DUAL CIRCUMFERENCE AND COLLINEAR SETS[☿]

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Abstract. We show that, if an n-vertex triangulation G of maximum degree Δ has a dual that contains a cycle of length ℓ, then G has a non-crossing straight-line drawing in which some set, called a *collinear set*, of $\Omega(\ell/\Delta^4)$ vertices lie on a line. Using the current lower bounds on the length of longest cycles in cubic 3-connected graphs, this implies that every n-vertex planar graph of maximum degree Δ has a collinear set of size $\Omega(n^{0.8}/\Delta^4)$.

1 Introduction

Throughout this paper, all graphs are simple and finite and have at least 4 vertices. For a planar graph G, we say that a set $S \subseteq V(G)$ is a *collinear set* if G has a non-crossing straight-line drawing in which the vertices of S are all collinear. A *plane graph* is a planar graph G along with a particular non-crossing drawing of G. The *dual* $G^\star$ of a plane graph G is the graph whose vertex set $V(G^\star)$ is the set of faces in G and in which $fg \in E(G^\star)$ if and only if the faces f and g of G have at least one edge in common. The *circumference*, $c(G)$, of a graph G is the length of the longest cycle in G. In Section 2, we prove the following theorem:

Theorem 1. *Let G be a triangulation of maximum degree Δ whose dual $G^\star$ has circumference ℓ. Then G has a collinear set of size $\Omega(\ell/\Delta^4)$.*

The dual of a triangulation is a 3-connected cubic planar graph. The study of the circumference of 3-connected cubic planar graphs has a long and rich history going back to at least 1884 when Tait [27] conjectured that every such graph is Hamiltonian. In 1946, Tait's conjecture was disproved by Tutte who gave a non-Hamiltonian 46-vertex example [28]. Repeatedly replacing vertices of Tutte's graph with copies of itself gives a family of graphs, $\langle G_i : i \in \mathbb{Z}\rangle$ in which G_i has $46 \cdot 45^i$ vertices and circumference at most $45 \cdot 44^i$. Stated another way, n-vertex members of the family have circumference $O(n^\alpha)$, for $\alpha = \log_{44}(45) < 0.9941$. The current best upper bound of this type is due to Grünbaum and Walther [18] who construct a 24-vertex non-Hamiltonian cubic 3-connected planar graph, resulting in a family of graphs in which n-vertex members have circumference $O(n^\alpha)$ for $\alpha = \log_{23}(22) < 0.9859$.

A series of results has steadily improved the lower bounds on the circumference of n-vertex (not necessarily planar) 3-connected cubic graphs. Barnette [5] showed that, for every n-vertex 3-connected cubic graph G, $c(G) = \Omega(\log n)$. Bondy and Simonovits [8] improved this bound to $e^{\Omega(\sqrt{\log n})}$ and conjectured that it can be improved to $\Omega(n^\alpha)$ for

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some $\alpha > 0$. Jackson [19] confirmed this conjecture with $\alpha = \log_2(1+\sqrt{5}) - 1 > 0.6942$. Billinksi et al. [6] improved this to the solution of $4^{1/\alpha} - 3^{1/\alpha} = 2$, which implies $\alpha > 0.7532$. The current record is held by Liu, Yu, and Zhang [22] who show that $\alpha > 0.8$.

It is known that any planar graph of maximum degree Δ can be triangulated so that the resulting triangulation has maximum degree $\lceil 3\Delta/2 \rceil + 11$ [21]. This fact, together with Theorem 1 and the result of Liu, Yu, and Zhang [22], implies the following corollary:

Corollary 1. *Every n-vertex planar graph of maximum degree Δ contains a collinear set of size $\Omega(n^{0.8}/\Delta^4)$.*

It is known that every planar graph G has a collinear set of size $\Omega(\sqrt{n})$ [9, 13] . Corollary 1 therefore improves on this bound for bounded-degree planar graphs and, indeed for the family of n-vertex planar graphs of maximum degree $\Delta \in O(n^{\delta})$, with $\delta < 0.075$. For example, the triangulations dual to Grünbaum and Walther's construction have maximum degree $\Delta \in O(\log n)$. As discussed below, this implies that there exists n-vertex triangulations of maximum degree $O(\log n)$ whose largest collinear set has size $O(n^{0.9859})$. Corollary 1 implies that every n-vertex planar graph of maximum degree $O(\log n)$ has a collinear set of size $\Omega(n^{0.8})$.

Recently, Dujmović et al. [14] have shown that every collinear set is *free*. That is, for any planar graph G, any collinear set $S \subseteq V(G)$, and any set $X \subset \mathbb{R}^2$ with $|X| = |S|$, there exists a non-crossing straight-line drawing of G in which the vertices of S are drawn on the points of X. Because of this, collinear sets have immediate applications in graph drawing and related areas. For applications of Corollary 1, including untangling [11, 23, 29, 17, 20, 9, 12, 13, 25], column planarity [3, 15, 12, 13], universal point subsets [16, 1, 12, 13], and partial simultaneous geometric drawings [15, 4, 2, 7, 13] the reader is referred to Dujmović [13] and Dujmović et al. [14, Section 1.1]. Corollary 1 gives improved bounds for all of these problems for planar graphs of maximum $\Delta \in o(n^{0.075})$.

For example, it is known that every n-vertex planar geometric graph can be untangled while keeping some set of $\Omega(n^{0.25})$ vertices fixed [9] and that there are n-vertex planar geometric graphs that cannot be untangled while keeping any set of $\Omega(n^{0.4948})$ vertices fixed [10]. Although asymptotically tight bounds are known for paths [11], trees [17], outerplanar graphs [17], planar graphs of treewidth two [25], and planar graphs of treewidth three [12], progress on the general case has been stuck for 10 years due to the fact that the exponent 0.25 comes from two applications of Dilworth's Theorem. Thus, some substantially new idea appears to be needed. By relating collinear/free sets to dual circumference, the current paper presents an effective new idea. Indeed, Corollary 1 implies that every bounded-degree n-vertex planar geometric graph can be untangled while keeping $\Omega(n^{0.4})$ vertices fixed. Even for bounded-degree planar graphs, $\Omega(n^{0.25})$ was the best previously-known lower bound.

Our work opens two avenues for further progress:

1. Lower bounds on the circumference of 3-connected cubic graphs are an active area of research. At the time of writing, the $\Omega(n^{0.8})$ lower bound of Liu, Yu, and Zhang [22] is less than a year old. Any further progress on these lower bounds will translate immediately to an improved bound in Corollary 1 and all its applications.

2. It is possible that the dependence on Δ can be removed from Theorem 1 and Corollary 1, thus making these results applicable to all planar graphs, regardless of maximum degree.

2 Proof of Theorem 1

Let G be a plane graph. We treat the vertices of G as points, the edges of G as closed curves, and the faces of G as closed sets (so that a face contains all the edges on its boundary and an edge contains both its endpoints). Whenever we consider subgraphs of G we treat them as having the same embedding as G. Similarly, if we consider a graph G' that is homeomorphic[1] to G then we assume that the edges of G'—each of which represents a path in G whose internal vertices all have degree 2—inherit their embedding from the paths they represent in G.

Finally, if we consider the dual $G^\star$ of G then we treat it as a plane graph in which each vertex f is represented as a point in the interior of the face f of G that it represents. The edges of $G^\star$ are embedded so that an edge fg is contained in the union of the two faces f and g of G, it intersects the interior of exactly one edge of G that is common to f and g, and this intersection consists of a single point.

A *proper good curve* C for a plane graph G is a Jordan curve with the following properties:

proper: for any edge xy of G, C either contains xy, intersects xy in a single point (possibly an endpoint), or is disjoint from xy; and

good: C contains at least one point in the interior of some face of G.

Da Lozzo et al. [12] show that proper good curves define collinear sets:

Theorem 2. *In a plane graph G, a set $S \subseteq V(G)$ is a collinear set if and only if there is a proper good curve for G that contains S.*

For a triangulation G, let $v(G)$ denote the size of a largest collinear set in G. We will show that, for any triangulation G of maximum degree Δ whose dual is $G^\star$, $v(G) = \Theta(c(G^\star)/\Delta^4)$ by relating proper good curves in G to cycles in $G^\star$.

As shown by Ravsky and Verbitsky [25, 24], the inequality $v(G) \le c(G^\star)$ is easy: If G is a triangulation that has a proper good curve C containing k vertices, then a slight deformation of C produces a proper good curve that contains no vertices. This curve intersects a cyclic sequence of faces $f_0, \dots, f_{k'-1}$ of G with $k' \ge k$. In this sequence, f_i and $f_{(i+1) \bmod k'}$ share an edge, for every $i \in \{0, \dots, k'-1\}$, so this sequence is a closed walk in the dual $G^\star$ of G. The properness of the original curve and the fact that each face of G is a triangle ensures that $f_i \ne f_j$ for any $i \ne j$, so this sequence is a cycle in $G^\star$ of length $k' \ge k$. Therefore, $c(G^\star) \ge v(G)$. From the result of Grünbaum and Walther described above, this implies that there are n-vertex triangulations G such that $v(G) = O(n^{0.9859})$.

The other direction, lower-bounding $v(G)$ in terms $c(G^\star)$ is more difficult. Not every cycle C of length ℓ in $G^\star$ can be easily transformed into a proper good curve containing

[1] We say that a graph G' is homeomorphic to G if G' can be obtained from G by repeatedly contracting an edge of G that is incident to a degree-2 vertex.

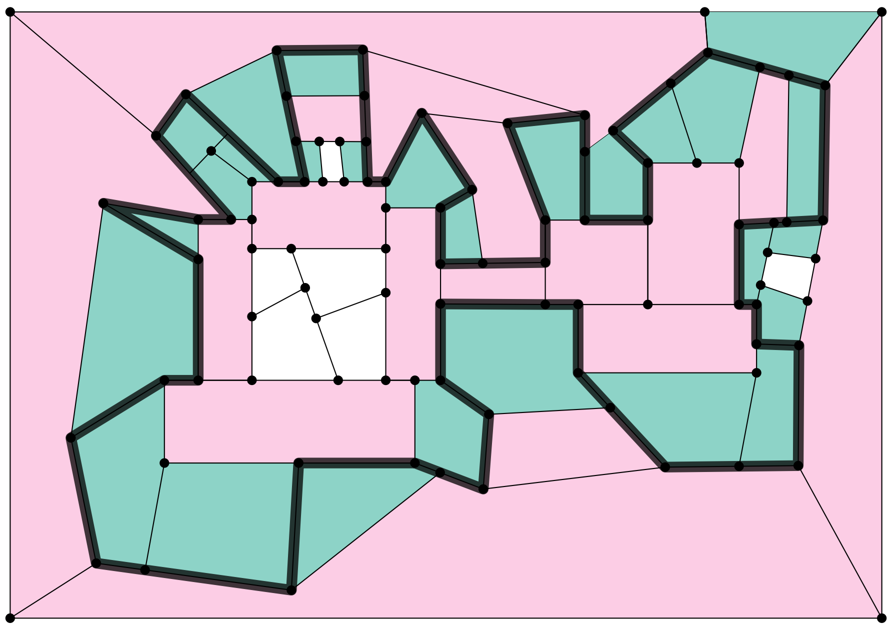

Figure 1: Faces of $G^\star$ that are pinched and caressed by C. C is bold, caressed faces are teal, pinched faces are pink, and untouched faces are unshaded.

a similar number of vertices in C. In the next section, we describe three parameters τ, ρ, and κ of a cycle C in $G^\star$ and show that C can always be transformed into a proper good curve containing $\Omega(\kappa)$ vertices of G.

2.1 Faces that are Touched, Pinched, and Caressed

Throughout the remainder of this paper, G is a triangulation whose dual is $G^\star$ and C is a cycle in $G^\star$. Refer to Figure 1 for the following definitions. We say that a face f of $G^\star$

1. is *touched* by C if $f \cap C \neq \emptyset$;
2. is *pinched* by C if $f \cap C$ is a cycle or has more than one connected component; and
3. is *caressed* by C if it is touched but not pinched by C.

Since C is almost always the cycle of interest, we will usually say that a face f of $G^\star$ is touched, pinched, or caressed, without specifically mentioning C. We will frequently use the values τ, ρ, and κ to denote the number of faces of $G^\star$ in some region that are τouched, ρinched or κaressed. Observe that, since every face that is touched is either pinched or caressed, we have the identity $\tau = \rho + \kappa$.

Lemma 1. *If C caresses κ faces of $G^\star$ then G has a proper good curve that contains at least $\kappa/4$ vertices so, by Theorem 2, $v(G) \geq \kappa/4$.*

Proof. Let F be the set of faces in $G^\star$ that are caressed by C. Each element $u \in F$ corresponds to a vertex of G so we will treat F as a set of vertices in G. Consider the subgraph

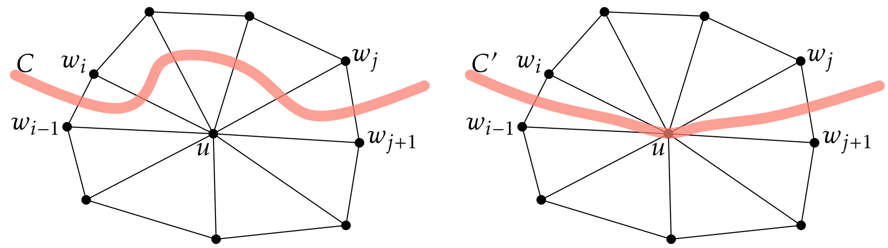

Figure 2: Transforming the dual cycle C into a proper good curve C' containing u.

$G[F]$ of G induced by F. The graph $G[F]$ is planar and has κ vertices. Therefore, by the 4-Colour Theorem [26], $G[F]$ contains an independent set $F' \subseteq F$ of size at least $\kappa/4$.

We claim that there is a proper good curve for G that contains all the vertices in F'. To see this, first observe that the cycle C in $G^\star$ already defines a proper good curve (that does not contain any vertices of G) that we also call C. We perform local modifications on C so that it contains all the vertices in F'.

For any vertex $u \in F'$, let $w_0,\dots,w_{d-1}$ denote the neighbours of u in cyclic order. The curve C intersects some contiguous subsequence $uw_i,\dots,uw_j$ of the edges adjacent to u. Since u is caressed, this sequence does not contain all edges incident to u. Therefore, the curve C crosses the edge $w_{i-1}w_i$, then crosses $uw_i,\dots,uw_j$, and then crosses the edge w_jw_{j+1}. We modify C by removing the portion between the first and last of these crossings and replacing it with a curve that contains u and is contained in the two faces $w_{i-1}uw_i$ and w_juw_{j+1}. (See Figure 2.)

After performing this local modification for each $u \in F'$ we have a curve C' that contains every vertex $u \in F'$. All that remains is to verify that C' is good and proper for G. That C' is good for G is obvious. That C' is proper for G follows from the following two observations: (i) C' does not contain any two adjacent vertices (since F' is an independent set); and (ii) if C' contains a vertex u, then it does not intersect the interior of any edge incident to u. □

Lemma 1 reduces our problem to finding a cycle in $G^\star$ that caresses many faces. It is tempting to hope that any sufficiently long cycle in $G^\star$ caresses many faces, but this is not true; Figure 3 shows that even a Hamiltonian cycle C in $G^\star$ may caress only four faces, two inside C and two outside of C. In this example, there is an obvious sequence of faces $f_0,\dots,f_k$, all contained in the interior of C where f_i shares an edge with f_{i+1} for each $i \in \{0,\dots,k-1\}$. The only caressed faces in the interior of C are the endpoints f_0 and f_k of this sequence.

Our strategy is to define a tree structure, T_0 on groups of faces contained in the interior of C and a similar structure, T_1 on groups of faces in the exterior of C. We will then show that every leaf of T_0 or T_1 contains a face caressed by C. In Figure 3, the tree T_0 is the path $f_0,\dots,f_k$ and, indeed, the leaves f_0 and f_k of this tree are caressed by C. After a non-trivial analysis of the trees T_0 and T_1, we will eventually show that, if C does not caress many faces, then T_0 and T_1 have many nodes, but few leaves. Therefore T_0 and T_1 have many degree-2 nodes. This abundance of degree-2 nodes makes it possible to perform a *surgery*

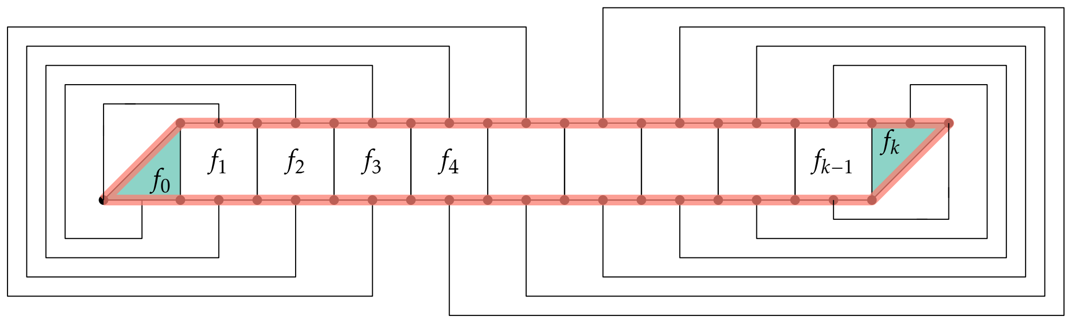

Figure 3: A Hamiltonian cycle C in $G^\star$ that caresses only four faces.

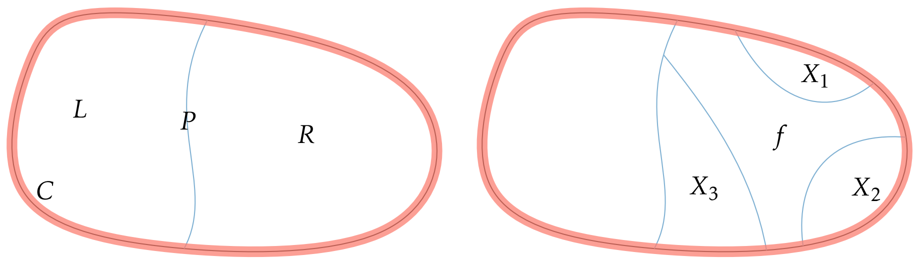

Figure 4: The proof of Lemma 2.

on C that increases the number of caressed faces. Performing this surgery repeatedly will then produce a curve C that caresses many faces.

A path $P = v_1, \ldots, v_r$ in $G^\star$ is a *chord path* (for C) if $v_1, v_r \in V(C)$ and $v_2, \ldots, v_{r-1} \notin V(C)$. Note that this definition implies that the interior vertices $v_2, \ldots, v_{r-1}$ of P are either all contained in the interior of C or all contained in the exterior of C.

Lemma 2. *Let P be a chord path for C and let L and R be the two faces of the graph formed by $P \cup C$ that each contain P in their boundary. Then R contains at least one face of $G^\star$ that is caressed by C.*

Proof. The proof is by induction on the number, t, of faces of $G^\star$ contained in R. If $t = 1$, then R is a face of $G^\star$ and it is caressed by C.

If $t > 1$, then consider the face f of $G^\star$ that is contained in R and has the first edge of P on its boundary. Refer to Figure 4. Since $t > 1$, $X = R \setminus f$ is non-empty. The set X may have several connected components $X_1, \ldots, X_k$, but each X_i has a boundary that contains a chord path P_i for C. We can therefore apply induction on P_1 (or any P_i) using $R = X_1$ in the inductive hypothesis. □

2.2 Auxiliary Graphs and Trees: H, $\tilde{H}$, T_0, and T_1

Refer to Figure 5. Consider the auxilliary graph H with vertex set $V(H) \subseteq V(G^\star)$ and whose edge set consist of the edges of C plus those edges of $G^\star$ that belong to any face pinched by C. Let $v_0, \ldots, v_{r-1}$ be the clockwise cyclic sequence of vertices on some face f

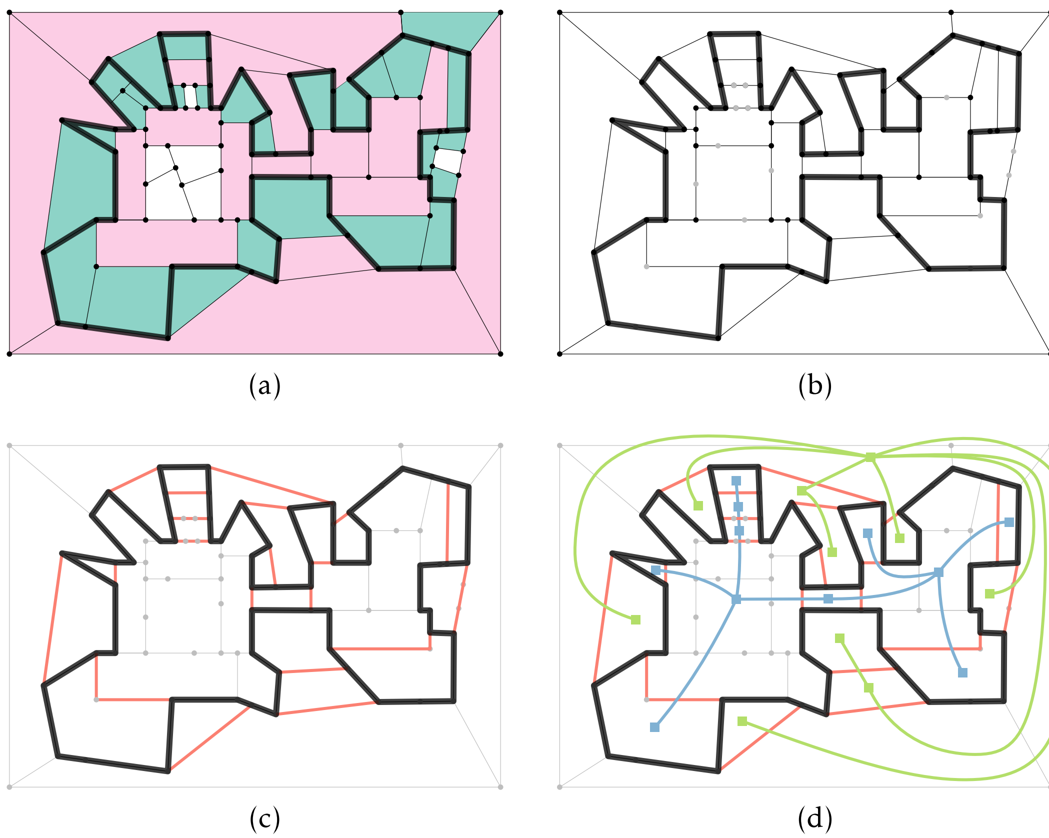

Figure 5: (a) the cycle C in $G^\star$ with faces classified as pinched or caressed; (b) the auxilliary graph H; (c) the auxilliary graph $\tilde{H}$ with keeper paths highlighted; (d) the trees T_0 and T_1.

of $G^\star$ that is pinched by C. We identify three kinds of vertices that are *special* with respect to f: (see Figure 6).

1. A vertex v_i is special of *Type A* if $v_{i-1}v_i$ is an edge of C and $v_i v_{i+1}$ is not an edge of C.
2. A vertex v_i is special of *Type B* if $v_{i-1}v_i$ is not an edge of C and $v_i v_{i+1}$ is an edge of C.
3. A vertex v_i is special of *Type Y* if v_i not incident to any edge of C and v_i has degree 3 in H.

We say that a chord path $v_i, \ldots, v_j$ is a *keeper* with respect to f if v_i is special of Type A, v_j is special of Type B, and none of $v_{i+1}, \ldots, v_{j-1}$ are special. We let $\tilde{H}$ denote the subgraph of H containing all the edges of C and all the edges of all paths that are keepers with respect to some pinched face f of $G^\star$.

It is worth emphasizing at this point that, by definition, every keeper is entirely contained in the boundary of at least one face f of $G^\star$. This property will be useful shortly.

Let $\tilde{H}'$ denote the graph that is homeormophic to $\tilde{H}$ but does not contain any degree 2 vertices. That is, $\tilde{H}'$ is the minor of $\tilde{H}$ obtained by repeatedly contracting an edge incident

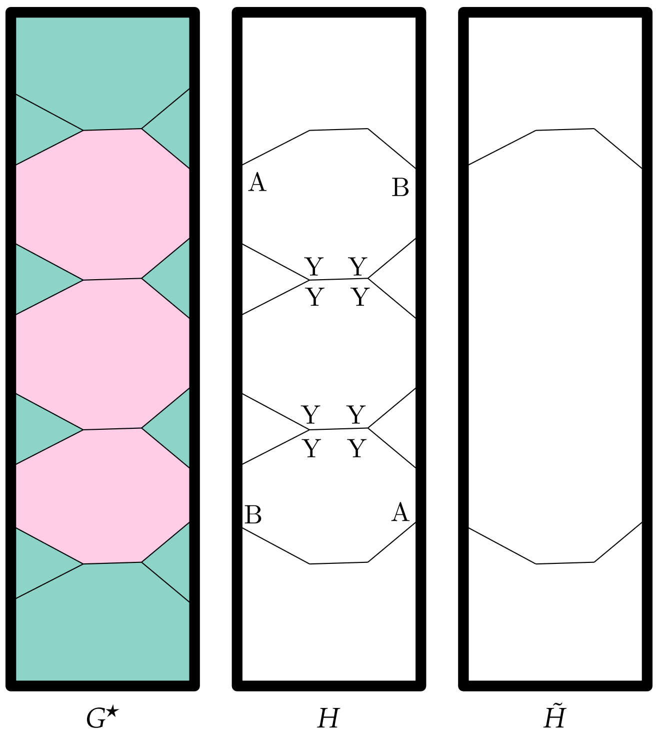

Figure 6: The graphs $G^\star$, H, and $\hat{H}$ and the classification of special vertices of types A, B, and Y.

a degree-2 vertex. The graph $\tilde{H}'$ naturally inherits an embedding from the embedding of $\tilde{H}$. This embedding partitions the edges of $\tilde{H}'$ into three sets:

1. The set B of edges that are contained in (the embedding of) C;
2. The set E_0 of edges whose interiors are contained in the interior of (the embedding of) C; and
3. The set E_1 of edges whose interiors are contained in the exterior of (the embedding of) C.

Observe that, for each $i \in \{0,1\}$, the graph H_i whose edges are exactly those in $B \cup E_i$ is outerplanar, since all vertices of H_i are on a single face, whose boundary is C. Let $H_i{}^\star$ be dual of H_i and let T_i be the subgraph of $H_i{}^\star$ whose edges are all those dual to the edges of E_i. From the outerplanarity of H_i, it follows that T_i is a tree.

Each vertex of T_i corresponds to a face of $\tilde{H}$. From this point onwards, we will refer to the vertices of T_i as *nodes* to highlight this fact, so that a node u of T_i is synonymous with the subset of $\mathbb{R}^2$ contained in the corresponding face of $\tilde{H}$. In the following, when we say that a node u of T_i contains a face f of $G^\star$ we mean that f is one of the faces of $G^\star$ whose union makes up u. The degree, δ_u of any node u in T_i is exactly equal to the number of keeper paths on the boundary of u.

The following lemma allows us to direct our effort towards proving that one of T_0 or T_1 has many leaves.

Lemma 3. *Each leaf u of T_i contains at least one face of $G^\star$ that is caressed by C.*

Proof. The edge of T_i incident to u corresponds to a chord path P. The graph $P \cup C$ has two faces with P on its boundary, one of which is u. The lemma now follows immediately from Lemma 2, with $R = u$. □

We will make use of the following well-known property of 3-connected plane graphs.

Lemma 4. *If G has $n \geq 4$ vertices then any two faces of $G^\star$ share at most one edge.*

Proof. Suppose that two faces f and g share two edges e_1 and e_2. Then e_1 and e_2 form an edge cutset of $G^\star$. If $G^\star$ contains at least four vertices, then two of the endpoints of e_1 and e_2 form a vertex cutset of $G^\star$ of size 2, contradicting the fact that $G^\star$ is 3-connected. That $G^\star$ contains at least four vertices follows from Euler's Formula, which gives the number of vertices in $G^\star$ as $2n-4 \geq 4$ for all $n \geq 4$. □

Note that, as should be evident from Figure 6, the number of faces in $\tilde{H}$ is not lower bounded by any function of the number of faces in H and therefore the number of nodes in T_0 and T_1 is not lower bounded by any function of ℓ. Indeed, a single face of $\tilde{H}$ may contain arbitrarily many faces of $G^\star$ that are touched by C. The following important lemma shows that, when this happens, the corresponding node in T_0 or T_1 either has high degree or contains many faces of $G^\star$ that are caressed by C. The latter case is obviously good for our purposes. The former case is also good because a vertex of degree δ in any tree creates $\delta-2$ leaves and, by Lemma 3, each leaf contains at least one caressed face.

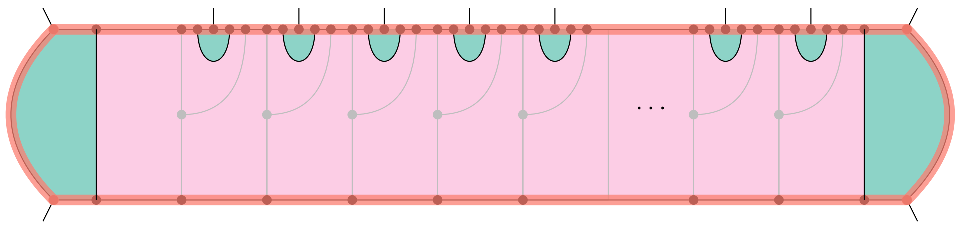

Figure 7: An example showing the tightness of Lemma 5.

For a node u of T_i, we let τ_u, ρ_u, κ_u, and δ_u denote the number of touched face of $T^\star$ in u, pinched faces of $G^\star$ in u, the number of caressed faces of $G^\star$ in u, and the degree of u in T_i, respectively.

Lemma 5. *For any node u of T_i, $\rho_u \le 2(\kappa_u + \delta_u)$.*

Before proving Lemma 5, we point out that the leading constant 2 is tight. Figure 7 shows an example in which all $\rho_u = 2k+1$ pinched faces of $G^\star$ are contained in a single (pink) node u of T_0 that contains $\kappa_u = 0$ caressed faces and has degree $\delta_u = k+2$.

Proof of Lemma 5. The proof is a discharging argument. We assign each pinched face in u a single unit of charge, so that the total charge is ρ_u. We then describe a discharging procedure that preserves the total charge and such that, after executing this procedure, the folowing conditions are satisfied:
(Post1) Each pinched face has no charge.
(Post2) Each caressed face has charge at most 2.
(Post3) Each keeper path has charge at most 2.
Since there is a bijection between keeper paths in u and edges of T_i incident to u, this proves the result.

The discharging procedure is made up of two routines, an *initialization procedure* and a *recursive procedure*. The recursive procedure takes inputs (L,R,P,c), where P is a chord path, $L,R \subseteq u$, $L \cap R = P$, L contains at least one face of $G^\star$, and $0 \le c \le 2$ is a charge that we think of as resting on P. The input (L,R,P,c) must satisfy the following conditions:
(Pre1) Each face of $G^\star$ in L that shares an edge with P is pinched.
(Pre2) If $c > 1$ then P is contained in the boundary of a single face of $G^\star$ that is contained in L.
The procedure guarantees that, after its completion, the charge of c that was resting on P has been moved into R, any other charges in L are undisturbed, and the faces contained in R satisfy (Post1)–(Post3).

Before defining the recursive procedure itself, we will show how it is used by the initialization procedure. This initialization procedure takes an arbitrary pinched face f contained in u. Since f is pinched, it has $r \ge 2$ chord paths $P_1,\dots,P_r$ on its boundary. For each $i \in \{1,\dots,r\}$, let L_i^- be the component of $u \setminus P_i$ that contains f, let $L_i = L_i^- \cup P_i$, and let $R_i = u \setminus L_{i^-}$. This initialization procedure guarantees that, after it runs, all the faces and chord paths in $u \setminus R_1$ satisfy (Post1)–(Post3) but does not modify charges on faces and keeper paths in R_1.

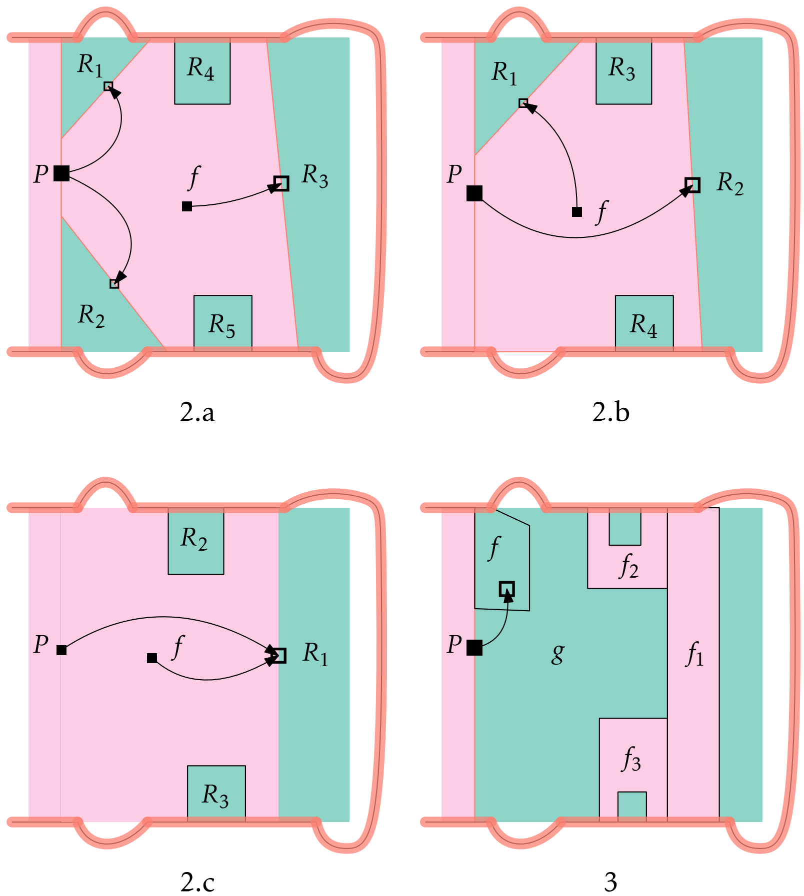

Figure 8: Discharging steps in the proof of Lemma 5.

The initialization procedure works as follows: Since f is pinched it has a charge of 1 so we move the charge from f onto P_2 and apply the recursive procedure to $(L_2, R_2, P_2, 1)$. Since f is pinched, this satisifies (Pre1) and since the final argument $c = 1$ this satisfies (Pre2). Once these recursive procedures are complete conditions (Post1)–(Post2) are satisfied for all faces in $f \cup R_2$.

Next, we apply the recursive procedure on $(L_i, R_i, P_i, 0)$ for each $i \in \{3, \ldots, r\}$. Since f is a pinched face, this satisifies (Pre1) and since the final argument $c = 0$ this satisfies (Pre2). Once the recursive procedure is complete conditions (Post1)–(Post2) are satisfied for all faces in $f \cup R_i$ and does affect any charges in R_1.

Since every face and keeper path contained in u is contained in R_i for at most one i, the initialization procedure produces a distribution of charges that satisifies (Post1)–(Post3) for $u \setminus R_1$, as required.

Next we describe the recursive discharging procedure that takes (L, R, P, c) satisifying (Pre1) and (Pre2) and moves charges in R, and the charge c resting on P, so that they satisfy (Post1)–(Post3). There are several cases to consider (see Figure 8):

1. R contains no face of $G^\star$ that is pinched by C. If R contains no face of $G^\star$ at all, then

$R = P$ is a keeper path, in which case we leave a charge of c on it and we are done. Otherwise R contains at least one face of $G^\star$ and Lemma 2 ensures that R contains at least one caressed face f. We move the charge from P onto f and we are done.

2. R contains a face f of $G^\star$ that is pinched by C and that shares at least one edge with P. We consider three subcases, each illustrated in Figure 8:

   (a) f contains neither endpoint of P. In this case, $R \setminus f$ has two distinct components, R_1^- and R_2^- each containing a distinct endpoint of P. For each $i \in \{1,2\}$, let P_i be the chord path that separates R_i^- from $u \setminus R_i^-$. Since f is pinched, f contains $r \geq 3$ chord paths $P_1,\dots,P_r$. Indeed, if P_1 and P_2 were the only chord paths on f, then f would be caressed. For each $i \in \{1,\dots,r\}$, let $L_i^- = u \setminus P_i$, let $L_i = L_i^- \cup P_i$, and let $R_i = u \setminus L_i^-$.

   We split the charge c on P evenly between P_1 and P_2 and apply the recursive procedure on $(L_i, R_i, P_i, c/2)$ for each $i \in \{1,2\}$. Next, we move the charge on f to P_3 and apply the recursive procedure on $(L_3, R_3, P_3, 1)$. Finally, we apply the recursive procedure on $(L_i, R_i, P_i, 0)$ for each $i \in \{4,\dots,r\}$.

   The recursive call $(L_1, R_1, P_1, c/2)$ satisfies (Pre1) because the path P_1 used in this recursive call is contained in the boundary of f and P. In particular each face of $G^\star$ contained in $u \setminus R_1$ that is incident to P_1 is either in L and incident to P or is the face f. The latter faces are pinched by (Pre1) and f is pinched by definition. The recursive call on $(L_1, R_1, P_1, c/2)$ also satisifies (Pre2) since $c \leq 2$, so $c/2 \leq 1$. The same argument shows that the recursive call on $(L_2, R_2, P_2, c/2)$ satisfies (Pre1) and (Pre2).

   For each $i \in \{3,\dots,r\}$, the recursive call on $(L_i, R_i, P_i, \star)$ satisfies (Pre1) because P_i is contained in f and f is pinched and satisfies (Pre2) because the final argument is 1 for $i = 3$ and 0 for $i \in \{4,\dots,r\}$.

   (b) f contains exactly one endpoint of P. In this case, $R \setminus f$ has one connected component R_1^- that contains an endpoint of P. Since f is pinched, f has $r \geq 2$ chord paths $P_1,\dots,P_r$ on its boundary, where P_1 separates R_1^- from $u \setminus R_1$. Define $L_1,\dots,L_r$, $R_1,\dots,R_r$, and $P_2,\dots,P_r$ as in the previous case.

   Because f is pinched, it has one unit of charge on it, that we move onto P_1 before calling the recursive procedure on $(L_1, R_1, P_1, 1)$. This satisfies (Pre1) for the same reasons described in the previous case and satisfies (Pre2) because the final argument is 1.

   The path P has a charge $c \leq 2$ which we move onto P_2 and call the recursive procedure on (L_2, R_2, P_2, c). This recursive call satisfies (Pre1) because f is pinched and it satisfies (Pre2) because P_2 is entirely contained in the boundary of f.

   Finally, for each $i \in \{3,\dots,r\}$, we call the recursive procedure on $(L_i, R_i, P_i, 0)$. Clearly each of these calls also satisfies (Pre1) and (Pre2).

   (c) f contains both endpoints of P. We claim that, in this case, P must be on the boundary of more than one face in L, otherwise P would be a keeper path. To see this, observe that the face f contains both the first edge e_1 and last edge e_2 of P. If $e_1 = e_2$ because P is a single edge, then it is certainly a keeper, which is not possible since P is in the interior of u. Otherwise, by Lemma 4, e_1 and e_2 are on the boundary of two different faces in L.

Therefore, by (Pre2) P has $c \le 1$ units of charge assigned to it. Now, since f is pinched, it has $r \ge 1$ chord paths $P_1,\dots,P_r$, other than P on its boundary. Define $L_1,\dots,L_r$ and $R_1,\dots,R_r$ as in the previous two cases. Now, P has a charge $c \le 1$ and, since it is pinched, f has a charge of 1. We move these $c+1$ units of charge from P and f onto P_1 and call the recursive procedure on $(L_1, R_1, P_1, c+1)$. This satisfies (Pre1) since f is pinched and satisifies (Pre2) since P_1 is entirely contained in the boundary of f.

For each $i \in \{2,\dots,r\}$ we then call the recursive procedure on $(L_i, R_i, P_1, 0)$. Clearly each of these calls satisfies (Pre1) and (Pre2).

3. R contains at least one pinched face of $G^\star$, but no pinched face in R shares an edge with P. We claim that there is a single face, g of H, contained in R, that contains all of P on its boundary. Indeed, edges of $G^\star$ not in C are in H only if they are on the boundary of some pinched face of $G^\star$. Since no pinched face of $G^\star$ in R shares an edge with P, none of the edges incident to internal vertices of P and contained in R are part of H. Therefore, P is on the boundary of a single face of H that is contained in R.

   Let f be the face of $G^\star$ that is contained in R and that contains the first edge of P. The face f is touched by C but not pinched, so it must be caressed. We move the c units of charge from P onto f.

   Now, R still contains one or more pinched faces $f_1,\dots,f_k$, such that each f_i shares part of a chord path P_i with g. Consider one such f_i and observe that $u \setminus f_i$ has $r_i \ge 2$ chord paths $P_{i,1},\dots,P_{i,r_i}$ on its boundary and use the convention that $P_{i,1} = P_i$. Define $L_{i,1},\dots,L_{i,r_i}$ and $R_{i,1},\dots,R_{i,r_i}$ in a manner analagous to $L_1,\dots,L_r$ and $R_1,\dots,R_r$ in the previous cases.

   On each such face f_i, we run the *initialization* procedure and this reorganizes the charges in $u \setminus R_{i,1}$ so that they satisfy (Post1)–(Post3) and does not modify charges in $L \cup g$. Doing this for each $i \in \{1,\dots,k\}$ completes the description of the discharging procedure.

To complete the proof, first observe that if u contains no pinched faces then the result is trivially true. Otherwise u contains a pinched face f such that one of the components R_1 of $u \setminus f$ contains no pinched faces. (The existence of such an f is established by choosing f so that the minimum number of faces in any component of $u \setminus f$ is minimum over all pinched faces f in u.) Since R_1 contains no pinched faces, it contains no charges, so it already satisfies (Post1)–(Post3). Running the initialization procedure on f will then redistribute charges so that they satisfy (Post1)–(Post3) for all faces and keeper paths in u. □

2.3 Bad Nodes

We say that a node of T_i is *bad* if it has degree 2 and contains no face of $G^\star$ that is caressed by C. We now move from studying individual nodes of T_0 and T_1 to studying global quantities associated with T_0 and T_1. From this point on, for each $i \in \{0,1\}$,

1. τ_i, ρ_i, and κ_i refer the total numbers of faces contained in nodes of T_i that are touched, pinched, and caressed by C, respectively;
2. n_i refers to the number of nodes of T_i;
3. $\delta_i = 2(n_i - 1)$ is the total degree of all nodes in T_i; and
4. b_i is the number of bad nodes in T_i.

Lemma 6. *If* $\kappa_i \leq \tau_i/6$ *then* $n_i \geq \tau_i/8$.

Proof. From Lemma 5 we know $\rho_i \leq 2(\kappa_i + \delta_i)$, so

$$\tau_i = \kappa_i + \rho_i \leq 3\kappa_i + 2\delta_i = 3\kappa_i + 4(n_i - 1) \leq \tau_i/2 + 4n_i \ ,$$

and reorganizing the left- and right-hand sides gives the desired result. □

Lemma 7. *For any* $0 < \epsilon < 1$, *if* $b_i \leq (1-\epsilon)n_i$, *then* $\kappa_i \geq \epsilon\tau_i/24$.

Proof. Partition the nodes of T_i into the following sets:

1. the set B of bad nodes;
2. the set N_1 of leaves;
3. the set $N_{\geq 3}$ of nodes having degree at least 3;
4. the set N_2 of nodes having degree 2 that are not bad.

Then

$$\begin{aligned}
b_i &= n_i - |N_1| - |N_{\geq 3}| - |N_2| \\
&> n_i - 2|N_1| - |N_2| && \text{since } |N_1| > |N_{\geq 3}| \\
&\geq n_i - 2\kappa_i - |N_2| && \text{(since, by Lemma 3, } \kappa_i \geq |N_1|) \\
&\geq n_i - 3\kappa_i && \text{(since each node in } N_2 \text{ contains a caressed face)}
\end{aligned}$$

Thus, we have

$$n_i - 3\kappa_i \leq b_i \leq (1-\epsilon)n_i$$

and rewriting gives

$$\kappa_i \geq \epsilon n_i/3 \ . \tag{1}$$

If $\kappa_i \geq \tau_i/6$, then the proof is complete since $\tau_i/6 > \tau_i/24$. On the other hand, if $\kappa_i \leq \tau_i/6$ then, by Lemma 6, $n_i \geq \tau_i/8$. Combining this with (1) gives

$$\kappa_i \geq \epsilon n_i/3 \geq \epsilon\tau_i/24 \ .$$ □

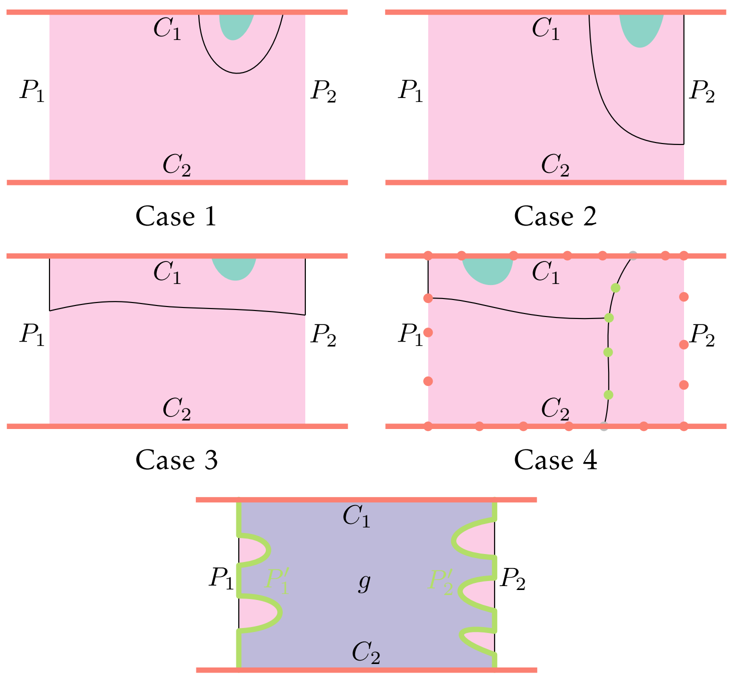

Figure 9: Cases in the proof of Lemma 8

2.4 Interactions Between Bad Nodes

We have now reached a point in which we know that the vast majority of nodes in T_0 and T_1 are bad nodes, otherwise Lemma 7 implies that a constant fraction of the faces touched by C are caressed by C. At this point, we are ready to study interactions between bad nodes of T_0 and bad nodes of T_1.

Lemma 8. *If u is a bad node then u is a face of $G^\star$.*

Proof. First observe that, since u is bad, it has degree 2, so $C \cap u$ has exactly two connected components C_1 and C_2. Thus u's boundary consists of C_1, C_2 and two chord paths P_1 and P_2. We first argue that there is a single face g of $G^\star$ that contains $C_1 \cup C_2$. If not, then $G^\star$ must contain a path P whose interior is in u and has both endpoints on the boundary of u. There are a few cases to consider:

1. P has both endpoints on C_i for some $i \in \{1,2\}$. In this case, P is a chord path and, by Lemma 2 u contains a face that is caressed by C, contradicting the assumption that u is a bad node.
2. P has one endpoint on C_i and one endpoint on P_j for some $i,j \in \{1,2\}$. In this case, $P \cup P_j$ contains a chord path with both endpoints on C_i, again contradicting the assumption that u is a bad node.
3. P has one endpoint on P_1 and one endpoint on P_2. In this case, $P \cup P_1 \cup P_2$ contains a chord path with both endpoints on C_1, again contradicting the assumption that u is a bad node.

4. P has one endpoint on C_1 and one endpoint on C_2. The path P is not a keeper, otherwise it would have split u into two nodes. Therefore, it must be the case that P contains an internal vertex. Let S_1 be the set of internal vertices of P and let S_2 be the set of vertices on the boundary of u, not including the endpoints of P. Since $G^\star$ is 3-connected, there is a path from S_1 to S_2 that does not contain either endpoint of P. The shortest such path, P', does not contain any edges of P. Again, using portions of P, P_1, P_2, and P' we can construct a chord path, contained in u, with both endpoints on C_1 or both endpoints on C_2, contradicting the assumption that u is a bad node.

This establishes that $C_1 \cup C_2$ is contained in the boundary of a single face g of $G^\star$. The boundary of g contains two disjoint paths P_1' and P_2' joining C_1 and C_2. We claim that P_i' is a keeper path, for each $i \in \{1,2\}$. Indeed, each internal x vertex of P_i' is either a vertex of P_j or is on the boundary of three faces: g and two faces that are not touched by C. In either case, x is not special of Type Y. Therefore P_i' has endpoints that are special of Type A and Type B with respect to the pinched face g and has no internal vertices that are special of Type Y, so P_i' is a keeper. Therefore $\{P_1', P_2'\} = \{P_1, P_2\}$ since, otherwise f would not be a face of $\tilde{H}$. Therefore $g = f$ so f is a face of $G^\star$. □

The following lemma shows that a bad node u in T_0 and a bad node w in T_1 share at most one edge of C.

Lemma 9. *Any two bad nodes u of T_i and w of T_j have at most one edge in common.*

Proof. By Lemma 8 u and w are each faces of $G^\star$. Therefore, by Lemma 4, u and w share at most one edge. □

2.5 Really Bad Nodes

At this point we will start making use of the assumption that the triangulation G has maximum degree Δ, which is equivalent to the assumption that each face of $G^\star$ has at most Δ edges on its boundary.

Observation 1. *If G has maximum degree Δ and C has length ℓ, then the number of faces τ of $G^\star$ touched by C is at least $2\ell/\Delta$. At least ℓ/Δ of these faces are in the interior of C and at least ℓ/Δ of these faces are in the exterior of C.*

Proof. Orient the edges of C counterclockwise so that, for each edge e of C, the face of $G^\star$ to the left of e is in C's interior and the face of $G^\star$ to the right of e is in C's exterior. Each face of $G^\star$ has at most Δ edges. Therefore, the number of faces to the right of edges in C is at least ℓ/Δ. The same is true for the number of faces of $G^\star$ to the left of edges in C. □

For each node u of T_i, we define $N(u)$ as the set of nodes in T_0 and T_1 (excluding u) that share an edge of $G^\star$ with u. Note that $N(u)$ contains the neighbours of u in T_i as well as nodes of T_{1-i} with which u shares an edge of C.

We say that a node u is *really bad* if u and all nodes in $N(u)$ are bad.

Lemma 10. *For each $i \in \{0,1\}$ and each $0 < \alpha < 1/24$, if G has maximum degree Δ, C has length ℓ, and the number κ, of faces of $G^\star$ caressed by C is at most $\alpha\ell/\Delta$, then the number b_i of really bad nodes in T_i is at least $n_i - \alpha(120\Delta + 72)n_i$.*

Proof. Without loss of generality, let $i = 0$. From Observation 1, we know that $\tau_0 \geq \ell/\Delta$. Therefore, $\kappa_0 \leq \kappa \leq \alpha\ell/\Delta \leq \alpha\tau_0 \leq \tau_0/6$ so, by Lemma 6, $n_0 \geq \tau_0/8$.

By Lemma 7, if $b_0 < (1 - 24\alpha)n_0$, then

$$\kappa > \kappa_0 \geq \alpha\tau_0 \geq \alpha\ell/\Delta \ .$$

This violates our assumption that $\kappa \leq \alpha\ell/\Delta$. Therefore, we may assume that $b_0 \geq (1 - 24\alpha)n_0$.

We now want to study how many of the bad nodes in T_0 are really bad. Let A be the set of nodes in T_0 that are not bad and partition A into A_1 (leaves), A_2 (degree-2 nodes) and $A_{\geq 3}$ (nodes of degree at least 3). We make use of the following inequality:

$$|A_1| = 2 + \sum_{w \in A_{\geq 3}} (\delta_w - 2) \geq \sum_{w \in A_{\geq 3}} (\delta_w - 2) \geq \sum_{w \in A_{\geq 3}} \delta_w/3 \ , \tag{2}$$

which is true because $x - 2 \geq x/3$ for all $x \geq 3$.

Now each node w in A can prevent at most δ_w bad nodes of T_0 from being really bad. We count this as follows:

$$\sum_{w \in A} \delta_w = \sum_{w \in A_1} \delta_w + \sum_{w \in A_2} \delta_w + \sum_{w \in A_{\geq 3}} \delta_w \leq |A_1| + 2|A_2| + 3|A_1| \ .$$

Now, A_1 contains leaves of T_0 and, by Lemma 3, each leaf of T_0 contains a caressed face. Therefore $|A_1| \leq \kappa$. Next, A_2 contains degree-2 nodes of T_0 that are not bad. If a node has degree-2 and contains no caressed face, then it is bad. Therefore each node in A_2 contains a caressed face. Therefore $|A_2| \leq \kappa$, so $|A_1| + 2|A_2| + 3|A_1| \leq 6\kappa$. Picking up where we left off:

$$\sum_{w \in A} \delta_w \leq |A_1| + 2|A_2| + 3|A_1| \leq 6\kappa \leq 6\alpha\ell/\Delta \leq 48\alpha n_0 \ ,$$

where the last inequality uses the fact that $n_0 \geq \tau_0/8 \geq \ell/(8\Delta)$. That is, the set A of non-bad nodes in T_0 prevents at most $48\alpha n_0$ bad nodes in T_0 from being really bad. Next we account for nodes in T_1 that prevent bad nodes in T_0 from being really bad.

Let A' be the set of nodes in T_1 that are not bad. For two nodes u in T_0 and w in T_1, $w \in N(u)$ if and only if w and u share an edge of C. The number of edges of C incident to a node w is at most $\Delta\tau_w$. Therefore, we can upper bound the number of bad nodes in T_0

that are prevented from being really bad by some node in T_1 as

$$\begin{aligned}
&\sum_{w\in A'} \Delta\tau_w \\
&\qquad = \sum_{w\in A'} \Delta(\rho_w+\kappa_w) && \text{(since } \tau_w=\rho_w+\kappa_w) \\
&\qquad \le \sum_{w\in A'} (3\Delta\kappa_w+2\Delta\delta_w) && \text{(by Lemma 5)} \\
&\qquad \le 3\Delta\kappa + \sum_{w\in A'} 2\Delta\delta_w \\
&\qquad < 3\Delta\kappa + 12\Delta\kappa && \text{(by defining } A'_1, A'_2, A'_{\ge 3} \text{ and arguing as above)} \\
&\qquad = 15\Delta\kappa \\
&\qquad \le 15\alpha\ell && \text{(since } \kappa \le \alpha\ell/\Delta\text{, by assumption)} \\
&\qquad \le 120\alpha\Delta n_0 && \text{(since } n_0 \ge \tau_0/8 \ge \ell/(8\Delta))
\end{aligned}$$

Therefore, the number of bad nodes in T_0 is b_0 and the number of these that are really bad is at least

$$b_0 - \alpha(120\Delta+48)n_0 \ge n_0 - \alpha(120\Delta+72)n_0 \ . \qquad \square$$

We say that a node u is *really really bad* if all the nodes in $N(u)$ are really bad. (Note that this implies that u is bad.) The following lemma extends Lemma 10 to really really bad nodes:

Lemma 11. *For each $i\in\{0,1\}$ and each $0<\alpha<1/24$, if G has maximum degree Δ, C has length ℓ, and the number κ, of faces of $G^\star$ caressed by C is at most $\alpha\ell/\Delta$, then the number b_i of really really bad nodes in T_i is at least $n_i - \alpha(\Delta+1)(120\Delta+72)n_i = n_i - O(\alpha\Delta^2)$.*

Proof. A node u is a *fringe node* if it is really bad but not really really bad. A node u is a *critical node* if it is bad but not really bad. Observe that every fringe node u is in $N(w)$ for some critical node w. To bound the number of fringe nodes, it therefore suffices to bound $\sum_w |N(w)| \le \sum_w \Delta$ where the sum is over all critical nodes and the inequality is due to Lemma 8, so $|N(w)|\le\Delta$ for any bad node w.

By Lemma 10, the number of nodes that are not really bad, and hence the number of critical nodes, is at most $\alpha(120\Delta+72)n_i$. Therefore, the number of fringe nodes is at most $\alpha\Delta(120\Delta+72)n_i$. Any node that is not really really bad is either a fringe node or is not really bad. Therefore, the number of nodes that are really really bad is at least

$$n_i - \alpha(\Delta+1)(120\Delta+72)n_i \ . \qquad \square$$

The following observation, illustrated in Figure 10, follows from the fact that all the nodes it considers are bad and that $\tilde{H}$ is a cubic graph, so each vertex of $\tilde{H}$ is on the boundary of 3 faces. The second part of the figure shows an example in which $\{a_1,\dots,a_r\}$ and $\{b_1,\dots,b_s\}$ are not disjoint. (In this example $b_1=a_3$.)

Observation 2. *Let $x_1,\dots,x_t$, $t\ge 1$ be a path in T_0 consisting entirely of really bad nodes. Then $C\cap\bigcup_{i=1}^t x_i$ consists of two paths C_a and C_b each having at least one edge and the subgraph*

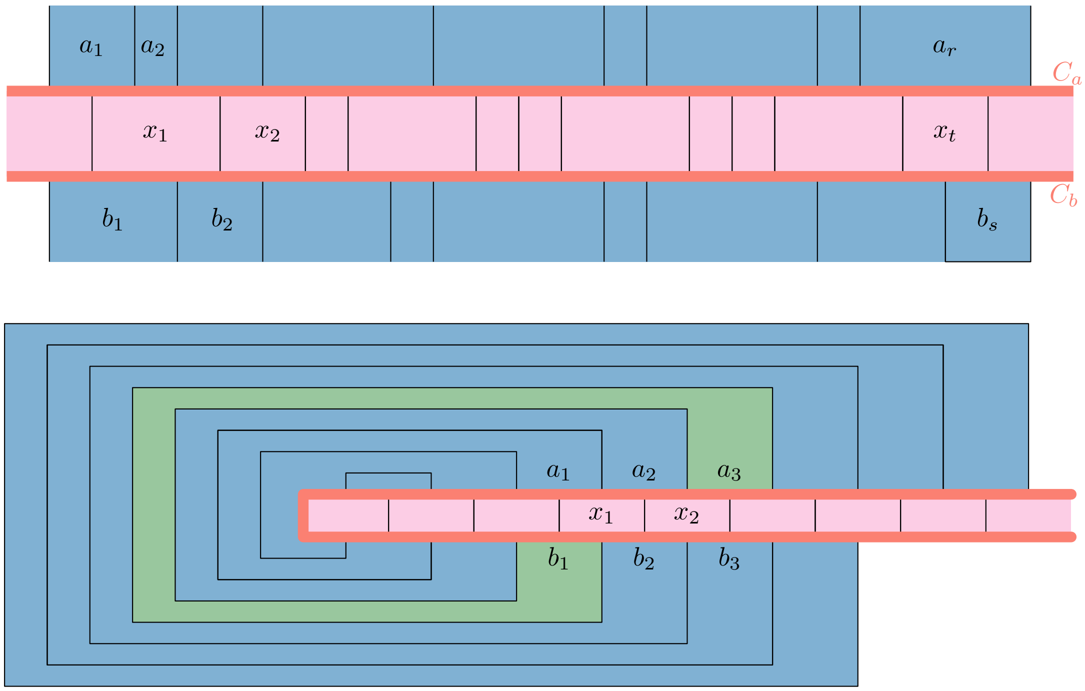

Figure 10: Two illustrations of Observation 2

of T_1 induced by $\bigcup_{i=1}^{t} N(x_i)$ is contained in two (not necessarily disjoint) paths $a_1,\dots,a_r$ and $b_1,\dots,b_s$, $r,s \geq 1$ where a_i contains an edge of C_a for each $i \in \{1,\dots,r\}$ and b_i contains an edge of C_b for each $i \in \{1,\dots,s\}$.

2.6 Tree/Cycle Surgery

We summarize the situation so far. By Lemma 1, finding a large collinear set is equivalent to finding a cycle in $G^\star$ that caresses many faces. By existing results on the circumference of cubic triconnected graphs, $G^\star$ has a cycle C_0 of length $\ell = \Omega(n^\alpha)$ for some $\alpha > 0.8$. Thus we assume that $G^\star$ has a cycle C_0 of length ℓ and we want to show the existence of a cycle C that caresses $\Omega(\ell/\Delta^4)$ faces.

Because each face of $G^\star$ has at most Δ edges, C_0 touches $\Omega(\ell/\Delta)$ faces (Observation 1). To complete the proof of Theorem 1 we must deal with the situation where C_0 caresses $o(\ell/\Delta^4)$ faces and therefore each of T_0 and T_1 has $o(\ell/\Delta^4)$ leaves (Lemma 3), $\Omega(\ell/\Delta)$ nodes (Lemma 6), and the fraction of really really bad nodes in T_0 and T_1 is $1-o(1/\Delta)$ (Lemma 11).

Figure 11 illustrates an extreme example of this situation. To handle cases like these, the only option is to perform surgery on the cycle C to increase the number of caressed faces. We achieve this by performing a surgery that increases the number of leaves in T_1. This surgery is quite delicate and requires a particular node u for which we have a good enough understanding of the faces of $\tilde{H}$ surrounding u so that we can make a local modification of C around $N(u)$ that is guaranteed to stricly increase the number of caressed faces.

Proof of Theorem 1. By Lemma 1, it suffices to prove the existence of a cycle C in $G^\star$ that

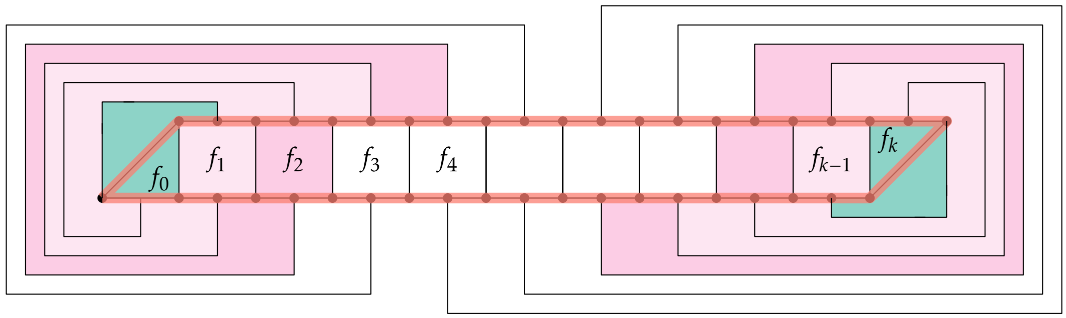

Figure 11: An example in which C caresses only 4 faces of $G^\star$, T_0 has only 2 non-bad nodes (in teal), 2 non-really bad nodes (in light pink), and 2 non-really really bad nodes (in pink).

caresses $\Omega(\ell/\Delta^4)$ faces. We begin by applying Lemma 11 with $\alpha = \epsilon/\Delta^3$. For sufficiently small, but constant, ϵ, Lemma 11 implies that $\kappa = \Omega(\ell/\Delta^4)$ or the number of nodes in T_0 that are not really really bad is at most $O(\epsilon n_0/\Delta)$. In the former case, C caresses $\Omega(\ell/\Delta^4)$ faces of $G^\star$ and we are done.

In the latter case, consider the forest obtained by removing all nodes of T_0 that are not really really bad. This forest has $(1-O(\epsilon/\Delta))n_0$ nodes. We claim that it also has $O(\epsilon n_0/\Delta)$ components. To see why this is so, let L be the set of leaves in T_0 and let S be the set of non-leaf nodes in T_0 that are not really really bad. Observe that it is sufficient to upper bound the number, k, of components in $T_0 - S$.

Since removing a degree d vertex from a graph increases the number of components by at most $d-1$, we have $k \le \sum_{u\in S}(\deg_{T_0}(u)-1)$. Since (S,L) is a partition of the nodes of T_0 that are not really really bad, we have $|L| \le |S|+|L| = O(\epsilon n_0/\Delta)$. Recall that a standard fact about trees is that the number of leaves in a tree G is exactly $2+\sum_u(\deg_T(u)-2$, where the sum runs over all non-leaf nodes u of G. Therefore,

$$|L| \ge \sum_{u\in S}(\deg_{T_0}(u)-2) = \sum_{u\in S}(\deg_{T_0}(u)-1) - |S| = k - |S| \ .$$

Therefore $k \le |S|+|L| = O(\epsilon n_0/\Delta)$, as claimed.

Thus the forest induced by all really really bad nodes of T_i has at most $O(\epsilon n_0/\Delta)$ components, each of which is a path. At least one of these paths contains $\Omega(\Delta/\epsilon)$ nodes. In particular, for a sufficiently small constant ϵ, one of these components, X, has at least 5Δ nodes.

Consider some node u in X, and let C_a and C_b be the two components of $u \cap C$. By Observation 2, the subgraph of T_1 induced by $N(u)$ consists of two paths $a_1,\dots,a_r$ and $b_1,\dots,b_s$ of really bad nodes where each $a_1,\dots,a_r$ contains an edge of C_a and each of $b_1,\dots,b_r$ contains an edge of C_b.

It follows from Lemma 9 that among any sequence of Δ consecutive nodes in X, at least one node has $r \ge 2$ and therefore $|N(u)| \ge 5$. Let u be any such node that is not among the first 2Δ or last 2Δ nodes of X. Such a u always exists because X contains at least 5Δ nodes.

Let $x_0 = u$. We now define notations for some of the nodes in the vicinity of u (refer to Figure 12):

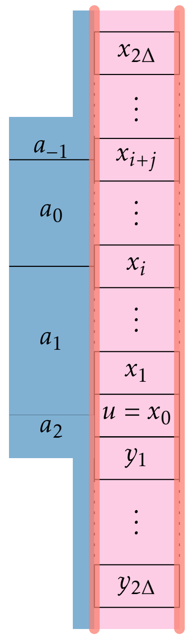

Figure 12: Nodes in the vicinity of $u = x_0$.

1. there is a path $x_{2\Delta},\ldots,x_1,x_0,y_1,\ldots,y_{2\Delta}$ in T_0 consisting entirely of really really bad nodes.

2. some really bad node a_1 of T_1 shares an edge with each of $x_0,\ldots,x_i$ for some $i \in \{1,\ldots,\Delta-4\}$.

3. some really bad node a_2 of T_1 shares an edge with a_1 and and edge with x_0.

4. some really bad node $a_0 \neq a_2$ of T_1 shares an edge with a_1 and with each of $x_i,\ldots,x_{i+j}$ for some $j \in \{0,\ldots,\Delta-4\}$.

The surgery we perform focuses on the nodes u and a_1. Consider the two components of $C \cap a_1$. At least one of these components, p, shares an edge with u. By Lemma 9, the other component, q, does not share an edge with u. Imagine removing u from T_0, thereby separating T_0 into a component T_x containing x_1 and a component T_y containing y_1. Equivalently, one can think of removing the edges of u from C separating C into two paths C_x and C_y on the boundary of T_x and T_y, respectively. Since q does not share an edge with u, $q \subseteq C_x$ or $q \subseteq C_y$. We treat these cases separately:

1. $q \subset C_x$. We transform this into Case 2, by redefinining u, x_1, y_1 and a_1 as follows: By Lemma 4 $a_1 \setminus C$ contains exactly two edges of $G^\star$ and exactly one of these edges, e, is not incident to u. Instead, e is incident to x_i. We set $u' = x_i$, $x_1' = x_{i-1}$, $y_1' = x_{i+1}$, and $a_1' = a_1$. Observe that a_1' connects the two components of $T_0 - u'$ and shares edges with u' and x_1'. This is exactly the situation considered in Case 2, next.

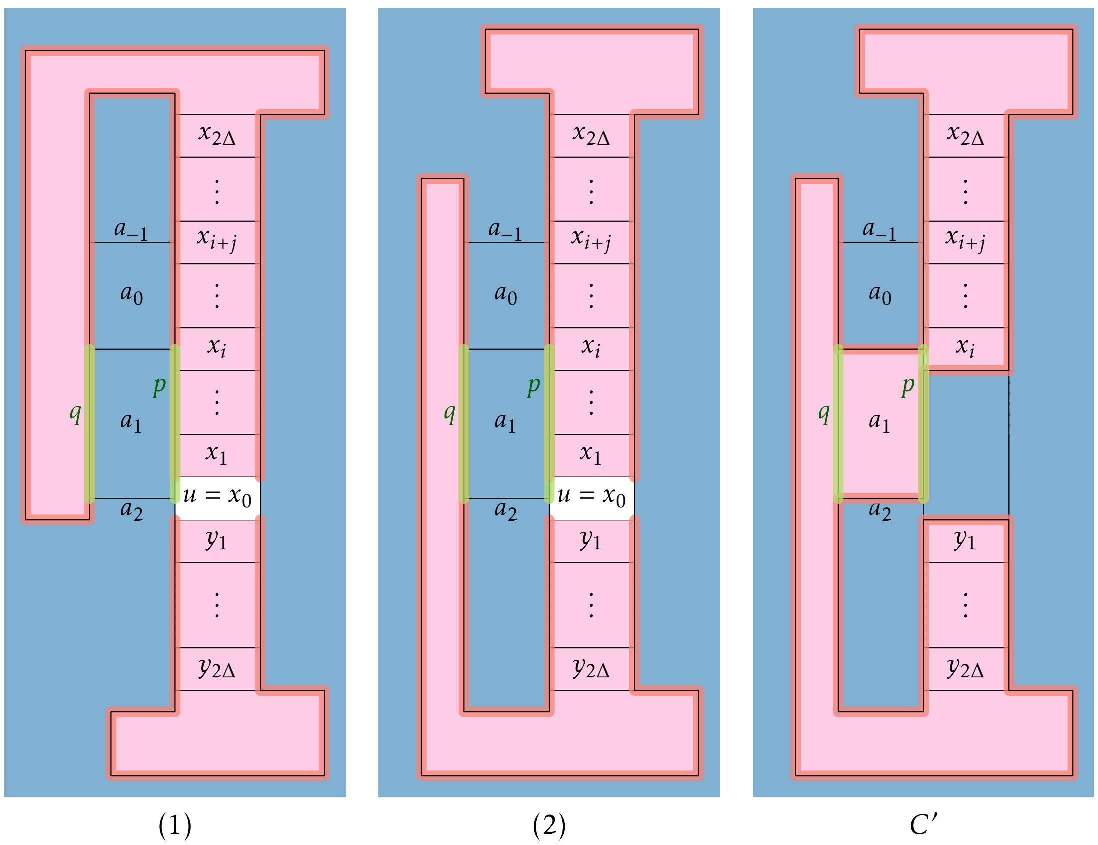

Figure 13: Cases 1 and 2 in the proof of Theorem 1 and the surgery performed in Case 2.

2. $q \subset C_y$. At this point it is helpful to think of T_0, T_1, and C as a partition of $\mathbb{R}^2$, where nodes of T_0 are coloured red, nodes of T_1 are coloured blue and C is the (purple) boundary between red and blue. To describe our modifications of C, we imagine changing the colours of nodes. The effect that such a recolouring has on C is immediately obvious: It produces a 1-dimensional set C' that contains every (purple) edge contained in the red-blue boundary. The set C' is a collection of vertices and edges of $G^\star$. Therefore, if C' is a simple cycle, then C' defines a new pair of trees T_0' and T_1'.

   Refer to the right two thirds of Figure 13 for a simple (and misleading) example of what follows. For a full example, refer to Figure 14. The surgery we perform recolours $x_0, x_1, \ldots, x_{i-1}$ blue and recolours a_1 red. Observe that, because $q \subset C_y$ and p contain an edge of x_i, this implies that the red subset of $\mathbb{R}^2$ is connected. Similarly, one can verify that $T_1 - \{a_1\}$ contains two components, one containing a_2 and one containing a_0 and b_1. The blue subset of $\mathbb{R}^2$ is connected because it contains a path from a_2 through u to b_1. Therefore the red and blue subsets of $\mathbb{R}^2$ are each connected and their common boundary C' is a simple cycle consisting of edges of $G^\star$. The new trees T_0' and T_1' are therefore well defined. We now make two claims that will complete our proof.

   Claim 1. *For each $i \in \{0,1\}$, and each node w of T_i that is not bad, $C \cap w = C' \cap w$. (Equivalently, for every face f of $G^\star$ that is not a bad node of T_0 or T_1, $C \cap f = C' \cap f$.)*

   Claim 2. *The face a_0 is caressed by C'.*

   These two claims complete the proof because, together, they imply that C' caresses at least one more face of $G^\star$ than C. Indeed, by definition, C did not caress any faces belonging to bad nodes. Therefore, the first claim implies that the faces of $G^\star$ caressed by C' are a superset of those caressed by C. The face a_1 is a bad node of T_i so it is not caressed by C but the second claim states that it is caressed by C'. Therefore C' caresses at least one more face than C.

   This surgery recolours at most $\Delta - 2 \leq \Delta$ nodes of T_0 and T_1, so the difference in length between C and C' is at most Δ^2. If we start with a cycle C of length ℓ, then we can perform this surgery at least $\ell/(4\Delta^2)$ times before the length of C decreases to less than $\ell' = \ell/2$. If at some point during this process, we are no longer able to perform this operation, it is because C caresses $\Omega(\ell'/\Delta^4) = \Omega(\ell/\Delta^4)$ faces of $G^\star$ and we are done. If the process runs to completion, then by its end, the number of faces caressed by C is at least $\ell/(4\Delta^2) \in \Omega(\ell/\Delta^2) \subset \Omega(\ell/\Delta^4)$ and we are also done.

   Thus, all that remains is to prove Claim 1 and Claim 2.

   To prove Claim 1, we observe that C and C' differ only on the boundaries of nodes that are recoloured. Thus, it is sufficient to show that all nodes in $R = \cup\{N(v) : v \in \{x_0, \ldots, x_{i-1}, a_1\}$ are bad. But this is immediate since $x_0, \ldots, x_{i-1}$ are really really bad and $a_1 \in N(x_0)$, so a_1 is really bad. Since every node in R share an edge with at least one of $\{x_0, \ldots, x_{i-1}, a_1\}$, every node in R is therefore bad, as required.

   To prove Claim 2 we consider the boundary of the face a_0 of $G^\star$ after the recolouring operation. This boundary consists of, in cyclic order:

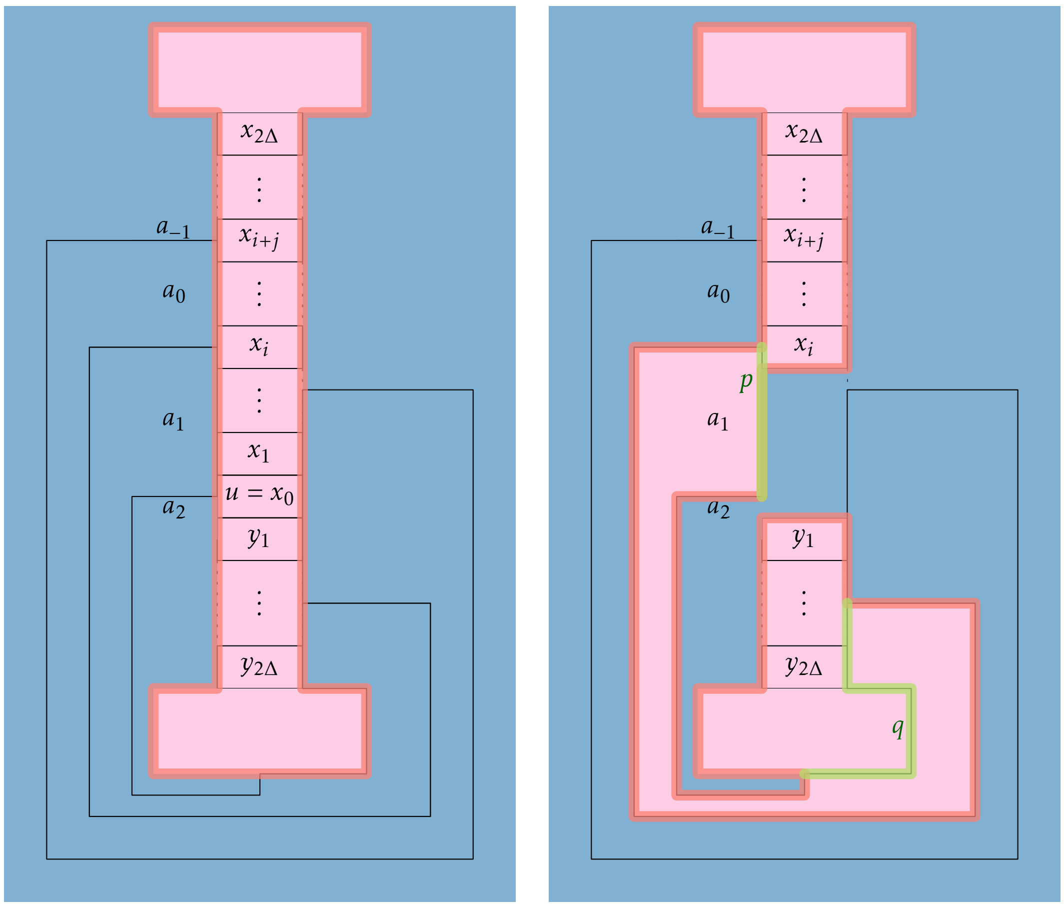

Figure 14: Performing surgery on C to obtain C' that caresses a_0.

(a) An edge p_0p_1 shared between a_0 and a_1. This edge is in C' since a_0 is in T_1' and a_1 is in T_0'. This edge has one endpoint, p_0, on the boundary of x_i (p_0 is also an endpoint of p).

(b) A path $p_1,\dots,p_\mu$ whose edges are shared with $x_i,\dots,x_{i+j}$. The nodes $x_i,\dots,x_{i+j}$ are in T_0 and are distinct from $x_0,\dots,x_{i-1}$, so these nodes are in T_0'. Therefore, $p_1,\dots,p_\mu$ is also contained in C'.

(c) An edge $p_\mu p_{\mu+1}$ shared between a_0 and another node $a_{-1} \neq a_1$ of T_1. The faces of a_{-1} are in T_1' because a_1 is the only face that moves from T_1 to T_0'. (a_1 is the only face whose colour goes from blue to red.) The edge $p_\mu p_{\mu+1}$ is therefore not contained in C'.

(d) A path $p_{\mu+1},\dots,p_\nu$ with $p_\nu = p_0$ that is contained in C. Let C_x' be the path obtained by removing all edges on the boundary of $x_1,\dots,x_{i-1}$ from C_x. Thus, the boundary of C is partitioned into four paths: C_y; a path P_1 that contains p; C_x'; and a path P_2 that does not contain p. Without loss of generality, assume that these four paths occur in the order C_y, P_1, C_x', P_2 when traversing C clockwise.

The path $p_{\mu+1},\dots,p_\nu$ ends at $p_\nu = p_0$, which is contained in C_y. This path must therefore either begin in P_2 or be entirely contained in C_y since, otherwise it would contain an edge of x_i, contradicting Lemma 4. The edges of P_2 are not in C'. Therefore $p_{\mu+1},\dots,p_\nu$ begins with a (possibly empty) sequence of edges $p_{\mu+1},\dots,p_{\mu+k}$ not contained in C' followed by a non-empty sequence $p_{\mu+k},\dots,p_\nu$ of edges that are contained in C'.

Therefore the intersection $C' \cap a_0$ is a path $p_{\mu+k},\dots,p_\nu,p_1,\dots,p_\mu$ so a_0 is caressed by C'. □

3 Discussion

It remains an open problem to eliminate the dependence of our results on the maximum degree, Δ, of G. The next significant step is to resolve the following conjecture:

Conjecture 1. *If G is a triangulation whose dual $G^\star$ has a cycle of length ℓ, then $G^\star$ has a cycle that caresses $\Omega(\ell)$ faces. (Therefore, by Lemma 1 and Theorem 2, G has a collinear set of size $\Omega(\ell)$.)*

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