A Note on Stabbing Convex Bodies with Points, Lines, and Flats

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Abstract

Consider the problem of constructing weak ε -nets where the stabbing elements are lines or k-flats instead of points. We study this problem in the simplest setting where it is still interesting—namely, the uniform measure of volume over the hypercube $[0,1]^d$. Specifically, a (k,ε) -net is a set of k-flats, such that any convex body in $[0,1]^d$ of volume larger than ε is stabbed by one of these k-flats. We show that for $k \geq 1$, one can construct (k,ε) -nets of size $O(1/\varepsilon^{1-k/d})$. We also prove that any such net must have size at least $\Omega(1/\varepsilon^{1-k/d})$. As a concrete example, in three dimensions all ε -heavy bodies in $[0,1]^3$ can be stabbed by $\Theta(1/\varepsilon^{2/3})$ lines. Note, that these bounds are sublinear in $1/\varepsilon$, and are thus somewhat surprising. The new construction also works for points providing a weak ε -net of size $O(\frac{1}{\varepsilon}\log^{d-1}\frac{1}{\varepsilon})$.

1. Introduction

Notations. Throughout, we use O_d , Ω_d , and Θ_d to hide constants depending on the dimension d. We use [n] to denote the set $\{1, \ldots, n\}$.

Range spaces and ε -nets. A *range space* is a pair $X = (\mathcal{U}, \mathcal{R})$, where \mathcal{U} is the *ground set* (finite or infinite) and \mathcal{R} is a (finite or infinite) family of subsets of \mathcal{U} . The elements of \mathcal{R} are *ranges*.

Suppose that \mathcal{U} is a finite set. For a parameter $\varepsilon \in (0,1)$, a subset $S \subseteq \mathcal{U}$ is an ε -net for the range space X, if for every range $\nu \in \mathcal{R}$ with $|\nu \cap \mathcal{U}| \geq \varepsilon |\mathcal{U}|$ has $\nu \cap S \neq \emptyset$. The ε -net theorem of Haussler and Welzl [HW87] implies the existence of ε -nets of size $O(\delta \varepsilon^{-1} \log \varepsilon^{-1})$, where δ is the VC dimension of the range space X. The use of ε -nets is widespread in computational geometry [Mat02, Har11].

Weak ε -nets. Consider the range space (P, \mathcal{C}) , where \mathcal{C} is the collection of all compact convex bodies in \mathbb{R}^d and $P \subset \mathbb{R}^d$ is a set of points. This range space has unbounded VC dimension—the standard ε -net constructions do not work in this case. The notion of **weak** ε -nets bypasses this issue by allowing the net S to use points outside of P. Specifically, any convex body Ξ that contains

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at least $\varepsilon |P|$ points of P must contain a point of S. The first construction of weak ε -nets in the plane was due to Bárány et al. [BFL90] of size $O(1/\varepsilon^{1026})$. For all $d \ge 1$, Alon et al. [ABFK92] were the first to construct weak ε -nets in \mathbb{R}^d whose size was bounded in terms of $1/\varepsilon$ and d. In 1995, Chazelle et al. [CEG+95] improved this bound to $O(\varepsilon^{-d}\log^{\zeta(d)}\varepsilon^{-1})$, where $\zeta(d) = O(2^d(d-1)!)$. In 2004, Matoušek and Wagner [MW04] gave an improved construction of weak ε -nets of size $O_d(\varepsilon^{-d}\log^{f(d)}\varepsilon^{-1})$, where $f(d) = O(d^2\log d)$. Recently, Rubin [Rub18, Rub21] gave an improved bound, showing the existence of weak ε -nets of size $O_d(\varepsilon^{-(d-0.5+\alpha)})$ for arbitrarily small $\alpha > 0$. For more detailed history of the problem, see the introduction of Rubin [Rub18, Rub21]. As for a lower bound, Bukh et al. [BMN09] gave constructions of point sets for which any weak ε -net must have size $\Omega_d(\varepsilon^{-1}\log^{d-1}\varepsilon^{-1})$. Closing this gap remains a major open problem. See [MV17] for a recent survey of ε -nets and related concepts.

 (k,ε) -nets and uniform measure. A natural extension of weak ε -nets is to allow the net S to contain other geometric objects. Given a collection of n points $P \subset \mathbb{R}^d$ and a parameter k, $0 \le k < d$, we define a (weak) (k,ε) -net to be a collection of k-flats S such that if Ξ is a convex body containing at least εn points of P, then there is a k-flat in S intersecting Ξ . Note that $(0,\varepsilon)$ -nets are exactly weak ε -nets.

In general, one would expect that as k increases, the size of the (k, ε) -net shrinks. For example, a $(1, \varepsilon)$ -net for a collection of points in \mathbb{R}^3 can be constructed by projecting the points down onto the xy-plane and applying Rubin's construction in the plane to obtain a weak ε -net S of size $O(\varepsilon^{-3/2-\alpha})$ [Rub18]. Lifting S up back into three dimensions results in a $(1, \varepsilon)$ -net of the same size, which is smaller than the best known weak ε -net size in \mathbb{R}^3 [MW04, Rub18, Rub21]. However, one might expect that a $(1, \varepsilon)$ -net of even smaller size is possible in \mathbb{R}^3 , as this construction uses a set of parallel lines (i.e., one would expect the lines in an optimal net to have multiple orientations).

Here, we study an even simpler version of the problem, where the ground set is the hypercube $[0,1]^d$. In particular, for $\varepsilon \in (0,1)$ and $0 \le k < d$, we are interested in computing the smallest set K of k-flats, such that if Ξ is a convex body with $\operatorname{vol}(\Xi \cap [0,1]^d) \ge \varepsilon$, then there is a k-flat in K which intersects Ξ . In the following, the set K is a (k,ε) -net for volume measure. We note that $[0,1]^d$ can be replaced with any arbitrary compact convex body in the definition – the size of the (k,ε) -net increases by roughly a factor of $d^{O(d)}$, see Lemma 2.3.

Deterministic and explicit constructions of ε **-nets.** The randomized algorithm for computing ε -nets, implied by the ε -net theorem, can be derandomized, but the resulting running time is exponential in the dimension. These algorithms work by repeatedly halving the input point set, using deterministic discrepancy constructions, until the set is of the desired size [Mat99, Cha01].

It is an open problem to compute ε -nets in deterministic polynomial time, in the dimension and $1/\varepsilon$, even for special cases. Previous such work on explicit efficient constructions of weak (and regular) ε -nets include:

(I) Axis-parallel boxes. Explicit constructions of $(0,\varepsilon)$ -nets for volume measure for axis-parallel boxes in \mathbb{R}^d , and is briefly mentioned in [BMN09]. In this case, one can construct a $(0,\varepsilon)$ -net for volume measure of size $2^{O(d\log d)}/\varepsilon$ using Van der Corput sets in two dimensions, and Halton-Hammersely sets in higher dimensions. These constructions are essentially described in [Mat99] (in the context of low-discrepancy point sets), the minor modifications required in the proofs are described in Appendix A.

- (II) Grid points and axis-parallel boxes. Linial et al. [LLSZ97] studied the problem of constructing explicit ε -nets for axis-parallel boxes, where the ground set is $[m]^d$, for some integer m > 0. The net size is $((m \log d)/\varepsilon)^{O(1)}$, and the construction time is $(md/\varepsilon)^{O(1)}$.
- (III) Halfplanes for vertices of the hypercube, and hypersphere. Rabani and Shpilka [RS10] showed that for $\{0,1\}^d$, and halfspaces, one can compute an ε -net of size $(d/\varepsilon)^{O(1)}$ (where the constant is dimension independent).

For $\varepsilon = \exp(-O(\sqrt{n}))$, they also show a construction for volume measure on the hypersphere, for halfspaces, with a similar upper bound.

1.1. Our results & paper organization

First, we show that any (k, ε) -net for volume measure must have size $\Omega_d(1/\varepsilon^{1-k/d})$ (Lemma 3.1). Perhaps surprisingly, we give a relatively simple construction of (k, ε) -nets for volume measure of size $O_d(1/\varepsilon^{1-k/d})$ for $k \ge 1$ (Theorem 4.4). For k = 0, when using points, the same construction works, but the net size increases to $O_d(\frac{1}{\varepsilon}\log^{d-1}\frac{1}{\varepsilon})$. As far as the authors are aware, this particular problem has not been addressed before.

Note that for the case of points and volume measure on the hypercube, it is enough to build a (regular) $O_d(\varepsilon)$ -net for ellipsoids (see Lemma 2.4 below). In particular, applying the known deterministic algorithm for computing ε -nets [Mat99, Cha01], it is not clear what the generated ε -net is, without running this construction algorithm outright (which seems quite challenging). In contrast, our algorithm enables us to output the *i*th point in the computed net in space and time polylogarithmic in $O(1/\varepsilon)$.

2. Preliminaries

2.1. Formal definition of (k, ε) -net

Definition 2.1. The affine hull of a point set $P = \{p_1, \dots, p_n\} \subseteq \mathbb{R}^d$ is the set

$$\left\{ \sum_{i} \alpha_{i} p_{i} \mid \forall i \ \alpha_{i} \in \mathbb{R} \ \text{and} \ \sum_{i} \alpha_{i} = 1 \right\}.$$

For $k = 0, \dots, d-1$, a k-flat is the affine hull of a set of k+1 (affinely independent) points.

Thus, a 0-flat is a point and a 1-flat is a line.

Definition 2.2. For parameters $\varepsilon \in (0,1)$ and $k \in \{0,1,\ldots,d-1\}$, a set K of k-flats is a (k,ε) -net for volume measure if for any convex body $\Xi \subseteq \mathbb{R}^d$ with $\operatorname{vol}(\Xi \cap [0,1]^d) \geq \varepsilon$, there exists a flat $\varphi \in K$ such that $\varphi \cap \Xi \neq \emptyset$.

2.2. Brunn-Minkowski inequality and unimodal functions

The Ξ be a convex body in \mathbb{R}^d . For a parameter $\alpha \in \mathbb{R}$, let $f(\alpha)$ denote the (d-1)-dimensional volume of Ξ intersected with the hyperplane $x = \alpha$. The Brunn-Minkowski inequality [Mat02, Har11] implies that the function $g(\alpha) = f(\alpha)^{1/(d-1)}$ is concave (over the range where it is not zero). In particular, g is **unimodal**. Namely, there exists a $\alpha \in \mathbb{R}$ such that g is non-decreasing on $(-\infty, \alpha]$ and non-increasing on $[\alpha, \infty)$. As such, the function f itself is unimodal. See Figure 2.1.

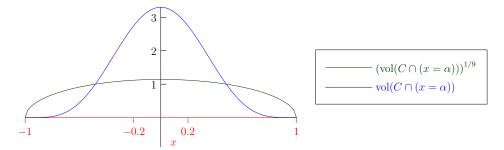


Figure 2.1: The slice volume, and its 1/9th power, for the unit radius ball Ξ in 10 dimensions. This is an example of the concavity implied by the Brunn-Minkowski inequality, which in turn implies that the slice function is unimodal.

2.3. Approximating convex bodies by ellipsoids

2.3.1. Replacing $[0,1]^d$ with other convex bodies

Lemma 2.3. Let C be an arbitrary compact convex body in \mathbb{R}^d with non-empty interior. Suppose there is a (k, ε) -net for the uniform measure on $[0,1]^d$ of size $T(\varepsilon, k, d)$. For a given integer k < d and $\varepsilon \in (0,1)$, there is a collection of k-flats K, of size $T(\Omega_d(\varepsilon), k, d)$, such that any convex body Ξ with $\mathsf{vol}(\Xi \cap C) \geq \varepsilon \, \mathsf{vol}(C)$ is intersected by a k-flat in K.

Proof: Assume without loss of generality that $\Xi \subseteq \mathcal{C}$. John's ellipsoid theorem [Mat02] implies that there exists a non-singular affine transformation \mathbf{M} , and a ball \mathbf{b} of diameter 1, such that $\mathbf{b}/d \subseteq \mathbf{M}(\mathcal{C}) \subseteq \mathbf{b} \subseteq [0,1]^d$, where \mathbf{b}/d is \mathbf{b} scaled by a factor of 1/d. We have that $\operatorname{vol}(\mathbf{b}) = c_d 2^{-d}$, where $c_d = 1/2^{\Theta(d \log d)}$ is the volume of the unit ball in \mathbb{R}^d . Additionally,

$$\operatorname{vol} \left([0,1]^d \right) = 1 = \frac{2^d}{c_d} \operatorname{vol}(\mathbf{b}) = \frac{(2d)^d}{c_d} \operatorname{vol}(\mathbf{b}/d) \leq \frac{(2d)^d}{c_d} \operatorname{vol}(\mathbf{M}(\mathcal{C})).$$

Set $\delta = c_d/(2d)^d$. Compute a (k, ε') -net K for $[0, 1]^d$, where $\varepsilon' = \varepsilon \delta$, which has size $T(\varepsilon', k, d)$. We claim that this is a (k, ε) -net with respect to $\mathbf{M}(\Xi)$. Indeed, consider any convex body $\Xi \subseteq \mathcal{C}$ with $\mathsf{vol}(\Xi \cap \mathcal{C}) \geq \varepsilon \, \mathsf{vol}(\mathcal{C})$. Since \mathbf{M} preserves the ratios of volumes, we have that

$$\operatorname{vol}(\mathbf{M}(\Xi) \cap [0,1]^d) \ge \operatorname{vol}(\mathbf{M}(\Xi) \cap \mathbf{M}(\mathcal{C})) \ge \varepsilon \operatorname{vol}(\mathbf{M}(\mathcal{C}))$$
$$\ge \varepsilon \delta \operatorname{vol}([0,1]^d) = \varepsilon' \operatorname{vol}([0,1]^d).$$

As such, one of the k-flats in K intersects $\mathbf{M}(\Xi)$. After applying the inverse transformation \mathbf{M}^{-1} to each k-flat in K, one of the k-flats in $\mathbf{M}^{-1}(K)$ intersects Ξ .

2.3.2. Its enough to hit ellipsoids

Lemma 2.4. Suppose there exists an (k, ε) -net for the volume measure over $[0, 1]^d$ for ellipsoids of size $T(\varepsilon, d)$. Then one can construct a (k, ε) -net for the volume measure over $[0, 1]^d$, for all convex bodies, of size $T(\varepsilon/d^d, d)$.

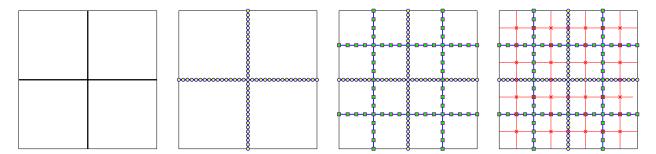


Figure 4.1: The multi-level grid, and its associated lines.

Proof: Consider any convex body Ξ , such that $\operatorname{vol}(\Xi \cap [0,1]^d) \geq \varepsilon$. Let \mathcal{E} be the ellipsoid of largest volume contained inside $\Xi \cap [0,1]^d$. By John's ellipsoid theorem, we have that $\mathcal{E} \subseteq \Xi \subseteq d\mathcal{E}$. In particular,

$$\operatorname{vol}(\mathcal{E}) = \frac{\operatorname{vol}(d\mathcal{E})}{d^d} \ge \frac{\operatorname{vol}(\Xi)}{d^d} \ge \frac{\varepsilon}{d^d}.$$

As such, any $(k, \varepsilon/d^d)$ -net for ellipsoids is a (k, ε) -net for general convex bodies.

3. Lower bound

Lemma 3.1. For a parameter $\varepsilon \in (0,1)$, and $k \in \{0,\ldots,d-1\}$, any (k,ε) -net for volume measure over $[0,1]^d$ must have size $\Omega_d(1/\varepsilon^{1-k/d})$.

Proof: Let K be a (k, ε) -net for volume measure. For each k-flat $\varphi \in K$, let $H(\varphi, r)$ be the locus of points in $[0, 1]^d$ within distance at most r from φ (for k = 1 in three dimensions, this is the intersection of $[0, 1]^3$ and the cylinder with radius r centered at the line φ). Note that a ball b with center c and radius r intersects a k-flat φ if and only if $c \in H(\varphi, r)$.

Fix $r = (\varepsilon/\mu)^{1/d}$, where μ is a constant to be determined shortly. We claim that by choosing μ appropriately, if K is a (k, ε) -net for volume measure, then the collection of objects $\{H(\varphi, r) \mid \varphi \in K\}$ covers $[0, 1]^d$. Indeed, suppose not. Then there exists a point $p \in [0, 1]^d$ not covered by any of the objects $H(\varphi, r)$. This implies that a ball b centered at p with radius r does not intersect any k-flat of K, and its volume is $c_d r^d = c_d \varepsilon/\mu$, where c_d is a constant that depends on d. Choose $\mu = c_d$ so that b has volume at least ε , but does intersect any k-flat of K. A contradiction to the required net property.

Hence, by the choice of r, any (k, ε) -net for volume measure must satisfy the condition that

$$\{H(\varphi,r)\mid \varphi\in K\}$$

covers $[0,1]^d$. For any k-flat φ , we have $\eta = \operatorname{vol}(H(\varphi,r)) = O_d(r^{d-k}) = O_d(\varepsilon^{1-k/d})$. Thus, to cover $[0,1]^d$, we have that $|K| \ge 1/\eta = \Omega_d(1/\varepsilon^{1-k/d})$.

4. Constructing (k, ε) -nets for volume measure

Here, we give a self-contained deterministic and explicit construction of (k, ε) -nets for volume measure for $k \in \{0, \ldots, d-1\}$. The constructed set size matches the lower bound of Lemma 3.1 up to constant factors for $k \ge 1$.

4.1. Preliminaries

For a number $x \in (0,1)$, lets its rank be the minimum i such that 2^ix is an integer. For example, rank(1/2) = 1, and rank(7/8) = 3. Thus, any binary string $s = s_1 \cdots s_i \in \{0,1\}^i$ of length i that ends in 1, corresponds to the number $\sum_{k=1}^i s_k/2^k$ of rank i. Let

$$\mathcal{B}_t = \{ x/2^t \mid x \in [2^t - 1] \}$$

be the set of all numbers in (0,1) of rank at most t. Observe that there are exactly 2^{i-1} numbers in \mathcal{B}_t of rank i, for $i=1,\ldots,t$.

4.2. Construction

The construction works recursively on the dimension d.

Base case: d = 1 and k = 0. Here a $(0, \varepsilon)$ -net for volume measure of size $O(1/\varepsilon)$ follows readily by spreading $2 + |1/\varepsilon|$ points uniformly on the interval $[0, 1]^1 = [0, 1]$.

Base case: d = k + 1 and k > 0. Here a $(d - 1, \varepsilon)$ -net for volume measure of size $d/\varepsilon^{1/d} = O(d/\varepsilon^{1-k/d})$ follows readily by overlaying a d-dimensional grid of size length $\varepsilon^{1/d}$ over $[0, 1]^d$. Each cell in this grid has volume ε . Thus, the net consists of the hyperplanes forming the grid¹.

Induction: d > k + 1. For i = 1, ..., d, and $\varphi \in \mathcal{B}_{\tau}$, where

$$\tau = \left\lceil \frac{1}{d} \log \frac{1}{\varepsilon} \right\rceil + 3 \lceil \log(3d) \rceil + 1, \tag{4.1}$$

consider the hyperplane $h(i,\varphi) \equiv (x_i = \varphi)$, and let $\ell = \text{rank}(\varphi)$. We recursively construct a (k, ε_{ℓ}) -net for volume measure on $h(i,\varphi)$ (which lies in d-1 dimensions), where

$$\varepsilon_{\ell} = \frac{2^{\ell} \varepsilon}{4d}.\tag{4.2}$$

Thus, hyperplanes with rank ℓ have a finer net on them than hyperplanes of rank $\ell+1$. We collect all such k-flats built on all of these hyperplanes of all ranks into a set K, which is the desired (k, ε) -net.

See Figure 4.1 for an illustration of the construction in two dimensions.

Intuition. The construction is based on quadtrees. Starting with the entire cube $[0,1]^d$, we construct d orthogonal hyperplanes which *split* the cube into 2^d cubes of side length 1/2. We refer to such hyperplanes as *splitting hyperplanes*. This splitting process is continued recursively. The rank of a hyperplane is thus the level of the recursion when it is being introduced. All the cubes at the *i*th level of the construction have side length $1/2^i$ and they form a grid. The number of cubes in this grid at the *i*th level is 2^{di} . Observe, that we recursively construct a net on each "wall" of a cell, where the density of the net is coarser as we go down the recursion.

¹This requires the convex bodies under consideration to be closed.

4.3. Analysis

Lemma 4.1. For $k \in [d-1]$, the constructed (k,ε) -net for volume measure has size at most $\beta(d)/\varepsilon^{1-k/d} = O_d(1/\varepsilon^{1-k/d})$, where $\beta(d) = 2^{O(d-k)}d^{6(d-k-1)+1}$

Proof: Let $T(\varepsilon,d)$ denote the size of a (k,ε) -net for volume measure over $[0,1]^d$ constructed above. The proof is by induction on d. When d=k+1, we have $T(\varepsilon,k+1) \leq (k+1)/\varepsilon^{1/(k+1)}$, by the base case described above. So assume $d \geq k+2$ and $T(\delta,d') \leq \beta(d')/\delta^{1-k/d'}$ for all d' < d, where $\beta(d')$ is a function and $\beta(k+1) = k+1$. We remind the reader that $\varepsilon_i = 2^i \varepsilon/(4d)$ and $\tau \leq \frac{1}{d} \log \frac{1}{\varepsilon} + 3 \log d + 8$. By the inductive hypothesis, the above construction produces a (k,ε) -net of size

$$|K| \leq \sum_{i=1}^{\tau} d2^{i-1} T(\varepsilon_{i}, d-1) \leq d \sum_{i=1}^{\tau} \frac{2^{i-1} \beta(d-1)}{\varepsilon_{i}^{1-k/(d-1)}}$$

$$\leq d \sum_{i=1}^{\tau} \frac{2^{i-1} \beta(d-1)}{(2^{i} \varepsilon/(4d))^{1-k/(d-1)}} \leq \frac{4d^{2} \beta(d-1)}{\varepsilon^{1-k/(d-1)}} \sum_{i=1}^{\tau} \frac{2^{i-1}}{2^{i-ik/(d-1)}}$$

$$= \frac{2d^{2} \beta(d-1)}{\varepsilon^{1-k/(d-1)}} \sum_{i=1}^{\tau} 2^{ik/(d-1)} \leq \frac{4d^{3} \beta(d-1)}{\varepsilon^{1-k/(d-1)}} \cdot 2^{\tau k/(d-1)}$$

$$\leq \frac{4d^{3} \beta(d-1)}{\varepsilon^{1-k/(d-1)}} \cdot \left(\frac{256d^{3}}{\varepsilon^{1/d}}\right)^{k/(d-1)} \leq \frac{1024d^{6} \beta(d-1)}{\varepsilon^{1-k/(d-1)+k/[d(d-1)]}}$$

$$= \frac{1024d^{6} \beta(d-1)}{\varepsilon^{1-k/d}}.$$

In particular, we obtain the recurrence $\beta(d)=1024d^6\beta(d-1)$, which solves to $\beta(d)=2^{O(d-k)}d^{6(d-k-1)+1}$ as $\beta(k+1)=k+1$.

Lemma 4.2. For k = 0, the constructed $(0, \varepsilon)$ -net for volume measure has size at most $\frac{\psi(d)}{\varepsilon} \log^{d-1} \frac{1}{\varepsilon}$, where $\psi(d) = (\log d)^{O(d^2)}$.

Proof: We follow the proof of Lemma 4.1. Let $T(\varepsilon,d)$ denote the size of a $(0,\varepsilon)$ -net for volume measure over $[0,1]^d$ constructed above. We have $T(\varepsilon,1) \leq 3/\varepsilon$. So assume $d \geq 2$, and $T(\delta,d') \leq \frac{\psi(d')}{\delta} \log^{d'-1} \frac{1}{\delta}$ for all d' < d, where $\psi(d')$ is a function with $\psi(1) = 3$. As a reminder, we have $\varepsilon_i = 2^i \varepsilon/(4d)$ and $\tau \leq 10 \log \frac{d}{\varepsilon}$. By the inductive hypothesis, the above construction produces a $(0,\varepsilon)$ -net for volume measure of size

$$\begin{split} |K| & \leq d \sum_{i=1}^{\tau} 2^{i-1} T(\varepsilon_i, d-1) \leq d \sum_{i=1}^{\tau} \frac{2^{i-1} \psi(d-1)}{\varepsilon_i} \log^{d-2} \frac{1}{\varepsilon_i} \\ & \leq d \sum_{i=1}^{\tau} \frac{4d \cdot 2^{i-1} \psi(d-1)}{2^i \varepsilon} \log^{d-2} \frac{4d}{2^i \varepsilon} \leq 4d^2 \psi(d-1) \cdot \tau \cdot \frac{1}{\varepsilon} \left(10 \log \frac{d}{\varepsilon} \right)^{d-2} \\ & \leq 4d^2 (10 \log d)^{d-1} \psi(d-1) \cdot \frac{1}{\varepsilon} \log^{d-1} \frac{1}{\varepsilon}. \end{split}$$

We obtain the recurrence $\psi(d) \leq 4d^2(10\log d)^{d-1}\psi(d-1)$, which solves to $\psi(d) = (\log d)^{O(d^2)}$.

Lemma 4.3. The constructed set K is a (k, ε) -net for volume measure over $[0, 1]^d$.

Proof: Let Ξ be a convex body contained in $[0,1]^d$ with volume at least ε . Assume, for the sake of contradiction, that Ξ is not stabbed by any of the k-flats of K. The constructed set being a net for the base cases of the construction (d = k + 1 or d = 1 and k = 0) are immediate.

So, let $h(\alpha)$ be the hyperplane orthogonal to the first axis which intersects the first axis at $\alpha \in \mathbb{R}$. Define the functions

$$f(\alpha) = \text{vol}(\Xi \cap h(\alpha))$$
 and $g(\alpha) = f(\alpha)^{1/(d-1)}$.

By the Brunn-Minkowski inequality, the function $g(\alpha)$ is concave and unimodal. Define the point $x^* \in [0,1]$ so that $x^* = \arg \max_{\alpha} f(\alpha)$.

Let $V(\Delta) = f(x^* + \Delta)$, and let $v(\Delta) = (V(\Delta))^{1/(d-1)}$. The function v, being a translation of g, is concave and unimodal. Let κ be the maximum index in $\llbracket \tau \rrbracket$, such that $\varepsilon_{\kappa} \leq V(0)$, see Eq. (4.2). Let $r_i \geq 0$ be the maximum number such that $V(r_i) = \varepsilon_i$, for $i = 1, \ldots, \kappa$. As we can assume that Ξ is smooth, it is easy to verify the rs are well defined.

Observe that if $r_i \geq 1/2^i$, then there is hyperplane orthogonal to the first axis that has a recursive construction of a net on it of level i, for ε_i , that lies in the range $[x^*, x^* + 1/2^i]$. This by induction would imply that the net intersects Ξ . We thus assume from this point on that

$$r_i < \frac{1}{2^i},$$

for all i. Observe that $r_1 \geq r_2 \geq \cdots \geq r_{\kappa}$, as $\varepsilon_1 < \varepsilon_2 < \cdots < \varepsilon_{\kappa}$ (more specifically, $\varepsilon_i = 2\varepsilon_{i-1}$ for all i).

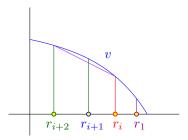


Figure 4.2: By the choice of $r_{\kappa} \leq \ldots \leq r_1$, we have $v(r_{\kappa}) \geq \ldots \geq v(r_1)$.

The concavity of $v(\cdot)$, see Figure 4.2, implies that

$$\frac{v(\boldsymbol{r}_{i+2})-v(\boldsymbol{r}_{i+1})}{\boldsymbol{r}_{i+2}-\boldsymbol{r}_{i+1}} \geq \frac{v(\boldsymbol{r}_{i+1})-v(\boldsymbol{r}_i)}{\boldsymbol{r}_{i+1}-\boldsymbol{r}_i} \qquad \Longrightarrow \qquad \frac{\boldsymbol{r}_{i+1}-\boldsymbol{r}_i}{\boldsymbol{r}_{i+2}-\boldsymbol{r}_{i+1}} \leq \frac{v(\boldsymbol{r}_{i+1})-v(\boldsymbol{r}_i)}{v(\boldsymbol{r}_{i+2})-v(\boldsymbol{r}_{i+1})},$$

as $r_{i+1} - r_i < 0$ and $v(r_{i+2}) - v(r_{i+1}) > 0$. Since $V(r_{i+1}) = \varepsilon_{i+1} = 2\varepsilon_i = 2V(r_i)$, we have that $v(r_{i+1}) = 2^{1/(d-1)}v(r_i)$. For $i < \kappa$, let $\ell_i = r_i - r_{i+1}$. Plugging this into the above, observe

$$\frac{\ell_i}{\ell_{i+1}} = \frac{\boldsymbol{r}_i - \boldsymbol{r}_{i+1}}{\boldsymbol{r}_{i+1} - \boldsymbol{r}_{i+2}} \le \frac{v(\boldsymbol{r}_{i+1}) - v(\boldsymbol{r}_i)}{v(\boldsymbol{r}_{i+2}) - v(\boldsymbol{r}_{i+1})} = \frac{(2^{1/(d-1)} - 1)v(\boldsymbol{r}_i)}{2^{1/(d-1)}(2^{1/(d-1)} - 1)v(\boldsymbol{r}_i)} = \frac{1}{2^{1/(d-1)}}.$$

Since $\ell_{\kappa-1} \leq r_{\kappa-1} \leq 1/2^{\kappa-1}$, we have

$$egin{aligned} m{r}_1 &= m{r}_\kappa + \sum_{i=1}^{\kappa-1} \ell_i \leq m{r}_\kappa + \ell_{\kappa-1} igg(1 + rac{1}{2^{1/(d-1)}} + rac{1}{2^{2/(d-1)}} + \cdots igg) \\ &\leq m{r}_\kappa + 2d\ell_{\kappa-1} \leq (2d+1)m{r}_{\kappa-1} \leq rac{2d+1}{2^{\kappa-1}} < rac{arepsilon^{1/d}}{4d^2}, \end{aligned}$$

as $\kappa \leq \tau$, and by the value of τ , see Eq. (4.1).

Let I_1 be the maximum interval, where the value of $V(x) \ge \varepsilon_1$ for any $x \in I_1$. By the above, we have that if the net does not intersect Ξ , then $||I_1|| \le 2r_1 \le 2\varepsilon^{1/d}/(4d^2)$.

We define I_2, \ldots, I_d in a similar fashion on the other axes, and the same argumentation would imply that $||I_j|| \leq 2\varepsilon^{1/d}/(4d^2)$, for all j. Furthermore, any plane orthogonal to the axes that avoids the box $B = I_1 \times I_2 \cdots \times I_d$ has an intersection with Ξ of volume at most ε_1 . We conclude that the total value of Ξ is at most

$$\operatorname{vol}(\Xi) \leq \operatorname{vol}(B) + \sum_{j=1}^d \int_{y \in [0,1] \backslash I_j} \operatorname{vol} \Bigl(\Xi \cap (x_j = y)\Bigr) dy \leq \prod_{j=1}^d \|I_j\| + d\varepsilon_1 < \varepsilon,$$

which is a contradiction to $vol(\Xi) \geq \varepsilon$.

Putting the above together, we get our main result.

Theorem 4.4. Given $\varepsilon \in (0,1)$ and $k \in \{1,\ldots,d-1\}$, the above is a deterministic and explicit construction of a (k,ε) -net for volume measure over $[0,1]^d$ of size $\beta(d)/\varepsilon^{1-k/d} = O_d(1/\varepsilon^{1-k/d})$, where $\beta(d) = 2^{O(d-k)} d^{6(d-k-1)+1}$.

For
$$k = 0$$
, the above construction has size $\frac{\psi(d)}{\varepsilon} \log^{d-1} \frac{1}{\varepsilon}$, where $\psi(d) = (\log d)^{O(d^2)}$.

Remark 4.5. (A) Our upper bound for the case of points matches the lower bound $\Omega_d(\frac{1}{\varepsilon}\log^{d-1}\frac{1}{\varepsilon})$ of Bukh et al. [BMN09] (which holds for somewhat different settings). This seems to be somewhat coincidental, as the ε -net theorem implies, in this case, a smaller weak ε -net for volume measure of size $O(\frac{d}{\varepsilon}\log\frac{1}{\varepsilon})$, via the reduction to ellipsoids, seeLemma 2.4.

(B) The construction here is orthogonal in nature. For the case of $(0, \varepsilon)$ -nets, the generated set is significantly larger than the Halton-Hammersely set (see Definition A.3 and Lemma A.4) which works for axis-aligned boxes. General convex bodies do not have the same predictable "behavior" of axis-aligned boxes, thus maybe explaining the need for a larger net.

5. Conclusions

The main open problem left by our work is bounding the size of (k, ε) -nets in the general case. That is, the input is a set P of n points in \mathbb{R}^d , and we would like to compute a minimum set of k-flats which stab all convex bodies containing at least εn points of P. As noted earlier, there is a (k, ε) -net of asymptotically the same size as of a weak ε -net in \mathbb{R}^{d-k} . This follows by projecting the point set to a subspace of dimension d-k, constructing a regular weak ε -net, and lifting the net back to the original space. Can one do better than this somewhat naive construction?

Note that it is easy to show a lower bound of size $\Omega(1/\varepsilon)$ for $(1,\varepsilon)$ -nets in the general case. Take a point set that consists of $\lceil 2/\varepsilon \rceil$ equally sized clusters of tightly packed points, such that no line passes through three clusters. Namely, our sublinear results in $1/\varepsilon$ are special for the uniform measure on the hypercube.

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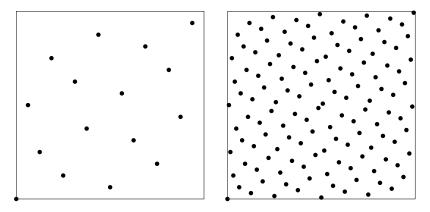


Figure A.1: The Van der Corput set with n = 16 (left) and n = 128 (right).

A. $(0,\varepsilon)$ -nets for axis-aligned boxes

Here we show the existence of a $(0, \varepsilon)$ -net of size $O(1/\varepsilon)$ that intersects any axis-aligned box B that has the property that $\operatorname{vol}(B \cap [0, 1]^2) \geq \varepsilon$. The following constructions are essentially described in [Mat99] (in the context of low-discrepancy point sets), however the proofs use similar tools. We give the proofs for completeness.

Definition A.1 (the Van der Corput set). For an integer α , let $\mathsf{bin}(\alpha) \in \{0,1\}^*$ denote the binary representation of α , and $\mathsf{rev}(\mathsf{bin}(\alpha))$ be the reversal of the string of digits in $\mathsf{bin}(\alpha)$. We define $\mathsf{br}(\alpha) \in [0,1]$ to be the bit-reversal of α , which is defined as the number obtained by concatenating "0." with the string $\mathsf{rev}(\mathsf{bin}(\alpha))$. For example, $\mathsf{br}(13) = 0.1011$. Formally, if $\alpha = \sum_{i=0}^{\infty} 2^i b_i$ with $b_i \in \{0,1\}$, then $\mathsf{br}(\alpha) = \sum_{i=0}^{\infty} b_i/2^{i+1}$.

For an integer n, the **Van** der Corput set is the collection of points p_0, \ldots, p_{n-1} , where $p_i = (i/n, br(i))$. See Figure A.1.

Lemma A.2. For a parameter $\varepsilon \in (0,1)$, there is a collection of $O(1/\varepsilon)$ points $P \subset [0,1]^2$ such that any axis-aligned box B with $\operatorname{vol}(B \cap [0,1]^2) \geq \varepsilon$ contains a point of P.

Proof: Let $n = \lceil 4/\varepsilon \rceil$. We claim that the Van der Corput set of size n is the desired point set P. Let B be a box contained in $[0,1]^2$ of width w and height h, with $wh \ge \varepsilon$. Let $q \ge 2$ be the smallest integer such that $1/2^q < h/2 \le 1/2^{q-1}$. By the choice of q, the projection of B onto the y-axis contains an interval of the form $I = [k/2^q, (k+1)/2^q)$ for some integer k. Let $B_I = B \cap \{(x,y) \in [0,1]^2 \mid y \in I\}$ be the box restricted to I along the y-axis. Observe that

$$\operatorname{vol}(B_I) = w/2^q = w/(4 \cdot 2^{q-2}) \ge wh/4 \ge \varepsilon/4 \iff w \ge 2^q \varepsilon/4.$$

Let $S = [0, 1] \times I$, so that each $p_j \in P \cap S$ has $\mathsf{br}(j) \in I$. In particular, the first q binary digits of $\mathsf{br}(j)$ are fixed. This implies that the q least significant binary digits of j are fixed. In other words, $P \cap S$ contains all points p_j such that $j \equiv \ell \pmod{2^q}$ for some integer ℓ —the x-coordinates of the points in P are regularly spaced in the strip S with distance $2^q/n$. If the width of B_I is at least $2^q/n$, then this implies that B contains a point of P in the strip S. Indeed, by the choice of n, $2^q/n \leq 2^q \varepsilon/4 \leq w$.

By extending the definition of the Van der Corput set to higher dimensions, the above proof also generalizes.

Definition A.3 (the Halton-Hammersely set). For a prime number ρ and an integer $\alpha = \sum_{i=0}^{\infty} \rho^i b_i$, with $b_i \in \{0, \dots, \rho - 1\}$, written in base ρ , define $\operatorname{br}_{\rho}(\alpha) = \sum_{i=0}^{\infty} b_i/\rho^{i+1}$. Note that $\operatorname{br}_2 = \operatorname{br}$ from Definition A.1.

For integers n and d, the **Halton-Hammersely set** is the collection of points

$$p_1,\ldots,p_{n-1},$$

where $p_i = (\mathsf{br}_{\rho_1}(i), \mathsf{br}_{\rho_2}(i), \dots, \mathsf{br}_{\rho_{d-1}}(i), i/n)$, and $\rho_1, \dots, \rho_{d-1}$ are the first d-1 prime numbers. (Making i/n the dth coordinate instead of the 1st coordinate simplifies future notation.)

Lemma A.4. For a parameter $\varepsilon \in (0,1)$, there is a collection of $2^{O(d \log d)}/\varepsilon$ points $P \subset [0,1]^d$ such that any axis-aligned box B with $\operatorname{vol}(B \cap [0,1]^d) \ge \varepsilon$ contains a point of P.

Proof: The proof is similar to Lemma A.2, with the Chinese remainder theorem as the additional tool.

Let $n = \lceil (2^{d-1}/\varepsilon) \cdot (d-1)\sharp \rceil$, where $k\sharp$ is the **primorial** function, defined as the product of the first k prime numbers. It is known that $k\sharp \leq \exp((1+o(1))k\log k)$, which implies $n = 2^{O(d\log d)}/\varepsilon$. We claim that the Halton-Hammersely set of size n is the desired point set P.

Denote the side lengths of the box B by s_1, \ldots, s_d , with $\prod_{i=1}^d s_i \geq \varepsilon$. For each $i=1,\ldots,d-1$, let q_i be the smallest integer such that $1/\rho_i^{q_i} < s_i/2 \leq 1/\rho_i^{q_i-1}$, where ρ_i is the ith prime number. By the choice of q_i , the projection of B onto the ith axis contains an interval of the form $I_i = [k_i/\rho_i^{q_i}, (k_i+1)/\rho_i^{q_i}]$ for some integer k_i . Let S denote the box $I_1 \times \ldots \times I_{d-1} \times [0,1]$ and $B_S = B \cap S$. Observe that

$$\operatorname{vol}(B_S) = s_d \prod_{i=1}^{d-1} \frac{1}{\rho_i^{q_i}} \ge s_d \prod_{i=1}^{d-1} \frac{s_i}{2\rho_i} \ge \frac{\varepsilon}{2^{d-1}} \prod_{i=1}^{d-1} \frac{1}{\rho_i} \iff s_d \ge \frac{\varepsilon}{2^{d-1}} \prod_{i=1}^{d-1} \rho_i^{q_i-1}.$$

Similar to Lemma A.2, we observe that the point $p_j \in P$ falls into S when $j \equiv \ell_i \pmod{\rho_i^{q_i}}$ for some integers $\ell_1, \ldots, \ell_{d-1}$. By the Chinese remainder theorem, there is exactly one number in the set $\left\{0, 1, \ldots, \prod_{i=1}^{d-1} \rho_i^{q_i} - 1\right\}$ (the dth coordinate of p_j) which satisfies these d-1 equations. In particular, the points in $P \cap S$ are spaced regularly along the dth axis with distance $\delta = (1/n) \prod_{i=1}^{d-1} \rho_i^{q_i}$. Once again, we argue that the length of B along the dth axis is at least δ , which implies the result. Indeed, by our choice of n we have that,

$$\delta = \frac{1}{n} \prod_{i=1}^{d-1} \rho_i^{q_i} \le \frac{\varepsilon}{2^{d-1}} \prod_{i=1}^{d-1} \rho_i^{q_i - 1} \le s_d.$$