# Almost sure convergence of the minimum bipartite matching functional in Euclidean space

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#### Abstract

Let  $L_N = L_{MBM}(X_1, \ldots, X_N; Y_1, \ldots, Y_N)$  be the minimum length of a bipartite matching between two sets of points in  $\mathbf{R}^d$ , where  $X_1, \ldots, X_N, \ldots$  and  $Y_1, \ldots, Y_N, \ldots$  are random points independently and uniformly distributed in  $[0, 1]^d$ . We prove that for  $d \geq 3$ ,  $L_N/N^{1-1/d}$  converges with probability one to a constant  $\beta_{MBM}(d) > 0$  as  $N \to \infty$ .

## 1 Introduction and statement of the result.

Given two sets of N points  $X = \{X_1, ..., X_N\}$  and  $Y = \{Y_1, ..., Y_N\}$  in  $\mathbb{R}^d$ , a bipartite matching of X and Y is a perfect matching M on the set  $X \cup Y$ , such that each pair in M is made of one point of X and one point of Y. The length of such a matching is defined to be the sum of the euclidean lengths of the edges formed by its pairs. The (euclidean) minimum bipartite matching problem (MBMP) then asks one to find a bipartite matching of X and Y whose length is as small as possible. We shall denote by  $L_{MBM}(X, Y)$  the length of a minimum bipartite matching of X and Y.

A related problem is the simple minimum matching problem (MMP), where one is asked to find a perfect matching of smallest euclidean length on a set  $X = \{X_1, ..., X_N\} \subset \mathbf{R}^d$ . The subadditive methods inaugurated by Beardwood, Halton and Hammersley (BHH) [4] and further developed in [9,10,12], show that a strong limit theorem applies to the length  $L_{MM}(X)$  of a simple minimum matching on X, when the points  $X_1, \ldots, X_N$  are random. The theorem states that for any dimension d, if  $X_1, \ldots, X_N, \ldots$  is a sequence of points distributed independently and uniformly in a bounded region  $\Omega \subset \mathbf{R}^d$ , then the ratio  $L_{MM}(X_1, \ldots, X_N)/N^{1-1/d}$  converges almost surely to  $\operatorname{Vol}(\Omega)^{1/d}\beta_{MM}(d)$ , where  $\operatorname{Vol}(\Omega)$  denotes the Lebesgues measure of  $\Omega$  and  $\beta_{MM}(d) > 0$  is a universal constant depending only upon d. The functional  $L_{MBM}$  does not satisfy this form of limit theorem in dimensions 1 and 2. For d = 1, the MBMP amounts to a sorting problem and it is not difficult to show that if X and Y both consist of N points independently and uniformly distributed in [0,1], there are constants  $0 < C_1 < C_2$  such that  $C_1\sqrt{N} \leq L_{MBM}(X,Y) \leq C_2\sqrt{N}$  with probability 1 - o(1) as  $N \to \infty$ . Moreover in that case the variance of  $L_{MBM}(X,Y)/\sqrt{N}$  does not converge to zero as  $N \to \infty$ . ( $L_{MBM}$  is not "self-averaging", in the statistical physics' terminology.) For d = 2 Ajtai et al. [1] proved a remarkable fact: if the sets X, Y are now distributed in  $[0,1]^2$ , then for some constants  $C_1, C_2$  indendent of N, one has  $C_1\sqrt{N\log N} \leq L_{MBM}(X,Y) \leq C_2\sqrt{N\log N}$  with probability 1 - o(1). Numerical simulations suggest that  $L_{MBM}(X,Y)/\sqrt{N\log N}$  converges to a non-random constant as  $N \to \infty$ , however this has not yet been proved.

In this article, we show that for any  $d \ge 3$  we recover a BHH theorem for the functional  $L_{MBM}$ .

**Theorem 1.1** Let  $X_1, ..., X_N, ...$  and  $Y_1, ..., Y_N, ...$  be two sequences of random points independently and uniformly distributed in  $[0, 1]^d$ , where  $d \ge 3$ , and let  $L_N = L_{MBM}(X_1, ..., X_N; Y_1, ..., Y_N)$ . There exists a constant  $\beta_{MBM}(d) > 0$ such that with probability one

$$\lim_{N \to \infty} L_N / N^{1 - 1/d} = \beta_{MBM}(d).$$

#### 2 Proof of Theorem 1.1.

To begin, we remark that to prove this theorem it will suffice to establish that  $L_N/N^{1-1/d}$  converges in mean value to a constant  $\beta_{MBM}(d)$ . This is a consequence of the following lemma [14]:

**Lemma 2.1** For any t > 0, one has

$$P(|\frac{L_N}{N^{1-1/d}} - E(\frac{L_N}{N^{1-1/d}})| > t) \le 2\exp(-\frac{N^{1-2/d}t^2}{8d})$$

This result follows from the application of Azuma's inequality [3] and the martingale difference method to  $L_N$ , in a way by now standard in the probabilistic theory of combinatorial optimisation [13]. Given the lemma, the theorem follows easily from the convergence of  $EL_N/N^{1-1/d}$  as  $N \to \infty$ , by applying the Borel-Cantelli lemma.

We have now to establish that for  $d \geq 3$  the quantity  $EL_N/N^{1-1/d}$  indeed converges to a constant  $\beta_{MBM}(d) > 0$ . To prove this we exploit the subadditivity properties of  $L_{MBM}$ , in the spirit of Steele's theory of subadditive Euclidean functionals [12]. Let us divide the unit cube  $[0,1]^d$  into disjoint similar subcubes  $Q_k$ ,  $k = 1, \ldots, m^d$  with edges of length 1/m, and compare the value of  $L_{MBM}(X, Y)$  to the sum

$$\sum_{k=1}^{m^a} L_k,\tag{1}$$

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where  $L_k$  is the value of the functional  $L_{MBM}$  for the set of points  $X_i$  and  $Y_i$ which belongs to  $Q_k$ . A difficulty arises as in general the  $Q_k$ 's do not contain the same number of points  $X_i$  and of points  $Y_i$ . (In fact the special properties of the MBMP in dimensions 1 and 2 originate from the fluctuations of the differences between these numbers around their mean value 0.) To give meaning to the sum (1) we need to generalize the functional  $L_{MBM}$  to matchings between two sets of different cardinalities. There are several ways to do this; we shall define  $L_{MBM}(X_1, \ldots, X_{N_1}; Y_1, \ldots, Y_{N_2})$  by imposing that the minimum matching contains as few unmatched points as possible. That is if  $N_1 > N_2$ , we leave  $N_1 - N_2$  points of X unmatched, whereas if  $N_1 < N_2$  we leave  $N_2 - N_1$  points of Y unmatched.

Although expression (1) now makes sense, it is still not possible to write a subadditivity inequality of the same form as the one studied in [12]. Indeed, such a form (which Steele calls "geometric subadditivity") implies an upper bound of the form  $CN^{1-1/d}$  for the functional at hand [13], and it is easy to see that no such bound applies to  $L_{MBM}(X, Y)$ . We shall however see that a geometric subadditivity property holds in the mean for the functional  $L_{MBM}$ . Suppose that the points  $X_1, \ldots, X_{N_1}, Y_1, \ldots, Y_{N_2}$  belong to an arbitrary cube Q having edge length a, and divide Q into disjoint cubes  $Q_p$ ,  $p = 1, \ldots, 2^d$  by splitting each edge in two halves. Construct in each  $Q_p$  an optimal matching in the sense just defined, between the  $n_{1,p}$  points  $X_i$  and the  $n_{2,p}$  points  $Y_i$  in  $Q_p$ , and denote its length by  $L_p$ . The points that are left unpaired are in number  $|n_{1,p} - n_{2,p}|$  in each  $Q_p$ , so if  $L_0$  denotes the length of an optimal matching for these points one has

$$L_{MBM}(X_1, \dots, X_{N_1}; Y_1, \dots, Y_{N_2}) \le \sum_{p=1}^{2^d} L_p + L_0$$
$$\le \sum_{p=1}^{2^d} L_p + \frac{1}{2} a \sqrt{d} \sum_{p=1}^{2^d} |n_{1,p} - n_{2,p}|,$$
(2)

where the last inequality is obtained by bounding  $L_0$  in an obvious way.

We shall apply this to  $Q = [0, 1]^d$ . Let  $Q_{p_1} p_1 = 1, \ldots 2^d$  be the cubes obtained in the above subdivision; let  $Q_{p_1p_2}$  be the cubes obtained by splitting in two halves the edges of each cube  $Q_{p_1}$ ; and so on. By repeating this operation K times, we get a subdivision with cubes  $Q_{p_1...p_K}$  whose edges are of length  $1/2^K$ . Let  $n_{1,p_1...p_K}$  and  $n_{2,p_1...p_K}$  be respectively the number of points  $X_i$  and  $Y_i$  in  $Q_{p_1...p_K}$ . Apply (2) first to the  $Q_{p_1,...p_{K-1}}$ 's, then to the  $Q_{p_1...p_{K-2}}$ 's, etc, keeping at each step only those points which are still unpaired. It is easy to convince oneself that the number of unpaired points in each  $Q_{p_1,...p_{K-k}}$  just after step k is given by  $|n_{1,p_1,...p_{K-k}} - n_{2,p_1,...p_{K-k}}|$ . After step k = K one obtains a matching between  $X_1, \ldots X_{N_1}$  and  $Y_1, \ldots Y_{N_2}$  where all the points but  $|N_1 - N_2|$ are matched. One is thus led to the following inequality:

$$L_{MBM}(X_1,\ldots,X_{N_1};Y_1,\ldots,Y_{N_2}) \leq \sum_{p_1\ldots p_K} L_{p_1\ldots p_K}$$

$$+\sum_{k=1}^{K} \frac{\sqrt{d}}{2^{k}} \sum_{p_{1}...p_{k}} |n_{1,p_{1}...p_{k}} - n_{2,p_{1}...p_{k}}|.$$
(3)

We now proceed to derive a subadditivity property for the mean value of  $L_{MBM}(X, Y)$ . We first consider the case where  $N_1 = \text{card}X$  and  $N_2 = \text{card}Y$  are not fixed integers but are independent Poisson random variables with the same mean value N, the elements of X and Y being chosen independently and uniformly in  $[0,1]^d$ . For a given k, the numbers  $n_{1,p_1,\dots,p_k}$  and  $n_{2,p_1,\dots,p_k}$  are then also independent Poisson random variables, with parameter  $N/2^{kd}$ . Let  $M(N) = EL_{MBM}(X_1,\dots,X_{N_1};Y_1,\dots,Y_{N_2})$ . It is immediate by homogeneity that

$$EL_{p_1\dots p_K} = 2^{-K} M(N/2^{Kd}).$$
(4)

Moreover from the well known properties of Poisson variables we have

$$E|n_{1,p_1\dots p_k} - n_{2,p_1\dots p_k}| \le \sqrt{2} \left(\frac{N}{2^{kd}}\right)^{1/2}.$$
 (5)

By taking mean values in (3) we obtain:

$$M(N) \le 2^{K(d-1)} M(N/2^{Kd}) + \sqrt{2dN} \sum_{k=1}^{K} 2^{k(d/2-1)}.$$
 (6)

This inequality has been obtained for a subdivision of  $[0,1]^d$  which consists in  $2^{Kd}$  similar cubes. Suppose now that we start from the subdivision  $\Sigma$  in  $m^d$  similar cubes  $Q_k \ k = 1, \ldots m^d$ , where m is an arbitrary integer. One can then reproduce the previous construction in the following manner. Let  $m = 2^K + r$  where  $0 \le r < 2^K$ . Consider the cube  $Q_0 = [0, 2^{K+1}/m]^d$  and form the natural subdivision  $\Sigma_0$  of  $Q_0$  by  $2^{(K+1)d}$  cubes  $Q_{p_0,\ldots,p_K}$  whose edges have length 1/m. We can proceed with  $Q_0$  and  $\Sigma_0$  to a K+1 steps construction similar to the one which led to (3). The only differences are that  $Q_0$  has edges of length  $2^{K+1}/m$  rather than 1, and that some of the  $Q_{p_0,\ldots,p_K}$ 's, namely those which belong to  $\Sigma_0$  but not to  $\Sigma$ , are empty. Nevertheless, we may write

$$L_{MBM}(X_{1}, \dots X_{N_{1}}; Y_{1}, \dots, Y_{N_{2}}) - \sum_{p=1}^{m^{d}} L_{k}$$

$$\leq \sum_{k=0}^{K} \frac{\sqrt{d}2^{K-k}}{m} \sum_{p_{0} \dots p_{k}} |n_{1, p_{0} \dots p_{k}} - n_{2, p_{0} \dots p_{k}}|$$

$$\leq \sum_{k=0}^{K} \frac{\sqrt{d}}{2^{k}} \sum_{p_{0} \dots p_{k}} |n_{1, p_{0} \dots p_{k}} - n_{2, p_{0} \dots p_{k}}|.$$
(7)

Now  $n_{1,p_0...p_k}$  and  $n_{2,p_0...p_k}$  are Poisson variables with parameter lower than  $2^{(K-k)d}N/m^d \le 2^{-kd}N$  so we still have

$$E|n_{1,p_0\dots p_k} - n_{2,p_0\dots p_k}| \le \sqrt{2} \left(\frac{N}{2^{kd}}\right)^{1/2}.$$
(8)

Taking average values one is led to

$$M(N) \le m^{d-1} M(N/m^d) + 2^d \sqrt{2dN} \sum_{k=0}^{K} 2^{k(d/2-1)}.$$
(9)

Dividing this last inequality by  $N^{1-1/d}$  and then replacing N by  $m^d N$ , we get

$$\frac{M(m^d N)}{(m^d N)^{1-1/d}} \le \frac{M(N)}{N^{1-1/d}} + \frac{2^d \sqrt{2d}}{N^{1/2-1/d}} \sum_{k=0}^K 2^{-k(d/2-1)}.$$
(10)

If d > 2, the sum on the r.h.s. of the last inequality is bounded above independently of N, and is divided by a positive power of N. Elementary analysis now shows that the ratio  $M(N)/N^{1-1/d}$  necessarily converges to a limit  $\beta_{MBM}(d)$  as  $N \to \infty$ . Indeed, let  $f(t) = M(t^d)/t^{d-1}$ . One verifies at once that f(t) satisfies

$$f(mt) \le f(t) + C/t^{d/2 - 1} \tag{11}$$

for all t > 0 and any integer m; f(t) is continuous, since M(N) is a continuous function of N. So the expression  $f(t) + C_d/t^{d/2-1}$  is bounded in [1, 2] and since  $[1, \infty[$  is the union of the intervals  $m[1, 2], m \ge 1$ , it follows from (11) that f(t) remains bounded as  $t \to \infty$ , thus  $\lim^* f(t) < \infty$ . Now define  $\beta = \lim_* f(t)$ . For any  $\epsilon > 0$ , chose  $t_0 \gg 1$  and  $\eta > 0$  such that  $f(t) + C_d/t^{d/2-1} < \beta + \epsilon$  for t in the interval  $I = [t_0 - \eta, t_0 + \eta]$ . Since the intervals  $mI, m \ge 1$  span a whole interval  $[A, \infty[$  for an A sufficiently large, it follows again from (11) that  $\lim^* f(t) \le \beta + \epsilon$ . Since  $\epsilon$  is arbitrary one has  $\lim^* f(t) = \beta$ , hence  $f(t) \to \beta$  as  $t \to \infty$ , from which it follows that  $\lim_{N\to\infty} M(N)/N^{1-1/d} = \beta$ . Q.E.D.

We have thus shown for  $d \geq 3$ , that one has

$$EL_{MBM}(X_1, \dots, X_{N_1}; Y_1, \dots, Y_{N_2}) \sim \beta^E_{MBMP}(d) N^{1-1/d}, \ N \to \infty$$
 (12)

when  $N_1$  and  $N_2$  are independent Poisson variables with parameter N. The same result for the mean value  $EL_N$ , where N is a fixed integer, follows then easily. Indeed, we have the obvious bound

$$|L_{MBM}(X_1, \dots, X_N; Y_1, \dots, Y_N) - L_{MBM}(X_1, \dots, X_{N_1}; Y_1, \dots, Y_{N_2})|$$
  
$$\leq \sqrt{d}(|N_1 - N| + |N_2 - N|), \qquad (13)$$

whence taking mean values,

$$|EL_N - EL_{MBM}(X_1, \dots, X_{N_1}; Y_1, \dots, Y_{N_2})| \le 2\sqrt{2dN},$$
 (14)

and dividing by  $N^{1-1/d}$  we deduce that

$$\lim_{N \to \infty} \frac{EL_N}{N^{1-1/d}} \to \beta_{MBM}(d).$$
(15)

Theorem 1.1 is now proved.

## 3 Concluding remarks.

1) Our decimation procedure does not give back the bounds proven by Ajtai *et al.* in d = 2, but a weaker  $O(\sqrt{N} \ln N)$  bound. It is believed that a self-averaging theorem applies also to the functional  $L_{MBM}$  in dimension 2 [11].

2) The estimation of the constants  $\beta_{MBM}(d)$  is also an interesting problem. A remarkable result of Talagrand [14] shows that one has  $\beta_{MBM}(d) = \sqrt{d/2e\pi}(1+O(\ln d/d))$  as  $d \to \infty$ . It is conjectured that a 1/d series expansion actually exists for  $\beta_{MBM}(d)$ .

3) Mézard and Parisi have obtained detailed analytic predictions for the random link versions of the MMP and the MBMP [8], where the distance matrix between the points  $X_i$  and  $Y_j$  is replaced by a matrix of independent and identically distributed entries. (Some of these predictions, for the random assignment problem, have been proven recently by Aldous [2].) Numerical studies [6,7] indicate that for the MMP and the MBMP, the random link model provides one with a very good "mean-field" approximation to the Euclidean model in the large d limit. Except for simpler combinatorial problems however [5], very few rigorous results are known for comparing the euclidean and the random link models.

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