# Almost sure convergence of the minimum bipartite matching functional in Euclidean space 

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#### Abstract

Let $L_{N}=L_{M B M}\left(X_{1}, \ldots, X_{N} ; Y_{1}, \ldots, Y_{N}\right)$ be the minimum length of a bipartite matching between two sets of points in $\mathbf{R}^{d}$, where $X_{1}, \ldots, X_{N}, \ldots$ and $Y_{1}, \ldots, Y_{N}, \ldots$ are random points independently and uniformly distributed in $[0,1]^{d}$. We prove that for $d \geq 3, L_{N} / N^{1-1 / d}$ converges with probability one to a constant $\beta_{M B M}(d)>0$ as $N \rightarrow \infty$.


## 1 Introduction and statement of the result.

Given two sets of $N$ points $X=\left\{X_{1}, \ldots, X_{N}\right\}$ and $Y=\left\{Y_{1}, \ldots, Y_{N}\right\}$ in $\mathbf{R}^{d}$, a bipartite matching of $X$ and $Y$ is a perfect matching $M$ on the set $X \cup Y$, such that each pair in $M$ is made of one point of $X$ and one point of $Y$. The length of such a matching is defined to be the sum of the euclidean lengths of the edges formed by its pairs. The (euclidean) minimum bipartite matching problem (MBMP) then asks one to find a bipartite matching of $X$ and $Y$ whose length is as small as possible. We shall denote by $L_{M B M}(X, Y)$ the length of a minimum bipartite matching of $X$ and $Y$.

A related problem is the simple minimum matching problem (MMP), where one is asked to find a perfect matching of smallest euclidean length on a set $X=\left\{X_{1}, \ldots, X_{N}\right\} \subset \mathbf{R}^{d}$. The subadditive methods inaugurated by Beardwood, Halton and Hammersley (BHH) [4] and further developed in [9, 10, 12], show that a strong limit theorem applies to the length $L_{M M}(X)$ of a simple minimum matching on $X$, when the points $X_{1}, \ldots, X_{N}$ are random. The theorem states that for any dimension $d$, if $X_{1}, \ldots, X_{N}, \ldots$ is a sequence of points distributed independently and uniformly in a bounded region $\Omega \subset \mathbf{R}^{d}$, then the ratio $L_{M M}\left(X_{1}, \ldots X_{N}\right) / N^{1-1 / d}$ converges almost surely to $\operatorname{Vol}(\Omega)^{1 / d} \beta_{M M}(d)$, where $\operatorname{Vol}(\Omega)$ denotes the Lebesgues measure of $\Omega$ and $\beta_{M M}(d)>0$ is a universal constant depending only upon $d$.

The functional $L_{M B M}$ does not satisfy this form of limit theorem in dimensions 1 and 2. For $d=1$, the MBMP amounts to a sorting problem and it is not difficult to show that if $X$ and $Y$ both consist of $N$ points independently and uniformly distributed in $[0,1]$, there are constants $0<C_{1}<C_{2}$ such that $C_{1} \sqrt{N} \leq L_{M B M}(X, Y) \leq C_{2} \sqrt{N}$ with probability $1-o(1)$ as $N \rightarrow \infty$. Moreover in that case the variance of $L_{M B M}(X, Y) / \sqrt{N}$ does not converge to zero as $N \rightarrow \infty$. ( $L_{M B M}$ is not "self-averaging", in the statistical physics' terminology.) For $d=2$ Ajtai et al. [1] proved a remarkable fact: if the sets $X, Y$ are now distributed in $[0,1]^{2}$, then for some constants $C_{1}, C_{2}$ indendent of $N$, one has $C_{1} \sqrt{N \log N} \leq L_{M B M}(X, Y) \leq C_{2} \sqrt{N \log N}$ with probability 1-o(1). Numerical simulations suggest that $L_{M B M}(X, Y) / \sqrt{N \log N}$ converges to a non-random constant as $N \rightarrow \infty$, however this has not yet been proved.

In this article, we show that for any $d \geq 3$ we recover a BHH theorem for the functional $L_{M B M}$.

Theorem 1.1 Let $X_{1}, \ldots, X_{N}, \ldots$ and $Y_{1}, \ldots, Y_{N}, \ldots$ be two sequences of random points independently and uniformly distributed in $[0,1]^{d}$, where $d \geq 3$, and let $L_{N}=L_{M B M}\left(X_{1}, \ldots, X_{N} ; Y_{1}, \ldots, Y_{N}\right)$. There exists a constant $\beta_{M B M}(d)>0$ such that with probability one

$$
\lim _{N \rightarrow \infty} L_{N} / N^{1-1 / d}=\beta_{M B M}(d)
$$

## 2 Proof of Theorem 1.1.

To begin, we remark that to prove this theorem it will suffice to establish that $L_{N} / N^{1-1 / d}$ converges in mean value to a constant $\beta_{M B M}(d)$. This is a consequence of the following lemma [14]:

Lemma 2.1 For any $t>0$, one has

$$
P\left(\left|\frac{L_{N}}{N^{1-1 / d}}-E\left(\frac{L_{N}}{N^{1-1 / d}}\right)\right|>t\right) \leq 2 \exp \left(-\frac{N^{1-2 / d} t^{2}}{8 d}\right)
$$

This result follows from the application of Azuma's inequality [3] and the martingale difference method to $L_{N}$, in a way by now standard in the probabilistic theory of combinatorial optimisation [13|. Given the lemma, the theorem follows easily from the convergence of $E L_{N} / N^{1-1 / d}$ as $N \rightarrow \infty$, by applying the Borel-Cantelli lemma.

We have now to establish that for $d \geq 3$ the quantity $E L_{N} / N^{1-1 / d}$ indeed converges to a constant $\beta_{M B M}(d)>0$. To prove this we exploit the subadditivity properties of $L_{M B M}$, in the spirit of Steele's theory of subadditive Euclidean functionals [12]. Let us divide the unit cube $[0,1]^{d}$ into disjoint similar subcubes $Q_{k}, k=1, \ldots, m^{d}$ with edges of length $1 / m$, and compare the value of $L_{M B M}(X, Y)$ to the sum

$$
\begin{equation*}
\sum_{k=1}^{m^{d}} L_{k} \tag{1}
\end{equation*}
$$

where $L_{k}$ is the value of the functional $L_{M B M}$ for the set of points $X_{i}$ and $Y_{i}$ which belongs to $Q_{k}$. A difficulty arises as in general the $Q_{k}$ 's do not contain the same number of points $X_{i}$ and of points $Y_{i}$. (In fact the special properties of the MBMP in dimensions 1 and 2 originate from the fluctuations of the differences between these numbers around their mean value 0.) To give meaning to the sum (1) we need to generalize the functional $L_{M B M}$ to matchings between two sets of different cardinalities. There are several ways to do this; we shall define $L_{M B M}\left(X_{1}, \ldots X_{N_{1}} ; Y_{1}, \ldots Y_{N_{2}}\right)$ by imposing that the minimum matching contains as few unmatched points as possible. That is if $N_{1}>N_{2}$, we leave $N_{1}-N_{2}$ points of $X$ unmatched, whereas if $N_{1}<N_{2}$ we leave $N_{2}-N_{1}$ points of $Y$ unmatched.

Although expression (1) now makes sense, it is still not possible to write a subadditivity inequality of the same form as the one studied in 12. Indeed, such a form (which Steele calls "geometric subadditivity") implies an upper bound of the form $C N^{1-1 / d}$ for the functional at hand 13], and it is easy to see that no such bound applies to $L_{M B M}(X, Y)$. We shall however see that a geometric subadditivity property holds in the mean for the functional $L_{M B M}$. Suppose that the points $X_{1}, \ldots X_{N_{1}}, Y_{1}, \ldots Y_{N_{2}}$ belong to an arbitrary cube $Q$ having edge length $a$, and divide $Q$ into disjoint cubes $Q_{p}, p=1, \ldots 2^{d}$ by splitting each edge in two halves. Construct in each $Q_{p}$ an optimal matching in the sense just defined, between the $n_{1, p}$ points $X_{i}$ and the $n_{2, p}$ points $Y_{i}$ in $Q_{p}$, and denote its length by $L_{p}$. The points that are left unpaired are in number $\left|n_{1, p}-n_{2, p}\right|$ in each $Q_{p}$, so if $L_{0}$ denotes the length of an optimal matching for these points one has

$$
\begin{array}{r}
L_{M B M}\left(X_{1}, \ldots X_{N_{1}} ; Y_{1}, \ldots, Y_{N_{2}}\right) \leq \sum_{p=1}^{2^{d}} L_{p}+L_{0} \\
\leq \sum_{p=1}^{2^{d}} L_{p}+\frac{1}{2} a \sqrt{d} \sum_{p=1}^{2^{d}}\left|n_{1, p}-n_{2, p}\right| \tag{2}
\end{array}
$$

where the last inequality is obtained by bounding $L_{0}$ in an obvious way.
We shall apply this to $Q=[0,1]^{d}$. Let $Q_{p_{1}} p_{1}=1, \ldots 2^{d}$ be the cubes obtained in the above subdivision; let $Q_{p_{1} p_{2}}$ be the cubes obtained by splitting in two halves the edges of each cube $Q_{p_{1}}$; and so on. By repeating this operation $K$ times, we get a subdivision with cubes $Q_{p_{1} \ldots p_{K}}$ whose edges are of length $1 / 2^{K}$. Let $n_{1, p_{1} \ldots p_{K}}$ and $n_{2, p_{1} \ldots p_{K}}$ be respectively the number of points $X_{i}$ and $Y_{i}$ in $Q_{p_{1} \ldots p_{K}}$. Apply (2) first to the $Q_{p_{1}, \ldots p_{K-1}}$ 's, then to the $Q_{p_{1} \ldots p_{K-2}}$ 's, etc, keeping at each step only those points which are still unpaired. It is easy to convince oneself that the number of unpaired points in each $Q_{p_{1}, \ldots p_{K-k}}$ just after step $k$ is given by $\left|n_{1, p_{1}, \ldots p_{K-k}}-n_{2, p_{1}, \ldots p_{K-k}}\right|$. After step $k=K$ one obtains a matching between $X_{1}, \ldots X_{N_{1}}$ and $Y_{1}, \ldots Y_{N_{2}}$ where all the points but $\left|N_{1}-N_{2}\right|$ are matched. One is thus led to the following inequality:

$$
L_{M B M}\left(X_{1}, \ldots X_{N_{1}} ; Y_{1}, \ldots Y_{N_{2}}\right) \leq \sum_{p_{1} \ldots p_{K}} L_{p_{1} \ldots p_{K}}
$$

$$
\begin{equation*}
+\sum_{k=1}^{K} \frac{\sqrt{d}}{2^{k}} \sum_{p_{1} \ldots p_{k}}\left|n_{1, p_{1} \ldots p_{k}}-n_{2, p_{1} \ldots p_{k}}\right| . \tag{3}
\end{equation*}
$$

We now proceed to derive a subadditivity property for the mean value of $L_{M B M}(X, Y)$. We first consider the case where $N_{1}=\operatorname{cardX}$ and $N_{2}=\operatorname{cardY}$ are not fixed integers but are independent Poisson random variables with the same mean value $N$, the elements of $X$ and $Y$ being chosen independently and uniformly in $[0,1]^{d}$. For a given $k$, the numbers $n_{1, p_{1}, \ldots p_{k}}$ and $n_{2, p_{1}, \ldots p_{k}}$ are then also independent Poisson random variables, with parameter $N / 2^{k d}$. Let $M(N)=E L_{M B M}\left(X_{1}, \ldots X_{N_{1}} ; Y_{1}, \ldots Y_{N_{2}}\right)$. It is immediate by homogeneity that

$$
\begin{equation*}
E L_{p_{1} \ldots p_{K}}=2^{-K} M\left(N / 2^{K d}\right) \tag{4}
\end{equation*}
$$

Moreover from the well known properties of Poisson variables we have

$$
\begin{equation*}
E\left|n_{1, p_{1} \ldots p_{k}}-n_{2, p_{1} \ldots p_{k}}\right| \leq \sqrt{2}\left(\frac{N}{2^{k d}}\right)^{1 / 2} \tag{5}
\end{equation*}
$$

By taking mean values in (3) we obtain:

$$
\begin{equation*}
M(N) \leq 2^{K(d-1)} M\left(N / 2^{K d}\right)+\sqrt{2 d N} \sum_{k=1}^{K} 2^{k(d / 2-1)} \tag{6}
\end{equation*}
$$

This inequality has been obtained for a subdivision of $[0,1]^{d}$ which consists in $2^{K d}$ similar cubes. Suppose now that we start from the subdivision $\Sigma$ in $m^{d}$ similar cubes $Q_{k} k=1, \ldots m^{d}$, where $m$ is an arbitrary integer. One can then reproduce the previous construction in the following manner. Let $m=2^{K}+r$ where $0 \leq r<2^{K}$. Consider the cube $Q_{0}=\left[0,2^{K+1} / m\right]^{d}$ and form the natural subdivision $\Sigma_{0}$ of $Q_{0}$ by $2^{(K+1) d}$ cubes $Q_{p_{0}, \ldots p_{K}}$ whose edges have length $1 / \mathrm{m}$. We can proceed with $Q_{0}$ and $\Sigma_{0}$ to a $K+1$ steps construction similar to the one which led to (3). The only differences are that $Q_{0}$ has edges of length $2^{K+1} / m$ rather than 1, and that some of the $Q_{p_{0} \ldots p_{K}}$ 's, namely those which belong to $\Sigma_{0}$ but not to $\Sigma$, are empty. Nevertheless, we may write

$$
\begin{align*}
& L_{M B M}\left(X_{1}, \ldots X_{N_{1}} ; Y_{1}, \ldots, Y_{N_{2}}\right)-\sum_{p=1}^{m^{d}} L_{k} \\
& \leq \sum_{k=0}^{K} \frac{\sqrt{d} 2^{K-k}}{m} \sum_{p_{0} \ldots p_{k}}\left|n_{1, p_{0} \ldots p_{k}}-n_{2, p_{0} \ldots p_{k}}\right| \\
& \quad \leq \sum_{k=0}^{K} \frac{\sqrt{d}}{2^{k}} \sum_{p_{0} \ldots p_{k}}\left|n_{1, p_{0} \ldots p_{k}}-n_{2, p_{0} \ldots p_{k}}\right| . \tag{7}
\end{align*}
$$

Now $n_{1, p_{0} \ldots p_{k}}$ and $n_{2, p_{0} \ldots p_{k}}$ are Poisson variables with parameter lower than $2^{(K-k) d} N / m^{d} \leq 2^{-k d} N$ so we still have

$$
\begin{equation*}
E\left|n_{1, p_{0} \ldots p_{k}}-n_{2, p_{0} \ldots p_{k}}\right| \leq \sqrt{2}\left(\frac{N}{2^{k d}}\right)^{1 / 2} \tag{8}
\end{equation*}
$$

Taking average values one is led to

$$
\begin{equation*}
M(N) \leq m^{d-1} M\left(N / m^{d}\right)+2^{d} \sqrt{2 d N} \sum_{k=0}^{K} 2^{k(d / 2-1)} . \tag{9}
\end{equation*}
$$

Dividing this last inequality by $N^{1-1 / d}$ and then replacing $N$ by $m^{d} N$, we get

$$
\begin{equation*}
\frac{M\left(m^{d} N\right)}{\left(m^{d} N\right)^{1-1 / d}} \leq \frac{M(N)}{N^{1-1 / d}}+\frac{2^{d} \sqrt{2 d}}{N^{1 / 2-1 / d}} \sum_{k=0}^{K} 2^{-k(d / 2-1)} . \tag{10}
\end{equation*}
$$

If $d>2$, the sum on the r.h.s. of the last inequality is bounded above independently of $N$, and is divided by a positive power of $N$. Elementary analysis now shows that the ratio $M(N) / N^{1-1 / d}$ necessarily converges to a limit $\beta_{M B M}(d)$ as $N \rightarrow \infty$. Indeed, let $f(t)=M\left(t^{d}\right) / t^{d-1}$. One verifies at once that $f(t)$ satisfies

$$
\begin{equation*}
f(m t) \leq f(t)+C / t^{d / 2-1} \tag{11}
\end{equation*}
$$

for all $t>0$ and any integer $m ; f(t)$ is continuous, since $M(N)$ is a continuous function of $N$. So the expression $f(t)+C_{d} / t^{d / 2-1}$ is bounded in $[1,2]$ and since $[1, \infty[$ is the union of the intervals $m[1,2], m \geq 1$, it follows from (11) that $f(t)$ remains bounded as $t \rightarrow \infty$, thus $\lim ^{*} f(t)<\infty$. Now define $\beta=\lim _{*} f(t)$. For any $\epsilon>0$, chose $t_{0} \gg 1$ and $\eta>0$ such that $f(t)+C_{d} / t^{d / 2-1}<\beta+\epsilon$ for $t$ in the interval $I=\left[t_{0}-\eta, t_{0}+\eta\right]$. Since the intervals $m I, m \geq 1 \mathrm{span}$ a whole interval $[A, \infty[$ for an $A$ sufficiently large, it follows again from (11) that $\lim ^{*} f(t) \leq \beta+\epsilon$. Since $\epsilon$ is arbitrary one has $\lim ^{*} f(t)=\beta$, hence $f(t) \rightarrow \beta$ as $t \rightarrow \infty$, from which it follows that $\lim _{N \rightarrow \infty} M(N) / N^{1-1 / d}=\beta$. Q.E.D.

We have thus shown for $d \geq 3$, that one has

$$
\begin{equation*}
E L_{M B M}\left(X_{1}, \ldots, X_{N_{1}} ; Y_{1}, \ldots, Y_{N_{2}}\right) \sim \beta_{M B M P}^{E}(d) N^{1-1 / d}, N \rightarrow \infty \tag{12}
\end{equation*}
$$

when $N_{1}$ and $N_{2}$ are independent Poisson variables with parameter $N$. The same result for the mean value $E L_{N}$, where $N$ is a fixed integer, follows then easily. Indeed, we have the obvious bound

$$
\begin{array}{r}
\left|L_{M B M}\left(X_{1}, \ldots X_{N} ; Y_{1}, \ldots Y_{N}\right)-L_{M B M}\left(X_{1}, \ldots X_{N_{1}} ; Y_{1}, \ldots Y_{N_{2}}\right)\right| \\
\leq \sqrt{d}\left(\left|N_{1}-N\right|+\left|N_{2}-N\right|\right), \tag{13}
\end{array}
$$

whence taking mean values,

$$
\begin{equation*}
\left|E L_{N}-E L_{M B M}\left(X_{1}, \ldots X_{N_{1}} ; Y_{1}, \ldots Y_{N_{2}}\right)\right| \leq 2 \sqrt{2 d N} \tag{14}
\end{equation*}
$$

and dividing by $N^{1-1 / d}$ we deduce that

$$
\begin{equation*}
\lim _{N \rightarrow \infty} \frac{E L_{N}}{N^{1-1 / d}} \rightarrow \beta_{M B M}(d) \tag{15}
\end{equation*}
$$

Theorem 1.1 is now proved.

## 3 Concluding remarks.

1) Our decimation procedure does not give back the bounds proven by Ajtai et al. in $d=2$, but a weaker $O(\sqrt{N} \ln N)$ bound. It is believed that a self-averaging theorem applies also to the functional $L_{M B M}$ in dimension 2 11].
2) The estimation of the constants $\beta_{M B M}(d)$ is also an interesting problem. A remarkable result of Talagrand 14] shows that one has $\beta_{M B M}(d)=\sqrt{d / 2 e \pi}(1+$ $O(\ln d / d))$ as $d \rightarrow \infty$. It is conjectured that a $1 / d$ series expansion actually exists for $\beta_{M B M}(d)$.
3) Mézard and Parisi have obtained detailed analytic predictions for the random link versions of the MMP and the MBMP |8|, where the distance matrix between the points $X_{i}$ and $Y_{j}$ is replaced by a matrix of independent and identically distributed entries. (Some of these predictions, for the random assignment problem, have been proven recently by Aldous [2].) Numerical studies [6, 7] indicate that for the MMP and the MBMP, the random link model provides one with a very good "mean-field" approximation to the Euclidean model in the large $d$ limit. Except for simpler combinatorial problems however [5], very few rigorous results are known for comparing the euclidean and the random link models.

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