# PERFECT DIFFERENCE SETS CONSTRUCTED FROM SIDON SETS

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ABSTRACT. A set  $\mathcal{A}$  of positive integers is a *perfect difference set* if every nonzero integer has an unique representation as the difference of two elements of  $\mathcal{A}$ . We construct dense *perfect difference sets* from dense Sidon sets. As a consequence of this new approach we prove that there exists a perfect difference set  $\mathcal{A}$  such that

$$A(x) \gg x^{\sqrt{2}-1-o(1)}$$

Also we prove that there exists a *perfect difference set*  $\mathcal{A}$  such that  $\limsup_{x\to\infty} A(x)/\sqrt{x} \ge 1/\sqrt{2}$ .

#### 1. INTRODUCTION

Let  $\mathbb{Z}$  denote the integers and  $\mathbb{N}$  the positive integers. For nonempty sets of integers  $\mathcal{A}$  and  $\mathcal{B}$ , we define the *difference set* 

$$\mathcal{A} - \mathcal{B} = \{ a - b : a \in \mathcal{A} \text{ and } b \in \mathcal{B} \}.$$

For every integer u, we denote by  $d_{\mathcal{A},\mathcal{B}}(u)$  the number of pairs  $(a,b) \in \mathcal{A} \times \mathcal{B}$ such that u = a - b. Let  $d_{\mathcal{A}}(u)$  the number of pairs  $(a,a') \in \mathcal{A} \times \mathcal{A}$  such that u = a - a'. The set  $\mathcal{A}$  is a *perfect difference set* if  $d_{\mathcal{A}}(u) = 1$  for every integer  $u \neq 0$ . Note that  $\mathcal{A}$  is a perfect difference set if and only  $d_{\mathcal{A}}(u) = 1$  for every positive integer u. For perfect difference sets, a simple counting argument shows that

$$A(x) \ll x^{1/2}$$

where the *counting function* A(x) counts the number of positive elements of  $\mathcal{A}$  not exceeding x.

It is not completely obvious that perfect difference sets exist, but the greedy algorithm produces [3] a perfect difference set  $\mathcal{A} \subseteq \mathbb{N}$  such that

$$A(x) \gg x^{1/3}$$

At the Workshop on Combinatorial and Additive Number Theory (CANT 2004) in New York in May, 2004, Seva Lev (see also [3]) asked if there exists a perfect difference set  $\mathcal{A}$  such that

$$A(x) \gg x^{\delta}$$
 for some  $\delta > 1/3$ .

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We answer this questions affirmatively by constructing perfect difference sets from classical Sidon sets.

We say that a set  $\mathcal{B}$  is a Sidon set if  $d_{\mathcal{B}}(u) \leq 1$  for all integer  $u \neq 0$ .

**Theorem 1.** For every Sidon set  $\mathcal{B}$  and every function  $\omega(x) \to \infty$ , there exists a perfect difference set  $\mathcal{A} \subseteq \mathbb{N}$  satisfying

$$A(x) \ge B(x/3) - \omega(x).$$

It is a difficult problem to construct dense infinite Sidon sets. Ruzsa [6] proved that there exists a Sidon set  $\mathcal{B}$  with  $B(x) \gg x^{\sqrt{2}-1-o(1)}$ . The following result follows easily.

**Theorem 2.** There exists a perfect difference set  $\mathcal{A} \subseteq \mathbb{N}$  such that

$$A(x) \gg x^{\sqrt{2}-1+o(1)}$$
.

Erdős [7] proved that the lower bound  $A(x) \gg x^{1/2}$  does not hold for any Sidon set  $\mathcal{A}$ , and so does not hold for perfect difference sets. However, Krückeberg [2] proved that there exists a Sidon set  $\mathcal{B}$  such that

$$\limsup_{x \to \infty} \frac{B(x)}{\sqrt{x}} \ge \frac{1}{\sqrt{2}}$$

We extend this result to perfect difference sets.

**Theorem 3.** There exists a perfect difference set  $\mathcal{A} \subset \mathbb{N}$  such that

$$\limsup_{x \to \infty} \frac{A(x)}{\sqrt{x}} \ge \frac{1}{\sqrt{2}}$$

Notice that an immediate application of Theorem 1 to Krückeberg's result would give only  $\limsup_{x\to\infty} A(x)x^{-1/2} \ge 1/\sqrt{6}$ .

## 2. Proof of Theorem 1

2.1. Sketch of the proof. The strategy of the proof is the following:

- Modify any dense Sidon set  $\mathcal{B}$  given by dilating it by 3 and removing a suitable *thin* subset of  $3 * \mathcal{B}$ .
- Complete the remainder set  $\mathcal{B}_0 = (3*\mathcal{B}) \setminus \{\text{removed set}\}\$  with a subset of the elements of a very sparse sequence  $\mathcal{U} = \{u_s\}$  by adding, if khas not appeared yet in the difference set, two elements  $u_{2k}, u_{2k+1}$  in the k-th step such that  $u_{2k+1} - u_{2k} = k$ .

2.2. The auxiliary sequence  $\mathcal{U}$ . For any strictly increasing function  $g : \mathbb{N} \to \mathbb{N}$  and  $k \ge 1$ , we define integers  $u_{2k}$  and  $u_{2k+1}$  by

$$\begin{cases} u_{2k} = 4^{g(k)} + \epsilon_k \\ u_{2k+1} = 4^{g(k)} + \epsilon_k + k \end{cases}$$

where

$$\epsilon_k = \begin{cases} 1 & \text{if } k \equiv 2 \pmod{3} \\ 0 & \text{otherwise} \end{cases}$$

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For all positive integers k we have

$$u_{2k+1} - u_{2k} = k.$$

Let  $\mathcal{U}_k = \{u_{2k}, u_{2k+1}\}$  and  $\mathcal{U}_{<\ell} = \bigcup_{k < \ell} \mathcal{U}_k$ . It will be useful to state some properties of the sequence  $\mathcal{U} = \{u_i\}_{i=2}^{\infty}$ .

**Lemma 1.** The sequence  $\mathcal{U} = \{u_i\}_{i=2}^{\infty}$  satisfies the following properties:

- (i) For all  $i \geq 2$ ,  $u_i \not\equiv 0 \pmod{3}$ .
- (ii) For all  $k \geq 2$ , for  $u \in \mathcal{U}_k$ , and for all  $u', u'', u''' \in \mathcal{U}_{< k}$ , we have u + u' > u'' + u'''.
- (iii) If  $k \ge 2$ ,  $u \in \mathcal{U}_k$ , and  $u' \in \mathcal{U}_{< k}$ , then u u' > u/2.

*Proof.* (i) By construction.

(ii) Since g(k) is strictly increasing we have  $k \leq g(k)$  and so

$$4k < 4^k < 4^{g(k)}$$

for all  $k \geq 2$ . It follows that

$$u'' + u''' \le 2u_{2k-1} \le 2(4^{g(k-1)} + k)$$
$$\le 2\left(4^{g(k)-1} + k\right) \le \frac{4^{g(k)}}{2} + 2k$$
$$< 4^{g(k)} < u < u + u'.$$

(iii) For  $k \ge 2$  we have

$$u' \le 4^{g(k-1)} + (k-1) + \epsilon_{k-1}$$
  
$$\le 4^{g(k)-1} + k$$
  
$$< 2 \cdot 4^{g(k)-1} = \frac{4^{g(k)}}{2}$$
  
$$\le u/2$$

and so u - u' > u/2.

2.3. Construction of the Sidon set  $\mathcal{B}_0$ . Take a Sidon set  $\mathcal{B}$  and consider the set  $\mathcal{B}' = 3 * \mathcal{B} = \{3b : b \in \mathcal{B}\}$ . Then  $\mathcal{B}'$  is a Sidon set such that  $b \equiv 0 \pmod{3}$  for all  $b \in \mathcal{B}'$  and  $\mathcal{B}'(x) = \mathcal{B}\left(\frac{x}{3}\right)$ .

The set  $\mathcal{B}_0$  will be the set  $\mathcal{B}' = 3 * \mathcal{B}$  after we remove all the elements  $b \in \mathcal{B}'$  that satisfy at least one of the followings conditions:

- **c1:** b = u u' + b' for some  $b' \in \mathcal{B}'$ , b > b' and  $u, u' \in \mathcal{U}$  such that  $u \in \mathcal{U}_r, u' \in \mathcal{U}_{< r}$  for some r.
- **c2:** b = u + u' b' for some  $b' \in \mathcal{B}$ ,  $b \ge b'$  and  $u, u' \in \mathcal{U}$ .
- **c3:** b = u + u' u'' for some  $u \in \mathcal{U}_r$ ,  $u' \in \mathcal{U}$ , and  $u'' \in \mathcal{U}_{< r}$  with  $u' \le u$ . **c4:**  $|b - u_i| \le i$  for some  $u_i \in \mathcal{U}$ .

2.4. The inductive step. We shall construct the set  $\mathcal{A}$  in Theorem 1 by adjoining terms to the *nice* Sidon set  $\mathcal{B}_0$  obtained above. More precisely, the sequence  $\mathcal{A}$  satisfying the conditions of the theorem will be

$$\mathcal{A} = \bigcup_{k=0}^{\infty} \mathcal{A}_k$$

where  $\mathcal{A}_k$  will be defined by  $\mathcal{A}_0 = \mathcal{B}_0$  and for,  $k \ge 1$ ,

$$\mathcal{A}_{k} = \begin{cases} \mathcal{A}_{k-1} \cup \mathcal{U}_{k} & \text{if } k \notin \mathcal{A}_{k-1} - \mathcal{A}_{k-1} \\ \mathcal{A}_{k-1} & \text{otherwise.} \end{cases}$$

**Lemma 2.** For every positive integer k we have

$$[-k,k] \subseteq \mathcal{A}_k - \mathcal{A}_k$$

and so

$$d_{\mathcal{A}}(n) \ge 1$$

for all integers n.

Proof. Clear.

2.5.  $\mathcal{A}$  is a Sidon set. First we state two lemmas.

**Lemma 3.** Let  $A_1$  and  $A_2$  be nonempty disjoint sets of integers and let  $A = A_1 \cup A_2$  For every integer n we have

$$d_A(n) = d_{A_1}(n) + d_{A_2}(n) + d_{A_1,A_2}(n) + d_{A_2,A_1}(n)$$

where

$$d_{A_i,A_j}(n) = \#\{(a,a') \in A_i \times A_j, \ a-a' = n\}.$$

*Proof.* This follows from the identity

$$(A_1 \cup A_2) \times (A_1 \cup A_2) = (A_1 \times A_1) \cup (A_2 \times A_2) \cup (A_1 \times A_2) \cup (A_2 \times A_1).$$

**Lemma 4.** If  $n \in A_{r-1} - U_r$  then

- (i) |n| > r, and so  $d_{\mathcal{U}_r,\mathcal{A}_{r-1}}(r) = d_{\mathcal{A}_{r-1},\mathcal{U}_r}(r) = 0$ .
- (ii)  $d_{\mathcal{A}_{r-1}}(n) = 0.$
- (iii)  $d_{\mathcal{U}_r,\mathcal{A}_{r-1}}(n) = 0.$

*Proof.* Write n = a - u, where  $a \in \mathcal{A}_{r-1}$  and  $u \in \mathcal{U}_r = \{u_{2r}, u_{2r+1}\}$ .

(i) If  $a = b \in \mathcal{B}_0$  we have that |b - u| > 2r > r because, by condition (c4), we have removed all elements b from  $\mathcal{B}$  such that  $|b - u_i| \le i$ .

If  $a = u' \in \mathcal{U}_{< r}$  then we apply Lemma 1 (iii) to conclude that

$$|u'-u| > \frac{u}{2} \ge \frac{4^{g(r)}}{2} > r.$$

(ii) Since  $\mathcal{A}_{r-1} \subseteq \mathcal{B}_0 \cup \mathcal{U}_{< r}$ , it follows that  $d_{\mathcal{A}_{r-1}}(n) \leq d_{\mathcal{B}_0 \cup \mathcal{U}_{< r}}(n) \leq d_{\mathcal{B}_0}(n) + d_{\mathcal{U}_{< r}}(n) + d_{\mathcal{B}_0, \mathcal{U}_{< r}}(n) + d_{\mathcal{U}_{< r}, \mathcal{B}_0}(n).$ If  $a = b \in \mathcal{B}_0$ , then n = b - u and

- (1)  $b \equiv 0 \pmod{3}$  but  $u \not\equiv 0 \pmod{3}$ , hence  $b u \not\equiv 0 \pmod{3}$  and  $d_{\mathcal{B}_0}(b-u) = 0$  (by Lemma 1 (i)),
- (2)  $d_{\mathcal{U}_{\leq r}}(b-u) = 0$  (by condition (c3)),
- (3)  $d_{\mathcal{B}_0,\mathcal{U}_{\leq r}}(b-u) = 0$  (by condition (c1)),
- (4)  $d_{\mathcal{U}_{< r}, \mathcal{B}_0}(b-u) = 0$  (by condition (c2)).

If  $a = u' \in \mathcal{U}_{< r}$ , then n = u' - u and

- (1)  $d_{\mathcal{B}_0}(u'-u) = 0$  (by condition (c1)),
- (2)  $d_{\mathcal{U}_{< r}}(u' u) = 0$  (by Lemma 1 (ii)),
- (3) If u' u = b u'' with  $u'' \in \mathcal{U}_{< r}$ , then Lemma 1 (iii) implies that  $0 < b = u' + u'' u \le 0$ , and so  $d_{B_0,\mathcal{U}_{< r}}(u' u) = 0$ .
- (4)  $d_{\mathcal{U}_{< r}, B_0}(u' u) = 0$  (by condition (c3)).
- (iii) Again, since  $\mathcal{A}_{r-1} \subseteq \mathcal{B}_0 \cup \mathcal{U}_{< r}$  we have that

$$d_{\mathcal{U}_r,\mathcal{A}_{r-1}}(n) \le d_{\mathcal{U}_r,\mathcal{B}_0}(n) + d_{\mathcal{U}_r,\mathcal{U}_{< r}}(n).$$

If  $a = b \in \mathcal{B}_0$  then  $d_{\mathcal{U}_r,\mathcal{B}_0}(b-u) = 0$  (by condition (c2)) and  $d_{\mathcal{U}_r,\mathcal{U}_{< r}}(b-u) = 0$  (by condition (c3)).

If  $a = u' \in \mathcal{U}_{< r}$  then  $d_{\mathcal{U}_r, \mathcal{B}_0}(u' - u) = 0$  (by condition (c3)). Finally, we have  $d_{\mathcal{U}_r, \mathcal{U}_{< r}}(u' - u) = 0$ , since if u' - u = u'' - u''',  $u'' \in \mathcal{U}_r$ ,  $u''' \in \mathcal{U}_{< r}$ , then 0 > u' - u = u'' - u''' > 0. This completes the proof.

**Lemma 5.** For every positive integer n we have

 $d_{\mathcal{A}}(n) \le 1$ 

and so A is a perfect difference set.

*Proof.* We will use induction to prove that, for every  $r \ge 0$ ,

 $d_{\mathcal{A}_r}(n) \leq 1$  for every nonzero integer n.

This is true for r = 0 because  $\mathcal{A}_0 = \mathcal{B}_0$  is a subset of a Sidon set.

We assume that the statement is true for r-1 and shall prove it for r. If  $d_{\mathcal{A}_{r-1}}(r) = 1$  then  $\mathcal{A}_r = \mathcal{A}_{r-1}$  and there is nothing to prove. Suppose that  $d_{\mathcal{A}_{r-1}}(r) = 0$ , and so  $\mathcal{A}_r = \mathcal{A}_{r-1} \cup \mathcal{U}_r$ . Since we have added two new elements  $u_{2r}, u_{2r+1}$  to  $\mathcal{A}_{r-1}$ , it is possible that there are *new* representations of a positive integer n so that  $d_{\mathcal{A}_r}(n) > 1$ . We shall prove that this cannot happen.

By Lemma 3, we can write

$$d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1}}(n) + d_{\mathcal{U}_r}(n) + d_{\mathcal{A}_{r-1},\mathcal{U}_r}(n) + d_{\mathcal{U}_r,\mathcal{A}_{r-1}}(n)$$

If n = r, then Lemma 4 (i) and the relation  $u_{2r+1} - u_{2r} = r$  imply that  $d_{\mathcal{A}_r}(r) = d_{\mathcal{A}_{r-1}}(r) + d_{\mathcal{U}_r}(r) + d_{\mathcal{A}_{r-1},\mathcal{U}_r}(r) + d_{\mathcal{U}_r,\mathcal{A}_{r-1}}(r) = 0 + 1 + 0 + 0 = 1$ If  $n \neq r$ , then

 $d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1}}(n) + d_{\mathcal{A}_{r-1},\mathcal{U}_r}(n) + d_{\mathcal{U}_r,\mathcal{A}_{r-1}}(n).$ 

If  $n \in \mathcal{A}_{r-1} - \mathcal{U}_r$  (the case  $n \in \mathcal{U}_r - \mathcal{A}_{r-1}$  is similar), then we can write n = a - u where  $a \in \mathcal{A}_{r-1}, \ u \in \mathcal{U}_r$ . Applying Lemma 4 (ii) and Lemma 4 (iii), we obtain

$$d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1},\mathcal{U}_r}(n).$$

If  $d_{\mathcal{A}_{r-1},\mathcal{U}_r}(n) \geq 2$ , then there exist  $a, a' \in \mathcal{A}_{r-1}$  such that  $a - u_{2r} = a' - u_{2r+1}$ . This implies that

$$a' - a = u_{2r+1} - u_{2r} = r \in \mathcal{A}_{r-1} - \mathcal{A}_{r-1}$$

which is false, so  $d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1},\mathcal{U}_r}(n) \leq 1$ . If  $n \notin (\mathcal{A}_{r-1} - \mathcal{U}_r) \cup (\mathcal{U}_r - \mathcal{A}_{r-1})$  then

$$d_{\mathcal{A}_r}(n) = d_{\mathcal{A}_{r-1}}(n) \le 1.$$

This completes the proof.

2.6. The counting function A(x). We have

$$A(x) \ge B_0(x) = B(x) - R(x) = B(x/3) - R(x)$$

where  $R = R_1 \cup R_2 \cup R_3 \cup R_4$  and  $R_i$  denotes the set of elements of B removed by condition  $(c_i)$ , i = 1, 2, 3, 4.

**Lemma 6.** Let U(x) denote the counting function of the set  $\mathcal{U}$ . For the sets  $R_1, R_2, R_3, R_4$  defined above, we have

(i)  $R_1(x) \le U^2(2x)$ . (ii)  $R_2(x) \le U^2(2x)$ . (iii)  $R_3(x) \le U^3(2x)$ (iv)  $R_4(x) \le 2U^2(2x) + U(2x)$ .

*Proof.* (i) We have

 $R_1(x) = \#\{b \in B : b \le x \text{ and } b \text{ satisfies condition (c1)}\}.$ 

Because B is a Sidon set, for every pair of integers  $u, u' \in \mathcal{U}$  there exists at most one pair of integers  $b, b' \in \mathcal{B}$  such that b - b' = u - u'. The condition  $x \ge b > b'$  implies that  $0 < u - u' \le x$ . On the other hand Lemma 1 (iii) implies that u - u' > u/2 and so u < 2x and

$$R_1(x) \le \#\{(u, u'), u' < u, u < 2x\} \le U^2(2x).$$

(ii) Again, because B is a Sidon set, for every pair  $u, u' \in \mathcal{U}$  there exists at most one pair  $b, b' \in \mathcal{B}$  such that b + b' = u + u'. The condition  $x \ge b \ge b'$  implies  $u, u' \le 2x$  and so

$$R_2(x) \le \#\{(u, u') \in \mathcal{U} \times \mathcal{U} : u \le 2x, u' \le 2x\} \le U^2(2x).$$

(iii) If  $u \in \mathcal{U}_r$ ,  $u'' \in \mathcal{U}_{< r}$ , then Lemma 1 (iii) implies that b = u + u' - u'' > u - u'' > u - u'' > u/2 and so

$$R_3(x) = \#\{b \in B : b \le x \text{ and } b \text{ satisfies condition (c3)}\}$$
$$\leq \#\{(u, u', u'') \in \mathcal{U} \times \mathcal{U} \times \mathcal{U} : u < 2x, u'' < u, u' \le u\}$$
$$< U(2x)^3.$$

(iv) We have

$$R_4(x) = \#\{b \in B : b \le x \text{ and } |b - u_i| \le i \text{ for some } u_i \in \mathcal{U}\}$$
$$\le \#\{n \in \mathbb{N} : n \le x \text{ and } |n - u_i| \le i \text{ for some } i\}.$$

If  $n \leq x$  and  $|n - u_i| \leq i$ , then  $u_i \leq n + i \leq x + i$ . Since  $u_2 = 4^{g(1)} \geq 4$ ,  $u_3 = 4^{g(1)+1} \geq 16$ , and, for  $i \geq 4$ ,

$$u_i \ge 4^{g((i-1)/2)} \ge 4^{(i-1)/2} = 2^{i-1} \ge 2i.$$

Therefore,  $u_i \leq x + i \leq x + u_i/2$  and so  $u_i \leq 2x$ . It follows that  $i \leq U(2x)$  and so

$$R_4(x) \le \#\{n \le x : |n - u_i| \le U(2x) \text{ and } u_i \le 2x\}$$
  
$$\le (2U(2x) + 1)U(2x) = 2U(2x)^2 + U(2x).$$

This completes the proof of the lemma.

Finally, given any function  $\omega(x) \to \infty$  we have that

$$A(x) \ge B(x/3) - \left(U(2x)^3 + 4U^2(2x) + U(2x)\right) \ge B(x/3) - \omega(x)$$

for any function  $g : \mathbb{N} \to \mathbb{N}$  and sequence  $\mathcal{U}$  growing fast enough. This completes the proof of Theorem 1.

### 3. Proof of Theorem 3

**Lemma 7.** If  $C_1$  and  $C_2$  are Sidon sets such that  $(C_i - C_i) \cap (C_j - C_j) = \{0\}$ ,  $(C_i + C_i) \cap (C_j + C_j) = \emptyset$  and  $(C_i + C_i - C_i) \cap C_j = \emptyset$  for  $i \neq j$ , then  $C_1 \cup C_2$  is a Sidon set.

*Proof.* Obvious.

**Lemma 8.** For each odd prime p there exist a Sidon set  $\mathcal{B}_p$  such that

(i)  $\mathcal{B}_p \subseteq [1, p^2].$ (ii)  $(\mathcal{B}_p - \mathcal{B}_p) \cap [-\sqrt{p}, \sqrt{p}] = \emptyset.$ (iii)  $|\mathcal{B}_p| > p - 2\sqrt{p}.$ 

Proof. Ruzsa [5] constructed, for each prime p, a Sidon set  $R_p \subseteq [1, p^2 - p]$ with  $|R_p| = p - 1$ . We consider the subset  $\mathcal{B}_p$  of  $R_p$  that we obtain by removing all elements  $b \in R_p$  such that  $0 < |b - b'| \le \sqrt{p}$  for some  $b' \in R_p$ . Since  $R_p$  is a Sidon set, it follows that we have removed at most  $\sqrt{p}$  elements from  $R_p$ , and so  $|\mathcal{B}_p| \ge |R_p| - \sqrt{p} = p - \sqrt{p} - 1 > p - 2\sqrt{p}$ .

Proof of Theorem 3. We shall construct an increasing sequence of finite set  $A_1 \subseteq A_2 \subseteq A_3 \subseteq \cdots$  such that  $\mathcal{A} = \bigcup_{k=1}^{\infty} A_k$  is a perfect difference set satisfying Theorem 3.

In the following,  $l_k$  will denote the largest integer in the set  $A_{k-1}$ , and  $p_k$  the least prime greater than  $4l_k^2$ . Let

$$A_1 = \{0, 1\}.$$

Then  $l_2 = 1$  and  $p_2 = 5$ . We define

$$A_{k} = \begin{cases} A_{k-1} \cup \left(\mathcal{B}_{p_{k}} + p_{k}^{2} + 2l_{k}\right) & \text{if } k \in A_{k-1} - A_{k-1} \\ A_{k-1} \cup \left(\mathcal{B}_{p_{k}} + p_{k}^{2} + 2l_{k}\right) \cup \{4p_{k}^{2}, 4p_{k}^{2} + k\} & \text{otherwise.} \end{cases}$$

We shall prove that the set  $\mathcal{A} = \bigcup_{k=1}^{\infty} A_k$  satisfies the theorem.

By construction,  $[1,k] \subseteq A_k - A_k$  for every positive integer k and so  $\mathcal{A} - \mathcal{A} = \mathbb{Z}$ .

We must prove that  $A_k$  is a Sidon set for every  $k \ge 1$ .

This is clear for k = 1. Suppose that  $A_{k-1}$  is a Sidon set. Let  $C_1 = A_{k-1}$ and  $C_2 = \mathcal{B}_{p_k} + p_k^2 + 2l_k$ . We shall show that

$$C_1 \cup C_2 = A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k)$$

is a Sidon set. Notice that

$$C_1 - C_1 \subseteq [-l_k, l_k] \subseteq [-\sqrt{p_k}, \sqrt{p_k}]$$
$$C_2 - C_2 = \mathcal{B}_{p_k} - \mathcal{B}_{p_k}$$
$$[-\sqrt{p_k}, \sqrt{p_k}] \cap (\mathcal{B}_{p_k} - \mathcal{B}_{p_k}) = \{0\}.$$

Then

$$(C_1 - C_1) \cap (C_2 - C_2) = \{0\}$$

Notice also that if  $x \in C_2 + C_2$  then  $x \ge 2p_k^2 + 4l_k$ , but  $C_1 + C_1 \subset [1, 2l_k]$ . Then

$$(C_1+C_1)\cap(C_2+C_2)=\emptyset$$

If  $x \in (C_1 + C_1 - C_1)$ , then  $x \leq 2l_k$ , but if  $x \in C_2$ , then  $x > 2l_k$ . Thus,

$$(C_1 + C_1 - C_1) \cap C_2 = \emptyset.$$

If  $x \in C_2 + C_2 - C_2$ , then  $x \ge 2(p_k^2 + 2l_k + 1) - (p_k^2 + p_k^2 + 2l_k) = 2l_k + 1$ , and if  $x \in C_1$ , then  $x \le l_k$ . Therefore,

$$(C_2 + C_2 - C_2) \cap C_1 = \emptyset.$$

Then  $A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k)$  is a Sidon set.

Now we must distiguish two cases:

If  $k \in A_{k-1} - A_{k-1}$  then  $A_k = A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k)$  and we have proved that it is a Sidon set.

If  $k \notin A_{k-1} - A_{k-1}$  then  $A_k = A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k) \cup \{4p_k^2, 4p_k^2 + k\}$ and we have to prove that it is also a Sidon set. In this case we take  $C_1 = A_{k-1} \cup (\mathcal{B}_{p_k} + p_k^2 + 2l_k)$  and  $C_2 = \{4p_k^2, 4p_k^2 + k\}$ . We can write

$$C_{1} - C_{1} = (A_{k-1} - A_{k-1}) \cup (\mathcal{B}_{p_{k}} - \mathcal{B}_{p_{k}}) \cup (A_{k-1} - (\mathcal{B}_{p_{k}} + p_{k}^{2} + 2l_{k})) \cup ((\mathcal{B}_{p_{k}} + p_{k}^{2} + 2l_{k}) - A_{k-1}).$$

If  $x \in (A_{k-1} - (\mathcal{B}_{p_k} + p_k^2 + 2l_k)) \cup ((\mathcal{B}_{p_k} + p_k^2 + 2l_k) - A_{k-1})$ , then  $|x| \ge p_k^2 + l_k > k$ .

If  $x \in (\mathcal{B}_{p_k} - \mathcal{B}_{p_k})$  then x = 0 or  $|x| > \sqrt{p_k} > k$ , then, since  $C_2 - C_2 = \{-k, 0, k\}$ , we have

$$(C_1 - C_1) \cap (C_2 - C_2) = \{0\}.$$

On the other hand if  $x \in C_2 + C_2$  then  $x \ge 8p_k^2$  but

$$C_1 + C_1 \subset [1, 2((p_k^2 - p_k) + p_k^2 + 2l_k)] \subset [1, 4p_k^2].$$

Then

$$(C_1 + C_1) \cap (C_2 + C_2) = \emptyset.$$
  
If  $x \in C_1 + C_1 - C_1$  then  $x \le 3p_k^2 + 2l_k < 4p_k^2$ . Thus,  
 $(C_1 + C_1 - C_1) \cap C_2 = \emptyset.$ 

Also we have that  $C_2 + C_2 - C_2 = 4p_k^2 + \{-k, 0, k, 2k\}$ , but if  $x \in C_1$  we have that  $x < 2p_k^2 + 2l_k < 2p_k^2 + \sqrt{p_k} < 4p_k^2 - k$ . Thus

$$(C_2 + C_2 - C_2) \cap C_1 = \emptyset.$$

To finish the proof of the theorem note that

$$\limsup_{x \to \infty} \frac{\mathcal{A}(x)}{\sqrt{x}} \ge \limsup_{k \to \infty} \frac{\mathcal{A}(2p_k^2 - p_k + l_k)}{\sqrt{2p_k^2 - p_k + l_k}} \ge$$
$$\limsup_{k \to \infty} \frac{|\mathcal{B}_{p_k}|}{\sqrt{2p_k^2 - p_k + l_k}} \ge \limsup_{k \to \infty} \frac{p_k - 2\sqrt{p_k}}{\sqrt{2p_k^2 - p_k + \sqrt{p_k}/2}} = \frac{1}{\sqrt{2}}.$$

### 4. Remarks and Open problems

4.1. The sequence  $t(\mathcal{A})$  associated to a perfect difference set. Any translation of a perfect difference set intersects to itself in exactly one element, and so we can define, for every perfect difference set  $\mathcal{A}$ , a sequence  $t(\mathcal{A})$  whose elements are given by  $t_n = \mathcal{A} \cap (\mathcal{A} - n)$  for all  $n \geq 1$ . The sequence  $t_n$  is very irregular, but the greedy algorithm used in [3] generates a perfect difference set such that  $t_n \ll n^3$ . Our method generates a dense Sidon set  $\mathcal{A}$ , but gives a very poor upper bound for the sequence  $t_n$ .

**Problem 1.** Does there exists perfect difference set such that  $t_n = o(n^3)$ ?

4.2. Sidon sets included in perfect difference sets. We have proved that any Sidon set can be perturbed slightly to become a subset of a perfect difference set. Every subset of a perfect difference set is a Sidon set. It is natural to ask if *every* Sidon set is a subset of a perfect difference set. The answer is negative. To construct a counterexample, we take a perfect difference set  $\mathcal{A}$  and consider the set  $\mathcal{B} = 2 * \mathcal{A} = \{2a : a \in \mathcal{A}\}$ . The set  $\mathcal{B}$  has the following properties:

- (i)  $\mathcal{B}$  is a Sidon set.
- (ii) If n is an even integer not in  $\mathcal{B}$ , then  $\mathcal{B} \cup \{n\}$  is not a Sidon set.

(iii) If m and m' are distinct odd integers not in  $\mathcal{B}$ , then  $\mathcal{B} \cup \{m, m'\}$  is not a Sidon set.

The Sidon set  $\mathcal{B}$  is not a subset of a perfect difference set. Since this construction is rather artificial, we wonder if almost all Sidon sets are subsets of perfect difference sets.

**Problem 2.** Determine when a Sidon set is a subset of a perfect difference set.

4.3. **Perfect** *h*-sumsets. Let  $\mathcal{A}$  be a set of of integers. For every integer u, we denote by  $r_{\mathcal{A}}^{h}(u)$  the number of *h*-tuples  $(a_1, \ldots, a_h) \in \mathcal{A}^h$ , such that

$$a_1 \leq \cdots \leq a_h$$

and

$$a_1 + \dots + a_h = u.$$

We say that  $\mathcal{A}$  is a *perfect h-sumset* or a *unique representation basis of* order h if  $r^h_{\mathcal{A}}(u) = 1$  for every integer u. Nathanson [4] proved that for every  $h \geq 2$  and for every function  $f : \mathbb{Z} \to \mathbb{N}_0 \cup \{\infty\}$  such that  $\limsup_{|u|\to\infty} f(u) \geq 1$  there exists a set of integers  $\mathcal{A}$  such that

$$r^h_A(u) = f(u)$$

for every integer u. In particular, the *perfect h-sumsets* correspond to the representation function  $f \equiv 1$ . Nathanson's construction produces a *perfect h-sumset* A with

$$A(x) \gg x^{1/(2h-1)}$$

and he asked for denser constructions.

It is easy to modify our approach to get a perfect 2-sumset  $\mathcal{A}$  with  $A(x) \gg x^{\sqrt{2}-1+o(1)}$ . But for  $h \geq 3$  our method cannot be adapted easily, and a more complicated construction is needed. We shall study perfect *h*-sumsets in a forthcoming paper [1].

4.4. Sums and differences. Let  $\mathcal{A}$  be a set of integers. For every integer u, we denote by  $d_A(u)$  and  $s_A(u)$  the number of solutions of

$$u = a - a'$$
 with  $a, a' \in \mathcal{A}$ 

and

$$u = a + a'$$
 with  $a, a' \in \mathcal{A}$  and  $a \leq a'$ ,

respectively. We say that  $\mathcal{A}$  is a *perfect difference sumset* if  $d_{\mathcal{A}}(n) = 1$  for all  $n \in \mathbb{N}$  and if  $s_{\mathcal{A}}(n) = 1$  for all  $n \in \mathbb{Z}$ .

We can extend Theorem 1 and Theorem 3 to perfect difference sumsets. Then it is a natural to ask if, for any two functions  $f_1 : \mathbb{N} \to \mathbb{N}$  and  $f_2 : \mathbb{Z} \to \mathbb{N}$ , there exists a set  $\mathcal{A}$  such that  $d_{\mathcal{A}}(n) = f_1(n)$  for all  $n \in \mathbb{N}$  and  $s_{\mathcal{A}}(n) = f_2(n)$  for all  $n \in \mathbb{Z}$ . (Note that perfect difference sumsets correspond to the functions  $f_1 \equiv 1$  and  $f_2 \equiv 1$ .) It is not difficult to guess that the answer is no. For example, if  $s_{\mathcal{A}}(n) = 2$  for infinitely many integers n, it is easy to see that  $d_{\mathcal{A}}(n) \geq 2$  for infinitely many integers n.

**Problem 3.** Give general conditions for functions  $f_1$  and  $f_2$  to assure that there exists a set  $\mathcal{A}$  such that  $d_{\mathcal{A}}(n) \equiv f_1(n)$  and  $s_{\mathcal{A}}(n) \equiv f_2(n)$ .

Is the condition  $\liminf_{u\to\infty}f_1(u)\geq 2$  and  $\liminf_{|u|\to\infty}f_2(u)\geq 2$  sufficient?

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