# Privileged users in zero-error transmission over a noisy channel 

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#### Abstract

The $k$-th power of a graph $G$ is the graph whose vertex set is $V(G)^{k}$, where two distinct $k$ tuples are adjacent iff they are equal or adjacent in $G$ in each coordinate. The Shannon capacity of $G, c(G)$, is $\lim _{k \rightarrow \infty} \alpha\left(G^{k}\right)^{\frac{1}{k}}$, where $\alpha(G)$ denotes the independence number of $G$. When $G$ is the characteristic graph of a channel $\mathcal{C}, c(G)$ measures the effective alphabet size of $\mathcal{C}$ in a zero-error protocol. A sum of channels, $\mathcal{C}=\sum_{i} \mathcal{C}_{i}$, describes a setting when there are $t \geq 2$ senders, each with his own channel $\mathcal{C}_{i}$, and each letter in a word can be selected from either of the channels. This corresponds to a disjoint union of the characteristic graphs, $G=\sum_{i} G_{i}$. It is well known that $c(G) \geq \sum_{i} c\left(G_{i}\right)$, and in $\mathbb{1}$ it is shown that in fact $c(G)$ can be larger than any fixed power of the above sum.

We extend the ideas of $\mathbb{1}$ and show that for every $\mathcal{F}$, a family of subsets of $[t]$, it is possible to assign a channel $\mathcal{C}_{i}$ to each sender $i \in[t]$, such that the capacity of a group of senders $X \subset[t]$ is high iff $X$ contains some $F \in \mathcal{F}$. This corresponds to a case where only privileged subsets of senders are allowed to transmit in a high rate. For instance, as an analogue to secret sharing, it is possible to ensure that whenever at least $k$ senders combine their channels, they obtain a high capacity, however every group of $k-1$ senders has a low capacity (and yet is not totally denied of service). In the process, we obtain an explicit Ramsey construction of an edge-coloring of the complete graph on $n$ vertices by $t$ colors, where every induced subgraph on $\exp (\Omega(\sqrt{\log n \log \log n}))$ vertices contains all $t$ colors.


## 1 Introduction

A channel $\mathcal{C}$ on an input alphabet $V$ and an output alphabet $U$ maps each $x \in V$ to some $S(x) \subset U$, such that transmitting $x$ results in one of the letters of $S(x)$. The characteristic graph of the channel

[^0]$\mathcal{C}, G=G(\mathcal{C})$, has a vertex set $V$, and two vertices $x \neq y \in V$ are adjacent iff $S(x) \cap S(y) \neq \emptyset$, i.e., the corresponding input letters are confusable in the channel. Clearly, a maximum set of predefined letters which can be transmitted in $\mathcal{C}$ without possibility of error corresponds to a maximum independent set in the graph $G$, and has cardinality $\alpha(G)$ (the independence number of $G$ ).

The strong product of two graphs, $G_{1}=\left(V_{1}, E_{1}\right)$ and $G_{2}=\left(V_{2}, E_{2}\right)$ is the graph, $G_{1} \cdot G_{2}$, on the vertex set $V_{1} \times V_{2}$, where two distinct vertices $\left(u_{1}, u_{2}\right) \neq\left(v_{1}, v_{2}\right)$ are adjacent iff for all $i=1,2$, either $u_{i}=v_{i}$ or $u_{i} v_{i} \in E_{i}$. In other words, the pairs of vertices in both coordinates are either equal or adjacent. This product is associative and commutative, hence we can define $G^{k}$ to be the $k$-th power of $G$, where two vertices $\left(u_{1}, \ldots, u_{k}\right) \neq\left(v_{1}, \ldots, v_{k}\right)$ are adjacent iff for all $i=1, \ldots, k$, either $u_{i}=v_{i}$ or $u_{i} v_{i} \in E(G)$.

Note that if $I, J$ are independent sets of two graphs, $G, H$, then $I \times J$ is an independent set of $G \cdot H$. Therefore, $\alpha\left(G^{n+m}\right) \geq \alpha\left(G^{n}\right) \alpha\left(G^{m}\right)$ for every $m, n \geq 1$, and by Fekete's lemma (cf., e.g., [4], p. 85), the limit $\lim _{n \rightarrow \infty} \alpha\left(G^{n}\right)^{\frac{1}{n}}$ exists and equals $\sup _{n} \alpha\left(G^{n}\right)^{\frac{1}{n}}$. This parameter, introduced by Shannon in [5], is the Shannon capacity of $G$, denoted by $c(G)$.

When sending $k$-letter words in the channel $\mathcal{C}$, two words are confusable iff the pairs of letters in each of their $k$-coordinates are confusable. Thus, the maximal number of $k$-letter words which can be sent in $\mathcal{C}$ without possibility of error is precisely $\alpha\left(G^{k}\right)$, where $G=G(\mathcal{C})$. It follows that for sufficiently large values of $k$, the maximal number of $k$-letter words which can be sent without possibility of error is roughly $c(G)^{k}$. Hence, $c(G)$ represents the effective alphabet size of the channel in zero-error transmission.

The sum of two channels, $\mathcal{C}_{1}+\mathcal{C}_{2}$, describes the setting where each letter can be sent from either of the two channels, and letters from $\mathcal{C}_{1}$ cannot be confused with letters from $\mathcal{C}_{2}$. The characteristic graph in this case is the disjoint union $G_{1}+G_{2}$, where $G_{i}$ is the characteristic graph of $\mathcal{C}_{i}$. Shannon showed in [5] that $c\left(G_{1}+G_{2}\right) \geq c\left(G_{1}\right)+c\left(G_{2}\right)$ for every two graphs $G_{1}$ and $G_{2}$, and conjectured that in fact $c\left(G_{1}+G_{2}\right)=c\left(G_{1}\right)+c\left(G_{2}\right)$ for all $G_{1}$ and $G_{2}$. This was disproved in [1], where the author gives an explicit construction of two graphs $G_{1}, G_{2}$ with a capacity $c\left(G_{i}\right) \leq k$, satisfying $c\left(G_{1}+G_{2}\right) \geq k^{\Omega\left(\frac{\log k}{\log \log k}\right)}$.

We extend the ideas of [1] and show that it is possible to construct $t$ graphs, $G_{i}(i \in[t]=$ $\{1,2, \ldots, t\}$ ), such that for every subset $X \subseteq[t]$, the Shannon capacity of $\sum_{i \in X} G_{i}$ is high iff $X$ contains some subset of a predefined family $\mathcal{F}$ of subsets of $[t]$. This corresponds to assigning $t$ channels to $t$ senders, such that designated groups of senders $F \in \mathcal{F}$ can obtain a high capacity by combining their channels $\left(\sum_{i \in F} \mathcal{C}_{i}\right)$, and yet every group of senders $X \subseteq[t]$ not containing any $F \in \mathcal{F}$ has a low capacity. In particular, a choice of $\mathcal{F}=\{F \subset[t]:|F|=k\}$ implies that every set $X$ of senders has a high Shannon capacity of $\sum_{i \in X} \mathcal{C}_{i}$ if $|X| \geq k$, and a low capacity otherwise. The following theorem, proved in Section 2 formalizes the claims above:

Theorem 1.1. Let $T=\{1, \ldots, t\}$ for some fixed $t \geq 2$, and let $\mathcal{F}$ be a family of subsets of $T$. For every (large) $n$ it is possible to construct graphs $G_{i}, i \in T$, each on $n$ vertices, such that the
following two statements hold for all $X \subseteq T$ :

1. If $X$ contains some $F \in \mathcal{F}$, then $c\left(\sum_{i \in X} G_{i}\right) \geq n^{1 /|F|} \geq n^{1 / t}$.
2. If $X$ does not contain any $F \in \mathcal{F}$, then

$$
c\left(\sum_{i \in X} G_{i}\right) \leq \mathrm{e}^{(1+o(1)) \sqrt{2 \log n \log \log n}}
$$

where the $o(1)$-term tends to 0 as $n \rightarrow \infty$.

As a by-product, we obtain the following Ramsey construction, where instead of forbidding monochromatic subgraphs, we require "rainbow" subgraphs (containing all the colors used for the edge-coloring). This is stated by the next proposition, which is proved in Section 3,
Proposition 1.2. For every (large) $n$ and $t \leq \sqrt{\frac{2 \log n}{(\log \log n)^{3}}}$ there is an explicit $t$-edge-coloring of the complete graph on $n$ vertices, such that every induced subgraph on

$$
\mathrm{e}^{(1+o(1)) \sqrt{8 \log n \log \log n}}
$$

vertices contains all $t$ colors.

This extends the construction of Frankl and Wilson [2] that deals with the case $t=2$ (using a slightly different construction).

## 2 Graphs with high capacities for unions of predefined subsets

The upper bound on the capacities of subsets not containing any $F \in \mathcal{F}$ relies on the algebraic bound for the Shannon capacity using representations by polynomials, proved in [1]. See also Haemers [3] for a related approach.

Definition. Let $\mathbb{K}$ be a field, and let $\mathcal{H}$ be a linear subspace of polynomials in $r$ variables over $\mathbb{K}$. A representation of a graph $G=(V, E)$ over $\mathcal{H}$ is an assignment of a polynomial $f_{v} \in \mathcal{H}$ and a value $c_{v} \in \mathbb{K}^{r}$ to every $v \in V$, such that the following holds: for every $v \in V, f_{v}\left(c_{v}\right) \neq 0$, and for every $u \neq v \in V$ such that $u v \notin E, f_{u}\left(c_{v}\right)=0$.

Theorem 2.1 ([1]). Let $G=(V, E)$ be a graph and let $\mathcal{H}$ be a space of polynomials in $r$ variables over a field $\mathbb{K}$. If $G$ has a representation over $\mathcal{H}$ then $c(G) \leq \operatorname{dim}(\mathcal{H})$.

We need the following simple lemma:
Lemma 2.2. Let $T=[t]$ for $t \geq 1$, and let $\mathcal{F}$ be a family of subsets of $T$. There exist sets $A_{1}, A_{2}, \ldots, A_{t}$ such that for every $X \subseteq T$ :

$$
X \text { does not contain any } F \in \mathcal{F} \Longleftrightarrow \bigcap_{i \in X} A_{i} \neq \emptyset
$$

Furthermore, $\left|\bigcup_{i=1}^{t} A_{i}\right| \leq\binom{ t}{\lfloor t / 2\rfloor}$.

Proof of lemma. Let $\mathcal{Y}$ denote the family of all maximal sets $Y$ such that $Y$ does not contain any $F \in \mathcal{F}$. Assign a unique element $p_{Y}$ to every $Y \in \mathcal{Y}$, and define:

$$
\begin{equation*}
A_{i}=\left\{p_{Y}: i \in Y, Y \in \mathcal{Y}\right\} \tag{1}
\end{equation*}
$$

Let $X \subseteq T$, and note that (11) implies that $\bigcap_{i \in X} A_{i}=\left\{p_{Y}: X \subseteq Y\right\}$. Thus, if $X$ does not contain any $F \in \mathcal{F}$, then $X \subseteq Y$ for some $Y \in \mathcal{Y}$, and hence $p_{Y} \in \bigcap_{i \in X} A_{i}$. Otherwise, $X$ contains some $F \in \mathcal{F}$ and hence is not a subset of any $Y \in \mathcal{Y}$, implying that $\bigcap_{i \in X} A_{i}=\emptyset$.

Finally, observe that $\mathcal{Y}$ is an anti-chain and that $\left|\bigcup_{i=1}^{t} A_{i}\right| \leq|\mathcal{Y}|$, hence the bound on $\left|\bigcup_{i=1}^{t} A_{i}\right|$ follows from Sperner's Theorem [6].

Proof of Theorem 1.1. Let $p$ be a large prime, and let $\left\{p_{Y}: Y \in \mathcal{Y}\right\}$ be the first $|\mathcal{Y}|$ primes succeeding $p$. Define $s=p^{2}$ and $r=p^{3}$, and note that, as $t$ and hence $|\mathcal{Y}|$ are fixed, by well-known results about the distribution of prime numbers, $p_{Y}=(1+o(1)) p<s$ for all $Y$, where the $o(1)$-term tends to 0 as $p \rightarrow \infty$.

The graph $G_{i}=\left(V_{i}, E_{i}\right)$ is defined as follows: its vertex set $V_{i}$ consists of all $\binom{r}{s}$ possible $s$-element subsets of $[r]$, and for every $A \neq B \in V_{i}$ :

$$
\begin{equation*}
(A, B) \in E_{i} \Longleftrightarrow|A \cap B| \equiv s \quad\left(\bmod p_{Y}\right) \text { for some } p_{Y} \in A_{i} \tag{2}
\end{equation*}
$$

Let $X \subseteq T$. If $X$ does not contain any $F \in \mathcal{F}$, then, by Lemma $2.2 \bigcap_{i \in X} A_{i} \neq \emptyset$, hence there exists some $q$ such that $q \in A_{i}$ for every $i \in X$. Therefore, for every $i \in X$, if $A, B$ are disconnected in $G_{i}$, then $|A \cap B| \not \equiv s(\bmod q)$. It follows that the graph $\sum_{i \in X} G_{i}$ has a representation over a subspace of the multi-linear polynomials in $|X| r$ variables over $\mathbb{Z}_{q}$ with a degree smaller than $q$. To see this, take the variables $x_{j}^{(i)}, i=1, \ldots,|X|, j=1, \ldots, r$, and assign the following polynomial to each vertex $A \in V_{i}$ :

$$
f_{A}(\bar{x})=\prod_{u \neq s}\left(u-\sum_{j \in A} x_{j}^{(i)}\right) .
$$

The assignment $c_{A}$ is defined as follows: $x_{j}^{\left(i^{\prime}\right)}=1$ if $i^{\prime}=i$ and $j \in A$, otherwise $x_{j}^{\left(i^{\prime}\right)}=0$. As every assignment $c_{A^{\prime}}$ gives values in $\{0,1\}$ to all $x_{j}^{(i)}$, it is possible to reduce every $f_{A}$ modulo the polynomials $\left(x_{j}^{(i)}\right)^{2}-x_{j}^{(i)}$ for all $i$ and $j$, and obtain multi-linear polynomials, equivalent on all the assignments $c_{A^{\prime}}$.

The following holds for all $A \in V_{i}$ :

$$
f_{A}\left(c_{A}\right)=\prod_{u \neq s}(u-s) \not \equiv 0 \quad(\bmod q),
$$

and for every $B \neq A$ :

$$
\begin{aligned}
B \in V_{i},(A, B) \notin E_{i} & \Longrightarrow f_{A}\left(c_{B}\right)=\prod_{u \neq s}(u-|A \cap B|) \equiv 0 \quad(\bmod q), \\
B \notin V_{i} & \Longrightarrow f_{A}\left(c_{B}\right)=\prod_{u \neq s} u \equiv 0 \quad(\bmod q),
\end{aligned}
$$

where the last equality is by the fact that $s \not \equiv 0(\bmod q)$, as $s=p^{2}$ and $p<q$. As the polynomials $f_{A}$ lie in the direct sum of $|X|$ copies of the space of multi-linear polynomials in $r$ variables of degree less than $q$, it follows from Theorem [2.1] that the Shannon capacity of $\sum_{i \in X} G_{i}$ is at most:

$$
|X| \sum_{i=0}^{q-1}\binom{r}{i} \leq t \sum_{i=0}^{q-1}\binom{r}{i}<t\binom{r}{q} .
$$

Recalling that $q=(1+o(1)) p$ and writing $t\binom{r}{q}$ in terms of $n=\binom{r}{s}$ gives the required upper bound on $c\left(\sum_{i \in X} G_{i}\right)$.

Assume now that $X$ contains some $F \in \mathcal{F}, F=\left\{i_{1}, \ldots, i_{|F|}\right\}$. We claim that the following set is an independent set in $\left(\sum_{i \in X} G_{i}\right)^{|F|}$ :

$$
\left\{\left(A^{\left(i_{1}\right)}, A^{\left(i_{2}\right)}, \ldots, A^{\left(i_{|F|}\right)}\right): A \subseteq[r],|A|=s\right\},
$$

where $A^{\left(i_{j}\right)}$ is the vertex corresponding to $A$ in $V_{i_{j}}$. Indeed, if $(A, A, \ldots, A)$ and $(B, B, \ldots, B)$ are adjacent, then for every $i \in F,|A \cap B| \equiv s\left(\bmod p_{Y}\right)$ for some $p_{Y} \in A_{i}$. However, $\bigcap_{i \in F} A_{i}=\emptyset$, hence there exist $p_{Y} \neq p_{Y}^{\prime}$ such that $|A \cap B|$ is equivalent both to $s\left(\bmod p_{Y}\right)$ and to $s\left(\bmod p_{Y}^{\prime}\right)$. By the Chinese Remainder Lemma, it follows that $|A \cap B|=s$ (as $|A \cap B|<p_{Y} p_{Y}^{\prime}$ ), thus $A=B$. Therefore, the Shannon capacity of $\sum_{i \in X} G_{i}$ is at least $\binom{r}{s}^{1 /|F|}=n^{1 /|F|}$.

## 3 Explicit construction for rainbow Ramsey graphs

Proof of Proposition 1.2, Let $p$ be a large prime, and let $p_{1}<\ldots<p_{t}$ denote the first $t$ primes succeeding $p$. We define $r, s$ as in the proof of Theorem $1.1 s=p^{2}, r=p^{3}$, and consider the complete graph on $n$ vertices, $K_{n}$, where $n=\binom{r}{s}$, and each vertex corresponds to an $s$-element subset of $[r]$. The fact that $t \leq \sqrt{\frac{2 \log n}{(\log \log n)^{3}}}$ implies that $t \leq\left(\frac{1}{2}+o(1)\right) \frac{p}{\log p}$, and hence, by the distribution of prime numbers, $p_{t}<2 p$ (with room to spare) for a sufficiently large value of $p$.

We define an edge-coloring $\gamma$ of $K_{n}$ by $t$ colors in the following manner: for every $A, B \in V$, $\gamma(A, B)=i$ if $|A \cap B| \equiv s\left(\bmod p_{i}\right)$ for some $i \in[t]$, and is arbitrary otherwise. Note that for every $i \neq j \in\{1, \ldots, t\}, s<p_{i} p_{j}$. Hence, if $|A \cap B| \equiv s\left(\bmod p_{i}\right)$ and $|A \cap B| \equiv s\left(\bmod p_{j}\right)$ for such $i$ and $j$, then by the Chinese Remainder Lemma, $|A \cap B|=s$, and in particular, $A=B$. Therefore, the coloring $\gamma$ is well-defined.

It remains to show that every large induced subgraph of $K_{n}$ has all $t$ colors according to $\gamma$. Indeed, this follows from the same consideration used in the proof of Theorem 1.1. To see this, let $G_{i}$ denote the spanning subgraph of $K_{n}$ whose edge set consists of all $(A, B)$ such that $\gamma(A, B) \neq i$. Each such pair satisfies $|A \cap B| \not \equiv s\left(\bmod p_{i}\right)$, hence $G_{i}$ has a representation over the multi-linear polynomials in $r$ variables over $\mathbb{Z}_{p_{i}}$ with a degree smaller than $p_{i}$ (define $f_{A}\left(x_{1}, \ldots, x_{r}\right.$ ) as is in the proof of Theorem 1.1 and take $c_{A}$ to be the characteristic vector of $A$ ). Thus, $c\left(G_{i}\right)<\binom{r}{p_{i}}$, and in particular, $\alpha\left(G_{i}\right)<\binom{r}{p_{i}}$. This ensures that every induced subgraph on at least $\binom{r}{p_{i}} \leq\binom{ r}{2 p}$ vertices contains an $i$-colored edge, and the result follows.

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