ell'}$ (which is well defined since $a < b_\ell$), we deduce that $(a_1, b_1), \ldots, (a_{\ell'}, b_{\ell'}), (a, b)$ is a jump sequence with $a_1 < i < b_1$ and of spread $b - a_1 > b_\ell - a_1$, again a contradiction. Hence, no inner jump (a, b) with a < s < b exists, as claimed.

Next, if there exists an inner jump (a, b) with a < j < b, let $(a'_1, b'_1), \ldots, (a'_{\ell'}, b'_{\ell'})$ be a jump sequence such that $a'_{\ell'} < j < b'_{\ell'}$ and maximizing its spread, and let $t = a'_1$. If no such jump exists, simply let t = j. By a symmetric argument, there is no inner jump (a, b) with a < t < b.

Recall that every jump sequence has spread strictly less than $2k^2 \cdot g_{31}(k, q-1)$. Thus, $s-i \leq 2k^2 \cdot g_{31}(k, q-1) - 1$ and $j-t \leq 2k^2 \cdot g_{31}(k, q-1) - 1$. It follows that

$$t-s \ge j-i-4k^2 \cdot g_{31}(k,q-1)+2 \ge 2.$$

In other words, [s+1, t-1] is not empty. Since $[s+1, t-1] \subseteq [i+1, j-1]$ and $M \cap [i+1, j-1] = \emptyset$, there is no outer jump (1, b) with $b \in [s+1, t-1]$ and there is no outer jump (a, r) with $a \in [s+1, t-1]$. Since we already established that there is no inner jump (a, b) with a < s < bor a < t < b, we deduce that the two indices s, t satisfy the third outcome of the claim. That is, s and t are non-consecutive indices with 1 < s < t < r such that $\{p_s, p_t\}$ separates in Gthe $p_s - p_t$ subpath of P from the other vertices of P.

As an easy corollary of Lemma 31, we obtain the following strengthening of Lemma 4.7 in [14].¹

Lemma 32. For all $k \in \mathbb{N}$, let $g_{32}(k) = 3(8k^3)^k$. Let G be a graph with no k-fan minor. Let P be a path in G of length at least $g_{32}(k)$ such that $V(G) \setminus V(P)$ is a stable set. Then there exist two non-consecutive internal vertices u, v of P such that $\{u, v\}$ separates in G the u-v subpath of P from the other vertices of P.

Proof. Note that $g_{32}(k) = g_{31}(k,k)$. The lemma follows by applying Lemma 31 to G and P, and noting that the first two outcomes of Lemma 31 are impossible since G has no k-fan minor.

Next, we introduce two lemmas about 3-connected graphs containing subdivisions of large fans as subgraphs. Given a graph G, we say that F is a *fan subdivision in* G if F is a subgraph of G isomorphic to a subdivision of a fan. Moreover, we say that F is a *maximal* fan subdivision in G if F is maximal with respect to subgraph inclusion. That is, for every fan subdivision F' in G such that $F \subseteq F' \subseteq G$, we have F = F'.

Lemma 33. For all $k \in \mathbb{N}$, let $g_{33}(k) = 8k^4 + 4k^3 + 10k$. If G is a 3-connected graph and F is a maximal fan subdivision in G such that at least $g_{33}(k)$ of the edges of the fan are subdivided, then G has an L_k , S_k or F_k minor.

Proof. Let F^* denote the *m*-fan such that F is a subdivision of F^* , where v_0 is the center of F^* and $v_1 \cdots v_m$ is the outer path of F^* .

In the following we consider the graph H obtained from G by performing the following two operations. First, we contract each component of G - V(F) into a vertex. Second, for each edge e of F^* that is subdivided at least once in F, we contract the corresponding path P of

¹The latter lemma works under the assumption that G does not have the graph consisting of two vertices linked by k parallel edges as a minor, which is more restrictive than just forbidding a k-fan minor. Nevertheless, the two proofs are based on a similar strategy.

F into a 2-edge path, that is, we leave just one subdivision vertex. We call this subdivision vertex v_i^1 if $e = v_0 v_i$ for some $i \in [m]$, and v_i^2 if $e = v_i v_{i+1}$ for some $i \in [m-1]$.

Hence, each vertex of H is of the form v_i, v_i^1, v_i^2 , or results from the contraction of a component of G - V(F). We denote by F' the fan subdivision in H that is the image of F, that is, which is obtained from F by the above contractions. Observe that F' is a maximal fan subdivision in H. Indeed, if some fan subdivision in H strictly contained F' then that fan subdivision could be mapped to a fan subdivision in G strictly containing F, contradicting the maximality of F.

We will establish the following key property of H:

(*) If u_i is a vertex of H of the form v_i^1 or v_i^2 , then there is an F'-path P_i in H of length at most 2 connecting u_i to another vertex u'_i of F' distinct from

its two neighbors in F' and from v_0 .

Suppose (*) does not hold for some v_i^1 . Then $\{v_0, v_i\}$ is a size-2 cutset of H separating v_i^1 from every vertex v_j with $j \notin \{0, i\}$ (here we implicitly use that $m \geq 2$, since F^* has at least $g_{33}(k) \geq 2$ edges). By the construction of H, the set $\{v_0, v_i\}$ is also a cutset of G separating v_i^1 from every vertex v_j with $j \notin \{0, i\}$. However, this contradicts the fact that G is 3-connected.

The remaining case is if (\star) does not hold for some v_i^2 . Here we first observe that v_i^2 is not adjacent to v_0 in H, because otherwise this would contradict the maximality of F' in H. For the same reason, there is no length-2 path from v_i^2 to v_0 in H going through a vertex in $V(H) \setminus V(F')$. Using these two observations, we can proceed similarly as in the proof for v_i^1 . This concludes the proof of (\star) .

Now, we color each edge of F' blue, and each remaining edge of H red. Consider the graph H^* obtained from H as follows. Every edge of the form $v_i^1 v_i$ is contracted to the vertex v_i , every edge of the form $v_i^2 v_i$ is contracted to the vertex v_i , and finally, for every vertex $w \in V(H) \setminus V(F')$, we select a neighbor of w distinct from v_0 in the current graph (which exists) and contract the corresponding edge. Finally, we delete all red edges incident to v_0 . Loops and parallel edges resulting from edge contractions are deleted as always, but if a red edge parallel to a blue edge is created, we keep the blue edge and delete the red edge. Thus, the blue subgraph of H^* is exactly the fan F^* . Let R^* denote the red subgraph of H^* . We regard R^* as a spanning subgraph of H^* , and thus R^* may have isolated vertices.

If R^* has a vertex of degree at least 2k + 1, then that vertex is not v_0 (since v_0 is not incident to any red edge), and it is then easily seen that H^* has an S_k minor. Thus we may assume that the maximum degree of R^* is at most 2k.

If R^* has a matching of size k^3 , then by Pigeonhole and Erdős-Szekeres [12], R^* has a matching $M = \{v_{a_i}v_{b_i} : i \in [k]\}$ of size k that satisfies one of the following three conditions:

- (1) $a_1 < a_2 < \dots < a_k < b_1 < b_2 < \dots < b_k$, or
- (2) $a_1 < a_2 < \dots < a_k < b_k < b_{k-1} < \dots < b_1$, or (3) $a_1 < b_1 < a_2 < b_2 < \dots < a_k < b_k$.

In the first two cases, we see that H^* has an L_k minor (obtained by combining M with the $v_{a_1}-v_{a_k}$ and $v_{b_1}-v_{b_k}$ subpaths of the outer path of H^*). In the third case, we see that H^* has an F_k minor. Hence we may assume that R^* has no matching of size k^3 .

It follows that R^* has a vertex cover of size at most $2k^3$. However, since R^* has maximum degree at most 2k, it follows in turn that at most $2k^3(2k+1)$ vertices of R^* have non-zero degrees in R^* .

Recall that v_i^1 and v_i^2 (if they exist) are the only 2 vertices of F' that are contracted to v_i in F^* . Since F^* has at least $g_{33}(k)$ edges that are subdivided in F' and $g_{33}(k)/2-2k^3(2k+1)=5k$, there exists $I \subseteq [m]$ with |I| = k such that the following holds:

- there is a vertex u_i of the form v_i^1 or v_i^2 in H, for each $i \in I$;
- v_i has degree 0 in R^* for all $i \in I$, and
- $|i-j| \ge 5$ for all $i, j \in I$ with $i \ne j$.

Now, consider an index $i \in I$ and its associated subdivision vertex u_i in H. By (\star) , there is an F'-path P_i in H of length at most 2 connecting u_i to another vertex u'_i of F' distinct from its two neighbors in F' and from v_0 . The (one or two) edges of P_i are red and are not incident to v_0 , and they disappeared in the edge contraction operations leading to the graph H^* . It follows that u'_i is very close to u_i in $F' - v_0$, namely u'_i must be one of v_{i-1}, v_{i+1} , or one of the subdivision vertices $v_{i-1}^1, v_{i+1}^1, v_{i-1}^2, v_i^2, v_{i+1}^2$ (if they exist). Since the paths P_i and P_j are vertex disjoint for all $i, j \in I$ with $i \neq j$ (which follows from

Since the paths P_i and P_j are vertex disjoint for all $i, j \in I$ with $i \neq j$ (which follows from the fact that v_i and v_j have degree 0 in R^*), and since $|i - j| \geq 5$, combining F' with these k paths we can see that H contains an F_k minor.

Let F be an m-fan with center v_0 and outer path $v_1 \cdots v_m$. Suppose that F is a subgraph of a graph G. We say that F is *reducible in* G if $m \ge 5$ and all vertices v_2, \ldots, v_{m-1} have degree exactly 3 in G. The F-reduction of G is the minor of G obtained by contracting the edges of the path $v_3 \cdots v_{m-1}$. Thus, the resulting graph has m - 4 fewer vertices than G.

A reducible fan subgraph in G is said to be maximal in G if it is not a proper subgraph of any other reducible fan subgraph of G. Observe that if F_1 and F_2 are two distinct maximal reducible fan subgraphs of G then F_1 and F_2 are almost vertex disjoint in the following sense: F_2 contains none of the internal vertices of the outer path of F_1 , and vice versa. We define the fan-reduction of G as the minor of G obtained by simultaneously performing all F-reductions for all maximal reducible fan subgraphs F of G. By the previous observation, this minor is well-defined. We say that G is fan-reduced if G does not contain a reducible fan subgraph. Observe that the fan-reduction of G is fan-reduced.

Lemma 34. For all $k \in \mathbb{N}$, let $g_{34}(k) = 20k^5 + 14k^4 + 2k^3 + 5k$. If G is a 3-connected fan-reduced graph containing a $g_{34}(k)$ -fan as a subgraph, then G contains an S_k , F_k or L_k minor.

Proof. Consider an *m*-fan subgraph F in G with center v_0 , outer path $v_1 \cdots v_m$, and $m = g_{34}(k)$. Let H be obtained from G by contracting each component of G - V(F) into a vertex. We color the edges of F blue and the remaining edges of H red as in the proof of Lemma 33, and define H^* in exactly the same way. The only difference here is that no edge of F needs to be contracted since F is already a fan. In the notation used in the proof of Lemma 33, here we have $F = F' = F^*$. Let R^* denote the red spanning subgraph of H^* .

If R^* has a vertex of degree at least 2k + 1 or a matching of size k^3 , then we find one of our target minors, exactly as in the proof of Lemma 33. Thus we may assume that this does not happen, implying that at most $2k^3(2k + 1)$ vertices of R^* have non-zero degrees in R^* .

Since $(m - 2k^3(2k + 1))/(2k^3(2k + 1) + 1) \ge 5k$ there is an index $i \in [m - 5k]$ such that none of $v_{i+1}, \ldots, v_{i+5k}$ is incident to a red edge in H^* . For each $\ell \in [k]$, there must be an index $j \in \{i + 5(\ell - 1) + 2, i + 5(\ell - 1) + 3, i + 5(\ell - 1) + 4\}$ such that v_j is incident to a red edge of H. Otherwise, $v_{i+5(\ell-1)+1}, \ldots, v_{i+5(\ell-1)+5}$ together with v_0 form a reducible fan in G. Since all red edges incident to v_j in H disappeared when constructing H^* , it follows that v_j is adjacent in H to a vertex $w_{\ell} \in V(H) \setminus V(F)$ such that the neighbors of w_{ℓ} in H are a subset of $\{v_0, v_{j-1}, v_j, v_{j+1}\}$. Furthermore, w_{ℓ} must be adjacent to at least three of these four vertices, since otherwise G would not be 3-connected. Now, combining F with the k vertices w_1, \ldots, w_k we see that H contains an F_k minor. \Box

Combining the two previous lemmas, we obtain the following lemma.

Lemma 35. For all $k \in \mathbb{N}$, let $g_{35}(k) = g_{34}(k)(g_{33}(k) + 1) + g_{33}(k)$. If G is a 3-connected, fan-reduced graph containing a subdivision of a $g_{35}(k)$ -fan as a subgraph, then G has an S_k , F_k or L_k minor.

Proof. Since G contains a $g_{35}(k)$ -fan subdivision, G contains a maximal m-fan subdivision F with $m \ge g_{35}(k)$. If at least $g_{33}(k)$ edges of the m-fan are subdivided in F, then, by

Lemma 33, G contains an L_k , S_k or F_k minor. Otherwise, F contains an m'-fan as a subgraph with $m' \ge (g_{35}(k) - g_{33}(k))/(g_{33}(k) + 1) = g_{34}(k)$, and by Lemma 34, G contains an L_k , S_k or F_k minor.

The next lemma is standard, we include the proof nevertheless for completeness.

Lemma 36. For all $k \in \mathbb{N}$, let $g_{36}(k) = k^{k^2+2}$. If G is a graph with a $g_{36}(k)$ -fan minor, then G contains a subdivision of a k-fan as a subgraph, or G contains an L_k minor.

Proof. Let G be a graph containing an m-fan F as minor with $m = g_{36}(k)$. Let v_0 be the center of F and $v_1 \cdots v_m$ be the outer path. Let $\{X_i \mid i \in \{0, 1, \ldots, m\}\}$ denote an F-model in G, with X_i denoting the vertex image of v_i .

For every edge $v_i v_j$ of F we choose vertices x_i^j, x_j^i of X_i, X_j , respectively, such that $x_i^j x_j^i \in E(G)$. Let T be a subtree of $G[X_0 \cup \{x_i^0 \mid i \in [m]\}]$ such that the leaves of T are exactly the vertices x_i^0 for $i \in [m]$. If T contains a vertex of degree at least k, then G contains a subdivision of a k-fan. Thus we may assume that T has maximum degree less than k.

Now, suppress all degree-2 vertices in T, giving a tree T'. Thus every non-leaf vertex of T' has degree between 3 and k-1 in T'. In particular, $k \ge 4$. Choose an arbitrary non-leaf vertex r of T'. Since T' has $m \ge (k-1)^{k^2+2}$ leaves and maximum degree at most k-1, it follows that there is a leaf of T' at distance at least $\log_{k-1} |T'| - 1 \ge \log_{k-1} (k-1)^{k^2+2} - 1 = k^2 + 1$ from r in T'.

Consider the path P' of T' from r to that leaf, minus the leaf, and let P denote the corresponding path of T. By construction, there are k^2 vertex-disjoint $V(P) - \{x_i^0 \mid i \in [m]\}$ paths in the graph $G[X_0 \cup \{x_i^0 \mid i \in [m]\}]$. Applying Erdős-Szekeres we then find an L_k minor in G.

Lemma 37. For all $k \in \mathbb{N}$, let $g_{37}(k) = g_{32}(g_{36}(g_{35}(g_{30}(k))))$. If G is a 3-connected, fanreduced graph with no \mathcal{U}_{∞}^k minor, then the maximum length of a path in G is at most $g_{37}(k)$.

Proof. By Lemmas 36, 35 and 30, we deduce that G has no m-fan minor, where $m = g_{36}(g_{35}(g_{30}(k)))$. Arguing by contradiction, suppose G has a path P of length more than $g_{37}(k) = g_{32}(m)$.

Let C_1, \ldots, C_p denote the components of G - V(P). Let H be the graph obtained from G by contracting each component C_i into a vertex c_i . Note that H has no m-fan minor, since H is a minor of G. By Lemma 32, applied to the graph H and path P, there exist two non-consecutive internal vertices u, v of P such that $\{u, v\}$ separates in H the uv-subpath of P from the other vertices of P. However, the same remains true in G, by construction of H. Therefore, $\{u, v\}$ is a cutset of G, contradicting the fact that G is 3-connected.

In the following we will use another reduction operation for 3-connected graphs. Let G be a 3-connected graph and let $h \ge 3$ be a fixed integer. Let T_1, \ldots, T_ℓ be an enumeration of all stable sets of G satisfying the following conditions for each $i \in [\ell]$,

- $|T_i| \ge h+1$,
- there exists $S_i \subseteq V(G)$ with $|S_i| \leq h$ such that for all $v \in T_i$, the set of neighbors of v in G is exactly S_i ,
- T_i is inclusion-wise maximal with respect to the above two properties.

Observe that by maximality, the sets T_1, \ldots, T_ℓ are pairwise disjoint. Let G' be the graph obtained from G by removing all vertices in T_i except h + 1 of them, for each $i \in [\ell]$. Clearly, G' does not depend on which h + 1 vertices remain in each T_i . We call G' the *h*-reduction of G. Note that, since G is 3-connected, G' is also 3-connected. If G' is the graph G itself, that is, no vertex was removed in the process, then we say that G is *h*-reduced.

Lemma 38. Let G be a 3-connected graph, let $h \ge 3$, and let G' be the h-reduction of G. Then $\tau(G') = \tau(G)$. *Proof.* Since G' is a subgraph of $G, \tau(G') \leq \tau(G)$. It remains to show that $\tau(G') \geq \tau(G)$.

Let T_1, \ldots, T_ℓ and S_1, \ldots, S_ℓ be as in the definition of *h*-reduction. Let *W* be a minimumsize vertex cover of *G'*. We claim $\bigcup_{i \in [\ell]} S_i \subseteq W$. By contradiction, suppose that there exists a vertex $w \in S_i \setminus W$ for some $i \in [\ell]$. Then all edges incident to *w* have to be covered with all h + 1 vertices of T_i remaining in *G'*. However, S_i has at most *h* vertices. Hence, replacing these h + 1 vertices of T_i with the at most *h* vertices of S_i in *W* gives a smaller vertex cover, a contradiction.

Now, we note that W is also a vertex cover of G, implying that $\tau(G') \ge \tau(G)$. To see this, observe that all edges of G that are not in G' are of the form vw with $v \in T_i$ and $w \in S_i$, and every such edge vw is covered by $w \in S_i \subseteq W$.

Let G be a connected graph and let T be a depth-first search (DFS) tree of G from some vertex r of G. We see T as being rooted at r, and define the usual notions of ancestors and descendants: w is an ancestor of v if w is on the r-v path in T, in which case we say that v is a descendant of w. Note that these relations are not strict: v is both an ancestor and a descendant of itself. By definition of DFS trees, all edges vw of G are such that either v is a strict ancestor of w in T or v is a strict descendant of w in T.

Lemma 39. For all $k, p \in \mathbb{N}$, let $g_{39}(k, p) = ((p+1)2^p + kp^3)^{p+1}$. Let G be a 3-connected graph such that the longest path in G has length at most p, G is p-reduced, and G has no S_k minor. Then $|V(G)| \leq g_{39}(k, p)$.

Proof. Let T be a DFS tree of G rooted at some vertex r of G. First we claim that for every vertex v of G, at most $(p+1)2^p$ children of v in T are leaves of T. Indeed, for each such leaf w, the neighborhood of w in G is a subset of the set X of ancestors of v in T. Since G is p-reduced, at most p+1 of these leaves have the same neighborhood in G. Moreover, $|X| \leq p$, since T has no path of length more than p, implying that there are at most 2^p choices for the neighborhood of w. This implies the claim.

Let

$$d = (p+1)2^p + k(p-1)\binom{p-1}{2} + 1.$$

If T has maximum degree at most d, then since T has at most p + 1 levels,

$$|V(G)| = |V(T)| \le \sum_{i=0}^{p} d^{i} = \frac{d^{p+1} - 1}{d - 1} \le d^{p+1} \le g_{39}(k, p),$$

as desired. Hence, it is enough to show that T has maximum degree at most d. For each $x \in V(T)$, we let T_x be the subtree of T rooted at x. Note that if x has at least two children, then the set of ancestors A of x is a cutset of G. Since G is 3-connected, $|A| \ge 3$. Partitioning the vertices of T into levels according to their distances from the root, it follows that there is only one vertex on each of the first 3 levels. We argue by contradiction and suppose that there is a vertex v of T having at least d children in T. Since $d \ge 2$, the set X of ancestors of v is a cutset of G with $|X| \ge 3$. This implies that v is at distance at least 2 from the root r of T.

Let w be the ancestor of v closest to r in T that is adjacent in G to at least one vertex in T_v . Let P be the w-v path in T. If w has a neighbor in G which is a strict descendant of v, we let v_0 denote a child of v whose subtree T_{v_0} contains a neighbor of w, and let w_0 denote such a neighbor. Otherwise, we just let $v_0 = w_0 = v$. Let C denote the cycle of G obtained by adding the edge ww_0 to the $w-w_0$ path of T.

Recall that at most $(p+1)2^p$ children of v are leaves of T. Enumerate the non-leaf children of v that are distinct from v_0 as v_1, \ldots, v_q ; thus, $q \ge d - (p+1)2^p - 1 = k(p-1)\binom{p-1}{2}$.

Fix some index $i \in [q]$, and let x_i denote a child of v_i in T. We will construct a special K_4 -model in G using the cycle C and some vertices of the subtree T_{v_i} . The four vertex images

of this K_4 -model are denoted V_i, X'_i, P^1_i, P^2_i . We proceed with their definitions in the next few paragraphs.

First, observe that every edge out of $V(T_{x_i})$ in $G - v_i$ has its other end in P, by our choice of w. Choose a vertex x'_i in $V(T_{x_i})$ having a neighbor p_i^2 in V(P), with p_i^2 as close to v on Pas possible (thus possibly $p_i^2 = v$).

Since G is 3-connected, there is an $\{x'_i\}-V(P)$ path Q_i in the graph $G - \{v_i, p_i^2\}$. Let p_i^1 denote the end of Q_i in V(P). Note that all vertices of $Q_i - p_i^1$ are in $V(T_{x_i})$. Also, p_i^1 is a strict ancestor of p_i^2 by our choice of p_i^2 .

For a walk W and vertices a, b of W, we write aWb to denote the a-b subwalk of W. If W_1 and W_2 are walks such that W_1 ends at the same vertex that W_2 starts, we let W_1W_2 denote the concatenation of W_1 and W_2 .

Next, let R_i be a $\{v_i\}$ - $(V(P) \cup V(Q_i))$ path in the graph $G - \{v, x'_i\}$, and let y_i denote its end distinct from v_i . We choose R_i so that y_i is as close as possible to V(P) in the graph $P \cup Q_i$. Let S_i denote the $v_i - x'_i$ path in T. If s_i is the last vertex of R_i contained in S_i , we replace R_i by $S_i s_i R_i$. The definitions of the four vertex images V_i, X'_i, P^1_i, P^2_i depend on whether $y_i \in V(P)$ or not.

First suppose that $y_i \in V(P)$. We define $V_i = V(R_i) \setminus \{y_i\}$ and $X'_i = (V(S_i) \setminus V(R_i)) \cup (V(Q_i) \setminus \{p_i^1\})$. Notice that there is an edge e_i of S_i with one end in V_i and the other in X'_i . The two sets P_i^1, P_i^2 will be a partition of the vertices of the cycle C, chosen as follows. If y_i is a strict ancestor of p_i^2 , let P_i^1 be the vertices of the $p_i^1 - y_i$ path of T, and let $P_i^2 = V(C) \setminus P_i^1$. If, on the other hand, y_i is a descendant of p_i^2 , let P_i^2 be the vertices of the $p_i^2 - y_i$ path of T, and let $P_i^1 = V(C) \setminus P_i^2$. This case is illustrated in Figure 12.

We now argue that the sets V_i, X'_i, P^1_i, P^2_i do form a K_4 -model in this case. These sets are connected, there is an edge between P^1_i and P^2_i (because of the cycle C), there is an edge between X'_i and P^j_i for $j \in [2]$ (because $p^j_i \in P^j_i$), there is an edge between V_i and X'_i (namely, e_i), and finally there is an edge between V_i and P^j_i for $j \in [2]$ (because one of v, y_i is in P^1_i and the other is in P^2_i). This concludes the case where $y_i \in V(P)$.

Next, suppose that $y_i \notin V(P)$. In this case, y_i is a vertex of $Q_i - p_i^1$. Consider an $\{v_i\} - V(Q_i)$ path R'_i in $G - \{v, y_i\}$. Note that, by our choice of R_i , the path R'_i avoids V(P), and thus all its vertices are in $V(T_{v_i})$. Furthermore, the end y'_i of R'_i distinct from v_i must be in the subpath $x'_iQ_iy_i - \{y_i\}$, again by our choice of R_i .

Define

$$V_i = (V(R_i) \setminus \{y_i\}) \cup (V(R'_i) \setminus \{y'_i\})$$
$$X'_i = V(x'_i Q_i y_i) \setminus \{y_i\}$$
$$P_i^1 = V(y_i Q_i p_i^1)$$
$$P_i^2 = V(C) \setminus \{p_i^1\}$$

Using the previous observations, one can check that V_i, X'_i, P_i^1, P_i^2 form a K_4 -model in this case as well. This case is illustrated in Figure 13.

This ends the definitions of the vertex images V_i, X'_i, P_i^1, P_i^2 . Observe that, in all cases, the only vertices of these sets *not* in the subtree T_{v_i} are the vertices of the cycle C.

Now, there are at most $\binom{p-1}{2}$ choices for p_i^1 and p_i^2 . Furthermore, when $y_i \in V(P)$, there are at most p-2 choices for vertex y_i . Seeing the possibility that $y_i \notin V(P)$ as another 'choice', and using that $q \ge k(p-1)\binom{p-1}{2}$, we conclude that there is a set I of k distinct indices $i \in [q]$ that have the same pair (p_i^1, p_i^2) , that agree on whether $y_i \in V(P)$, and furthermore that have the same vertex y_i in case $y_i \in V(P)$. Letting $P^j = \bigcup_{i \in I} P_i^j$ for $j \in [2]$, we then see that P^1, P^2 together with the sets V_i, X'_i for $i \in I$ define an S_k -model in G, a contradiction.



FIGURE 12. The case $y_i \in V(P)$ of the proof of Lemma 39.



FIGURE 13. The case $y_i \in V(Q_i)$ of the proof of Lemma 39.

Lemma 40. For all $k \in \mathbb{N}$, let $g_{40}(k) = g_{39}(k, g_{37}(k))$. If G is a 3-connected, fan-reduced graph having no \mathcal{U}_{∞}^k minor, then $\tau(G) \leq g_{40}(k)$.

Proof. By Lemma 37, the maximum length of a path in G is at most $p = g_{37}(k)$ since G is 3-connected, and does not have a \mathcal{U}_{∞}^k minor. Let G' be the p-reduction of G. Notice that G' is 3-connected, has no S_k minor and the length of a longest path in G' is bounded by p. Hence,

by Lemma 39, $\tau(G') \leq |V(G')| \leq g_{39}(k, p)$. Now, by Lemma 38,

$$\tau(G) = \tau(G') \le g_{39}(k, p) = g_{39}(k, g_{37}(k)) = g_{40}(k).$$

8. Finishing the proof

Recall that to prove our main result, Theorem 1, it suffices to establish the existence of the functions g_{45} and g_{46} from Lemma 29. We do this in Lemmas 45 and 46 at the end of this section. Before doing so, we require a few more lemmas. The *wheel* W_n is the graph obtained by adding a universal vertex to a cycle of length n.

Lemma 41. $f_{\infty}(W_n) \leq 4$, for all $n \geq 3$.

Proof. Let v_0 be the universal vertex of W_n and $W_n - v_0 = C = v_1 \cdots v_n v_1$. Let d be an arbitrary distance function on W_n . Define S to be the set of inclusion-wise minimal subsets S of E(C) such that S is not flattenable in (W_n, d) . Let d' be d restricted to E(C). Let S_1 be the sets in S that are not flattenable in (C, d'), and let $S_2 = S \setminus S_1$.

Fix $S \in S_2$ and let \vec{S} be an orientation of S such that \vec{S} is flat in (C, d'). Let the length function of $\langle W_n, d; \vec{S} \rangle$ be l, and Z be a negative directed cycle in $\langle W_n, d; \vec{S} \rangle$. Since S is flattenable in (C, d'), Z must use the vertex v_0 . By renaming vertices, we may assume that Z is of the form $v_0v_1\cdots v_kv_0$. Let $P = v_1\cdots v_k$ and $Q = v_k\cdots v_nv_1$. We abuse notation and regard P, Q, and C as subsets of edges or arcs whenever convenient.

Since \vec{S} is flat in (C, d'), $l(C) \ge 0$. Combining this with l(Z) < 0 gives

$$d(v_0v_1) + d(v_0v_k) < l(Q) \le d(Q) \text{ and } d(v_0v_1) + d(v_0v_k) < l(P) \le d(P).$$
(8)

Let H_1 and H_2 be the subgraphs of W_n induced by $\{v_0, v_1, \ldots, v_k\}$ and $\{v_0, v_k, \ldots, v_n, v_1\}$, respectively. Let d_i be the restriction of d to H_i . Clearly, each (H_i, d_i) can be covered by two flat sets F_i^1, F_i^2 . By (8), every negative directed cycle W in $\langle W_n, d; F_i^j \rangle$ can be shortened to a negative directed cycle W' in $\langle H_i, d_i; F_i^j \rangle$ for all $i, j \in [2]$. Therefore, F_i^j is also flat in (W_n, d) for all $i, j \in [2]$. Thus, (W_n, d) has a flat cover of size 4.

We may therefore assume that $S_2 = \emptyset$. That is, every set in S is not flattenable in (C, d'). Let U be the set of edges of W_n incident to v_0 . Note that U is flattenable in (W_n, d) by Lemma 6. If $S_1 = \emptyset$, then E(C) is flattenable in (W_n, d) , and so $E(W_n)$ is the union of two flattenable sets, E(C) and U. Therefore, we may assume $S_1 \neq \emptyset$ and choose $T \in S_1$. Let $X \subseteq E(C)$. Observe that if $\sum_{e \in X} d(e) \leq \frac{1}{2}d(C)$, then X is flattenable in (C, d'). It follows that for every $X \subseteq E(C)$, at least one of X or $E(C) \setminus X$ is flattenable in (C, d'). Since T is not flattenable in (C, d'), $E(C) \setminus T$ is flattenable in (C, d'). Since $S_2 = \emptyset$, $E(C) \setminus T$ is flattenable in (W_n, d) . By minimality, T is the union of two flattenable sets T_1 and T_2 of (W_n, d) . Thus, $E(W_n) = (E(C) \setminus T) \cup T_1 \cup T_2 \cup U$, as required. \Box

We now generalize Lemma 41. This generalization is analogous to Lemma 28 for 2-connected treewidth-2 graphs.

Lemma 42. Let H be a graph obtained by gluing 2-connected graphs G_1, \ldots, G_m on distinct edges of the wheel W_n , such that H has no S_k minor. Let $M = \max_{i \in [m]} f_{\infty}(G_i)$. Then $f_{\infty}(H) \leq (k+7)M$.

Proof. Let $W_n - v_0 = C = v_1 \cdots v_n$. We proceed by induction on |V(H)|. By Lemma 11, we may assume that H has minimum degree at least 3. Let E_0 be the set of glued edges incident to v_0 . If $|E_0| \ge k$, then W_n has a k-glumpkin minor. By Lemma 25, H contains an S_k minor, which is a contradiction. Thus, $|E_0| \le k - 1$.

Let d be an arbitrary distance function on H, and d_W be the restriction of d to W_n . By Lemma 41, (W_n, d_W) has a flat cover of size 4, say F_1, F_2, F_3, F_4 . Let F_0 be the set of arcs of $D(W_n)$ incident to v_0 . For each $i \in [4]$, let Γ_i^+, Γ_i^- be such that $\Gamma_i^+ \cup \Gamma_i^- = F_i \setminus F_0$ and $(v_{j+1}, v_j) \notin \Gamma_i^+$, $(v_j, v_{j+1}) \notin \Gamma_i^-$ for all $j \in \mathbb{Z}/n\mathbb{Z}$. Since every two arcs of Γ_i^{\pm} are both forward or both backward arcs of every directed cycle of $D(W_n)$, (Γ_i^{\pm}, F_i) is a frame of (W_n, d_W) for all $i \in [4]$. Let H' be the graph obtained from W_n by only gluing along glued edges belonging to E(C). By Lemma 6 and Lemma 22, $f_{\infty}(H') \leq 1 + 8M$. Since $|E_0| \leq k - 1$, Lemma 12 implies that

$$f_{\infty}(H) \le f_{\infty}(H') + (k-1)(M-1) \le (k+7)M.$$

We now apply our results about wheels to fan-reduced graphs. Recall that every graph can be obtained from its fan-reduction by replacing fan gadgets by fans.

Lemma 43. Let F be a reducible fan of a graph G, and let G' be the F-reduction of G. Then $f_{\infty}(G) \leq f_{\infty}(G') + 4$.

Proof. Let v_0 be the center of F, and $v_1 \cdots v_k$ be its outer path. When performing the F-reduction, we rename vertices such that v_0 is still the center and $v_1v_2v_{k-1}v_k$ is the outer path of the reduced fan. Let W_{k-2} be the wheel graph on k-1 vertices, where v_0 is the universal vertex, and $v_2v_3\cdots v_{k-1}v_2$ is the outer cycle. Let H be the graph obtained by performing the 3-sum of G' with W_{k-2} along the clique $v_0v_2v_{k-1}$. Note that G is obtained from H by deleting the edge v_2v_{k-1} . Hence, $f_{\infty}(G) \leq f_{\infty}(H)$. By Lemma 41, $f_{\infty}(W_{k-2}) \leq 4$. Therefore, applying Lemma 12,

$$f_{\infty}(G) \le f_{\infty}(H) \le f_{\infty}(G') + f_{\infty}(W_{k-2}) \le f_{\infty}(G') + 4.$$

Lemma 44. Let G be a graph, G' be the fan-reduction of G, and t be the number of reduced fans in G'. Then, $t \leq \tau(G')$ and $f_{\infty}(G) \leq 5\tau(G')$.

Proof. Suppose F' is a reduced fan in G', where v_0 is the center and $v_1 \cdots v_4$ is the outer path. Note that every vertex cover of G' must use at least one of v_2 or v_3 . Since $\{v_2, v_3\}$ is disjoint from all other reduced fans, we conclude that $t \leq \tau(G')$. For the second part, first observe that $f_{\infty}(G') \leq \tau(G')$, by Lemma 7. By repeatedly applying Lemma 43 to each maximal reducible fan of G,

$$f_{\infty}(G) \le f_{\infty}(G') + 4t \le 5\tau(G').$$

Lemma 45. For all $k \in \mathbb{N}$, let $g_{45}(k) = 5g_{40}(k)$. If G is a 3-connected graph with no \mathcal{U}_{∞}^k minor, then $f_{\infty}(G) \leq g_{45}(k)$.

Proof. Let G' be the fan-reduction of G. By Lemmas 44 and 40,

$$f_{\infty}(G) \le 5\tau(G') \le 5g_{40}(k) = g_{45}(k).$$

Lemma 46. For all $k, M \in \mathbb{N}$, let $g_{46}(k, M) = (2k + 11)Mg_{40}(k)$. Let G be a 3-connected graph and let H be a graph obtained by gluing 2-connected graphs G_1, \ldots, G_m on distinct edges of G such that H has no \mathcal{U}^k_{∞} minor. Let $M = \max_{i \in [m]} f_{\infty}(G_i)$. Then $f_{\infty}(H) \leq g_{46}(k, M)$.

Proof. We proceed by induction on |E(H)|. By Lemma 11, we may assume that H has minimum degree at least 3. Let \mathcal{F} be the set of maximal reducible fans in G. Let G' be the fan-reduction of G and let \mathcal{F}' be the set of reduced fans in G'. If F is a fan with center v_0 and outerpath $v_1 \cdots v_m$, we define $I(F) = V(F) \setminus \{v_0, v_1, v_m\}$. Let X' be a vertex cover of G'and set $X = X' \setminus \bigcup_{F' \in \mathcal{F}'} I(F')$. We regard X as a subset of vertices of G. Let Γ be the set of glued edges of G and Γ_X be the set of edges of Γ incident to a vertex in X.

If $|\Gamma_X| > (k-1)\tau(G')$, then there is a vertex $x \in X$ incident to at least k glued edges xy_1, \ldots, xy_k . Since G is 3-connected, there is a tree in G-x containing $\{y_1, \ldots, y_k\}$. Therefore, G contains a k-glumpkin minor that is obtained by contracting the tree to a single vertex. By Lemma 25, H contains an S_k minor, which is a contradiction. Hence, $|\Gamma_X| \leq (k-1)\tau(G')$.

Let $F \in \mathcal{F}$ with center v_0 and outerpath $v_1 \cdots v_m$. Let F^+ be the graph obtained from F by adding the edge v_1v_m (if it is not already present) and gluing all G_i whose glued edge is contained in E(F).

Let G^X be obtained from G by gluing all G_i whose glued edge belongs to Γ_X and replacing each $F \in \mathcal{F}$ by a triangle, Δ_F . Let H^+ be obtained from G^X by simultaneously taking the clique-sum of F^+ and G^X along Δ_F for all $F \in \mathcal{F}$. Notice that H is a subgraph of H^+ . By Lemma 44, $f_{\infty}(G) \leq 5\tau(G')$. Since $|\Gamma_X| \leq (k-1)\tau(G')$, by Lemma 12

 $f_{\infty}(G^X) \le f_{\infty}(G) + (k-1)(M-1)\tau(G') \le (k+4)M\tau(G').$

Since G' is a 3-connected fan-reduced graph not containing a \mathcal{U}_{∞}^k minor, by Lemma 40, $\tau(G') \leq g_{40}(k)$. By Lemma 42, $f_{\infty}(F^+) \leq (k+7)M$, for all $F \in \mathcal{F}$. Finally, $|\mathcal{F}| \leq \tau(G')$, by Lemma 44. Putting this altogether,

$$f_{\infty}(H) \leq f_{\infty}(H^{+}) \\ \leq f_{\infty}(G^{X}) + (k+7)M\tau(G') \\ \leq (k+4)M\tau(G') + (k+7)M\tau(G') \\ = (2k+11)M\tau(G') \\ \leq (2k+11)Mg_{40}(k) \\ = q_{46}(k,M).$$

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UNAVOIDABLE MINORS FOR GRAPHS WITH LARGE $\ell_p\text{-}\mathrm{DIMENSION}$

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