

# Monochromatic triangles in two-colored plane

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## Abstract

We prove that for any partition of the plane into a closed set  $C$  and an open set  $O$  and for any configuration  $T$  of three points, there is a translated and rotated copy of  $T$  contained in  $C$  or in  $O$ .

Apart from that, we consider partitions of the plane into two sets whose common boundary is a union of piecewise linear curves. We show that for any such partition and any configuration  $T$  which is a vertex set of a non-equilateral triangle there is a copy of  $T$  contained in the interior of one of the two partition classes. Furthermore, we give the characterization of these “polygonal” partitions that avoid copies of a given equilateral triple.

These results support a conjecture of Erdős, Graham, Montgomery, Rothschild, Spencer and Straus, which states that every two-coloring of the plane contains a monochromatic copy of any nonequilateral triple of points; on the other hand, we disprove a stronger conjecture by the same authors, by providing non-trivial examples of two-colorings that avoid a given equilateral triple.

## 1 Introduction

Euclidean Ramsey theory addresses the problems of the following kind: assume that a finite configuration  $X$  of points is given; for what values of  $c$  and  $d$  is it true that every coloring of the  $d$ -dimensional Euclidean space by  $c$  colors contains a monochromatic congruent copy of  $X$ ? The first systematic treatise on this theory appears in 1973 in a series of papers [2, 3, 4] by Erdős, Graham, Montgomery, Rothschild, Spencer and Straus. Since that time, many strong

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results have been obtained in this field, often related to high-dimensional configurations (see, e.g., [5, 7, 8, 9] or the survey [6]); however, there are basic ‘low-dimensional’ problems that remain open.

In this paper, we consider the special case when  $d = 2$ ,  $c = 2$  and  $|X| = 3$ ; in other words, we study the configurations of three points in the Euclidean plane colored by two colors. We use the term *triangle* to refer to any set of three points, including collinear triples of points, which we call *degenerate* triangles. An  $(a, b, c)$ -triangle is a triangle whose edges, in anti-clockwise order, have respective lengths  $a$ ,  $b$  and  $c$ . A  $(1, 1, 1)$ -triangle is also called a *unit triangle*.

We say that a set of points  $X \subseteq \mathbb{R}^2$  is a *copy* of a set of points  $Y \subseteq \mathbb{R}^2$ , if  $X$  can be obtained from  $Y$  by translations and rotations in the plane. A *coloring* is a partition of  $\mathbb{R}^2$  into two sets  $\mathfrak{B}$  and  $\mathfrak{W}$ . The elements of  $\mathfrak{B}$  and  $\mathfrak{W}$  are called *black points* and *white points*, respectively. We use the term *boundary of  $\chi$*  to refer to the common boundary of the sets  $\mathfrak{B}$  and  $\mathfrak{W}$ . Given a coloring  $\chi = (\mathfrak{B}, \mathfrak{W})$ , we say that a set of points  $X$  is *monochromatic*, if  $X \subseteq \mathfrak{B}$  or  $X \subseteq \mathfrak{W}$ .

We say that a coloring  $\chi$  *contains* a triangle  $T$ , if there exists a monochromatic set  $T'$  which is a copy of  $T$ ; otherwise, we say that  $\chi$  *avoids*  $T$ .

A coloring that avoids the unit triangle is easy to obtain: consider a coloring  $\chi^*$  that partitions the plane into alternating half-open strips of width  $\frac{\sqrt{3}}{2}$ ; formally, a point  $(x, y)$  is black if and only if  $n\sqrt{3} < y \leq (n + \frac{1}{2})\sqrt{3}$  for some integer  $n$ . It can be easily checked that  $\chi^*$  avoids the unit triangle. We can even change the color of some of the points on the boundaries of the strips without creating any monochromatic unit triangle. Erdős et al. [4, Conjecture 1] have conjectured that this is essentially the only example of colorings avoiding a given triangle:

**Conjecture 1.1** (Erdős et al. [4]). *For every triangle  $T$  and every coloring  $\chi$ , if  $\chi$  avoids  $T$ , then  $T$  is an equilateral  $(l, l, l)$ -triangle and  $\chi$  is equal to an  $l$ -times scaled copy of the coloring  $\chi^*$  defined above, up to possible modifications of the colors of the points on the boundary of the strips.*

In Section 3 of this paper, we present a counterexample to this conjecture, and define a general class of colorings (which includes  $\chi^*$  as a special case) that avoid the unit triangle.

On the other hand, the following conjecture by Erdős et al. [4, Conjecture 3] remains open:

**Conjecture 1.2** (Erdős et al. [4]). *Every coloring  $\chi$  contains every nonequilateral triangle  $T$ .*

In the past, it has been shown that Conjecture 1.2 holds for special types of triangles  $T$  (see, e.g., [4, 6, 10]). Our approach is different: we prove that the conjecture is valid for a restricted class of colorings  $\chi$  and arbitrary  $T$ . In Section 2, we show that every coloring that partitions  $\mathbb{R}^2$  into a closed set and an open set contains every triangle  $T$ . Then, in Section 3, we consider *polygonal* colorings, whose boundary is a union of piecewise linear curves (see page 6 for

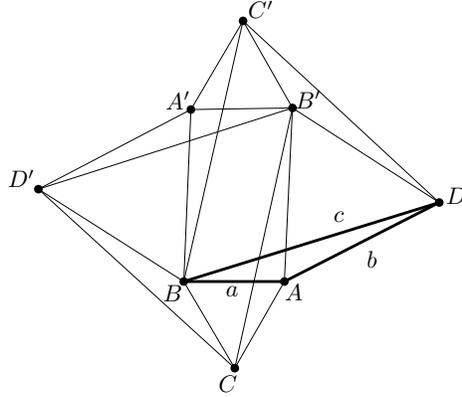


Figure 1: The illustration of the proof of Lemma 1.3

the precise definition). We show that Conjecture 1.2 holds for the polygonal colorings, but there are polygonal counterexamples to the stronger Conjecture 1.1. In fact, we are able to characterize all these polygonal counterexamples.

The following lemma from [4] offers a useful insight into the topic of monochromatic triangles in two-colored plane:

**Lemma 1.3.** *Let  $\chi$  be a coloring of the plane. The following holds:*

- (i) *If  $\chi$  contains an  $(a, a, a)$ -triangle for some  $a > 0$ , then  $\chi$  contains an  $(a, b, c)$ -triangle, for every  $b, c > 0$  such that  $a, b, c$  satisfy the (possibly degenerate) triangle inequality.*
- (ii) *If  $\chi$  contains an  $(a, b, c)$ -triangle, then  $\chi$  contains an  $(x, x, x)$ -triangle for some  $x \in \{a, b, c\}$ .*

*Proof.* The essence of the proof is the configuration in Figure 1. The configuration consists of two  $(a, a, a)$ -triangles  $ABC$  and  $A'B'C'$ , two  $(b, b, b)$ -triangles  $ADB'$  and  $A'D'B$  and two  $(c, c, c)$ -triangles  $BDC'$  and  $B'D'C$ . To prove the first part of the lemma, assume, for a given  $\chi$ , that there is a monochromatic  $(a, a, a)$ -triangle  $ABC$ , and choose arbitrary  $b$  and  $c$  satisfying triangle inequality with  $a$ . Assume that  $A, B$  and  $C$  are all black. Furthermore, assume for contradiction that no  $(a, b, c)$ -triangle is monochromatic. Considering the configuration in Fig. 1, we deduce that the points  $B', D$  and  $D'$  are all white, otherwise one of the  $(a, b, c)$ -triangles  $BAD, CAB'$  and  $CBD'$  would be monochromatic. Then,  $A'$  is black, due to  $B'A'D'$ , and  $C'$  is white, due to  $C'A'B$ . It follows that  $C'B'D$  is monochromatic, a contradiction.

The second part is proved by an analogous argument: assume that  $BAD$  is an all-white monochromatic triangle and that the statement does not hold. Then  $B', C$  and  $C'$  are all black, due to  $ADB', ABC$  and  $BDC'$ .  $A'$  is white, due to  $A'B'C'$ ;  $D'$  is black, due to  $A'D'B$ , and  $B'D'C$  is monochromatic.

This concludes the proof.  $\square$

From Lemma 1.3, we obtain directly the following facts:

**Corollary 1.4.** *For every coloring  $\chi$  the following holds:*

- (i)  $\chi$  contains every triangle if and only if  $\chi$  contains every equilateral triangle.
- (ii)  $\chi$  contains every non-equilateral triangle if and only if there is an  $a_0 > 0$  such that  $\chi$  contains the equilateral  $(a, a, a)$ -triangle for all values of  $a > 0$  different from  $a_0$ .
- (iii)  $\chi$  contains an  $(a, b, c)$ -triangle if and only if  $\chi$  contains a  $(b, a, c)$ -triangle.

## 2 Coloring by closed and open sets

The aim of this section is to prove the following result:

**Theorem 2.1.** *Let  $\chi = (\mathfrak{B}, \mathfrak{W})$  be a coloring such that  $\mathfrak{B}$  is closed and  $\mathfrak{W}$  is open. Then  $\chi$  contains every triangle  $T$ .*

By Corollary 1.4, it suffices to prove Theorem 2.1 for the case when  $T$  is an arbitrary equilateral triangle. Moreover, since scaling does not affect the topological properties of  $\mathfrak{B}$  and  $\mathfrak{W}$ , we only need to consider the case when  $T$  is the unit triangle. Before stating the proof, we introduce a definition and prove an auxiliary result.

**Definition 2.2.** Let  $\varepsilon > 0$ . An  $(a, b, c)$ -triangle whose edge-lengths satisfy  $1 - \varepsilon \leq a, b, c \leq 1 + \varepsilon$  is called an  $\varepsilon$ -almost unit triangle.

Suppose that an orthogonal coordinate system is given in the plane. For  $a > 0$ , let  $Q(a)$  be the closed square with vertices  $(a, a), (-a, a), (-a, -a), (a, -a)$ .

**Proposition 2.3.** *Let  $Q(3) = \mathfrak{B} \cup \mathfrak{W}$  be a decomposition of the square  $Q(3)$  into two disjoint sets such that there is no monochromatic unit triangle in  $Q(3)$ . Then for every  $\varepsilon > 0$  both  $\mathfrak{B}$  and  $\mathfrak{W}$  contain an  $\varepsilon$ -almost unit triangle.*

*Proof.* Let  $\varepsilon$  be a given positive number. Assume that we are given a partition  $\mathfrak{B} \cup \mathfrak{W} = Q(3)$  such that  $Q(3)$  does not contain any monochromatic unit triangle. For contradiction, assume that one of the classes, wlog the class  $\mathfrak{B}$ , does not contain any  $\varepsilon$ -almost unit triangle.

There is a white point  $S$  and a black point  $R$  in  $Q(1)$  such that  $|R - S| < \varepsilon$  (otherwise the whole square  $Q(1)$  would be monochromatic). Let  $\mathcal{C}$  be the unit circle centered at  $S$ . For every  $\alpha \in \mathbb{R}$ , let  $K(\alpha)$  denote the point of  $\mathcal{C}$  with coordinates  $(x_S + \cos(\alpha), y_S + \sin(\alpha))$ , where  $(x_S, y_S)$  are the coordinates of  $S$ .

Note that the distance between  $R$  and any point on  $\mathcal{C}$  is always in the interval  $(1 - \varepsilon, 1 + \varepsilon)$ ; thus, for every  $\alpha$ , the points  $K(\alpha)$  and  $K(\alpha + \frac{\pi}{3})$  must have different colors, otherwise they would form a monochromatic white unit triangle with  $S$  or a monochromatic black  $\varepsilon$ -almost unit triangle with  $R$ .

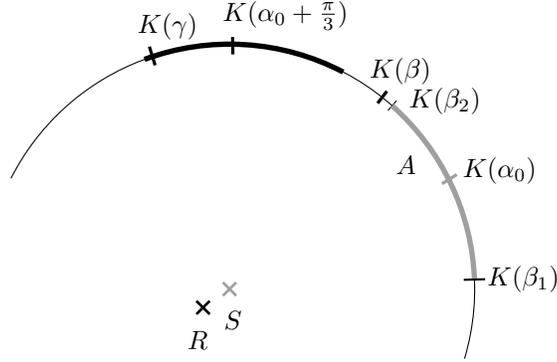


Figure 2: Illustration of the proof of Proposition 2.3

Let  $K(\alpha_0)$  be a white point, then  $K(\alpha_0 + \frac{\pi}{3})$  is black (see Fig. 2). Note that for every  $\alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)$  the distance between  $K(\alpha)$  and  $K(\alpha_0 + \frac{\pi}{3})$  is in the interval  $(1 - \varepsilon, 1 + \varepsilon)$ , so the whole arc  $\{K(\alpha); \alpha \in (\alpha_0 - \varepsilon, \alpha_0 + \varepsilon)\}$  is white. Let  $A = \{K(\alpha); \alpha \in (\beta_1, \beta_2)\}$  be the maximal open white arc of  $\mathcal{C}$  containing the point  $K(\alpha_0)$ . Then the whole arc  $\{K(\alpha); \alpha \in (\beta_1 + \frac{\pi}{3}, \beta_2 + \frac{\pi}{3})\}$  is black. By definition of  $A$ , there exists  $\beta \in (\beta_2, \beta_2 + \frac{\varepsilon}{2})$  such that  $K(\beta)$  is black. There also exists  $\gamma \in (\beta_2 + \frac{\pi}{3} - \frac{\varepsilon}{2}, \beta_2 + \frac{\pi}{3})$  such that  $K(\gamma)$  is black. But then  $(\gamma - \beta) \in (\frac{\pi}{3} - \varepsilon, \frac{\pi}{3})$ , so the distance between the black points  $K(\beta)$  and  $K(\gamma)$  is in the interval  $(1 - \varepsilon, 1)$ , hence the three points  $R, K(\beta), K(\gamma)$  form a black  $\varepsilon$ -almost unit triangle—a contradiction.  $\square$

We are now ready to prove the main result of this section.

*Proof of Theorem 2.1.* Let  $\chi = (\mathfrak{B}, \mathfrak{W})$  be a coloring, with  $\mathfrak{B}$  closed. By Corollary 1.4, it is sufficient to show that  $\chi$  contains the unit triangle. Assume, for contradiction, that this is not the case. Let  $\mathfrak{B}_0 = Q(3) \cap \mathfrak{B}$  and let  $\mathfrak{W}_0 = Q(3) \cap \mathfrak{W}$ . Clearly, neither  $\mathfrak{B}_0$  nor  $\mathfrak{W}_0$  contain the unit triangle, so by Proposition 2.3, both these sets contain  $\varepsilon$ -almost unit triangles for every  $\varepsilon > 0$ . In particular, the set  $\mathfrak{B}_0$  contains, for every  $n \in \mathbb{N}$ , a  $\frac{1}{n}$ -almost unit triangle  $X_n Y_n Z_n$ .

Since  $\mathfrak{B}_0$  is a compact set, the set  $\mathfrak{B}_0^3 = \mathfrak{B}_0 \times \mathfrak{B}_0 \times \mathfrak{B}_0$  is compact as well. The sequence  $\{(X_n, Y_n, Z_n); n \in \mathbb{N}\}$  is an infinite sequence of points in  $\mathfrak{B}_0^3$ , so there exists a convergent subsequence  $\{(X_{n_k}, Y_{n_k}, Z_{n_k}); k \in \mathbb{N}\}$ . Let  $(X, Y, Z) \in \mathfrak{B}_0^3$  be its limit. Then  $X, Y, Z \in \mathfrak{B}$  are limits of the sequences  $\{X_{n_k}; k \in \mathbb{N}\}$ ,  $\{Y_{n_k}; k \in \mathbb{N}\}$ , and  $\{Z_{n_k}; k \in \mathbb{N}\}$ , respectively. The Euclidean distance is a continuous function of two variables, so  $|X - Y| = \lim_{k \rightarrow \infty} |X_{n_k} - Y_{n_k}| = 1$ , similarly  $|Y - Z| = |Z - X| = 1$ . Thus,  $\{X, Y, Z\}$  is a black unit triangle in  $Q(3)$ , which is a contradiction.  $\square$

### 3 Polygonal colorings

Throughout this section,  $\mathcal{C}(A)$  denotes the unit circle with center  $A$ , and  $\mathcal{D}(A)$  denotes the closed unit disc with center  $A$ .

In this section, we consider *polygonal* colorings of the plane, defined as follows:

**Definition 3.1.** A coloring  $\chi = (\mathfrak{B}, \mathfrak{W})$  is said to be *polygonal*, if it satisfies the following conditions (see an example in Fig. 3):

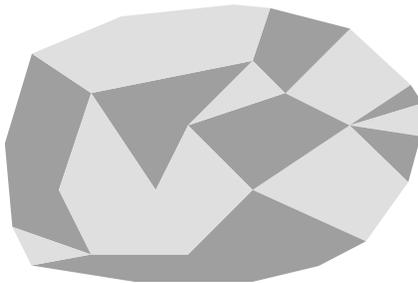


Figure 3: Example of a polygonal coloring

- Each of the two sets  $\mathfrak{B}$  and  $\mathfrak{W}$  is contained in the closure of its interior.
- The boundary of  $\chi$  (denoted by  $\Delta$ ) is a union of straight line segments (called *boundary segments*). Two boundary segments may only intersect at their endpoints. We allow these segments to be unbounded, i.e., a boundary segment may in fact be a half-line or a line. An endpoint of a boundary segment is called a *boundary vertex*. We may assume that if exactly two boundary segments meet at a boundary vertex, then the two segments do not form a straight angle, because otherwise they could be replaced with a single boundary segment. Note that with this condition, the boundary segments and boundary vertices of  $\chi$  are determined uniquely.
- Every bounded region of the plane is intersected by only finitely many boundary segments (which implies that every bounded region contains only finitely many boundary vertices).

Note that these conditions imply that a sufficiently small disc around an interior point of a boundary segment is separated by the boundary segment into two halves, one of which is colored black and the other white. Note also that we make no assumptions about the colors of the points on the boundary  $\Delta$ .

We say that a coloring  $\chi'$  is a *twin* of a coloring  $\chi$  if the two colorings have the same boundary and they assign the same colors to the points outside this boundary.

The main aim of this section is to prove that every polygonal coloring contains every nonequilateral triangle, and to characterize the polygonal colorings that avoid an equilateral triangle. To achieve this, we need the following definition:

**Definition 3.2.** A coloring  $\chi = (\mathfrak{B}, \mathfrak{W})$  is called *zebra-like* if it has the following form: the boundary of  $\chi$  is a disjoint union of infinitely many continuous curves  $\mathcal{L}_i; i \in \mathbb{Z}$  with the following properties (see Fig. 4):

- (a) There is a unit vector  $\vec{x}$  such that for every  $i \in \mathbb{Z}$ ,  $\mathcal{L}_i + \vec{x} = \mathcal{L}_i$ . In other words, the  $\mathcal{L}_i$  are invariant upon a translation of length 1.
- (b) For every  $i \in \mathbb{Z}$ , the curve  $\mathcal{L}_{i+1}$  is a translated copy of  $\mathcal{L}_i$ . Moreover, there is a unit vector  $\vec{y}$  orthogonal to  $\vec{x}$ , so that

$$\mathcal{L}_{i+1} = \mathcal{L}_i + \frac{1}{2}\vec{x} + \frac{\sqrt{3}}{2}\vec{y}.$$

In other words, for an arbitrary boundary point  $X \in \mathcal{L}_i$ , the points  $Y = X + \vec{x}$  and  $Z = X + \frac{1}{2}\vec{x} + \frac{\sqrt{3}}{2}\vec{y}$  belong to the boundary as well. Note that  $XYZ$  is a unit triangle, and that  $Y \in \mathcal{L}_i$  and  $Z \in \mathcal{L}_{i+1}$ .

- (c) For every  $i \in \mathbb{Z}$ , the interior of the region delimited by  $\mathcal{L}_i \cup \mathcal{L}_{i+1}$  is colored with a different color than the interior of the region delimited by  $\mathcal{L}_{i-1} \cup \mathcal{L}_i$ .
- (d) For two points  $A$  and  $B$ , let  $\theta_{AB}$  denote the size of the acute angle formed by the segment  $AB$  and the vector  $\vec{x}$ . For every  $i \in \mathbb{Z}$  and every two points  $A \in \mathcal{L}_i$  and  $B \in \mathcal{L}_{i+1}$ , the following holds:  $\|AB\| > 1$  if and only if  $\theta_{AB} < \frac{\pi}{3}$ .

This last condition can also be stated in the following equivalent form: Let  $A \in \mathcal{L}_i$  be an arbitrary point on the boundary. Let  $B_1 = A - \frac{1}{2}\vec{x} + \frac{\sqrt{3}}{2}\vec{y}$  and  $B_2 = A + \frac{1}{2}\vec{x} + \frac{\sqrt{3}}{2}\vec{y}$  (the two points  $B_1, B_2$  belong to  $\mathcal{L}_{i+1}$  by the previous conditions), and let  $A' = A + \sqrt{3}\vec{y}$  (so that  $A' \in \mathcal{L}_{i+2}$ ). Under these assumptions, the portion of  $\mathcal{L}_{i+1}$  between  $B_1$  and  $B_2$  is contained inside of the closed lens-shaped region  $\mathcal{D}(A) \cap \mathcal{D}(A')$  and no other point of  $\mathcal{L}_{i+1}$  is inside this region.

We stress that a zebra-like coloring is not necessarily polygonal.

### 3.1 The result

The following theorem is the main result of this section:

**Theorem 3.3.** *For a polygonal coloring  $\chi$ , the following conditions are equivalent:*

- (C1) *The coloring  $\chi$  is a zebra-like polygonal coloring.*
- (C2) *The coloring  $\chi$  has a twin  $\chi'$  which avoids the unit triangle.*

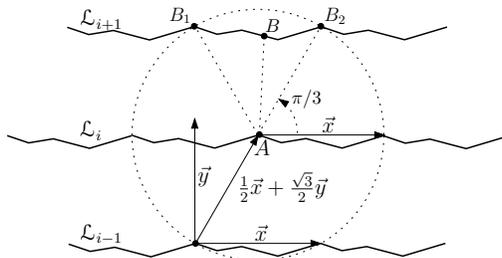


Figure 4: The boundary of a zebra-like coloring

(C3) For every monochromatic unit triangle  $ABC$ , at least one of the three points  $A, B$  and  $C$  belongs to the boundary of  $\chi$ .

Clearly, the condition (C2) of Theorem 3.3 implies the condition (C3), so we only need to prove that (C1) implies (C2) and that (C3) implies (C1).

The proof is organized as follows: we first prove that (C3) $\Rightarrow$ (C1). This part of the proof proceeds in several steps: first of all, we use the condition (C3) to describe the set  $\Delta(\chi) \cap \mathcal{C}(A)$ , where  $A$  is a boundary point. Then we apply a continuity argument to extend this information into a global description of  $\chi$ .

Next, in Theorem 3.19, we prove that every (not necessarily polygonal) zebra-like coloring has a twin that avoids the unit triangle, which shows that (C1) $\Rightarrow$ (C2), completing the proof of Theorem 3.3.

In the last part of this section, we show that Theorem 3.3 implies that every polygonal coloring contains a monochromatic copy  $T$  of a given non-equilateral triangle, with the vertices of  $T$  avoiding the boundary.

## 3.2 The proof

We begin with an auxiliary lemma:

**Lemma 3.4.** *Let  $q_1, q_2, q_3$  be (not necessarily distinct) lines in the plane, not all three parallel. Then exactly one of the following possibilities holds:*

1. *The lines  $q_1, q_2, q_3$  intersect at a common point and every two of them form an angle  $\frac{\pi}{3}$ .*
2. *There exist only finitely many unit triangles  $ABC$  such that  $A \in q_1, B \in q_2$  and  $C \in q_3$ .*

*Proof.* It can be easily checked that the two conditions cannot hold simultaneously: in fact, if the three lines satisfy the first condition, then for every point  $A \in q_1$  whose distance from the other two lines is at most 1 there are points  $B \in q_2$  and  $C \in q_3$  such that  $ABC$  is a unit triangle. We now show that at least one of the two conditions holds.

Since the three lines are not all parallel, we may assume that neither  $q_1$  nor  $q_2$  is parallel to  $q_3$ . Consider a Cartesian coordinate system whose  $y$ -axis is

$q_3$ . There exist real numbers  $a_1, a_2, b_1, b_2$  such that for  $i \in \{1, 2\}$  we have  $q_i = \{(x, y) \in \mathbb{R}^2; y = a_i x + b_i\}$ . Let  $ABC$  be a unit triangle with  $A = (x_1, y_1) \in q_1$ ,  $B = (x_2, y_2) \in q_2$  and  $C \in q_3$ , and assume that  $A, B, C$  are in the counter-clockwise order (the other case is symmetric). Then  $C = (\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}) + \frac{\sqrt{3}}{2}(y_1-y_2, x_2-x_1)$ . The point  $C$  lies on  $q_3$ , which implies the following equality:

$$\frac{x_1+x_2}{2} + \frac{\sqrt{3}(y_1-y_2)}{2} = 0 \quad (1)$$

Points  $A$  and  $B$  are at the distance 1, from which we get

$$(x_1-x_2)^2 + (y_1-y_2)^2 = 1 \quad (2)$$

By combining (1) and (2) and eliminating  $y_1, y_2$  we get

$$\left(\frac{x_1+x_2}{2}\right)^2 = \frac{3}{4}(1-(x_1-x_2)^2),$$

which yields

$$x_1^2 + x_2^2 - x_1x_2 = \frac{3}{4}. \quad (3)$$

Substituting  $y_1 = a_1x_1 + b_1$  and  $y_2 = a_2x_2 + b_2$  into (1) gives

$$\frac{1+\sqrt{3}a_1}{2}x_1 + \frac{1-\sqrt{3}a_2}{2}x_2 + \frac{\sqrt{3}}{2}(b_1-b_2) = 0 \quad (4)$$

If both  $\frac{1+\sqrt{3}a_1}{2}$  and  $\frac{1-\sqrt{3}a_2}{2}$  are equal to zero, then the equality (4) degenerates and we get that  $a_1 = -\frac{1}{\sqrt{3}}$ ,  $a_2 = \frac{1}{\sqrt{3}}$  and  $b_1 = b_2$ , so the first case of the statement holds.

In the other case, suppose (wlog) that  $\frac{1+\sqrt{3}a_1}{2} \neq 0$ . From (4) we can obtain that  $x_1 = cx_2 + d$  for some reals  $c, d$ . By substituting it into (3) we get a quadratic equation for the variable  $x_2$ , where the leading coefficient is equal to  $c^2 - c + 1 = (c - \frac{1}{2})^2 + \frac{3}{4} > 0$ , so there exist at most two possible values for  $x_2$ , thus at most two possible locations of  $B$  and at most four possible unit triangles  $ABC$ .  $\square$

Throughout the rest of this section, we assume that  $\chi$  is a fixed polygonal coloring satisfying the condition (C3) of Theorem 3.3. Every boundary segment can be regarded as a common edge of two (possibly unbounded) polygonal regions, one of which is white and the other black. We choose an orientation of the boundary segments in the following way: a boundary segment with endpoints  $A$  and  $B$  is directed from  $A$  to  $B$  if the white region adjacent to this segment is on the left hand side from the point of view of an observer walking from  $A$  to  $B$ .

**Definition 3.5.** A boundary point  $A \in \Delta$  is called *feasible*, if  $A$  is not a boundary vertex, and the unit circle  $\mathcal{C}(A)$  does not contain any boundary vertex. An *infeasible* point is a point on the boundary that is not feasible.

We may easily see that every bounded subset of the plane contains only finitely many infeasible points.

The first step in the proof of the main result is the description of the set of all the boundary points at the unit distance from a given feasible point  $A$ .

Let  $A$  be a fixed feasible point, let  $s$  be the boundary segment containing  $A$ . The set  $\Delta \cap \mathcal{C}(A)$  is finite, by the definition of polygonal coloring; on the other hand, this set is nonempty, otherwise we could find two points  $B, C$  of  $\mathcal{C}(A)$  such that  $ABC$  is a unit triangle, with  $B$  and  $C$  in the interior of the same color class. By shifting the triangle  $ABC$  slightly in a suitable direction, we would obtain a monochromatic unit triangle avoiding the boundary, which is forbidden by the condition (C3).

In the following arguments, we will use a Cartesian coordinate system whose origin is the point  $A$ , and whose  $x$ -axis is parallel to  $s$  and has the same orientation. We shall assume that the  $x$ -axis and the segment  $s$  is directed left-to-right and the  $y$ -axis is directed bottom-to-top. Assuming this coordinate system, we let  $P(\alpha, A)$  denote the point of  $\mathcal{C}(A)$  with coordinates  $(\cos(\alpha), \sin(\alpha))$ . If no ambiguity arises, we write  $P(\alpha)$  instead of  $P(\alpha, A)$ .

**Lemma 3.6.** *Let  $B = P(\alpha)$  be an arbitrary element of  $\Delta \cap \mathcal{C}(A)$ , let  $t$  be the boundary segment containing  $B$  (the segment  $t$  is determined uniquely, because  $A$  is a feasible point). Then the segments  $s$  and  $t$  are parallel.*

*Proof.* For contradiction, assume that  $s$  and  $t$  are not parallel, let  $\sigma \in (0, \pi)$  be the angular slope of  $t$  with respect to the coordinate system established above, i.e.,  $\sigma$  is the angle formed by the lines containing  $s$  and  $t$ .

First of all, note that the point  $C = P(\alpha + \frac{\pi}{3})$  lies on the boundary  $\Delta$ ; otherwise, a sufficiently small translation of the unit triangle  $ABC$  in a suitable direction would yield a counterexample to condition (C3) (here we use the assumption that  $s$  and  $t$  are not parallel). Let  $u$  be the boundary segment containing  $C$ , and let  $\tau$  be the angular slope of  $u$ .

Secondly, we may deduce that  $\{\sigma, \tau\} = \{\frac{\pi}{3}, \frac{2\pi}{3}\}$ , and the three lines containing  $s$ ,  $t$  and  $u$  all meet at one point. If this were not the case, then by Lemma 3.4 there would be only finitely many unit triangles with vertices belonging to the three segments  $s$ ,  $t$  and  $u$ . Thus, we could find a unit triangle  $A'B'C'$  with  $A' \in s$ ,  $B' \in t$  and  $C' \notin \Delta$ , which is impossible, by the argument presented in the previous paragraph. By repeating this argument with  $\{\alpha + \frac{i\pi}{3}; i = 1, \dots, 5\}$  in place of  $\alpha$ , we obtain the following conclusions:

- The six points  $\{P(\alpha + \frac{i\pi}{3}); i = 1, \dots, 6\}$  all belong to the boundary  $\Delta$ .
- The lines passing through the boundary segments containing these six points all meet at one point.
- The boundary segments containing  $P(\alpha)$ ,  $P(\alpha + \frac{2\pi}{3})$  and  $P(\alpha + \frac{4\pi}{3})$  all have the same slope.

This is a contradiction, because three parallel segments intersecting a circle in three distinct points cannot belong to a single line, and two parallel lines do not intersect.  $\square$

**Lemma 3.7.**  $P(\frac{\pi}{2}) \notin \Delta$ ,  $P(-\frac{\pi}{2}) \notin \Delta$ .

*Proof.* For contradiction, assume that  $B = P(\frac{\pi}{2}) \in \Delta$  (the case of  $P(-\frac{\pi}{2})$  is symmetric), let  $t$  denote the boundary segment containing  $B$ . Let  $C = P(\frac{\pi}{6})$ . We distinguish the following cases:

- The segment  $t$  has the same orientation as the segment  $s$ . In this case, by applying a rotation around the center  $C$  and then, if  $C \in \Delta$ , a suitable translation, we may transform the triple  $ABC$  into a monochromatic triple with vertices avoiding the boundary, contradicting (C3).
- The segments  $s$  and  $t$  have opposite orientations (i.e.,  $t$  is oriented right-to-left, which means that there is a white region touching  $t$  from below); furthermore, either  $C \in \Delta$  or  $C$  is in the interior of the white color. In such case, we may rotate the configuration  $ABC$  around the center of the segment  $AB$  to obtain a unit triangle in the interior of the white color.
- The segments  $s$  and  $t$  have opposite orientations and the point  $C$  is in the interior of the black color. Let  $\theta$  be the maximal angle with the properties that for every  $\alpha \in (\frac{\pi}{2}, \frac{\pi}{2} + \theta)$  the point  $P(\alpha)$  lies in the interior of the white color and for every  $\alpha \in (\frac{\pi}{6}, \frac{\pi}{6} + \theta)$  the point  $P(\alpha)$  lies in the interior of the black color. The value of  $\theta$  is well defined, and by the previous assumptions,  $0 < \theta < \frac{\pi}{3}$ . Let  $B' = P(\frac{\pi}{2} + \theta)$  and  $C' = P(\frac{\pi}{6} + \theta)$ . By the maximality of  $\theta$ , at least one of the two points lies on the boundary, and the boundary segment passing through this point is directed left-to-right (see Fig. 5). As in the first case of this proof, we may rotate and translate the configuration  $AB'C'$  to obtain a monochromatic unit triangle.

In all cases we get a contradiction. □

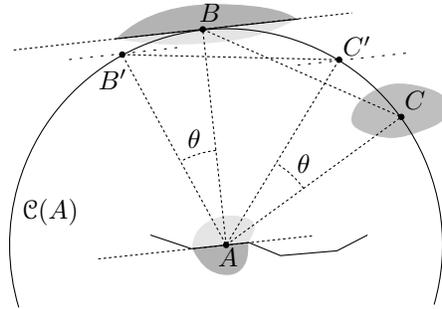


Figure 5: Illustration of the proof of Lemma 3.7

The previous two lemmas imply that if  $A$  is a feasible point, then no boundary segment is tangent to  $\mathcal{C}(A)$ .

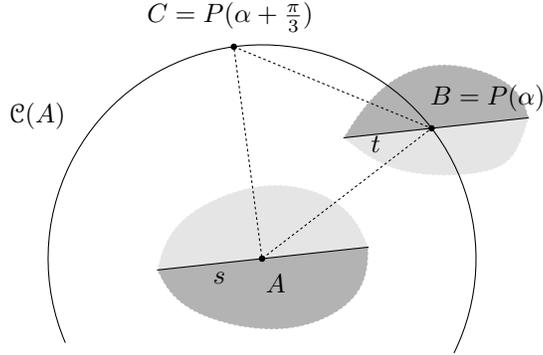


Figure 6: Illustration of the proof of Lemma 3.9

**Lemma 3.8.** *Let  $B = P(\alpha) \in \Delta$  be a point on the boundary, let  $t$  be the boundary segment containing this point. If  $\alpha \in (\frac{\pi}{6}, \frac{5\pi}{6})$  or  $\alpha \in (\frac{7\pi}{6}, \frac{11\pi}{6})$ , then  $s$  and  $t$  have opposite orientation. If  $|\alpha| < \frac{\pi}{6}$  or  $|\alpha - \pi| < \frac{\pi}{6}$ , then  $s$  and  $t$  have the same orientation.*

*Proof.* We first consider the case  $\alpha \in (\frac{\pi}{6}, \frac{5\pi}{6})$  or  $\alpha \in (\frac{7\pi}{6}, \frac{11\pi}{6})$ . The proof is analogous to the proof of the first part of Lemma 3.7: if  $t$  had the same orientation as  $s$ , we could take  $C = P(\frac{\pi}{3} + \alpha)$  and then by rotating and translating the unit triangle  $ABC$  we would get a contradiction. Note that the condition  $\alpha \in (\frac{\pi}{6}, \frac{5\pi}{6}) \cup (\frac{7\pi}{6}, \frac{11\pi}{6})$  guarantees that  $C$  is either the leftmost or the rightmost point of the triangle  $ABC$ , so whenever we start rotating the triangle  $ABC$  around  $C$ , the two points  $A, B$  move into the interior of the same color.

The case  $|\alpha| < \frac{\pi}{6}$  or  $|\alpha - \pi| < \frac{\pi}{6}$  can be proven analogously.  $\square$

**Lemma 3.9.**  *$P(\alpha) \in \Delta$  if and only if  $P(\alpha + \frac{\pi}{3}) \in \Delta$ .*

*Proof.* It suffices to prove one implication, the other case is symmetric. Assume that for some  $\alpha$  we have  $P(\alpha) \in \Delta$  and  $P(\frac{\pi}{3} + \alpha) \notin \Delta$ . Let  $B = P(\alpha)$ ,  $C = P(\frac{\pi}{3} + \alpha)$ , and let  $t$  be the boundary segment containing  $B$ . We consider the following cases:

- If  $s$  and  $t$  have opposite orientation, we may rotate  $ABC$  around the center of  $AB$  to obtain a monochromatic unit triangle in the interior of one color (see Fig. 6). Here we use the fact that  $\alpha \neq \frac{\pi}{2}$ , which follows from Lemma 3.7.
- If  $s$  and  $t$  have the same orientation, a small translation in a suitable direction transforms  $ABC$  into a monochromatic unit triangle.

In both cases we get a contradiction.  $\square$

**Lemma 3.10.** *For every  $\theta$  there is exactly one value of  $\alpha \in [\theta, \theta + \frac{\pi}{3})$  such that  $P(\alpha) \in \Delta$ .*

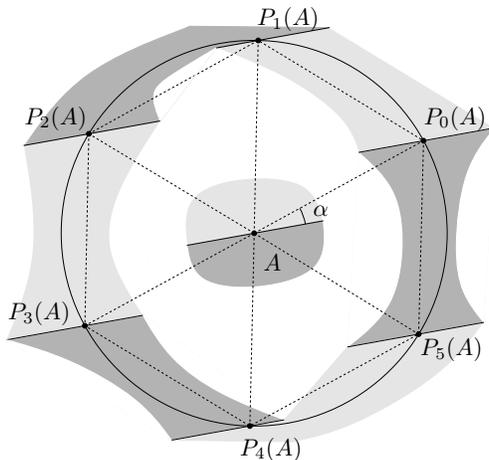


Figure 7: Illustration of Claim 3.11

*Proof.* By Lemma 3.9, if the statement holds for some value of  $\theta$ , it holds for all other values of  $\theta$  as well. Thus, it is enough to prove the lemma for  $\theta = \frac{\pi}{2}$ .

Clearly, there is at least one  $\alpha \in [\frac{\pi}{2}, \frac{5\pi}{6}]$  such that  $P(\alpha) \in \Delta$ ; otherwise, the set  $\mathcal{C}(A) \cap \Delta$  would be empty, which is impossible.

Assume that there are  $\alpha$  and  $\alpha'$  such that  $\frac{\pi}{2} \leq \alpha < \alpha' < \frac{5\pi}{6}$  with  $P(\alpha) \in \Delta$  and  $P(\alpha') \in \Delta$ . Let us fix  $\alpha$  and  $\alpha'$  as small as possible. Let  $t$  and  $t'$  be the boundary segments containing  $P(\alpha)$  and  $P(\alpha')$ . The circle  $\mathcal{C}(A)$  consists of alternating black and white arcs and one of these arcs has  $P(\alpha)$  and  $P(\alpha')$  for endpoints. It follows that one of the segments  $t, t'$  has the same orientation as the segment  $s$ , contradicting Lemma 3.8.  $\square$

Before we proceed with the proof of the main result, we summarize the lemmas proved so far (and introduce some related notation) in the following claim (see Fig. 7):

**Claim 3.11.** *Let  $A \in \Delta$  be an arbitrary feasible point. The circle  $\mathcal{C}(A)$  intersects the boundary  $\Delta$  at exactly six points, which form the vertex set of a regular hexagon. These six points will be denoted by  $P_0(A), \dots, P_5(A)$ , where  $P_i(A) = P(\alpha + \frac{i\pi}{3}, A)$  with  $\alpha \in (-\frac{\pi}{6}, \frac{\pi}{6})$  (this determines  $P_i(A)$  uniquely). The boundary segments containing the six points  $P_i(A)$  are all parallel to the boundary segment  $s$  containing the point  $A$ . The boundary segments containing the points  $P_0(A)$  and  $P_3(A)$  have the same orientation as  $s$ , whereas the boundary segments containing  $P_1(A), P_2(A), P_4(A)$  and  $P_5(A)$  have opposite orientation.*

Now we use Claim 3.11 to get more global information about the boundary.

**Lemma 3.12.** *Let  $u_1$  and  $u_2$  be two boundary segments that share a common endpoint  $X$ . The size of the convex angle formed by these two segments is greater than  $\frac{2\pi}{3}$ .*

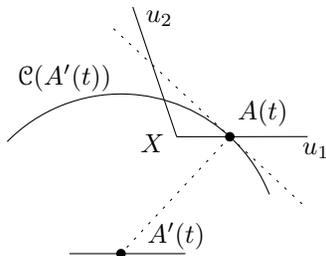


Figure 8: Illustration of the proof of Lemma 3.12

*Proof.* For contradiction, assume that for some  $u_1, u_2$  and  $X$ , the statement of the lemma does not hold (see Fig. 8). We may assume that the convex angle determined by  $u_1$  and  $u_2$  does not contain any other boundary segment with endpoint  $X$ . Furthermore, we may assume that the segment  $u_1$  is directed from  $X$  to the other endpoint.

For  $0 < t < |u_1|$ , let  $A(t) \in u_1$  denote the point with  $|A(t) - X| = t$  and let  $A'(t) = P_4(A(t))$ . There exists  $\varepsilon > 0$  such that for all  $0 < t < \varepsilon$  the points  $A(t)$  are feasible, the points  $A'(t)$  are feasible as well and lie on a common boundary segment. By our assumption, the convex angles between the ray  $A(t)A'(t)$  and the segments  $u_1, u_2$  directed from  $X$  are both greater than  $\frac{\pi}{2}$ . It follows that if  $t$  is sufficiently small, the tangent to the circle  $\mathcal{C}(A(t))$  at  $A(t)$  intersects both segments  $u_1, u_2$  and so does the circle  $\mathcal{C}(A(t))$ , contradicting Claim 3.11.  $\square$

An important consequence of Lemma 3.12 is that no three boundary segments share a common endpoint. Hence, every connected component of the boundary is either an infinite piecewise linear curve, or a simple closed piecewise linear curve (i.e. the boundary of a simple polygon). We will call these curves *boundary components* or simply *components*.

**Definition 3.13.** Let  $A$  be a point on the boundary. For  $t \in \mathbb{R}$ , let  $A(t)$  denote the point of the same boundary component as  $A$ , such that the directed length of the part of the boundary starting at  $A$  and ending at  $A(t)$  is equal to  $t$ .  $A(t)$  is clearly a continuous function of  $t$ . If  $A(t)$  is a feasible point, we let  $p_i(t) = P_i(A(t))$ , for  $i = 0, \dots, 5$ .

It is easy to see that the functions  $p_i$  are continuous on a sufficiently small neighborhood of every value of  $t$  for which  $A(t)$  is a feasible point. Our next aim is to show that these functions can be extended into continuous functions by suitably defining the values of  $p_i(t)$  when  $A(t)$  is not feasible. It is not obvious that the functions  $p_i$  can be extended in this way: the definition of  $P_i(A(t))$  uses the Cartesian system whose  $x$ -axis is parallel with the boundary segment containing  $A(t)$ . Hence, if  $A_1$  and  $A_2$  are two feasible points belonging to two distinct boundary segments of the same boundary component, it might not be immediately clear that  $P_i(A_1)$  belongs to the same boundary component as  $P_i(A_2)$ . The next lemma shows that these technical difficulties can be overcome.

**Lemma 3.14.** *Let  $A(t_0)$  be an infeasible point. For every  $i = 0, \dots, 5$ , there is a point  $P_i \in \Delta$  such that*

$$\lim_{t \rightarrow t_0^-} p_i(t) = P_i = \lim_{t \rightarrow t_0^+} p_i(t)$$

*This means that if we define  $p_i(t_0) = P_i$ , then  $p_i$  is continuous at  $t_0$ .*

*Proof.* It is sufficient to prove the lemma for  $i = 0$ , because  $p_i(t)$  is clearly a continuous function of  $A(t)$  and  $p_0(t)$ . Since every boundary segment contains only finitely many infeasible points, we may choose a sufficiently small  $\varepsilon > 0$ , such that for every  $t$  from the open interval  $(t_0 - \varepsilon, t_0)$  the points  $A(t)$  are feasible and they all belong to a single boundary segment  $u_1$ , and similarly, for every  $t' \in (t_0, t_0 + \varepsilon)$  the points  $A(t')$  are feasible, and they belong to a single boundary segment  $u_2$ . If the segments  $u_1$  and  $u_2$  are distinct, then  $A(t_0)$  is their common endpoint. Note that for  $t \in (t_0 - \varepsilon, t_0)$ , the points  $p_0(t)$  all belong to a single boundary segment  $v_1$ , otherwise some of the  $A(t)$  would not be feasible. By Claim 3.11, the segment  $v_1$  is parallel and consistently oriented with  $u_1$ . Similarly, for  $t' \in (t_0, t_0 + \varepsilon)$  the points  $p_0(t')$  belong to a single boundary segment  $v_2$ , parallel and consistently oriented with  $u_2$ . We do not know yet whether  $v_1$  and  $v_2$  appear consecutively on the same component of the boundary.

Let  $B = \lim_{t \rightarrow t_0^-} p_0(t)$  (clearly, the limit exists, because the points  $\{p_0(t); t \in (t_0 - \varepsilon, t_0)\}$  form an open segment whose endpoint is  $B$ ). See Fig. 9.

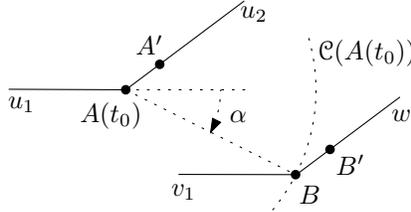


Figure 9: Illustration of the proof of Lemma 3.14

For  $t \in (t_0 - \varepsilon, t_0)$ , let us fix  $\alpha \in (-\frac{\pi}{6}, \frac{\pi}{6})$  such that  $p_0(t) = P(\alpha, A(t))$ , i.e.,  $\alpha$  is the (signed) measure of the angle between the segment  $u_1$  and the segment  $A(t)p_0(t)$ . Note that  $\alpha$  does not depend on the choice of  $t$ . The circle  $\mathcal{C}(A(t_0))$  intersects the boundary at  $B$ . Let  $w$  be the boundary segment starting at  $B$  and directed away from  $B$ . By Lemma 3.12, the convex angles determined by  $v_1$  and  $w$  and by  $u_1$  and  $u_2$  have size at least  $\frac{2\pi}{3}$ , which implies that the convex angle  $\alpha'$  between  $u_2$  and  $BA(t_0)$  is acute and the convex angle between  $w$  and  $BA(t_0)$  is obtuse. Thus, for  $t' \in (t_0, t_0 + \varepsilon)$  the circle  $\mathcal{C}(A')$  (where  $A' = A(t')$ ) intersects the segment  $w$  at a point  $B' = p_i(t')$ . From Claim 3.11 it follows that  $w$  is parallel to  $u_2$ . Also, the segment  $A'B'$  is parallel to the segment  $A(t_0)B$ , which is in turn parallel to any of the segments  $A(t)p_0(t)$ , for  $t \in (t_0 - \varepsilon, t_0)$ .

To finish the proof of this lemma, we need to show that  $B' = p_0(t')$  (as opposed to  $B' = p_i(t')$  for some  $i \neq 0$ ), i.e., we need to prove that the angle  $\alpha'$  determined by the segment  $u_2$  and the segment  $A'B'$  falls into the range  $(-\frac{\pi}{6}, \frac{\pi}{6})$ . We have observed that  $\alpha' \in (-\frac{\pi}{2}, \frac{\pi}{2})$ . This leaves us with the following three possibilities: either  $B' = p_5(t')$ , or  $B' = p_1(t')$ , or  $B' = p_0(t')$ . However, the former two possibilities are ruled out by the fact that the segment  $w$  is oriented consistently with the segment  $u_2$ . This concludes the proof.  $\square$

**Lemma 3.15.** *Let  $i \in \{0, \dots, 5\}$ , let  $A \in \Delta$  be an arbitrary boundary point. All the unit segments of the form  $A(t)p_i(t)$  have the same slope, independently of the choice of  $t$ .*

*Proof.* The slope of  $A(t)p_i(t)$  (as a function of  $t$ ) is constant in a neighborhood of every  $t$  for which  $A(t)$  is feasible. Moreover, this slope is a continuous function of  $t$ , which follows from Lemma 3.14. Hence the function is constant on the whole range.  $\square$

Lemma 3.15 shows that every translation that maps a feasible point  $A$  to the point  $P_i(A)$  also maps the boundary component containing  $A$  onto the boundary component containing  $P_i(A)$  (which may be the same component). Composing such translations (or their inverses) we conclude that the translations that send  $P_i(A)$  to  $P_j(A)$  have the same component-preserving property.

For the proof of Lemma 3.17, we will need a slight extension of Claim 3.11 to infeasible points:

**Claim 3.16.** *Let  $A \in \Delta$  be an arbitrary infeasible point.*

- (i) *At each of the six points  $P_0(A), P_1(A), \dots, P_5(A)$  the circle  $\mathcal{C}(A)$  properly crosses the corresponding boundary component, i.e., in a sufficiently small neighborhood of such point, the circle  $\mathcal{C}(A)$  separates the boundary component into two portions, one lying inside  $\mathcal{C}(A)$  and the other one lying outside  $\mathcal{C}(A)$ .*
- (ii) *There are no more proper crossings of  $\mathcal{C}(A)$  with boundary components. (However,  $\mathcal{C}(A)$  may touch the boundary at some other points.)*
- (iii) *The boundary components containing the points  $P_0(A)$  and  $P_3(A)$  have the same orientation as the component containing  $A$ , whereas the boundary components containing  $P_1(A), P_2(A), P_4(A)$  and  $P_5(A)$  have opposite orientation.*

*Proof.* The first two statements follow from the fact that  $\mathcal{C}(A)$  has the same number of proper crossings with the boundary as the circle  $\mathcal{C}(A(t))$ , where  $A(t)$  is a feasible point sufficiently close to  $A$ . The third statement follows from Claim 3.11 applied to the point  $A(t)$ .  $\square$

**Lemma 3.17.** *Let  $A \in \Delta$  be an arbitrary boundary point. For the sake of brevity, let us write  $P_i$  instead of  $P_i(A)$ ,  $\mathcal{C}$  instead of  $\mathcal{C}(A)$  and  $\mathcal{D}$  instead of  $\mathcal{D}(A)$  in the statement and proof of this lemma. The point  $P_1$  belongs to the*

same boundary component as  $P_2$ , the point  $P_0$  belongs to the same boundary component as  $A$  and  $P_3$ , and the point  $P_4$  belongs to the same boundary component as  $P_5$ . The four portions of the boundary that connect  $P_1$  with  $P_2$ ,  $P_0$  with  $A$ ,  $A$  with  $P_3$ , and  $P_4$  with  $P_5$  are all translated copies of a single piecewise linear curve. These four portions of the boundary are all contained in the closed unit disc with center  $A$ .

*Proof.* It suffices to show that the boundary component that enters inside  $\mathcal{D}$  at  $P_1$  leaves  $\mathcal{D}$  at  $P_2$ . The rest of the statement follows from Lemma 3.15.

Let  $\mathcal{L}$  be the boundary component that contains  $P_1$ . Let us follow  $\mathcal{L}$  from  $P_1$  in the direction of its orientation, i.e., into the interior of the unit disc  $\mathcal{D}$ , and let  $X$  be the first point where  $\mathcal{L}$  leaves  $\mathcal{C}$ . We observe the following:

- $X$  is neither  $P_3$  nor  $P_5$ , because in these points, the boundary is oriented into the interior of the disc  $\mathcal{D}$ .
- $X$  is not the point  $P_0$ : if  $X = P_0$ , then the translation  $P_0 \mapsto A$  would map the fragment of the boundary between  $P_1$  and  $P_0$  onto a fragment directed from  $P_2$  to  $A$ . Similarly, the translation  $P_1 \mapsto A$  would map the fragment  $P_1P_0$  onto a fragment directed from  $P_5$  to  $A$ . This is impossible, because two different boundary fragments of equal length cannot both end at  $A$ .
- $X$  is not  $P_4$ : if  $X$  were equal to  $P_4$ , we would consider the boundary component that enters into the interior of  $\mathcal{C}$  at the point  $P_3$ . Since this boundary component cannot intersect the boundary fragment between  $P_1$  and  $P_4$ , it must leave the interior of  $\mathcal{C}$  at the point  $P_2$ . However, this is symmetric to the previous case and leads to contradiction in the same way.
- Having excluded all other possibilities, we know that  $X = P_2$ .

Let  $U$  denote the fragment of  $\mathcal{L}$  between  $P_1$  and  $P_2$ . By definition, this fragment properly crosses  $\mathcal{C}$  only at its endpoints. Applying a symmetric argument, we find that the boundary fragment from  $P_5$  to  $P_4$  (which is a translated copy of  $U$ ) properly crosses  $\mathcal{C}$  only in its endpoints. Translating  $U$  appropriately, we obtain the boundary fragments connecting  $P_3$  with  $A$  and  $A$  with  $P_0$ . This concludes the proof.  $\square$

From the previous lemmas, we readily obtain the following claim.

**Claim 3.18.** *The condition (C3) of Theorem 3.3 implies the condition (C1).*

*Proof.* We check that the coloring  $\chi$  satisfies the conditions of Definition 3.2. Let  $\vec{x}$  denote the unit vector  $\overrightarrow{AP_0}$  and let  $\vec{y}$  be a unit vector orthogonal to  $\vec{x}$ . By Lemma 3.17, every component of the boundary is a piecewise linear  $\vec{x}$ -periodic curve and if  $\mathcal{L}$  is a boundary component, then any other component is a translate of  $\mathcal{L}$  by an integral multiple of the vector  $\overrightarrow{AP_1} = \frac{1}{2}\vec{x} + \frac{\sqrt{3}}{2}\vec{y}$ . Let  $\vec{z}$  denote this last vector and let  $\mathcal{L}_i = \mathcal{L}_0 + i\vec{z}$ ,  $i \in \mathbb{Z}$ , where  $\mathcal{L}_0$  is a boundary component chosen arbitrarily. We have  $\Delta = \bigcup_{i \in \mathbb{Z}} \mathcal{L}_i$ . The condition (d) of Definition 3.2 follows from Lemma 3.17.  $\square$

It remains to show that the condition (C1) implies (C2). This is the easier part of the proof. In fact, we prove a more general claim:

**Theorem 3.19.** *Every zebra-like coloring has a twin that avoids the unit triangle.*

*Proof.* Let  $\chi$  be a zebra-like coloring, let  $\mathcal{L}_i$ ,  $\vec{x}$  and  $\vec{y}$  be as in Definition 3.2. Let  $\vec{z} = \frac{1}{2}\vec{x} + \frac{\sqrt{3}}{2}\vec{y}$ . Let  $\chi'$  be the twin coloring of  $\chi$  such that the points of  $\mathcal{L}_i$  are black in  $\chi'$  if  $i$  is even and white if  $i$  is odd.

Observe that by the definition of the coloring, the color of a point  $P$  is equal to the color of  $P + \vec{x}$  and different from the color of  $P + \vec{z}$ . Now assume that  $ABC$  is a monochromatic unit triangle, wlog the three points are black. By the previous observation, no edge of the triangle forms an angle of size  $\frac{\pi}{3}$  (or  $\frac{2\pi}{3}$ ) with the vector  $\vec{x}$ . It follows that exactly one of the three edges (wlog the edge  $AB$ ) forms with  $\vec{x}$  an angle whose size falls into the range  $(\frac{\pi}{3}, \frac{2\pi}{3})$ .

We claim that the three points  $A, B, C$  all belong to a single connected component of the black color: otherwise one of the two edges  $AC$  and  $BC$  would have to intersect (at least) two curves  $\mathcal{L}_i$  and  $\mathcal{L}_{i+1}$ . By the definition of the coloring, the distance between the two points of intersection is greater than 1, contradicting the fact that  $ABC$  is a unit triangle.

We now deduce that  $\|AB\| < 1$ : let  $\ell$  be the line containing the segment  $AB$ . Note that the line  $\ell$ , as well as any other line not parallel with  $\vec{x}$ , must intersect all the curves  $\mathcal{L}_i$ . Let  $A'B'$  be the segment obtained as the convex hull of the intersection of  $\ell$  with the closure of the black component containing  $A$  and  $B$ . By the definition of the coloring,  $\|A'B'\| \leq 1$ . Moreover, since the two points  $A'$  and  $B'$  belong to two adjacent boundary curves  $\mathcal{L}_i$  and  $\mathcal{L}_{i+1}$ , they have different colors. Hence, the segment  $AB$  is a proper subset of the segment  $A'B'$ , and  $\|AB\| < 1$ . This shows that  $ABC$  is not a unit triangle — a contradiction.  $\square$

This concludes the proof of Theorem 3.3. Next, we present a simple corollary, which shows that every polygonal coloring of the plane contains any nonequilateral triangle.

### 3.3 Nonequilateral triangles

The following result is a direct consequence of Theorem 3.3, by an easy modification of the proof of Lemma 1.3.

**Theorem 3.20.** *Let  $XYZ$  be a nonequilateral triangle, let  $\chi$  be a polygonal coloring. There is a monochromatic copy  $X'Y'Z'$  of the configuration  $XYZ$ , such that none of the three points  $X', Y'$  and  $Z'$  belongs to the boundary of  $\chi$ .*

*Proof.* Let  $a, b$  and  $c$  be the lengths of the three edges of  $XYZ$ . Wlog, assume that  $a \neq b$ . From Theorem 3.3 it follows that no polygonal coloring can simultaneously avoid copies of equilateral triangles of two different sizes. Hence, we may assume that  $\chi$  contains a monochromatic equilateral triangle  $ABC$  with edges of length  $a$  whose vertices avoid the boundary of  $\chi$ . Assume that the three

points  $A$ ,  $B$  and  $C$  are all black. Consider the configuration of eight points on Fig. 1. As discussed in the proof of the first part of Lemma 1.3, every coloring of the five points  $D$ ,  $A'$ ,  $B'$ ,  $C'$  and  $D'$  yields a monochromatic  $(a, b, c)$ -triangle. Furthermore, we may assume that the eight points all avoid the boundary of  $\chi$ , otherwise we might shift the configuration slightly to move the points away from the boundary, without changing the color of  $ABC$  (recall that  $A$ ,  $B$  and  $C$  already belong to the interior of the black color). This concludes the proof.  $\square$

## 4 Concluding remarks

The Conjecture 1.2 remains wide open, despite the indirect support from the results of this paper, as well as from earlier research. It might happen that the validity of this conjecture would depend on the particular choice of set-theoretical axioms. Such issues do not arise in this paper, since our proof techniques are very elementary. Unfortunately, these elementary techniques do not offer much hope for broad generalizations. It might nevertheless be possible to extend our results about polygonal colorings to some broader class of colorings, e.g., the colorings by monochromatic regions bounded by continuous curves. Colorings of this kind have already been studied in the context of the related problem of the chromatic number of the plane (see [11]).

The zebra-like colorings provide a hitherto unknown example of colorings that avoid an equilateral triangle. We are not aware of any other examples of colorings avoiding a given triangle, but we do not dare to make any conjectures about the uniqueness of our construction, because our understanding of non-polygonal colorings is rather limited.

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