# $k$-noncrossing and $k$-nonnesting graphs and fillings of Ferrers diagrams 

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#### Abstract

We give a correspondence between graphs with a given degree sequence and fillings of Ferrers diagrams by nonnegative integers with prescribed row and column sums. In this setting, $k$-crossings and $k$-nestings of the graph become occurrences of the identity and the antiidentity matrices in the filling. We use this to show the equality of the numbers of $k$-noncrossing and $k$-nonnesting graphs with a given degree sequence. This generalizes the analogous result for matchings and partition graphs of Chen, Deng, Du, Stanley, and Yan, and extends results of Klazar to $k>2$. Moreover, this correspondence reinforces the links recently discovered by Krattenthaler between fillings of diagrams and the results of Chen et al.


## 1 Introduction

Let $G$ be a graph on $[n]$; unless otherwise stated, we allow multiple edges and isolated vertices, but no loops. Two edges $\{i, j\}$ and $\{k, l\}$ are a crossing if $i<k<j<l$ and they are a nesting if $i<k<l<j$. If we draw the vertices of $G$ on a line and represent the corresponding edges by arcs above the line, crossings and nestings have the obvious geometric meaning. A graph without crossings (respectively, nestings) is called noncrossing (resp., nonnesting). Klazar 10 proves the equality between the numbers of noncrossing and nonnesting simple graphs, counted by order, and also between the numbers of noncrossing and nonnesting graphs without isolated vertices, counted by size. The purpose of this paper is to study analogous results for sets of $k$ pairwise crossing and $k$ pairwise nested edges.

A $k$-crossing is a set of $k$ edges every two of them being a crossing, that is, edges $\left\{i_{1}, j_{1}\right\}, \ldots,\left\{i_{k}, j_{k}\right\}$ such that $i_{1}<i_{2}<\cdots<i_{k}<j_{1}<\cdots<j_{k}$. A $k$-nesting is a set of $k$ edges pairwise nested, that is, $\left\{i_{1}, j_{1}\right\}, \ldots,\left\{i_{k}, j_{k}\right\}$ such that $i_{1}<i_{2}<\cdots<i_{k}<$ $j_{k}<\cdots<j_{1}$. A graph with no $k$-crossing is called $k$-noncrossing and a graph with no $k$-nesting is called $k$-nonnesting. The largest $k$ for which a graph $G$ has a $k$-crossing (respectively, a $k$-nesting) is denoted $\operatorname{cross}(G)$ (resp., nest $(G)$ ). The aim of this paper is to show that the number of $k$-noncrossing graphs equals the number of $k$-nonnesting graphs, counted by order, size, and degree sequences. This problem was originally posed

[^0]by Martin Klazar and we learned of it at the Homonolo 2005 workshop [2]; the case where the number of vertices of the graph is $2 k+1$ was proved by A. Pór (unpublished). Our main result (Theorem (3.3) states that the numbers of $k$-noncrossing and $k$-nonnesting graphs with a given degree sequence are the same.

Chen et al. 4] prove the equality of the numbers of $k$-noncrossing and $k$-nonnesting graphs for two subclasses of graphs, namely for perfect matchings and for partition graphs, also counted by degree sequences (under a different but equivalent terminology). A perfect matching is a graph where each vertex has degree one, and a partition graph is a graph that is a disjoint union of monotone paths, that is, where each vertex has at most one edge to its right and at most one to its left. The latter correspond in a natural way to set partitions, hence the result can be stated in terms of these. The paper [4] also contains other identities and enumerative results on $k$-noncrossing and $k$ nonnesting matchings and partitions. Krattenthaler [12] deduces most of these from his more general results on fillings of Ferrers diagrams. In this paper we also use fillings of diagrams to prove results about graphs. The difference is that whereas in [12], and also in [7], the results about graphs follow from general theorems by restricting the shape of the diagram, here we show that the results about graphs are in fact equivalent to those about fillings with arbitrary shapes.

The main idea is to encode graphs by fillings of Ferrers diagrams in such a way that $k$-crossings and $k$-nestings are easy to recognize. A $k$-noncrossing ( $k$-nonnesting) graph becomes a filling of a diagram that avoids the identity (antiidentity) matrix of order $k$, and the degree sequence of the graph can be recovered from the shape of the diagram and the row and column sums of the filling. Then proving that there are as many $k$-noncrossing as $k$-nonnesting graphs is equivalent to showing that the numbers of fillings avoiding these two matrices are the same. This idea generalizes easily to other subgraphs in addition to crossing and nestings, and allows us to show that the study of fillings of Ferrers diagrams with forbidden configurations is equivalent to the study of graphs avoiding certain subgraphs, in the sense defined in Section 3

The structure of the paper is as follows. In Section we show that the equality of the numbers of $k$-noncrossing and $k$-nonnesting graphs counted by size and order is already in the literature, although not explicitly stated in this form. We introduce some notation on pattern avoiding fillings of Ferrers diagrams and we rephrase results of Krattenthaler [12] and Jonsson and Welker [8 in terms of $k$-noncrossing and $k$ nonnesting graphs. Section 3 introduces a new correspondence between graphs and fillings of diagrams that keeps track of degree sequences. Then we discuss why, from the perspective of pattern avoiding, graphs and fillings of diagrams are equivalent objects. In particular, showing that the number of $k$-noncrossing graphs with a fixed degree sequence equals the number of such $k$-nonnesting graphs is equivalent to proving a result on fillings of diagrams with restrictions on the row and column sums. Our proof is an adaptation of the one in 1 to allow arbitrary entries in the filling, and this is the content of Section (4. We conclude with some remarks and open questions.

## 2 Fillings of diagrams

We start by setting some notation on fillings of Ferrers diagrams. Let $\lambda=\left(\lambda_{1}, \lambda_{2}, \ldots, \lambda_{k}\right)$ be an integer partition. The Ferrers diagram of shape $\lambda$ (or simply a diagram) is the arrangement of square cells, left-justified and from top to bottom, having $\lambda_{i}$ cells in row $i$, for $i$ with $1 \leq i \leq k$. For a Ferrers diagram $T$ of shape $\lambda$ with rows indexed from top to bottom and columns from left to right, a filling $L$ of $T$ consists of assigning a nonnegative integer to each cell of the diagram. We say that a cell is empty if it has been assigned the integer 0 . Let $M$ be an $s \times t 0-1$ matrix. We say that the filling contains $M$ if there is a selection of rows $\left(r_{1}, \ldots, r_{s}\right)$ and columns $\left(c_{1}, \ldots, c_{t}\right)$ of $T$ such that if $M_{i, j}=1$ then the cell $\left(r_{i}, c_{j}\right)$ of $T$ is nonempty and moreover the cell $\left(r_{s}, c_{t}\right)$ is in the diagram (in other words, we require that the matrix $M$ is fully contained in $T$ ). We say that the filling avoids $M$ if there is no such selection of rows and columns. If a filling $L$ contains $M$, by an occurrence of $M$ we mean the set of cells of $T$ that correspond to the 1's in $M$. We are mainly concerned about diagrams avoiding the identity matrix $I_{t}$ and the antiidentity matrix $J_{t}$; the latter is the matrix with 1 's in the main antidiagonal and 0 's elsewhere. As an example of these concepts, Figure $\mathbb{1}$ shows a filling of a diagram of shape $(7,6,5,4,3,2,1)$ that contains the matrices $I_{3}$ and $J_{2}$ but avoids $J_{3}$. (For clarity, we omit the zeros corresponding to the empty cells.)


Figure 1: Left: a filling of a diagram that contains $I_{3}$ and $J_{2}$ but avoids $J_{3}$. Right: the graph determined by the filling has a 3 -nesting and several 2 -crossings, but no 3 -crossing.

Studying fillings of diagrams avoiding matrices is a natural generalization of pattern avoiding permutations, as explained in [15. We explore two types of connections between graphs and fillings of diagrams. The first one is straightforward, being essentially the adjacency matrix, and it has been used in [7, [12] to derive results on $k$-noncrossing maximal graphs and $k$-noncrossing and $k$-nonnesting matchings and partitions.

Suppose $G$ is a graph on $[n]$ and consider a diagram $\Delta$ of shape $(n-1, n-2, \ldots, 2,1)$. Then if there are $d \geq 0$ edges joining vertices $i$ and $j$, with $i<j$, fill the cell of column $i$ and row $n-j+1$ with $d$. Let this filling of the diagram be called $\Delta(G)$. Obviously the sum of the entries of $\Delta(G)$ is the number of edges of $G$ and the number of vertices is just one plus the number of rows of $\Delta$. If $G$ is a simple graph, then $\Delta(G)$ is a $0-1$ filling. If the edges $\left\{i_{1}, j_{1}\right\}, \ldots,\left\{i_{k}, j_{k}\right\}$ are a $k$-nesting of $G$, then $\Delta(G)$ contains the $k \times k$ identity matrix $I_{k}$ in columns $i_{1}, \ldots, i_{k}$ and rows $n-j_{1}+1, \ldots, n-j_{k}+1$. Similarly, if $G$ contains a $k$-crossing, then $\Delta(G)$ contains the antiidentity matrix $J_{k}$ (the condition
$i_{k}<j_{1}$ guarantees that the matrix is indeed contained in the diagram).
Krattenthaler [12] derives many of the results of Chen et al. 44 for matchings and partitions by specializing to $\Delta(G)$ his results on fillings of diagrams avoiding large identity or antiidentity matrices. His Theorem 13 gives a generalization to arbitrary graphs which is implicitly included in the remark after it; we explicitly state his result here (see also the comment after Theorem 3.3 in the next section). The following is a weaker version of [12, Theorem 13]

Theorem 2.1 For any diagram $T$ and any integer $m$, consider fillings of $T$ with nonnegative integers adding up to $m$. Then for each $k>1$, the number of such fillings that do not contain the identity matrix $I_{k}$ equals the number of fillings that do not contain the antiidentity matrix $J_{k}$.

By restricting to $T=\Delta$ we immediately get the following.
Corollary 2.2 The number of $k$-noncrossing graphs with $n$ vertices and $m$ edges equals the number of $k$-nonnesting such graphs.

Actually, from the statement of [12, Theorem 13] one gets a stronger result. For this we need to introduce weak $k$-crossings and weak $k$-nestings. The edges $\left\{i_{1}, j_{1}\right\}, \ldots,\left\{i_{k}, j_{k}\right\}$ are a weak $k$-crossing if $i_{1} \leq i_{2} \leq \cdots \leq i_{k}<j_{1} \leq \cdots \leq j_{k}$; similarly, they are a weak $k$-nesting if $i_{1} \leq i_{2} \leq \cdots \leq i_{k}<j_{k} \leq \cdots \leq j_{1}$. Let cross* $^{*}(G)$ (respectively, nest* $\left.(G)\right)$ be the largest $k$ for which $G$ has a weak $k$-crossing (resp., weak $k$-nesting). Then the following is a corollary of the full version of [12, Theorem 13].

Corollary 2.3 The number of graphs with $n$ vertices and $m$ edges with $\operatorname{cross}(G)=r$ and nest* $(G)=s$ equals the number of such graphs with $\operatorname{cross}^{*}(G)=s$ and nest $(G)=r$.

Ideally, one would like to have an analogous result proving the symmetry of the distribution of $\operatorname{cross}(G)$ and nest $(G)$ for all graphs. This is known to be true for matchings and partition graphs [4. Theorem 1 and Corollary 4]; actually, for these graphs weak crossings (respectively, weak nestings) are the same as crossings (resp., nestings). For simple graphs, the result would follow if Problem 2 in [12] has a positive answer for the diagram $\Delta$.

The bijection used to prove [12, Theorem 13] does not preserve the values of the entries of the filling, so we cannot deduce from it the corresponding result for simple graphs. However, this follows from a result of Jonsson and Welker. They deal with fillings not of diagrams, but of stack polyominoes. A stack polyomino consists of taking a diagram, reflecting it through the vertical axis, and gluing it to another (unreflected) diagram. The content of a stack polyomino is the multiset of the lengths of its columns. The definitions of fillings and containment of matrices in stack polyominoes are analogous to those for diagrams. The following is Corollary 6.5 of [8]. (The particular case where $m$ below is maximal was proved in [7].)

Theorem 2.4 The number of $0-1$ fillings of a stack polyomino with $m$ nonzero entries that avoid the matrix $I_{k}$ depends only on the content of the polyomino and not on the ordering of the columns.

By a simple reflection argument we get the following for the triangular diagram $\Delta$ of shape ( $n-1, n-2, \ldots, 2,1$ ): the number of $0-1$ fillings of $\Delta$ with $m$ non-zero entries and that avoid the matrix $I_{k}$ is the same as those that avoid the matrix $J_{k}$. Hence we have the following in terms of graphs.

Corollary 2.5 The number of $k$-noncrossing simple graphs on $n$ vertices and $m$ edges equals the number of such $k$-nonnesting simple graphs.

In the next section we deal with graphs with a fixed degree sequence. For this we need to consider diagrams of arbitrary shapes, since the correspondence between graphs and fillings is no longer restricted to the triangular diagram $\Delta$.

## 3 Degree sequences and fillings with prescribed row and column sums

The left-right degree sequence of a graph on $[n]$ is the sequence $\left(\left(l_{i}, r_{i}\right)\right)_{1 \leq i \leq n}$, where $l_{i}$ (resp., $r_{i}$ ) is the left (resp., right) degree of vertex $i$; by the left (resp., right) degree of $i$ we mean the number of edges that join $i$ to a vertex $j$ with $j<i$ (resp., $j>i$ ). Obviously $l_{i}+r_{i}$ is the degree of vertex $i$ (loops are not allowed). For instance, if $r_{i} \leq 1$ and $l_{i} \leq 1$ for all $i$, then the graph is either a matching or a partition graph, perhaps with some isolated vertices. If a graph $G$ has $D$ as its left-right degree sequence, we say that $G$ is a graph on $D$. A useful way of thinking of left-right degree sequences is drawing for each vertex $i, l_{i}$ half-edges going left and $r_{i}$ half-edges going right. Then a graph is just a way of matching these half-edges; recall that we allow multiple edges. For completeness we mention here that a sequence $\left(\left(l_{i}, r_{i}\right)\right)_{1 \leq i \leq n}$ is the left-right degree sequence of some graph on $[n]$ if and only if

$$
\begin{equation*}
\sum_{i=1}^{n} l_{i}=\sum_{i=1}^{n} r_{i} \text { and } \sum_{i=1}^{k} l_{i} \leq \sum_{i=1}^{k-1} r_{i}, \quad \forall k \in[n] . \tag{1}
\end{equation*}
$$

This and the next section are devoted to proving that for each left-right degree sequence $D$ there are as many $k$-noncrossing graphs on $D$ as $k$-nonnesting. We stress that the fact that we allow multiple edges is essential, since if we restrict to simple graphs the result does not hold. For instance, one can check that there is one simple nonnesting graph with left-right degree sequence $(0,2),(0,2),(1,1),(2,0),(2,0)$, but no such noncrossing simple graph. However, it turns out that there is a bijection between $k$ noncrossing and $k$-nonnesting simple graphs that preserves left degrees (or right degrees, but not both simultaneously). This follows from the following result of Rubey 14 , Theorem 4.2] applied to the filling $\Delta(G)$ by noting that the sum of the entries in row $n-j+1$ of $\Delta(G)$ corresponds to the left degree of vertex $j$. Rubey's result is for moon polyominoes, but we state the version for stack polyominoes. (A weaker version of this result was proved by Jonsson [7] Corollary 26].)

Theorem 3.1 For any stack polyomino $\Lambda$ with $s$ rows and for any sequence $\left(d_{1}, \ldots, d_{s}\right)$ of nonnegative integers, the number of $0-1$ fillings of $\Lambda$ that avoid $I_{k}$ and have $d_{i}$ nonzero entries in row $i$ depends only on the content of $\Lambda$ and not on the ordering of the columns.

By the same reflection argument as at the end of Section 2 we obtain the following corollary.

Corollary 3.2 Let $\left(l_{2}, \ldots, l_{n}\right)$ be a sequence of nonnegative integers. Then the number of $k$-noncrossing simple graphs on $[n]$ with vertex $i$ having left degree $l_{i}$ for $2 \leq i \leq n$ is the same as the number of such $k$-nonnesting simple graphs.

The main result of this paper says that by allowing multiple edges we can simultaneously fix left and right degrees.

Theorem 3.3 For any left-right degree sequence $D$, the number of $k$-noncrossing graphs on $D$ equals the number of $k$-nonnesting graphs on $D$.

This result generalizes to arbitrary graphs some of the results of 4], which are only for partition graphs but also taking into account degree sequences (with different terminology). To approach Theorem 3.3 we could use again the filling $\Delta(G)$ of the previous section and fix the sums of the entries in each row and column. In this setting Theorem [3.3] is again implicitly included in the remark after [12] Theorem 13] by keeping track of the changes in the partitions involved in the proof of that theorem. (I am grateful to Christian Krattenthaler for this observation.) However, our approach consists of encoding graphs not by the triangular diagram $\Delta$ but by an arbitrary diagram whose shape depends on the degree sequence. By doing this we actually show that not only results on $k$-noncrossing and $k$-nonnesting graphs can be deduced from results on fillings of Ferrers diagrams avoiding $I_{k}$ and $J_{k}$, but that actually these two families of results are completely equivalent. Moreover, we have an analogous assertion for arbitrary matrices (see Theorem 3.7).

We start with an easy lemma that follows immediately from the fact that the edges in a $k$-crossing or a $k$-nesting must be vertex-disjoint.

Lemma 3.4 The number of $k$-noncrossing (resp., $k$-nonnesting) graphs with left-right degree sequence

$$
\left(l_{1}, r_{1}\right), \ldots,\left(l_{n}, r_{n}\right)
$$

is the same as that of those with left-right degree sequence

$$
\left(l_{1}, r_{1}\right), \ldots,\left(l_{i-1}, r_{i-1}\right),\left(l_{i}, 0\right),\left(0, r_{i}\right),\left(l_{i+1}, r_{i+1}\right), \ldots,\left(l_{n}, r_{n}\right)
$$

for any $i$ with $1 \leq i \leq n$.
Hence it is enough to prove Theorem 3.3 for left-right degree sequences whose elements $\left(l_{i}, r_{i}\right)$ are such that either $l_{i}$ or $r_{i}$ is 0 . We call these graphs left-right graphs; note though that we do not require that the degrees of the vertices alternate between right
and left. The case where both left and right degrees are 0 corresponds to an isolated vertex.

We now describe a bijection between left-right graphs and fillings of Ferrers diagrams of arbitrary shape; this bijection has the property that the left-right degree sequence of the graph can be recovered from the shape and filling of the diagram. Let $G$ be a left-right graph. If the degree of vertex $i$ is of the form $\left(0, r_{i}\right)$ we say that $i$ is opening, and if it is of the form $\left(l_{i}, 0\right)$ we say that $i$ is closing. An isolated vertex is both opening and closing. Let $i_{1}, \ldots, i_{c}$ be the closing vertices of $G$ and let $j_{1}, \ldots, j_{o}$ be the opening ones. For each closing vertex $i$, let $p(i)$ be the number of vertices $j$ with $j<i$ that are opening. We consider a diagram $T(G)$ of shape $\left(p\left(i_{c}\right), p\left(i_{c-1}\right), \ldots, p\left(i_{1}\right)\right.$ ), and if there are $d$ edges going from the opening vertex $j_{s}$ to the closing vertex $i_{r}$, we fill the cell in column $s$ and row $c-r+1$ with the integer $d$ (see Figure (2). Thus graphs with left degrees $l_{1}, \ldots, l_{c}$ and right degrees $r_{1}, \ldots, r_{o}$ correspond to fillings of this diagram with nonnegative entries such that the sum of the entries in row $i$ is $l_{i}$ and the sum of the entries in column $j$ is $r_{j}$. Conversely, any filling of a diagram arises in this way. Indeed, given a filling $L$ of a diagram $T$, the shape of $T$ gives the ordering of the opening and closing vertices of the graph, the row and column sums give the left and right degrees (it is easy to see that they must satisfy equation (11)), and the entries of the filling give the edges of the graph. Given a graph $G$, we denote by $L(G)$ the filling of $T(G)$ corresponding to $G$. Similarly, given a filling $L$ of a diagram, we denote by $G(L)$ the left-right graph corresponding to this filling.

In this setting, it is immediate to check that again $k$-crossings of $G$ correspond to occurrences of $I_{k}$ in $L(G)$ and $k$-nestings to occurrences of $J_{k}$.


Figure 2: A filling $L$ of a diagram with row sums 4, 2, 3, 2 and column sums 2, 2, 3, 2, 2, and the corresponding graph $G(L)$.

By a diagram with prescribed row and column sums we mean a diagram and two sequences $\left(\rho_{i}\right)$ and $\left(\gamma_{j}\right)$ of nonnegative integers such that the only fillings allowed for this diagram are those where the row and column sums are given by the sequences ( $\rho_{i}$ ) and $\left(\gamma_{j}\right)$. Given two matrices $M$ and $N$, we say that they are equirestrictive if for all diagrams $T$ with prescribed row and column sums, the number of fillings of $T$ that avoid $M$ equals the number of fillings of $T$ that avoid $N$. With this notation, Theorem 3.3] is an immediate consequence of the following result, the proof of which is the content of the next section.

Theorem 3.5 The identity matrix $I_{k}$ and the antiidentity matrix $J_{k}$ are equirestrictive.
Before moving to the proof of Theorem [3.5] let us make some remarks and point out
some consequences of the proof. We start by further exploring the bijection between left-right graphs and fillings of diagrams.

Let $G$ and $H$ be graphs on $[n]$ and $[h]$, respectively, with $h \leq n$. For the rest of this section we assume that $H$ is simple (but $G$ can have multiple edges as usual). We say that $G$ contains $H$ if there is an order-preserving injection $\sigma:[h] \rightarrow[n]$ such that if $\{i, j\}$ is an edge of $H$ then $\{\sigma(i), \sigma(j)\}$ is an edge of $G$. For instance, a $k$-noncrossing graph is a graph that does not contain the graph on [2k] with edges $\{1, k+1\},\{2, k+2\}, \ldots,\{k, 2 k\}$.

A $0-1$ matrix $M$ with $s$ rows and $t$ columns can also be viewed as a filling of the diagram of shape $(t, t, \stackrel{(s)}{\stackrel{( }{)}, t) \text {. By the correspondence between graphs and fillings }}$ of diagrams described above, we have that $M$ gives a graph $G(M)$ with $t$ opening vertices and $s$ closing vertices and such that all opening vertices appear before the closing vertices. Let us call such a graph a split graph, a particular case being the graph of a $k$-crossing or a $k$-nesting. As a consequence of the previous discussion we have that in terms of containment of substructures (matrices or split graphs), fillings of diagrams and graphs are equivalent objects.

Theorem 3.6 For any split graph $H$ there is a matrix $M(H)$ such that a left-right graph $G$ contains $H$ if and only if the filling $L(G)$ contains $M(H)$. And conversely, for each matrix $M$ there is a split graph $H(M)$ such that a filling $L$ of a diagram $T$ contains $M$ if and only if the graph $G(L)$ contains $H(M)$.

Observe now that Lemma 3.4 can be generalized by substituting " $k$-noncrossing graphs" with "graphs that do not contain the split graph $H$ ". Hence the following.

Theorem 3.7 Let $H$ and $H^{\prime}$ be two split graphs. Then for any left-right degree sequence $D$ there are as many graphs on $D$ avoiding $H$ as graphs on $D$ avoiding $H^{\prime}$ if and only if for each diagram with prescribed row and column sums there are as many fillings avoiding $M(H)$ as fillings avoiding $M\left(H^{\prime}\right)$.

Following the notation for matrices, we say that two split graphs $H$ and $H^{\prime}$ are equirestrictive if for any left-right degree sequence $D$, there are as many graphs on $D$ avoiding $H$ as graphs on $D$ avoiding $H^{\prime}$. All the split graphs that are known to be equirestrictive are obtained from the graph of a $k$-crossing or a $k$-nesting by using Proposition 4.1 from the next section. This proposition states that if $M$ and $N$ are equirestrictive matrices, then for any other matrix $A$ the matrices

$$
\left(\begin{array}{cc}
M & 0 \\
0 & A
\end{array}\right) \quad \text { and } \quad\left(\begin{array}{cc}
N & 0 \\
0 & A
\end{array}\right)
$$

defined by blocks are also equirestrictive. This has the following implications in terms of graphs. Given a split graph $H$ on $[h]$, a $(k-H)$-crossing is a graph on $2 k+h$ such that the graph induced by the vertices $[k] \cup\{k+h+1, \ldots, 2 k+h\}$ is a $k$-crossing, the graph induced by $\{k+1, \ldots, k+h\}$ is $H$, and there are no other edges. A $(k-H)$-nesting is defined similarly. Then by combining Theorems 3.5 and 3.7 and Proposition 4.1 we deduce the following.

Corollary 3.8 For any split graph $H$ and any nonnegative integer $k,(k-H)$-crossings and $(k-H)$-nestings are equirestrictive.

Observe that if we take $H$ to be an $h$-nesting, a $(k-H)$-nesting is a $(k+h)$-nesting, so it follows that a $k$-nesting, a $k$-crossing, and any combination of a $t$-crossing "over" a $(k-t)$-nesting are equirestrictive. However, it is not true that $t$-nestings over $(k-t)$ crossings are equirestrictive, not even within matchings, as observed in the remark after Theorem 1 of [5]. This implies also that there is no analogous version of Proposition 4.1 where $A$ is the top-left block and $M$ and $N$ are the bottom-right blocks of the matrix.

Finally, we comment on the results known for $0-1$ fillings of diagrams with row and column sums equal to 1 . Our correspondence translates these results into results for matchings and partition graphs, as we next explain. In the literature, two permutation matrices $M$ and $N$ are called shape-Wilf-equivalent if for each diagram $T$ with row and column sums set to 1 , the number of fillings avoiding $M$ equals the number of fillings avoiding $N$. (In view of this notation, we could have chosen the name graph-Wilf-equivalent instead of equirestrictive.) Let $P$ be a $t \times t$ permutation matrix. The split graph corresponding to $P$ is a matching (these are sometimes called permutation matchings). Now if two permutation matrices $P$ and $P^{\prime}$ are shape-Wilf-equivalent, then by straightforward application of Theorem 3.7 we have that for all graphs whose left and right degrees are one, the number of graphs avoiding the matching $H(P)$ equals the number of graphs avoiding the matching $H\left(P^{\prime}\right)$. Since graphs with left and right degrees one are exactly partition graphs, it turns out that shape-Wilf-equivalence is equivalent to the matchings $H(P)$ and $H\left(P^{\prime}\right)$ being equirestrictive among partition graphs, counted by left-right degree sequences.

There are not many pairs of permutation matrices known to be shape-Wilf-equivalent. Backelin, West, and Xin [1] show that $I_{k}$ and $J_{k}$ are shape-Wilf-equivalent; in graph theoretic terms, this gives another alternative proof of the equality between $k$-noncrossing and $k$-nonnesting partition graphs from 4]. Let us mention here that Krattenthaler [12] deduces both the result of Chen et al. and that of Backelin, West, and Xin from his Theorem 3, but for the first one he sets $T=\Delta$ and for the second he restricts the number of non-empty cells in the filling (and takes arbitrary shapes). Since these two apparently unrelated results are in fact equivalent, it is obvious that they must follow from the same theorem, but it is interesting that they do in different ways. Another observation is that by Lemma 3.4 and its generalization to split graphs, if we know that two split graphs are equirestrictive within matchings, then they are so within partition graphs. For instance, a bijective proof of the equality of the numbers of $k$-noncrossing and $k$-nonnesting matchings would immediately give a bijection for $k$-noncrossing and $k$-nonnesting partition graphs.

In addition to the matrices $I_{t}$ and $J_{t}$ and the ones that follow from Proposition 4.1 the only other pair of matrices known to be shape-Wilf-equivalent are (see [15])

$$
M(213)=\left(\begin{array}{ccc}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{array}\right) \quad \text { and } \quad M(132)=\left(\begin{array}{ccc}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}\right)
$$

The graph theoretic version of this result has been independently proved by Jelínek 5]. It is not known to us if $M(231)$ and $M(132)$ are also equirestrictive, or more generally if there is a pair of shape-Wilf-equivalent permutation matrices that are not equirestrictive.

Lastly, let us mention that all the discussion of this section can be carried out with almost no changes to the case where the matrix we want to avoid in the filling can have arbitrary nonnegative entries; this corresponds to avoiding split graphs with multiple edges. The interested reader will have no problems in filling in the details.

## 4 Proof of Theorem 3.5

This section is devoted to the proof of Theorem 3.5] We show that we can adapt to our setting the proof of [1], which is for shape-Wilf-equivalence, that is, row and column sums equal to 1 ; we include the details for the sake of completeness. (Actually, [1] contains two proofs of the analogous of our Theorem 3.5 for shape-Wilf-equivalence; the proof we adapt is the first one.) This bijection has been further studied in 3]. Here we show that it extends, in a quite straightforward way, to arbitrary fillings. This gives a result stronger than Theorem [3.5] the consequences of which in graph theoretic terms have already been pointed out at the end of the previous section. Let us also mention that Theorem 3.5 can also be proved using the techniques of [12].

From now on $\bar{T}$ denotes a diagram with prescribed row and column sums. When we say that a cell is above (or below, to the right, to the left) of another cell we always mean strictly. If we say that a cell is weakly above (below, etc.) we mean not above (not below, etc.)

If $A$ and $B$ are two matrices, by $[A \mid B]$ we mean the matrix having $A$ and $B$ as blocks, that is,

$$
\left(\begin{array}{cc}
A & 0 \\
0 & B
\end{array}\right)
$$

Proposition 4.1 Let $M$ and $N$ be a pair of equirestrictive matrices and let $A$ be any matrix. Then the matrices $[M \mid A]$ and $[N \mid A]$ are also equirestrictive.

Proof. Let $L$ be a filling of the diagram $\bar{T}$ that avoids $[M \mid A]$. Let $T^{\prime}$ be the set of cells $(i, j)$ of $T$ such that the cells to the right and below $(i, j)$ contain the matrix $A$. $T^{\prime}$ is a diagram, since if $(i, j)$ is in $T^{\prime}$ all the cells weakly above and weakly to the left of it are also in $T^{\prime}$. Now set the row and column sums of $T^{\prime}$ according to the restriction of $L$ to $T^{\prime}$, call it $L^{\prime}$, giving a diagram $\bar{T}^{\prime}$. Now $L^{\prime}$ is a filling of $\overline{T^{\prime}}$ that avoids $M$, so by assumption there is a bijection between such fillings and the ones that avoid $N$. Change the entries of $L$ corresponding to $T^{\prime}$ to obtain a filling of $\bar{T}$ that avoids $[N \mid A]$.

The bijection in the other direction goes just in the same way.
Let $F_{t}$ be the matrix $\left[J_{t-1} \mid I_{1}\right]$. The proof of the following proposition takes the rest of this section.

Proposition 4.2 For all $t, F_{t}$ and $J_{t}$ are equirestrictive.
We get as a corollary a stronger version of Theorem [3.5]

Corollary 4.3 For all $t,\left[I_{t} \mid A\right]$ and $\left[J_{t} \mid A\right]$ are equirestrictive.
Proof. By Proposition 4.1 it is enough to show that $I_{t}$ and $J_{t}$ are equirestrictive. The proof is by induction on $t$; clearly $I_{1}$ and $J_{1}$ are equirestrictive. By Proposition 4.2 it is enough to show that $I_{t}$ and $F_{t}$ are equirestrictive, and this follows by the induction hypothesis combined with Proposition 4.1

A sketch of the proof of Proposition 4.2 is as follows. We first define two maps between fillings that transform occurrences of $F_{t}$ into occurrences of $J_{t}$, and conversely, and use them to define to algorithms that transform a filling avoiding $F_{t}$ into a filling avoiding $J_{t}$, and conversely. The fact that these two algorithms are inverses of each other follows from a series of lemmas.

For any filling $L$, given two occurrences $G_{1}$ and $G_{2}$ of $J_{t}$ in $L$, we say that $G_{1}$ precedes $G_{2}$ if the first entry in which they differ, from left to right, is either higher in $G_{1}$ or it is at the same height and the one in $G_{1}$ is to the left. So two occurrences are either equal or comparable.

The order for the occurrences of $F_{t}$ goes the other way around, i.e., we look at the first entry in which they differ, from right to left, and the lower entries have preference, and if they are at the same height, the one more to the right goes first.

Let $L$ be a filling with the first occurrence of $J_{t}$ in rows $r_{1}, \ldots, r_{t}$ and columns $c_{1}, \ldots, c_{t}$. Let $\phi(L)$ be the result of substracting 1 from each cell $\left(r_{s}, c_{s}\right), 1 \leq s \leq t$ and adding 1 to each cell $\left(r_{s}, c_{s-1}\right), 2 \leq s \leq t$ and to cell ( $r_{1}, c_{t}$ ). Since row and column sums have not been altered, $\phi(L)$ is a filling of $\bar{T}$. So we have changed an occurrence of $J_{t}$ to an occurrence of $F_{t}$. Define $\psi$ as the inverse procedure, that is, $\psi$ takes a filling of the diagram, looks for the first occurrence of $F_{t}$, and replaces it by an occurrence of $J_{t}$.

We define the algorithms $A 1$ and $A 2$ in the following way. Algorithm $A 1$ starts with a filling avoiding $F_{t}$ and applies $\phi$ successively until there is no occurrence of $J_{t}$. The result (provided the algorithm finishes) is a filling that avoids $J_{t}$. Similarly, algorithm $A 2$ starts with a filling avoiding $J_{t}$ and applies $\psi$ until there are no occurrences of $F_{t}$ left. We claim that $A 1$ and $A 2$ are inverse of each other. We prove this through a series of analogous lemmas. It is enough to prove the following claims.

- That both algorithms end. (Lemmas 4.5 and 4.11)
- That $\psi\left(\phi^{n}(L)\right)=\phi^{n-1}(L)$ for all $n$. (Lemma 4.9)
- That $\phi\left(\psi^{n}(L)\right)=\psi^{n-1}(L)$ for all $n$. (Lemma 4.15)

In order to prove these claims, we need to investigate some properties of the maps $\phi$ and $\psi$. We start by studying the map $\phi$.

Let us first introduce some notation. Let $L$ be a filling of the diagram and let $a_{1}, \ldots, a_{t}$ be the cells of the first $J_{t}$ in $L$, listed from left to right; say they are $\left(r_{1}, c_{1}\right), \ldots,\left(r_{t}, c_{t}\right)$. So in each cell $a_{i}$ there is a positive integer, possibly greater than one. Let $b_{1}, \ldots, b_{t}$ be the cells $\left(r_{2}, c_{1}\right),\left(r_{3}, c_{2}\right), \ldots,\left(r_{t}, c_{t-1}\right)$ and $\left(r_{1}, c_{t}\right)$; hence, $b_{1}, \ldots, b_{t}$ are the cells corresponding to the occurrence of $F_{t}$ that is created after applying $\phi$ to $L$. So cell $b_{i}$ is in the same row as $a_{i+1}$ and in the same column as $a_{i}$, for $i$ with $1 \leq i \leq t-1$.

Consider now the following two paths of cells determined by $a_{1}, \ldots, a_{t}$ and $b_{1}, \ldots, b_{t}$ (see Figure 3). The path $A$ starts at the leftmost cell in the row of $a_{1}$, continues to the right until it reaches the column of $a_{2}$, then takes this column up until it hits cell $a_{2}$, then turns right until reaching the column of $a_{3}$, goes up until $a_{3}$, then turns right again, and so on, until it reaches cell $a_{t}$, at which point continues up until the top of the diagram. The path $B$ is defined in a similar manner. It starts at the leftmost cell of the row of cell $b_{1}$, and goes right until it hits $b_{1}$. Then it turns up until the row of $b_{2}$, where it turns and continues to the right until hitting $b_{2}$. Then it goes up until the row of $b_{3}$, and then turns to the right until $b_{3}$, and so on, until reaching $b_{t-1}$, at which point it goes up until reaching the top of the diagram. Since $a_{1}, \ldots, a_{t}$ are the first occurrence of $J_{t}$, the cells that are both to the right of $B$ and to the left of $A$ are empty, or, in other words, this region of the diagram avoids $J_{1}$. We denote this region by $E$. The choice of the first $J_{t}$ also imposes some other less trivial bounds on the longest $J_{i}$ 's that can be found in some other areas determined by $E$. Note that in the next lemma the area left of $E$ includes the path $B$.


Figure 3: The regions $A, B$, and $E$

Lemma 4.4 With the above notation, the following hold for any filling $L$ and for the corresponding $\phi(L)$.
(i) For all $i$ with $1 \leq i \leq t-1$, there is no $J_{i}$ below $b_{i}$ and to the left of $E$.
(ii) For all $i$ with $1 \leq i \leq t-1$, there is no $J_{t-i}$ above and to the right of $b_{i}$ and to the left of $E$.
(iii) For all $i, j$ with $1 \leq i<j \leq t-1$, the rectangle determined by $b_{i}$ and $b_{j}$ contains no $J_{j-i}$ to the left of $E$; that is, there is no $J_{j-i}$ below $b_{j}$, above $b_{i}$, to the right of $b_{i}$, and to the left of $E$.

Proof. The arguments below apply to both $L$ and $\phi(L)$ since they do not use the entries in cells $b_{i}$.
(i) Assume there was such a $J_{i}$. Then this $J_{i}$ together with $a_{i+1}, \ldots, a_{t}$ would form a $J_{t}$ contradicting the choice of $a_{1}$.
(ii) Suppose there was such a $J_{t-i}$. Then $a_{1}, \ldots, a_{i}$ followed by this $J_{t-i}$ form a $J_{t}$ that contradicts the choice of $a_{i+1}$.
(iii) Again, if there was such a $J_{j-i}$, combined with $a_{1}, \ldots, a_{i-1}$ and $a_{j+1}, \ldots, a_{t}$, it would create a $J_{t}$ contradicting the choice of $a_{i}$.

Lemma 4.5 There is no $J_{t}$ in $\phi(L)$ in the rows above $a_{1}$
Proof. We argue by contradiction. Let $G$ be an occurrence of $J_{t}$ in $\phi(L)$. Since $\phi$ picked $a_{1}$ as the topmost cell being the left-bottom cell of a $J_{t}, G$ must use at least one of the cells $b_{1}, \ldots, b_{t-1}$. The idea is to substitute these cells $b_{i}$, and possibly others, by some of the cells $a_{i}$, to find an occurrence of $J_{t}$ in $L$ in the rows above $a_{1}$, hence contradicting the choice of $a_{1}$.

Now for each cell $b_{i}$ which belong to $G$, find the largest integer $j$ such that all cells of $G$ above $b_{i}$ and weakly below $b_{j-1}$ lie left of $E$. In this way it is possible to find two sequences $i_{1}, \ldots, i_{s}$ and $j_{1}, \ldots, j_{s}$ with the following properties:

- $i_{k}<j_{k}, 1 \leq i_{k-1}<i_{k}$, and $j_{k-1}<j_{k} \leq t$ for all $k$;
- $b_{i_{k}}$ is in $G$;
- if $b_{l}$ is in $G$, then $i_{k} \leq l \leq j_{k}-1$ for some $k$;
- all cells of $G$ above $b_{i_{k}}$ and weakly below $b_{j_{k}-1}$ are to the left of $E$, and $j_{k}$ is the largest integer with this property.

Now we show that we can replace the cells of $G$ that fall left of $E$ and are contained in the rectangles determined by $b_{i_{k}}$ and $b_{j_{k}}$ by some of the $a_{i}$, giving an instance of $J_{t}$ contained in $L$ and above $a_{1}$. We need to distinguish two cases, according to whether $j_{s}=t$ or not. Assume first that $j_{s} \neq t$. For each $k$, consider the rectangles determined by $b_{i_{k}}$ and $b_{j_{k}}$. By Lemma 4.4(iii), there are at most $j_{k}-i_{k}-1$ elements of $G$ in this rectangle and to the left of $E$. Replace these cells, together with $b_{i_{k}}$, by a (possibly proper) subset of $a_{i_{k}+1}, \ldots, a_{j_{k}}$. After doing this for each $k$, we still have an occurrence of $J_{t}$ starting above $a_{1}$, but now it is contained in the original filling $L$, contradicting the hypothesis. Now assume that $j_{s}=t$. For $k<s$, do the same substitutions as in the
previous case; for $k=s$, we have by Lemma 4.4()(ii) that there are at most $t-i_{k}-1$ cells of $G$ left of $E$ and above $b_{i_{k}}$. Replace these cells and $b_{i_{k}}$ by a subset of $a_{i_{k}+1}, \ldots, a_{t}$. Again we obtain an occurrence of $J_{t}$ in $L$ that starts above $a_{1}$, a contradiction.

This lemma alone shows that algorithm $A 1$ terminates. Indeed, after one application of $\phi$ all the cells in the row of $a_{1}$ and to the left of $a_{1}$ are empty (because of the choice of $a_{1}$ ), and the cell $a_{1}$ has decreased its value by one. So the leftmost cell of the first occurrence of $J_{t}$ in $\phi(L)$ is either $a_{1}$, or it is to the right of $a_{1}$, or it is below $a_{1}$. But since the value in cell $a_{1}$ decreases and cells to the left of $a_{1}$ stay empty, eventually there will be no occurrence of $J_{t}$ whose leftmost cell is $a_{1}$. So the selection of $J_{t}$ 's goes from top to bottom and from left to right, so for some $n$ the filling $\phi^{n}(L)$ is free of $J_{t}$ 's.

It is not the case that if we apply $\phi$ to an arbitrary filling $L$ of $\bar{T}$ we have that $\psi(\phi(L))=L$. But algorithm $A 1$ starts with a filling that avoids $F_{t}$ and the successive applications of $\phi$ create occurrences of $F_{t}$ from top to bottom and from left to right. We need to show that in this situation after each application of $\phi$, the first occurrence of $F_{t}$ is precisely the one created by $\phi$. The next lemmas are devoted to proving this.

Lemma 4.6 If $L$ contains no $F_{t}$ with at least one square below $a_{1}$, then $\phi(L)$ contains no such $F_{t}$.

Proof. The proof is similar to the one of the previous lemma. Let $G$ be an occurrence of $F_{t}$ in $\phi(L)$ with at least one cell below $a_{1}$. Since $L$ had no such occurrence, $G$ contains at least one of the cells $b_{i}$. The bottom-right cell of $G$ is below $a_{1}$, and it cannot be to the right of $a_{t-1}$, otherwise this cell together with $a_{1}, \ldots, a_{t-1}$ would form an $F_{t}$ in $L$. By an argument similar to the one in the previous lemma, we change all cells $b_{i}$ of $G$, and possibly others, to some of the cells $a_{i}$, so that at the end we have an occurrence of $J_{t-1}$ that together with the bottom-right cell of $G$ gives an occurrence of $F_{t}$ that contradicts the hypothesis.

For each $b_{i}$ that is in $G$, look for the smallest $j$ such that all cells in $G$ that are left of $b_{i}$ and weakly to the right of $b_{j+1}$ are left of $E$. By doing this we find integers $i_{1}, \ldots, i_{s}$ and $j_{1}, \ldots, j_{s}$ with the following properties:

- $i_{k}>j_{k}, t-1 \geq i_{k-1}>i_{k}$, and $j_{k-1}>j_{k} \geq 0$ for all $k$;
- $b_{i_{k}}$ is in $G$ for all $k$ with $1 \leq k \leq s$;
- $j_{k}$ is the smallest integer such that all cells of $G$ that are left of $b_{i_{k}}$ and weakly to the right of $b_{j_{k}+1}$ are to the left of $E$;
- if $b_{l}$ is in $G$, then $j_{k}+1 \leq l \leq i_{k}$ for some $k$.

We have to distinguish whether $j_{s}=0$ or not. Assume first $j_{s} \neq 0$. Since by Lemma 4.4(iii) there are at most $i_{k}-j_{k}-1$ cells of $G$ in the rectangle determined by $b_{j_{k}}$ and $b_{i_{k}}$, these cells, together with $b_{i_{k}}$, can be replaced by a (possibly proper) subset of $a_{j_{k}+1}, \ldots, a_{i_{k}}$. By doing this for all $k$, we have an occurrence of $J_{t-1}$ in $L$ that together with the right-bottom cell of $G$ contradicts the hypothesis. If $j_{s}=0$, then we
do the same substitutions for all $k \neq s$; for $k=s$, we have by Lemma 4.4(i) that there are at most $i_{s}-1$ cells of $G$ left of $E$ and below $b_{i_{s}}$, so we can substitute those and $b_{i_{s}}$ by $a_{1}, \ldots, a_{i_{s}}$. After these substitutions, the result is again an occurrence of $F_{t}$ in $L$ that contains a cell below $a_{1}$, contradicting the hypothesis.

The following is easy but we state it for the sake of completeness.
Lemma 4.7 If $L$ contains no $F_{t}$ with a cell to the right of $a_{t}$ and below $a_{2}$, then $\phi(L)$ contains no such $F_{t}$.

Proof. Again we argue by contradiction. Suppose $G$ is an $F_{t}$ in $\phi(L)$ that contains a cell to the right of $a_{t}$ and below $a_{2}$. This cell together with $a_{2}, \ldots, a_{t}$ gives an occurrence of $F_{t}$ in $L$ that contradicts to the assumption.

Lemma 4.8 For each $k$ with $1 \leq k \leq t-1$, there is no $J_{k}$ in $\phi(L)$ above $a_{1}$ and to the left of and below $a_{k+1}$.

Proof. Let $G$ be an occurrence of such a $J_{k}$. If $G$ contains none of $b_{1}, \ldots, b_{k-1}$, then $G$ followed by $a_{k+1}, \ldots, a_{t}$ forms a $J_{t}$ in $L$ that is above $a_{1}$, and this contradicts the choice of $a_{1}$. Hence, $G$ uses some $b_{i}$ for $1 \leq i \leq k-1$. By an argument analogous to that of the proof of Lemma 4.5 we can substitute the cells $b_{i}$ that are in $G$ and possibly others by some $a_{i}$ 's so that we get an occurrence of $J_{k}$ in $L$ that is below $a_{k+1}$ and above $a_{1}$. This followed by $a_{k+1}, \ldots, a_{t}$, gives an $J_{t}$ in $L$ that contradicts the choice of $a_{1}$.

The following lemma is just a combination of the previous and induction; it implies that the inverse of algorithm $A 1$ is $A 2$.

Lemma 4.9 (i) If $L$ does not contain any occurrence of $F_{t}$ below $a_{1}$, then the first occurrence of $F_{t}$ in $\phi(L)$ is $b_{1}, \ldots, b_{t}$.
(ii) If $L$ is a filling that avoids $F_{t}$, then $\psi\left(\phi^{n}(L)\right)=\phi^{n-1}(L)$.

Proof. For the first statement, let $f_{1}, \ldots, f_{t}$ be the first occurrence of $F_{t}$ in $\phi(L)$, with the elements ordered from left to right. Recall that $b_{1}, \ldots, b_{t}$ is an occurrence of $F_{t}$ in $\phi(L)$; we need to show that $f_{i}=b_{i}$ for all $i$. By Lemma 4.6, $f_{t}$ is in the same row as $b_{t}$. By Lemma 4.7 $f_{t}$ cannot be to the right of $a_{t}$, hence $f_{t}=b_{t}$. Now use induction on $t-i$. Suppose we know $f_{i+1}=b_{i+1}, \ldots, f_{t}=b_{t}$. It is enough now to show that $f_{i}$ lies in the same row as $b_{i}$, since all the cells to the right of $b_{i}$ but left of $b_{i+1}$ lie in $E$, which we know contains only empty cells. But now Lemma 4.8 guarantees that there is no $J_{i}$ below $b_{i}$, to the left of $b_{i+1}$, and above $b_{t}$, as required.

For the second statement, it follows by Lemma 4.6and induction on $n$ that the filling $\phi^{n}(L)$ contains no $F_{t}$ whose lowest cell is below the lowest cell of the first occurrence of $J_{t}$. Hence the previous statement applied to $\phi^{n}(L)$ gives immediately that $\psi\left(\phi^{n}(L)\right)=$ $\phi^{n-1}(L)$.

So the inverse of algorithm $A 1$ is $A 2$. Now we only need to prove the converse. The proof follows exactly the same steps and we content ourselves by stating and proving the corresponding lemmas. Actually in this case some proofs are slightly simpler.

We keep the notation as above. Let $L$ be now a filling of $\bar{T}$ and let $b_{1}, \ldots, b_{t}$ be the first occurrence of $F_{t}$ and let $a_{1}, \ldots, a_{t}$ be the occurrence of $J_{t}$ in $\psi(L)$ created after applying $\psi$ to $L$. Consider again the region $E$ as defined above. By the choice of $b_{1}, \ldots, b_{t}$ as the first occurrence of $F_{t}$ in $L$, all the cells of $E$ are again empty.

Lemma 4.10 For all $i, j$ with $1 \leq i<j \leq t$, the rectangle determined by $a_{i}$ and $a_{j}$ contains no $J_{j-i}$ to the right of $E$ in either $L$ or $\psi(L)$; that is, there is no $J_{j-i}$ below $a_{j}$, above $a_{i}$, to the left of $a_{j}$, and to the right of $E$.

Proof. Suppose there was such a $J_{j-i}$. Then $b_{1}, \ldots, b_{i-1}$, followed by this $J_{j-i}$ and then followed by $b_{j}, \ldots, b_{t}$ gives an occurrence of $F_{t}$ in $L$ that contradicts the choice of $b_{j-1}$.

Lemma 4.11 There is no $F_{t}$ in $\psi(L)$ with at least one cell in a row below $a_{1}$.
Proof. Suppose there is such an $F_{t}$. Its right-bottom cell is below $a_{1}$ and also weakly to the left of $b_{t-1}$, since otherwise $b_{1}, \ldots, b_{t-1}$ and this cell would form an $F_{t}$ contradicting the choice of $b_{t}$. Let $G$ be this occurrence of $F_{t}$ except the right-bottom cell. $G$ must contain some of the cells $a_{1}, \ldots, a_{t}$. As in the previous lemmas, the idea is to substitute the $a_{i}$ in $G$ together with other cells by some of the $b_{i}$ so that we obtain an occurrence of $F_{t}$ in $L$ contradicting the choice of $b_{t}$. Find integers $i_{1}, \ldots, i_{s}$ and $j_{1}, \ldots, j_{s}$ with the following properties:

- $i_{k}<j_{k}, 1 \leq i_{k-1}<i_{k}$, and $j_{k-1}<j_{k} \leq t-1$ for all $k$;
- $a_{i_{k}}$ is in $G$ for all $k$ with $1 \leq k \leq s$;
- $j_{k}$ is the largest integer such that all cells of $G$ that are to the right of $a_{i_{k}}$ and weakly to the left of $a_{j_{k}-1}$ are to the right of $E$;
- if $a_{l}$ is in $G$, then $i_{k} \leq l \leq j_{k}-1$ for some $k$.

Now, by Lemma 4.10, there are at most $j_{k}-i_{k}-1$ elements of $G$ in the rectangle determined by $a_{i_{k}}$ and $a_{j_{k}}$. Together with $a_{i_{k}}$, they account for at most $j_{k}-i_{k}$ elements of $G$; substitute them for a subset of $b_{i_{k}}, \ldots, b_{j_{k}-1}$. Doing this for all $k$, we get an occurrence of $F_{t}$ in $L$ that contains a cell below $a_{1}$, hence contradicting the choice of $b_{1}, \ldots, b_{t}$ as the first $F_{t}$ in $L$.

Lemma 4.12 If $L$ contains no $J_{t}$ that is above $a_{1}$, then $\psi(L)$ contains no such $J_{t}$.
Proof. Let $G$ be such a $J_{t} ; G$ must contain some of the cells $a_{i}$. Find integers $i_{1}, \ldots, i_{s}$ and $j_{1}, \ldots, j_{s}$ with the following properties:

- $i_{k}>j_{k}, t \geq i_{k-1}>i_{k}$, and $j_{k-1}>j_{k} \geq 1$ for all $k$;
- $a_{i_{k}}$ is in $G$ for all $k$ with $1 \leq k \leq s$;
- $j_{k}$ is the smallest integer such that all cells of $G$ that are below $a_{i_{k}}$ and weakly above $a_{j_{k}+1}$ are to the right of $E$;
- if $a_{l}$ is in $G$, then $j_{k}+1 \leq l \leq i_{k}$ for some $k$.

As in the proof of the previous lemma, it is possible to substitute the elements of $G$ contained in the rectangles determined by $a_{i_{k}}$ and $a_{j_{k}}$, plus the cell $a_{i_{k}}$, by (a subset of) the elements $b_{j_{k}}, \ldots, b_{i_{k}-1}$. These substitutions give a $J_{t}$ in $L$ that is above $a_{1}$, contrary to the hypothesis.

Lemma 4.13 If $L$ contains no $J_{t}$ with a cell to the left of $a_{1}$ and below $a_{2}$, then neither does $\psi(L)$.

Proof. If this were the case, the leftmost cell of this $J_{t}$ together with $b_{1}, \ldots, b_{t-1}$ would give a $J_{t}$ contradicting the hypotheses

Lemma 4.14 If $L$ contains no $J_{t}$ above $a_{1}$, there is no $J_{t-r}$ in $\psi(L)$ above $a_{r+1}$ such that the lowest cell of this $J_{t-r}$ is weakly to the left of $a_{r+1}$.

Proof. Suppose $G$ is an occurrence of such a $J_{t-r}$. $G$ must contain some of the cells $a_{r+2}, \ldots, a_{t}$, otherwise $b_{1}, \ldots, b_{r}$ followed by $G$ would form a $J_{t}$ contradicting the hypothesis. Find integers $i_{1}, \ldots, i_{s}$ and $j_{1}, \ldots, j_{s}$ with the following properties:

- $i_{k}>j_{k}, t \geq i_{k-1}>i_{k}$, and $j_{k-1}>j_{k} \geq r-1$ for all $k$;
- $a_{i_{k}}$ is in $G$ for all $k$ with $1 \leq k \leq s$;
- $j_{k}$ is the smallest integer such that all cells of $G$ that are below $a_{i_{k}}$ and weakly above $a_{j_{k}+1}$ are to the right of $E$;
- if $a_{l}$ is in $G$, then $j_{k}+1 \leq l \leq i_{k}$ for some $k$.

As before, the rectangle determined by $a_{i_{k}}$ and $a_{j_{k}}$ contains at most $i_{k}-j_{k}-1$ cells of $G$; these cells, together with $a_{i_{k}}$, can be replaced by a subset of $b_{j_{k}}, \ldots, b_{i_{k}-1}$. After all these substitutions we get an occurrence of $J_{t-r}$ in $L$ that combined with $b_{1}, \ldots, b_{r}$ gives an occurrence of $J_{t}$ in $L$ contradicting the hypothesis.

Lemma 4.15 (i) If $L$ does not contain any occurrence of $J_{t}$ above $b_{t}$, then the first occurrence of $J_{t}$ in $\phi(L)$ is $a_{1}, \ldots, a_{t}$.
(ii) If $L$ is a filling that avoids $J_{t}$, then $\phi\left(\psi^{n}(L)\right)=\psi^{n-1}(L)$.

Proof. For the first statement, let $d_{1}, \ldots, d_{t}$ be the first occurrence of $J_{t}$ in $\psi(L)$, with cells listed from left to right. We want to show that $a_{i}=d_{i}$ for all $i$ with $1 \leq i \leq t$. By Lemma 4.12 $d_{1}$ is in the same row as $a_{1}$, and by Lemma 4.13 it is weakly to the right of $a_{1}$, hence $d_{1}=a_{1}$. Now we proceed by induction on $i$. Suppose $d_{1}=a_{1}, \ldots, d_{i}=a_{i}$. By Lemma 4.14 we have that the only $J_{t-i}$ in $\psi(L)$ that is weakly above and weakly to the left of $a_{i+1}$ is $a_{i+1}, \ldots, a_{t}$, hence $d_{i+1}=a_{i+1}$, as needed.

For the second statement, by induction and Lemma 4.12 we get that $\psi^{n}(L)$ satisfies the hypothesis of part (i), hence it follows that $\phi\left(\psi^{n}(L)\right)=\psi^{n-1}(L)$.

## 5 Concluding remarks

In his paper [12], Krattenthaler speaks of a "bigger picture" that would englobe several recent results on pattern avoiding fillings of diagrams. We believe that our correspondence between graphs and fillings of diagrams also belongs to this picture and that it may shed some light in the understanding of it. We have shown that for each statement in pattern avoiding fillings there is a statement about graphs avoiding certain split graphs. So we can claim that in some sense the resources available to attack either problem have doubled. An example of this are the "repeated" results in the literature mentioned at the end of Section 3

For completeness, we mention here a result by Bousquet-Mélou and Steingrimsson 3] that can be cast in terms of $k$-noncrossing and $k$-nonnesting graphs. They restrict to diagrams with self-conjugate shape and row and column sums are set to 1 , and they only consider symmetric $0-1$ fillings (that is, symmetric with respect to the main diagonal of the diagram). For these fillings, they show that $I_{t}$ and $J_{t}$ are equirestrictive. In terms of matchings, this says that for each left-right degree sequence, the number of $k$-noncrossing symmetric matchings is the same as the number of $k$-nonnesting ones, where a matching on $[2 n]$ is symmetric if it equals its reflection through the vertical axis that goes between vertices $n$ and $n+1$. Similar results for symmetric graphs can be deduced from [12, Theorem 15].

Let us finish by going back to our initial motivation of studying $k$-noncrossing and $k$-nonnesting graphs. Even if our main question has been answered positively, it is fair to say that it has not been solved in the most satisfactory way; ideally we would like to find a bijective proof in graph theoretic terms. Note that due to its roundabout character, our proof of Theorem 3.5 does not give a clear bijection, neither in terms of graphs nor of fillings. Also the proofs of Corollaries [2.2, 2.5, and 3.2 do not provide bijections in graph theoretic terms. A bijective proof of Theorem 3.3 for $k=2$ has recently been found by Jelínek, Klazar, and de Mier [6].

Other interesting questions related to $k$-crossings and $k$-nestings of graphs include, as mentioned before, to determine whether the pairs $(\operatorname{cross}(G)$, nest $(G))$ are symmetrically distributed among all graphs. This is already known for matchings and partition graphs [4]. One would also hope for a wide generalization of Theorem 3.3 stating that the number of graphs with $r k$-crossings and $s k$-nestings equals the number of graphs with $s k$-crossings and $r k$-nestings. Again, the case $k=2$ is known for matchings 11 and partition graphs 9. Unfortuntely, for $k=3$ this is not true even for matchings; for instance, Marc Noy [13] checked that there are more matchings with six edges and only one 3 -crossing than with only one 3 -nesting.

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