An Approximation Scheme for Cake Division with a Linear Number of Cuts

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Abstract. In the cake cutting problem, $n \geq 2$ players want to cut a cake into n pieces so that every player gets a 'fair' share of the cake by his own measure.
We prove the following result: For every $\varepsilon > 0$, there exists a cake division scheme for n players that uses at most $c_\varepsilon n$ cuts, and in which each player can enforce to get a share of at least $(1-\varepsilon)/n$ of the cake according to his own private measure.

1 Introduction

The second paragraph of the poem *"The voice of the lobster"* by Lewis Carrol [2] gives a classical example for the unfair division of a common resource:

"I passed by his garden, and marked, with one eye,
How the owl and the panther were sharing a pie.
The panther took pie-crust, and gravy, and meat,
While the owl had the dish as its share of the treat."

Note that pie-crust, gravy, and meat might be of completely different value to the owl and to the panther. Is there any protocol which enables owl and panther to divide the food into two pieces such that both will get at least half of it by their own measure? The answer to this question is yes, and there is a fairly simple and fairly old solution due to Hugo Steinhaus [8] from 1948: The owl cuts the food into two pieces, and the panther chooses its piece out of the two. The owl is sure to get at least half the food if it cuts two equal pieces by its measure. The panther is sure to get at least half the food by its measure by choosing the better half.

In a more general and a more mathematical formulation, there is a certain resource $\mathcal{C}$ (hereinafter referred to as: *the cake*), and there are n players $1, \ldots, n$. Every player p $(1 \leq p \leq n)$ has his own measure μ_p on the subsets of $\mathcal{C}$. These measures satisfy $\mu_p(X) \geq 0$ for all $X \subseteq \mathcal{C}$, and $\mu_p(X) + \mu_p(X') = \mu_p(X \cup X')$ for all disjoint subsets $X, X' \subseteq \mathcal{C}$. For every $X \subseteq \mathcal{C}$ and for every λ with $0 \leq \lambda \leq 1$, there exists a piece $X' \subseteq X$ such that $\mu_p(X') = \lambda \cdot \mu_p(X)$. The cake $\mathcal{C}$ is to be divided among the n players according to some fixed *protocol*, i.e., a step by step interactive procedure that can issue queries to the players whose answers may affect future decisions. We only consider protocols that satisfy the following properties.

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- If the participants obey the protocol, then each player ends up with his piece of cake after finitely many steps.
- Each time a player is required to make a cut, he must be able to do this in complete isolation and without interaction of the other players.
- The protocol has no reliable information about the measure μ_p of player p. These measures are considered private information.

The first condition simply keeps every execution of the protocol finite. The second condition does not forbid coalitions, but it protects players from intimidation. Moreover, it eliminates any form of the moving knife procedure (Stromquist [9]). The third condition states that players cannot be trusted to reveal their true preferences. Similar and essentially equivalent conditions are stated in the papers by Woodall [11], Even & Paz [3], and Robertson & Webb [5, 6].

A *strategy* of a player is an adaptive sequence of moves consistent with the protocol. For a real number β with $0 \leq \beta \leq 1$ and some fixed protocol P, a β-*strategy* of a player is a strategy that will guarantee him at least a fraction β of the cake according to his own measure, independently from the strategies of the other $n-1$ players. (So, even if the other $n-1$ players all plot up against the nth player, the nth player in this case will still be able to get a fraction β.) A protocol is called β-*fair*, if every player has a β-strategy. A protocol for n players is called *perfectly fair*, if every player has a $\frac{1}{n}$-strategy.

Even & Paz [3] show that for $n \geq 3$ players, there does not exist a perfectly fair protocol that makes only $n-1$ cuts. Moreover, [3] describe a perfectly fair protocol for $n \geq 3$ players that uses only $n \log_2(n)$ cuts. Tighter results are known for small values of n: For $n = 2$ players, the Steinhaus protocol yields a perfectly fair protocol with a single cut. For $n = 3$ and $n = 4$ players, Even & Paz [3] present perfectly fair protocols that make at most 3 and 4 cuts, respectively. Webb [10] presents a perfectly fair protocol for $n = 5$ players with 6 cuts, and he shows that no perfectly fair protocol exists that uses only 5 cuts. For any $n \geq 2$, Robertson & Webb [6] design $1/(2n-2)$-fair protocols that make only $n-1$ cuts, and they show that this result is best possible for $n-1$ cuts. The result in [6] was rediscovered independently by Krumke et al [4]. For more information on this problem and on many other of its variants, we refer the reader to the books by Brams & Taylor [1] and by Robertson & Webb [7].

The central open problem in this area is whether there exist perfectly fair n-player protocols that only use $O(n)$ cuts. This problem was explicitly formulated by Even & Paz [3], and essentially goes back to Steinhaus [8]. The general belief is that no such protocol exists. We will not settle this problem in this paper, but we will design protocols with $O(n)$ cuts that come arbitrarily close to being $\frac{1}{n}$-fair. Our main result is as follows.

Theorem 1 *For every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ and a cake division scheme for n players such that*

- *each player can enforce to get a share of at least $(1-\varepsilon)/n$ of the cake, and*
- *altogether at most $c_\varepsilon n$ cuts are made.*

This seems to be the strongest possible result one can prove without settling the general question. The protocol is defined and explained in Section 2, and its fairness is analyzed in Section 3.

(S0) If there are $n \leq 2t-1$ players,
then the cake is divided according to the Even & Paz protocol.
STOP.

(S1) Each of the first $2t$ players p ($p = 1, \ldots, 2t$) makes an arbitrary cut c_p through the cake.

(S2) The first $2t$ players are divided into two groups L' and R' with $|L'| = |R'| = t$ such that for every $\ell \in L'$ and for every $r \in R'$ we have $c_\ell \leq c_r$.

(S3) Let $c^* = \max\{c_p : \ p \in L'\}$.
The cut c^* divides the cake $\mathcal{C}$ into a left piece $\mathcal{C}_L$ and a right piece $\mathcal{C}_R$.

(S4) Every player p in L' chooses an integer x_p with $\lceil n/2 \rceil \leq x_p \leq n$.
Every player p in R' chooses an integer x_p with $0 \leq x_p \leq \lceil n/2 \rceil$.
Every player $p \notin L' \cup R'$ chooses an integer x_p with $0 \leq x_p \leq n$.

(S5) The players are divided into two non-empty groups L and R, such that
(i) $|L| \geq t$ and $|R| \geq t$,
(ii) $x_p \geq |L|$ holds for every player $p \in L$,
(iii) $x_p \leq |L|$ holds for every player $p \in R$.

(S6) The players in L recursively share the left piece $\mathcal{C}_L$.
The players in R recursively share the right piece $\mathcal{C}_R$.

Fig. 1. The protocol $P(t)$ for n players

2 The Protocol

In this section, we define a recursive protocol $P(t)$ that is based on an integer parameter $t \geq 1$. Without loss of generality we assume that the cake $\mathcal{C}$ is the unit interval $[0, 1]$, and that all pieces generated during the execution of the protocol are subintervals of $[0, 1]$. The steps (S0)–(S6) of protocol $P(t)$ are described in Figure 1. The protocol $P(t)$ is based on a divide-and-conquer approach that is similar to that of Even & Paz [3].

Let us start with some simple remarks on $P(t)$. It is irrelevant for our arguments and for our analysis, whether the cuts in step (S1) are done in parallel or sequentially, and whether one player knows or does not know about the cuts of the other players. The same holds for the selection of the numbers x_p in step (S4). If there are two or more feasible partitions $L' \cup R'$ in step (S2), then the protocol selects an arbitrary such partition.

Next, we will discuss the exact implementation of step (S5). Let $y_1 \geq y_2 \geq \cdots \geq y_n$ be an enumeration of the integers $x_1, \ldots, x_n$. Consider the function $g(j) := y_j - j$ for $1 \leq j \leq n$. Since the t integers of players in L' are all greater or equal to $\lceil n/2 \rceil$, we have $y_t \geq \lceil n/2 \rceil \geq t$. Hence, $g(t) \geq 0$. Since the t integers of players in R' are all less or equal to $\lceil n/2 \rceil$, we have $y_{n-t+1} \leq \lceil n/2 \rceil \leq n - t$. Hence, $g(n - t + 1) < 0$. We define the splitting index s as

$$s \;=\; \min\{j :\; g(j) \geq 0 \text{ and } g(j+1) < 0 \text{ and } t \leq j \leq n - t\}.$$

We define the set L of players in such a way that the two multi-sets $\{y_i :\; 1 \leq i \leq s\}$ and $\{x_p :\; p \in L\}$ are identical. Moreover, we define $R = \{1, \ldots, n\} - L$. Then $|L| = s \geq t$ and $|R| = n - s \geq t$. Since $y_s \geq s$, we have $x_p \geq |L|$ for every $p \in L$. Since $y_{s+1} < s + 1$, we have $x_p \leq |L|$ for every $p \in R$. Hence, the conditions (i)–(iii) of step (S5) are indeed satisfied by these groups L and R.

In the rest of this section, we prove an upper bound on the number of cuts in the protocol $P(t)$.

Lemma 2 *If the cake is divided among n players according to protocol $P(t)$, then the players altogether make at most $2t \cdot (n - 1)$ cuts.*

Proof. By induction on the number n of players. If $n \leq 2t - 1$ is small, protocol $P(t)$ becomes the Even & Paz protocol. Hence, there are at most $n \log_2(n) \leq 2t \cdot (n - 1)$ cuts. For $n \geq 2t$, there are $2t$ cuts made in step (S1). Moreover, by the inductive assumption there are at most $2t(|L| - 1)$ and at most $2t(|R| - 1)$ cuts made in the recursion in step (S6). Altogether, this yields at most $2t(|L| + |R| - 1) = 2t \cdot (n - 1)$ cuts. □

3 Proof of Fairness

In this section, we prove that the protocol $P(t)$ is $(1 - \frac{1}{t})$-fair.

Lemma 3 *Let $n \geq 2t$. Then every player p ($p = 1, \ldots, n$) can enforce that at the end of step (S4)*

$$x_p \;=\; \lceil (n - 1) \cdot \mu_p(\mathcal{C}_L)/\mu_p(\mathcal{C}) \rceil.$$

Proof. The statement trivially holds for the players $p = 2t + 1, \ldots, n$, since in step (S4) these players are free to choose x_p arbitrarily between 0 and n. Hence, consider a player p with $p = 1, \ldots, 2t$. We claim that a good strategy for player p is to make his cut c_p in step (S1) in such a way that $\mu_p([0, c_p]) = \mu_p(\mathcal{C}) \cdot \lceil n/2 \rceil / n$. We distinguish two cases.

In the first case, we assume that step (S2) puts player p into group L'. Then $c_p \leq c^*$ and $[0, c_p] \subseteq [0, c^*] = \mathcal{C}_L$. Hence,

$$\begin{aligned}(n - 1) \cdot \frac{\mu_p(\mathcal{C}_L)}{\mu_p(\mathcal{C})} &\geq (n - 1) \cdot \frac{1}{\mu_p(\mathcal{C})} \cdot \mu_p(\mathcal{C}) \cdot \frac{\lceil n/2 \rceil}{n} \\ &= (1 - \frac{1}{n})\lceil n/2 \rceil \quad > \quad \lceil n/2 \rceil - 1.\end{aligned}$$

Therefore, in this case the value $\lceil (n-1)\cdot \mu_p(\mathcal{C}_L)/\mu_p(\mathcal{C})\rceil$ is greater or equal to $\lceil n/2\rceil$, and indeed constitutes a feasible choice for a player p from L' in step (S4).

In the second case, we assume that step (S2) puts player p into group R'. This implies $c_p \le c^*$ and $\mathcal{C}_L = [0, c^*] \subseteq [0, c_p]$. Hence,

$$\begin{aligned}(n-1)\cdot\frac{\mu_p(\mathcal{C}_L)}{\mu_p(\mathcal{C})} &\le (n-1)\cdot\frac{1}{\mu_p(\mathcal{C})}\cdot\mu_p(\mathcal{C})\cdot\frac{\lceil n/2\rceil}{n}\\ &= (1-\frac{1}{n})\lceil n/2\rceil \quad\le\quad \lceil n/2\rceil.\end{aligned}$$

In this case the value $\lceil (n-1)\cdot \mu_p(\mathcal{C}_L)/\mu_p(\mathcal{C})\rceil$ is less or equal to $\lceil n/2\rceil$, and constitutes a feasible choice for any player from R' in step (S4). □

Lemma 4 *Let $n \ge 1$. Then every player p ($p = 1, \ldots, n$) can enforce to get at least a fraction* $\min\left\{\frac{1}{n},\ \frac{t-1}{t(n-1)}\right\}$ *of the cake $\mathcal{C}$.*

Proof. Player p behaves according to Lemma 3 and chooses $x_p = \lceil (n-1)\cdot \mu_p(\mathcal{C}_L)/\mu_p(\mathcal{C})\rceil$. We prove by induction on the number n of players that this ensures him a fraction $\min\left\{\frac{1}{n},\ \frac{t-1}{t(n-1)}\right\}$ of the cake $\mathcal{C}$.

For $n \le 2t-1$, the statement is trivial since p gets a fraction $1/n$ in step (S0). For the inductive step, we consider $n \ge 2t$ and we distinguish two cases. If in step (S6) the protocol assigns player p to the group L, then by properties (i) and (ii) from step (S5) we have $x_p \ge |L| \ge t$. By the inductive assumption, player p receives at least

$$\begin{aligned}\min\left\{\frac{1}{|L|},\ \frac{t-1}{t(|L|-1)}\right\}\mu_p(\mathcal{C}_L) &= \frac{t-1}{t(|L|-1)}\cdot\mu_p(\mathcal{C}_L)\\ &\ge \frac{t-1}{t(x_p-1)}\cdot\mu_p(\mathcal{C}_L)\\ &\ge \frac{t-1}{t}\cdot\frac{\mu_p(\mathcal{C})}{(n-1)\mu_p(\mathcal{C}_L)}\cdot\mu_p(\mathcal{C}_L)\\ &= \frac{t-1}{t(n-1)}\cdot\mu_p(\mathcal{C}).\end{aligned}$$

Here the first equation follows from $|L| \ge t$, and the first inequality follows from $x_p \ge |L|$. The second inequality holds since $x_p - 1 \le (n-1)\mu_p(\mathcal{C}_L)/\mu_p(\mathcal{C})$ by the choice of x_p. This completes the first case.

In the second case, we assume that step (S6) assigns player p to the group R. Then by properties (i) and (iii) from step (S5) we have $|R| \ge t$ and $|L| \ge x_p$. Therefore, $1/|R| \ge (t-1)/t(|R|-1)$. Then by the inductive assumption, player p receives at least

$$\begin{aligned}\frac{t-1}{t(|R|-1)}\cdot\mu_p(\mathcal{C}_R) &\ge \frac{t-1}{t(n-x_p-1)}\cdot\mu_p(\mathcal{C}_R)\\ &\ge \frac{t-1}{t(n-(n-1)\mu_p(\mathcal{C}_L)/\mu_p(\mathcal{C})-1)}\cdot(\mu_p(\mathcal{C})-\mu_p(\mathcal{C}_L))\\ &= \frac{t-1}{t(n-1)}\cdot\mu_p(\mathcal{C}).\end{aligned}$$

Here the first inequality follows from $x_p \leq |L| = n - |R|$. The second inequality holds since $x_p \geq (n-1)\mu_p(\mathcal{C}_L)/\mu_p(\mathcal{C})$ by the choice of x_p. This completes the second case, and also the inductive proof. □

Finally, let us prove Theorem 1. We use the protocol $P(t)$ with $t = \lceil 1/\varepsilon \rceil$. By Lemma 2, the total number of cuts is at most $2\lceil \frac{1}{\varepsilon} \rceil \cdot (n-1)$ and hence grows linearly in the number n of players. By Lemma 4, every player may enforce to get at least a fraction

$$\min\left\{\frac{1}{n}, \frac{t-1}{t(n-1)}\right\} \quad \geq \quad (1-\frac{1}{t}) \cdot \frac{1}{n} \quad \geq \quad (1-\varepsilon) \cdot \frac{1}{n}$$

of the cake $\mathcal{C}$. This completes the proof of Theorem 1.

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