# PERFECT DIFFERENCE SETS CONSTRUCTED FROM SIDON SETS 

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#### Abstract

A set $\mathcal{A}$ of positive integers is a perfect difference set if every nonzero integer has an unique representation as the difference of two elements of $\mathcal{A}$. We construct dense perfect difference sets from dense Sidon sets. As a consequence of this new approach we prove that there exists a perfect difference set $\mathcal{A}$ such that $$
A(x) \gg x^{\sqrt{2}-1-o(1)}
$$


Also we prove that there exists a perfect difference set $\mathcal{A}$ such that $\limsup _{x \rightarrow \infty} A(x) / \sqrt{x} \geq 1 / \sqrt{2}$.

## 1. Introduction

Let $\mathbb{Z}$ denote the integers and $\mathbb{N}$ the positive integers. For nonempty sets of integers $\mathcal{A}$ and $\mathcal{B}$, we define the difference set

$$
\mathcal{A}-\mathcal{B}=\{a-b: a \in \mathcal{A} \text { and } b \in \mathcal{B}\}
$$

For every integer $u$, we denote by $d_{\mathcal{A}, \mathcal{B}}(u)$ the number of pairs $(a, b) \in \mathcal{A} \times \mathcal{B}$ such that $u=a-b$. Let $d_{\mathcal{A}}(u)$ the number of pairs $\left(a, a^{\prime}\right) \in \mathcal{A} \times \mathcal{A}$ such that $u=a-a^{\prime}$. The set $\mathcal{A}$ is a perfect difference set if $d_{\mathcal{A}}(u)=1$ for every integer $u \neq 0$. Note that $\mathcal{A}$ is a perfect difference set if and only $d_{\mathcal{A}}(u)=1$ for every positive integer $u$. For perfect difference sets, a simple counting argument shows that

$$
A(x) \ll x^{1 / 2}
$$

where the counting function $A(x)$ counts the number of positive elements of $\mathcal{A}$ not exceeding $x$.

It is not completely obvious that perfect difference sets exist, but the greedy algorithm produces [3] a perfect difference set $\mathcal{A} \subseteq \mathbb{N}$ such that

$$
A(x) \gg x^{1 / 3}
$$

At the Workshop on Combinatorial and Additive Number Theory (CANT 2004) in New York in May, 2004, Seva Lev (see also [3]) asked if there exists a perfect difference set $\mathcal{A}$ such that

$$
A(x) \gg x^{\delta} \text { for some } \delta>1 / 3
$$

[^0]We answer this questions affirmatively by constructing perfect difference sets from classical Sidon sets.

We say that a set $\mathcal{B}$ is a Sidon set if $d_{\mathcal{B}}(u) \leq 1$ for all integer $u \neq 0$.
Theorem 1. For every Sidon set $\mathcal{B}$ and every function $\omega(x) \rightarrow \infty$, there exists a perfect difference set $\mathcal{A} \subseteq \mathbb{N}$ satisfying

$$
A(x) \geq B(x / 3)-\omega(x)
$$

It is a difficult problem to construct dense infinite Sidon sets. Ruzsa 6] proved that there exists a Sidon set $\mathcal{B}$ with $B(x) \gg x^{\sqrt{2}-1-o(1)}$. The following result follows easily.

Theorem 2. There exists a perfect difference set $\mathcal{A} \subseteq \mathbb{N}$ such that

$$
A(x) \gg x^{\sqrt{2}-1+o(1)}
$$

Erdős [7] proved that the lower bound $A(x) \gg x^{1 / 2}$ does not hold for any Sidon set $\mathcal{A}$, and so does not hold for perfect difference sets. However, Krückeberg [2] proved that there exists a Sidon set $\mathcal{B}$ such that

$$
\limsup _{x \rightarrow \infty} \frac{B(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}
$$

We extend this result to perfect difference sets.
Theorem 3. There exists a perfect difference set $\mathcal{A} \subset \mathbb{N}$ such that

$$
\limsup _{x \rightarrow \infty} \frac{A(x)}{\sqrt{x}} \geq \frac{1}{\sqrt{2}}
$$

Notice that an immediate application of Theorem to Krückeberg's result would give only $\lim \sup _{x \rightarrow \infty} A(x) x^{-1 / 2} \geq 1 / \sqrt{6}$.

## 2. Proof of Theorem 1

2.1. Sketch of the proof. The strategy of the proof is the following:

- Modify any dense Sidon set $\mathcal{B}$ given by dilating it by 3 and removing a suitable thin subset of $3 * \mathcal{B}$.
- Complete the remainder set $\mathcal{B}_{0}=(3 * \mathcal{B}) \backslash\{$ removed set $\}$ with a subset of the elements of a very sparse sequence $\mathcal{U}=\left\{u_{s}\right\}$ by adding, if $k$ has not appeared yet in the difference set, two elements $u_{2 k}, u_{2 k+1}$ in the $k$-th step such that $u_{2 k+1}-u_{2 k}=k$.
2.2. The auxiliary sequence $\mathcal{U}$. For any strictly increasing function $g$ : $\mathbb{N} \rightarrow \mathbb{N}$ and $k \geq 1$, we define integers $u_{2 k}$ and $u_{2 k+1}$ by

$$
\begin{cases}u_{2 k} & =4^{g(k)}+\epsilon_{k} \\ u_{2 k+1} & =4^{g(k)}+\epsilon_{k}+k\end{cases}
$$

where

$$
\epsilon_{k}= \begin{cases}1 & \text { if } k \equiv 2 \quad(\bmod 3) \\ 0 & \text { otherwise }\end{cases}
$$

For all positive integers $k$ we have

$$
u_{2 k+1}-u_{2 k}=k
$$

Let $\mathcal{U}_{k}=\left\{u_{2 k}, u_{2 k+1}\right\}$ and $\mathcal{U}_{<\ell}=\bigcup_{k<\ell} \mathcal{U}_{k}$. It will be useful to state some properties of the sequence $\mathcal{U}=\left\{u_{i}\right\}_{i=2}^{\infty}$.

Lemma 1. The sequence $\mathcal{U}=\left\{u_{i}\right\}_{i=2}^{\infty}$ satisfies the following properties:
(i) For all $i \geq 2, u_{i} \neq 0(\bmod 3)$.
(ii) For all $k \geq 2$, for $u \in \mathcal{U}_{k}$, and for all $u^{\prime}, u^{\prime \prime}, u^{\prime \prime \prime} \in \mathcal{U}_{<k}$, we have $u+u^{\prime}>u^{\prime \prime}+u^{\prime \prime \prime}$.
(iii) If $k \geq 2, u \in \mathcal{U}_{k}$, and $u^{\prime} \in \mathcal{U}_{<k}$, then $u-u^{\prime}>u / 2$.

Proof. (i) By construction.
(ii) Since $g(k)$ is strictly increasing we have $k \leq g(k)$ and so

$$
4 k<4^{k} \leq 4^{g(k)}
$$

for all $k \geq 2$. It follows that

$$
\begin{aligned}
u^{\prime \prime}+u^{\prime \prime \prime} & \leq 2 u_{2 k-1} \leq 2\left(4^{g(k-1)}+k\right) \\
& \leq 2\left(4^{g(k)-1}+k\right) \leq \frac{4^{g(k)}}{2}+2 k \\
& \leq 4^{g(k)} \leq u<u+u^{\prime}
\end{aligned}
$$

(iii) For $k \geq 2$ we have

$$
\begin{aligned}
u^{\prime} & \leq 4^{g(k-1)}+(k-1)+\epsilon_{k-1} \\
& \leq 4^{g(k)-1}+k \\
& <2 \cdot 4^{g(k)-1}=\frac{4^{g(k)}}{2} \\
& \leq u / 2
\end{aligned}
$$

and so $u-u^{\prime}>u / 2$.
2.3. Construction of the Sidon set $\mathcal{B}_{0}$. Take a Sidon set $\mathcal{B}$ and consider the set $\mathcal{B}^{\prime}=3 * \mathcal{B}=\{3 b: b \in \mathcal{B}\}$. Then $\mathcal{B}^{\prime}$ is a Sidon set such that $b \equiv 0$ $(\bmod 3)$ for all $b \in \mathcal{B}^{\prime}$ and $B^{\prime}(x)=B\left(\frac{x}{3}\right)$.

The set $\mathcal{B}_{0}$ will be the set $\mathcal{B}^{\prime}=3 * \mathcal{B}$ after we remove all the elements $b \in \mathcal{B}^{\prime}$ that satisfy at least one of the followings conditions:
c1: $b=u-u^{\prime}+b^{\prime}$ for some $b^{\prime} \in \mathcal{B}^{\prime}, b>b^{\prime}$ and $u, u^{\prime} \in \mathcal{U}$ such that $u \in \mathcal{U}_{r}, u^{\prime} \in \mathcal{U}_{<r}$ for some $r$.
c2: $b=u+u^{\prime}-b^{\prime}$ for some $b^{\prime} \in \mathcal{B}, b \geq b^{\prime}$ and $u, u^{\prime} \in \mathcal{U}$.
c3: $b=u+u^{\prime}-u^{\prime \prime}$ for some $u \in \mathcal{U}_{r}, u^{\prime} \in \mathcal{U}$, and $u^{\prime \prime} \in \mathcal{U}_{<r}$ with $u^{\prime} \leq u$.
c4: $\left|b-u_{i}\right| \leq i$ for some $u_{i} \in \mathcal{U}$.
2.4. The inductive step. We shall construct the set $\mathcal{A}$ in Theorem 1 by adjoining terms to the nice Sidon set $\mathcal{B}_{0}$ obtained above. More precisely, the sequence $\mathcal{A}$ satisfying the conditions of the theorem will be

$$
\mathcal{A}=\bigcup_{k=0}^{\infty} \mathcal{A}_{k}
$$

where $\mathcal{A}_{k}$ will be defined by $\mathcal{A}_{0}=\mathcal{B}_{0}$ and for, $k \geq 1$,

$$
\mathcal{A}_{k}= \begin{cases}\mathcal{A}_{k-1} \cup \mathcal{U}_{k} & \text { if } k \notin \mathcal{A}_{k-1}-\mathcal{A}_{k-1} \\ \mathcal{A}_{k-1} & \text { otherwise }\end{cases}
$$

Lemma 2. For every positive integer $k$ we have

$$
[-k, k] \subseteq \mathcal{A}_{k}-\mathcal{A}_{k}
$$

and so

$$
d_{\mathcal{A}}(n) \geq 1
$$

for all integers $n$.
Proof. Clear.
2.5. $\mathcal{A}$ is a Sidon set. First we state two lemmas.

Lemma 3. Let $A_{1}$ and $A_{2}$ be nonempty disjoint sets of integers and let $A=A_{1} \cup A_{2}$ For every integer $n$ we have

$$
d_{A}(n)=d_{A_{1}}(n)+d_{A_{2}}(n)+d_{A_{1}, A_{2}}(n)+d_{A_{2}, A_{1}}(n)
$$

where

$$
d_{A_{i}, A_{j}}(n)=\#\left\{\left(a, a^{\prime}\right) \in A_{i} \times A_{j}, a-a^{\prime}=n\right\}
$$

Proof. This follows from the identity

$$
\left(A_{1} \cup A_{2}\right) \times\left(A_{1} \cup A_{2}\right)=\left(A_{1} \times A_{1}\right) \cup\left(A_{2} \times A_{2}\right) \cup\left(A_{1} \times A_{2}\right) \cup\left(A_{2} \times A_{1}\right)
$$

Lemma 4. If $n \in \mathcal{A}_{r-1}-\mathcal{U}_{r}$ then
(i) $|n|>r$, and so $d_{\mathcal{U}_{r}, \mathcal{A}_{r-1}}(r)=d_{\mathcal{A}_{r-1}, \mathcal{U}_{r}}(r)=0$.
(ii) $d_{\mathcal{A}_{r-1}}(n)=0$.
(iii) $d_{\mathcal{U}_{r}, \mathcal{A}_{r-1}}(n)=0$.

Proof. Write $n=a-u$, where $a \in \mathcal{A}_{r-1}$ and $u \in \mathcal{U}_{r}=\left\{u_{2 r}, u_{2 r+1}\right\}$.
(i) If $a=b \in \mathcal{B}_{0}$ we have that $|b-u|>2 r>r$ because, by condition (c4), we have removed all elements $b$ from $\mathcal{B}$ such that $\left|b-u_{i}\right| \leq i$.

If $a=u^{\prime} \in \mathcal{U}_{<r}$ then we apply Lemma (iii) to conclude that

$$
\left|u^{\prime}-u\right|>\frac{u}{2} \geq \frac{4^{g(r)}}{2}>r
$$

(ii) Since $\mathcal{A}_{r-1} \subseteq \mathcal{B}_{0} \cup \mathcal{U}_{<r}$, it follows that

$$
d_{\mathcal{A}_{r-1}}(n) \leq d_{\mathcal{B}_{0} \cup \mathcal{U}_{<r}}(n) \leq d_{\mathcal{B}_{0}}(n)+d_{\mathcal{U}_{<r}}(n)+d_{\mathcal{B}_{0}, \mathcal{U}_{<r}}(n)+d_{\mathcal{U}_{<r}, \mathcal{B}_{0}}(n)
$$

If $a=b \in \mathcal{B}_{0}$, then $n=b-u$ and
(1) $b \equiv 0(\bmod 3)$ but $u \not \equiv 0(\bmod 3)$, hence $b-u \not \equiv 0(\bmod 3)$ and $d_{\mathcal{B}_{0}}(b-u)=0$ (by Lemma (i)),
(2) $d_{\mathcal{U}_{<r}}(b-u)=0$ (by condition (c3)),
(3) $d_{\mathcal{B}_{0}, \mathcal{U}_{<r}}(b-u)=0$ (by condition (c1)),
(4) $d_{\mathcal{U}_{<r}, \mathcal{B}_{0}}(b-u)=0$ (by condition (c2)).

If $a=u^{\prime} \in \mathcal{U}_{<r}$, then $n=u^{\prime}-u$ and
(1) $d_{\mathcal{B}_{0}}\left(u^{\prime}-u\right)=0$ (by condition (c1)),
(2) $d_{\mathcal{U}_{<r}}\left(u^{\prime}-u\right)=0$ (by Lemma (ii)),
(3) If $u^{\prime}-u=b-u^{\prime \prime}$ with $u^{\prime \prime} \in \mathcal{U}_{<r}$, then Lemma (iii) implies that $0<b=u^{\prime}+u^{\prime \prime}-u \leq 0$, and so $d_{B_{0}, \mathcal{U}_{<r}}\left(u^{\prime}-u\right)=0$.
(4) $d_{\mathcal{U}_{<r}, B_{0}}\left(u^{\prime}-u\right)=0$ (by condition (c3)).
(iii) Again, since $\mathcal{A}_{r-1} \subseteq \mathcal{B}_{0} \cup \mathcal{U}_{<r}$ we have that

$$
d_{\mathcal{U}_{r}, \mathcal{A}_{r-1}}(n) \leq d_{\mathcal{U}_{r}, \mathcal{B}_{0}}(n)+d_{\mathcal{U}_{r}, \mathcal{U}_{<r}}(n) .
$$

If $a=b \in \mathcal{B}_{0}$ then $d_{\mathcal{U}_{r}, \mathcal{B}_{0}}(b-u)=0$ (by condition (c2)) and $d_{\mathcal{U}_{r}, \mathcal{U}_{<r}}(b-u)=0$ (by condition (c3)).

If $a=u^{\prime} \in \mathcal{U}_{<r}$ then $d_{\mathcal{U}_{r}, \mathcal{B}_{0}}\left(u^{\prime}-u\right)=0$ (by condition (c3)). Finally, we have $d_{\mathcal{U}_{r}, \mathcal{U}_{<r}}\left(u^{\prime}-u\right)=0$, since if $u^{\prime}-u=u^{\prime \prime}-u^{\prime \prime \prime}, u^{\prime \prime} \in \mathcal{U}_{r}, u^{\prime \prime \prime} \in \mathcal{U}_{<r}$, then $0>u^{\prime}-u=u^{\prime \prime}-u^{\prime \prime \prime}>0$. This completes the proof.

Lemma 5. For every positive integer $n$ we have

$$
d_{\mathcal{A}}(n) \leq 1
$$

and so $\mathcal{A}$ is a perfect difference set.
Proof. We will use induction to prove that, for every $r \geq 0$,

$$
d_{\mathcal{A}_{r}}(n) \leq 1 \quad \text { for every nonzero integer } n \text {. }
$$

This is true for $r=0$ because $\mathcal{A}_{0}=\mathcal{B}_{0}$ is a subset of a Sidon set.
We assume that the statement is true for $r-1$ and shall prove it for $r$.
If $d_{\mathcal{A}_{r-1}}(r)=1$ then $\mathcal{A}_{r}=\mathcal{A}_{r-1}$ and there is nothing to prove. Suppose that $d_{\mathcal{A}_{r-1}}(r)=0$, and so $\mathcal{A}_{r}=\mathcal{A}_{r-1} \cup \mathcal{U}_{r}$. Since we have added two new elements $u_{2 r}, u_{2 r+1}$ to $\mathcal{A}_{r-1}$, it is possible that there are new representations of a positive integer $n$ so that $d_{A_{r}}(n)>1$. We shall prove that this cannot happen.

By Lemma 3 we can write

$$
d_{\mathcal{A}_{r}}(n)=d_{\mathcal{A}_{r-1}}(n)+d_{\mathcal{U}_{r}}(n)+d_{\mathcal{A}_{r-1}, \mathcal{U}_{r}}(n)+d_{\mathcal{U}_{r}, \mathcal{A}_{r-1}}(n)
$$

If $n=r$, then Lemma 4 (i) and the relation $u_{2 r+1}-u_{2 r}=r$ imply that

$$
d_{\mathcal{A}_{r}}(r)=d_{\mathcal{A}_{r-1}}(r)+d_{\mathcal{U}_{r}}(r)+d_{\mathcal{A}_{r-1}, \mathcal{U}_{r}}(r)+d_{\mathcal{U}_{r}, \mathcal{A}_{r-1}}(r)=0+1+0+0=1
$$

If $n \neq r$, then

$$
d_{\mathcal{A}_{r}}(n)=d_{\mathcal{A}_{r-1}}(n)+d_{\mathcal{A}_{r-1}, \mathcal{U}_{r}}(n)+d_{\mathcal{U}_{r}, \mathcal{A}_{r-1}}(n) .
$$

If $n \in \mathcal{A}_{r-1}-\mathcal{U}_{r}$ (the case $n \in \mathcal{U}_{r}-\mathcal{A}_{r-1}$ is similar), then we can write

$$
n=a-u \text { where } a \in \mathcal{A}_{r-1}, u \in \mathcal{U}_{r}
$$

Applying Lemma (ii) and Lemma (iii), we obtain

$$
d_{\mathcal{A}_{r}}(n)=d_{\mathcal{A}_{r-1}, \mathcal{U}_{r}}(n) .
$$

If $d_{\mathcal{A}_{r-1}, \mathcal{U}_{r}}(n) \geq 2$, then there exist $a, a^{\prime} \in \mathcal{A}_{r-1}$ such that $a-u_{2 r}=a^{\prime}-u_{2 r+1}$. This implies that

$$
a^{\prime}-a=u_{2 r+1}-u_{2 r}=r \in \mathcal{A}_{r-1}-\mathcal{A}_{r-1}
$$

which is false, so $d_{\mathcal{A}_{r}}(n)=d_{\mathcal{A}_{r-1}, \mathcal{U}_{r}}(n) \leq 1$.
If $n \notin\left(\mathcal{A}_{r-1}-\mathcal{U}_{r}\right) \cup\left(\mathcal{U}_{r}-\mathcal{A}_{r-1}\right)$ then

$$
d_{\mathcal{A}_{r}}(n)=d_{\mathcal{A}_{r-1}}(n) \leq 1 .
$$

This completes the proof.
2.6. The counting function $A(x)$. We have

$$
A(x) \geq B_{0}(x)=B(x)-R(x)=B(x / 3)-R(x)
$$

where $R=R_{1} \cup R_{2} \cup R_{3} \cup R_{4}$ and $R_{i}$ denotes the set of elements of $B$ removed by condition ( $c_{i}$ ), $i=1,2,3,4$.

Lemma 6. Let $U(x)$ denote the counting function of the set $\mathcal{U}$. For the sets $R_{1}, R_{2}, R_{3}, R_{4}$ defined above, we have
(i) $R_{1}(x) \leq U^{2}(2 x)$.
(ii) $R_{2}(x) \leq U^{2}(2 x)$.
(iii) $R_{3}(x) \leq U^{3}(2 x)$
(iv) $R_{4}(x) \leq 2 U^{2}(2 x)+U(2 x)$.

Proof. (i) We have

$$
R_{1}(x)=\#\{b \in B: b \leq x \text { and } b \text { satisfies condition (c1) }\} .
$$

Because $B$ is a Sidon set, for every pair of integers $u, u^{\prime} \in \mathcal{U}$ there exists at most one pair of integers $b, b^{\prime} \in \mathcal{B}$ such that $b-b^{\prime}=u-u^{\prime}$. The condition $x \geq b>b^{\prime}$ implies that $0<u-u^{\prime} \leq x$. On the other hand Lemma (iii) implies that $u-u^{\prime}>u / 2$ and so $u<2 x$ and

$$
R_{1}(x) \leq \#\left\{\left(u, u^{\prime}\right), u^{\prime}<u, u<2 x\right\} \leq U^{2}(2 x) .
$$

(ii) Again, because $B$ is a Sidon set, for every pair $u, u^{\prime} \in \mathcal{U}$ there exists at most one pair $b, b^{\prime} \in \mathcal{B}$ such that $b+b^{\prime}=u+u^{\prime}$. The condition $x \geq b \geq b^{\prime}$ implies $u, u^{\prime} \leq 2 x$ and so

$$
R_{2}(x) \leq \#\left\{\left(u, u^{\prime}\right) \in \mathcal{U} \times \mathcal{U}: u \leq 2 x, u^{\prime} \leq 2 x\right\} \leq U^{2}(2 x)
$$

(iii) If $u \in \mathcal{U}_{r}, u^{\prime \prime} \in \mathcal{U}_{<r}$, then Lemma (iii) implies that $b=u+u^{\prime}-u^{\prime \prime}>$ $u-u^{\prime \prime}>u / 2$ and so

$$
\begin{aligned}
R_{3}(x) & =\#\{b \in B: b \leq x \text { and } b \text { satisfies condition }(\mathrm{c} 3)\} \\
& \leq \#\left\{\left(u, u^{\prime}, u^{\prime \prime}\right) \in \mathcal{U} \times \mathcal{U} \times \mathcal{U}: u<2 x, u^{\prime \prime}<u, u^{\prime} \leq u\right\} \\
& \leq U(2 x)^{3} .
\end{aligned}
$$

(iv) We have

$$
\begin{aligned}
R_{4}(x) & =\#\left\{b \in B: b \leq x \text { and }\left|b-u_{i}\right| \leq i \text { for some } u_{i} \in \mathcal{U}\right\} \\
& \leq \#\left\{n \in \mathbb{N}: n \leq x \text { and }\left|n-u_{i}\right| \leq i \text { for some } i\right\} .
\end{aligned}
$$

If $n \leq x$ and $\left|n-u_{i}\right| \leq i$, then $u_{i} \leq n+i \leq x+i$. Since $u_{2}=4^{g(1)} \geq 4$, $u_{3}=4^{g(1)+1} \geq 16$, and, for $i \geq 4$,

$$
u_{i} \geq 4^{g((i-1) / 2)} \geq 4^{(i-1) / 2}=2^{i-1} \geq 2 i
$$

Therefore, $u_{i} \leq x+i \leq x+u_{i} / 2$ and so $u_{i} \leq 2 x$. It follows that $i \leq U(2 x)$ and so

$$
\begin{aligned}
R_{4}(x) & \leq \#\left\{n \leq x:\left|n-u_{i}\right| \leq U(2 x) \text { and } u_{i} \leq 2 x\right\} \\
& \leq(2 U(2 x)+1) U(2 x)=2 U(2 x)^{2}+U(2 x) .
\end{aligned}
$$

This completes the proof of the lemma.
Finally, given any function $\omega(x) \rightarrow \infty$ we have that

$$
A(x) \geq B(x / 3)-\left(U(2 x)^{3}+4 U^{2}(2 x)+U(2 x)\right) \geq B(x / 3)-\omega(x)
$$

for any function $g: \mathbb{N} \rightarrow \mathbb{N}$ and sequence $\mathcal{U}$ growing fast enough. This completes the proof of Theorem [1]

## 3. Proof of Theorem 3

Lemma 7. If $C_{1}$ and $C_{2}$ are Sidon sets such that $\left(C_{i}-C_{i}\right) \cap\left(C_{j}-C_{j}\right)=\{0\}$, $\left(C_{i}+C_{i}\right) \cap\left(C_{j}+C_{j}\right)=\emptyset$ and $\left(C_{i}+C_{i}-C_{i}\right) \cap C_{j}=\emptyset$ for $i \neq j$, then $C_{1} \cup C_{2}$ is a Sidon set.

Proof. Obvious.
Lemma 8. For each odd prime $p$ there exist a Sidon set $\mathcal{B}_{p}$ such that
(i) $\mathcal{B}_{p} \subseteq\left[1, p^{2}\right]$.
(ii) $\left(\mathcal{B}_{p}-\mathcal{B}_{p}\right) \cap[-\sqrt{p}, \sqrt{p}]=\emptyset$.
(iii) $\left|\mathcal{B}_{p}\right|>p-2 \sqrt{p}$.

Proof. Ruzsa [5] constructed, for each prime $p$, a Sidon set $R_{p} \subseteq\left[1, p^{2}-p\right]$ with $\left|R_{p}\right|=p-1$. We consider the subset $\mathcal{B}_{p}$ of $R_{p}$ that we obtain by removing all elements $b \in R_{p}$ such that $0<\left|b-b^{\prime}\right| \leq \sqrt{p}$ for some $b^{\prime} \in R_{p}$. Since $R_{p}$ is a Sidon set, it follows that we have removed at most $\sqrt{p}$ elements from $R_{p}$, and so $\left|\mathcal{B}_{p}\right| \geq\left|R_{p}\right|-\sqrt{p}=p-\sqrt{p}-1>p-2 \sqrt{p}$.

Proof of Theorem [3. We shall construct an increasing sequence of finite set $A_{1} \subseteq A_{2} \subseteq A_{3} \subseteq \cdots$ such that $\mathcal{A}=\cup_{k=1}^{\infty} A_{k}$ is a perfect difference set satisfying Theorem 3.

In the following, $l_{k}$ will denote the largest integer in the set $A_{k-1}$, and $p_{k}$ the least prime greater than $4 l_{k}^{2}$. Let

$$
A_{1}=\{0,1\}
$$

Then $l_{2}=1$ and $p_{2}=5$. We define

$$
A_{k}= \begin{cases}A_{k-1} \cup\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right) & \text { if } k \in A_{k-1}-A_{k-1} \\ A_{k-1} \cup\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right) \cup\left\{4 p_{k}^{2}, 4 p_{k}^{2}+k\right\} & \text { otherwise }\end{cases}
$$

We shall prove that the set $\mathcal{A}=\cup_{k=1}^{\infty} A_{k}$ satisfies the theorem.
By construction, $[1, k] \subseteq A_{k}-A_{k}$ for every positive integer $k$ and so $\mathcal{A}-\mathcal{A}=\mathbb{Z}$.

We must prove that $A_{k}$ is a Sidon set for every $k \geq 1$.
This is clear for $k=1$. Suppose that $A_{k-1}$ is a Sidon set. Let $C_{1}=A_{k-1}$ and $C_{2}=\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}$. We shall show that

$$
C_{1} \cup C_{2}=A_{k-1} \cup\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right)
$$

is a Sidon set. Notice that

$$
\begin{gathered}
C_{1}-C_{1} \subseteq\left[-l_{k}, l_{k}\right] \subseteq\left[-\sqrt{p_{k}}, \sqrt{p_{k}}\right] \\
C_{2}-C_{2}=\mathcal{B}_{p_{k}}-\mathcal{B}_{p_{k}} \\
{\left[-\sqrt{p_{k}}, \sqrt{p_{k}}\right] \cap\left(\mathcal{B}_{p_{k}}-\mathcal{B}_{p_{k}}\right)=\{0\}}
\end{gathered}
$$

Then

$$
\left(C_{1}-C_{1}\right) \cap\left(C_{2}-C_{2}\right)=\{0\}
$$

Notice also that if $x \in C_{2}+C_{2}$ then $x \geq 2 p_{k}^{2}+4 l_{k}$, but $C_{1}+C_{1} \subset\left[1,2 l_{k}\right]$. Then

$$
\left(C_{1}+C_{1}\right) \cap\left(C_{2}+C_{2}\right)=\emptyset
$$

If $x \in\left(C_{1}+C_{1}-C_{1}\right)$, then $x \leq 2 l_{k}$, but if $x \in C_{2}$, then $x>2 l_{k}$. Thus,

$$
\left(C_{1}+C_{1}-C_{1}\right) \cap C_{2}=\emptyset
$$

If $x \in C_{2}+C_{2}-C_{2}$, then $x \geq 2\left(p_{k}^{2}+2 l_{k}+1\right)-\left(p_{k}^{2}+p_{k}^{2}+2 l_{k}\right)=2 l_{k}+1$, and if $x \in C_{1}$, then $x \leq l_{k}$. Therefore,

$$
\left(C_{2}+C_{2}-C_{2}\right) \cap C_{1}=\emptyset
$$

Then $A_{k-1} \cup\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right)$ is a Sidon set.
Now we must distiguish two cases:
If $k \in A_{k-1}-A_{k-1}$ then $A_{k}=A_{k-1} \cup\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right)$ and we have proved that it is a Sidon set.

If $k \notin A_{k-1}-A_{k-1}$ then $A_{k}=A_{k-1} \cup\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right) \cup\left\{4 p_{k}^{2}, 4 p_{k}^{2}+k\right\}$ and we have to prove that it is also a Sidon set. In this case we take $C_{1}=A_{k-1} \cup\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right)$ and $C_{2}=\left\{4 p_{k}^{2}, 4 p_{k}^{2}+k\right\}$. We can write

$$
\begin{aligned}
C_{1}-C_{1}= & \left(A_{k-1}-A_{k-1}\right) \cup\left(\mathcal{B}_{p_{k}}-\mathcal{B}_{p_{k}}\right) \\
& \cup\left(A_{k-1}-\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right)\right) \\
& \cup\left(\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right)-A_{k-1}\right)
\end{aligned}
$$

If $x \in\left(A_{k-1}-\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right)\right) \cup\left(\left(\mathcal{B}_{p_{k}}+p_{k}^{2}+2 l_{k}\right)-A_{k-1}\right)$, then $|x| \geq$ $p_{k}^{2}+l_{k}>k$.

If $x \in\left(\mathcal{B}_{p_{k}}-\mathcal{B}_{p_{k}}\right)$ then $x=0$ or $|x|>\sqrt{p_{k}}>k$, then, since $C_{2}-C_{2}=$ $\{-k, 0, k\}$, we have

$$
\left(C_{1}-C_{1}\right) \cap\left(C_{2}-C_{2}\right)=\{0\} .
$$

On the other hand if $x \in C_{2}+C_{2}$ then $x \geq 8 p_{k}^{2}$ but

$$
C_{1}+C_{1} \subset\left[1,2\left(\left(p_{k}^{2}-p_{k}\right)+p_{k}^{2}+2 l_{k}\right)\right] \subset\left[1,4 p_{k}^{2}\right] .
$$

Then

$$
\left(C_{1}+C_{1}\right) \cap\left(C_{2}+C_{2}\right)=\emptyset .
$$

If $x \in C_{1}+C_{1}-C_{1}$ then $x \leq 3 p_{k}^{2}+2 l_{k}<4 p_{k}^{2}$. Thus,

$$
\left(C_{1}+C_{1}-C_{1}\right) \cap C_{2}=\emptyset .
$$

Also we have that $C_{2}+C_{2}-C_{2}=4 p_{k}^{2}+\{-k, 0, k, 2 k\}$, but if $x \in C_{1}$ we have that $x<2 p_{k}^{2}+2 l_{k}<2 p_{k}^{2}+\sqrt{p_{k}}<4 p_{k}^{2}-k$. Thus

$$
\left(C_{2}+C_{2}-C_{2}\right) \cap C_{1}=\emptyset .
$$

To finish the proof of the theorem note that

$$
\begin{array}{r}
\limsup _{x \rightarrow \infty} \frac{\mathcal{A}(x)}{\sqrt{x}} \geq \limsup _{k \rightarrow \infty} \frac{\mathcal{A}\left(2 p_{k}^{2}-p_{k}+l_{k}\right)}{\sqrt{2 p_{k}^{2}-p_{k}+l_{k}}} \geq \\
\limsup _{k \rightarrow \infty} \frac{\left|\mathcal{B}_{p_{k}}\right|}{\sqrt{2 p_{k}^{2}-p_{k}+l_{k}}} \geq \limsup _{k \rightarrow \infty} \frac{p_{k}-2 \sqrt{p_{k}}}{\sqrt{2 p_{k}^{2}-p_{k}+\sqrt{p_{k}} / 2}}=\frac{1}{\sqrt{2}} .
\end{array}
$$

## 4. Remarks and Open problems

4.1. The sequence $t(\mathcal{A})$ associated to a perfect difference set. Any translation of a perfect difference set intersects to itself in exactly one element, and so we can define, for every perfect difference set $\mathcal{A}$, a sequence $t(\mathcal{A})$ whose elements are given by $t_{n}=\mathcal{A} \cap(\mathcal{A}-n)$ for all $n \geq 1$. The sequence $t_{n}$ is very irregular, but the greedy algorithm used in 3] generates a perfect difference set such that $t_{n} \ll n^{3}$. Our method generates a dense Sidon set $\mathcal{A}$, but gives a very poor upper bound for the sequence $t_{n}$.
Problem 1. Does there exists perfect difference set such that $t_{n}=o\left(n^{3}\right)$ ?
4.2. Sidon sets included in perfect difference sets. We have proved that any Sidon set can be perturbed slightly to become a subset of a perfect difference set. Every subset of a perfect difference set is a Sidon set. It is natural to ask if every Sidon set is a subset of a perfect difference set. The answer is negative. To construct a counterexample, we take a perfect difference set $\mathcal{A}$ and consider the set $\mathcal{B}=2 * \mathcal{A}=\{2 a: a \in \mathcal{A}\}$. The set $\mathcal{B}$ has the following properties:
(i) $\mathcal{B}$ is a Sidon set.
(ii) If $n$ is an even integer not in $\mathcal{B}$, then $\mathcal{B} \cup\{n\}$ is not a Sidon set.
(iii) If $m$ and $m^{\prime}$ are distinct odd integers not in $\mathcal{B}$, then $\mathcal{B} \cup\left\{m, m^{\prime}\right\}$ is not a Sidon set.
The Sidon set $\mathcal{B}$ is not a subset of a perfect difference set. Since this construction is rather artificial, we wonder if almost all Sidon sets are subsets of perfect difference sets.

Problem 2. Determine when a Sidon set is a subset of a perfect difference set.
4.3. Perfect $h$-sumsets. Let $\mathcal{A}$ be a set of of integers. For every integer $u$, we denote by $r_{\mathcal{A}}^{h}(u)$ the number of $h$-tuples $\left(a_{1}, \ldots, a_{h}\right) \in \mathcal{A}^{h}$, such that

$$
a_{1} \leq \cdots \leq a_{h}
$$

and

$$
a_{1}+\cdots+a_{h}=u .
$$

We say that $\mathcal{A}$ is a perfect $h$-sumset or a unique representation basis of order $h$ if $r_{\mathcal{A}}^{h}(u)=1$ for every integer $u$. Nathanson [4] proved that for every $h \geq 2$ and for every function $f: \mathbb{Z} \rightarrow \mathbb{N}_{0} \cup\{\infty\}$ such that $\lim \sup _{|u| \rightarrow \infty} f(u) \geq$ 1 there exists a set of integers $\mathcal{A}$ such that

$$
r_{\mathcal{A}}^{h}(u)=f(u)
$$

for every integer $u$. In particular, the perfect $h$-sumsets correspond to the representation function $f \equiv 1$. Nathanson's construction produces a perfect $h$-sumset $\mathcal{A}$ with

$$
A(x) \gg x^{1 /(2 h-1)}
$$

and he asked for denser constructions.
It is easy to modify our approach to get a perfect 2 -sumset $\mathcal{A}$ with $A(x) \gg$ $x^{\sqrt{2}-1+o(1)}$. But for $h \geq 3$ our method cannot be adapted easily, and a more complicated construction is needed. We shall study perfect $h$-sumsets in a forthcoming paper [1].
4.4. Sums and differences. Let $\mathcal{A}$ be a set of integers. For every integer $u$, we denote by $d_{A}(u)$ and $s_{A}(u)$ the number of solutions of

$$
u=a-a^{\prime} \text { with } a, a^{\prime} \in \mathcal{A}
$$

and

$$
u=a+a^{\prime} \text { with } a, a^{\prime} \in \mathcal{A} \text { and } a \leq a^{\prime},
$$

respectively. We say that $\mathcal{A}$ is a perfect difference sumset if $d_{\mathcal{A}}(n)=1$ for all $n \in \mathbb{N}$ and if $s_{\mathcal{A}}(n)=1$ for all $n \in \mathbb{Z}$.

We can extend Theorem 1 and Theorem 3 to perfect difference sumsets. Then it is a natural to ask if, for any two functions $f_{1}: \mathbb{N} \rightarrow \mathbb{N}$ and $f_{2}: \mathbb{Z} \rightarrow$ $\mathbb{N}$, there exists a set $\mathcal{A}$ such that $d_{\mathcal{A}}(n)=f_{1}(n)$ for all $n \in \mathbb{N}$ and $s_{\mathcal{A}}(n)=$ $f_{2}(n)$ for all $n \in \mathbb{Z}$. (Note that perfect difference sumsets correspond to the functions $f_{1} \equiv 1$ and $f_{2} \equiv 1$.) It is not difficult to guess that the answer is no. For example, if $s_{\mathcal{A}}(n)=2$ for infinitely many integers $n$, it is easy to see that $d_{\mathcal{A}}(n) \geq 2$ for infinitely many integers $n$.

Problem 3. Give general conditions for functions $f_{1}$ and $f_{2}$ to assure that there exists a set $\mathcal{A}$ such that $d_{\mathcal{A}}(n) \equiv f_{1}(n)$ and $s_{\mathcal{A}}(n) \equiv f_{2}(n)$.

Is the condition $\liminf _{u \rightarrow \infty} f_{1}(u) \geq 2$ and $\liminf _{|u| \rightarrow \infty} f_{2}(u) \geq 2$ sufficient?

## References

[1] J. Cilleruelo and M. B. Nathanson, Dense sets of integers with prescribed representation functions, Preprint., 2006.
[2] F. Krückeberg, B2-Folgen und verwandte Zahlenfolgen, J. Reine Angew. Math. 206 (1961), 53-60.
[3] V. F. Lev, Reconstructing integer sets from their representation functions, Electron. J. Combin. 11 (2004), no. 1, Research Paper 78, 6 pp. (electronic).
[4] M. B. Nathanson, Every function is the representation function of an additive basis for the integers, Port. Math. (N.S.) 62 (2005), no. 1, 55-72.
[5] I. Z. Ruzsa, Solving a linear equation in a set of integers. I, Acta Arith. 65 (1993), no. 3, 259-282.
[6] , An infinite Sidon sequence, J. Number Theory 68 (1998), no. 1, 63-71.
[7] A. Stöhr, Gelöste und ungelöste Fragen über Basen der natürlichen Zahlenreihe. I, II, J. Reine Angew. Math. 194 (1955), 40-65, 111-140.

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