Title: A Note on Disjoint Arborescences

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A Note on Disjoint Arborescences

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Abstract

Recently Kamiyama, Katoh, and Takizawa have shown a theorem on packing arc-disjoint arborescences that is a proper extension of Edmonds' theorem on disjoint spanning branchings. We show a further extension of their theorem, which makes clear an essential rôle of a reachability condition played in the theorem. The right concept required for the further extension is "convexity" instead of "reachability."

1. Introduction: a theorem of Kamiyama, Katoh, and Takizawa

Recently Kamiyama, Katoh, and Takizawa [3] have shown a theorem (KKT theorem for short in the sequel) on packing arc-disjoint arborescences that is a proper extension of Edmonds' theorem [2] on disjoint spanning branchings, which is described as follows. (The precise definitions of terms used here will be given later.)

Let $G = (V, A)$ be a directed graph with a vertex set V and an arc set A. For any vertex $v \in V$ we denote by $R^+_G(v)$ the set of vertices reachable from v by directed paths in G. Given roots r_i $(i \in I)$, KKT theorem gives a characterization of the existence of a set of arc-disjoint arborescences H_i $(i \in I)$ such that for each $i \in I$ arborescence H_i has a root r_i and exactly spans $R^+_G(r_i)$.

In this note we show a further extension of KKT theorem, which makes clear an essential rôle played by a reachability condition in the theorem. The right concept required for the further extension is "convexity" instead of "reachability."

For more information about disjoint arborescences, their extensions, and related topics see [4, Part V] and [1].

2. An extension of KKT theorem

Let $G = (V, A)$ be a directed graph with a vertex set V and an arc set A. Each arc $a \in A$ has a *tail* denoted by $\partial^+ a$ and a *head* denoted by $\partial^- a$. For any vertex v the *in-degree* of v is equal to the number of arcs that have v as their heads. A *branching* in G is a subgraph $H = (U, B)$ of G without any cycle such that every vertex u in U has in-degrees at most one in H. Each connected component of branching H has a unique vertex, called a *root*, that has the in-degree equal to zero in H. A connected branching is called an *arborescence*, which has a single root.

For any vertex $v \in V$ we denote by $R^+_G(v)$ the set of vertices reachable from v by directed paths in G and by $R^-_G(v)$ the set of vertices from which v is reachable by a directed path in G. Also define for any $W \subseteq V$

$$R^+_G(W) = \bigcup\{R^+_G(v) \mid v \in W\}, \quad R^-_G(W) = \bigcup\{R^-_G(v) \mid v \in W\}. \tag{2.1}$$

A vertex subset W is called a *convex set* in G if we have $W = R^+_G(W) \cap R^-_G(W)$, i.e., for every directed (possibly closed) path P from a vertex in W to a vertex in W all the intermediate vertices of P also lie in W. The concept of convexity plays an essential rôle in our result, which replaces the rôle of reachability from roots in KKT theorem [3]. It should be noted that for any convex set U in G and the vertex set W of any strongly connected component of G that satisfy $U \cap W \neq \emptyset$, we must have $U \supseteq W$.

Suppose that we are given a finite index set I and, for each $i \in I$, a specified vertex $r_i \in V$. Here we may allow $r_i = r_j$ for some distinct $i, j \in I$. For each $i \in I$ we are also given a convex set $U_i \subseteq V$ such that $r_i \in U_i$. For any $v \in V$ define

$$I(v) = \{i \in I \mid v \in U_i\}. \tag{2.2}$$

We assume that $I(v) \neq \emptyset$ for all $v \in V$.

Now we are ready to state our main theorem, which is an extension of KKT theorem. It should be noted that replacing U_i by $R^+_G(r_i)$ for all $i \in I$ in our theorem yields KKT theorem. Our proof employs KKT theorem recursively. For any vertex subset $Z \subseteq V$ denote by $G[Z]$ the subgraph of G induced by Z.

Theorem 2.1: *The following two statements are equivalent.*

(a) *There exist arc-disjoint arborescences $H_i = (U_i, B_i)$ $(i \in I)$ such that for each $i \in I$ arborescence H_i has a root r_i.*

(b) *For each $v \in V$ there exist arc-disjoint directed paths P_i $(i \in I(v))$ such that for each $i \in I(v)$ path P_i is from r_i to v.*

(Proof) ((a) $\Rightarrow$ (b)): This implication is easy.

((b) $\Rightarrow$ (a)): Suppose (b) holds.

Consider the decomposition of graph G into strongly connected components, which defines a partial order $\preceq$ on the set of strongly connected components as follows. For two strongly connected components H and H' we have $H \preceq H'$ if and only if there exists a directed path from H' to H. Let $W \subseteq V$ be the vertex set of a strongly connected component that is minimal with respect to the partial order $\preceq$. In other words, W is the vertex set of a strongly connected component in G such that $R_G^+(W) = W$.

Define

$$I(W) = \bigcup\{I(v) \mid v \in W\}(= \{i \in I \mid W \subseteq U_i\}), \tag{2.3}$$

$$U_i(W) = U_i \cap R_G^-(W) \qquad (i \in I(W)), \tag{2.4}$$

$$V(W) = \bigcup\{U_i(W) \mid i \in I(W)\}. \tag{2.5}$$

Then consider the subgraph $\hat{G} = G[V(W)]$ of G induced by $V(W)$. Because of the convexity of U_i $(i \in I)$, definitions (2.3)–(2.5), and assumption (b) we can show the following two facts.

Fact 1: For each $i \in I(W)$ $U_i(W)$ is exactly the set of vertices that can be reached from r_i by directed paths in $\hat{G}$, i.e., $R_{\hat{G}}^+(r_i) = U_i(W)$.

Fact 2: For any $v \in V(W)$ and any directed path P in G from r_i $(i \in I(W))$ to v all the intermediate vertices of P lie in $U_i(W)$.

It follows from these two facts that assumption (b) (appropriately modified) also holds for graph $\hat{G}$ with index set $I(W)$ and convex (reachable) sets $R_{\hat{G}}^+(r_i) = U_i(W)$ $(i \in I(W))$. More precisely, the following (*) holds.

(*) for each $v \in V(W)$ there exist arc-disjoint directed paths P_i $(i \in I(v) \cap I(W))$ such that for each $i \in I(v) \cap I(W)$ path P_i is from r_i to v in $\hat{G}$.

Hence from KKT theorem there exist arc-disjoint arborescences $\hat{H}_i = (U_i(W), \hat{B}_i)$ $(i \in I(W))$ such that each arborescence $\hat{H}_i$ $(i \in I(W))$ has a root r_i.

Define

$$B_i^W = \hat{B}_i \cap \delta^- W \qquad (i \in I(W)), \tag{2.6}$$

where $\delta^- W$ is the set of arcs $a \in A$ with $\partial^- a \in W$. (Here note that we may have $\partial^+ a \in W$.) For all $i \in I \setminus I(W)$ define $B_i^W = \emptyset$. Then put

$$G \leftarrow G \setminus W, \tag{2.7}$$

$$U_i \leftarrow U_i \setminus W \quad (i \in I), \tag{2.8}$$

$$I \leftarrow I \setminus \{i \in I \mid r_i \in W\}, \tag{2.9}$$

where $G \setminus W$ is the graph obtained by removing from G the vertices of W and the arcs incident to W. Note that if $G \setminus W$ has desired arc-disjoint arborescences $H_i' = (U_i \setminus W, B_i')$ $(i \in I)$ restricted on $G \setminus W$, then $H_i = (U_i, B_i' \cup B_i^W)$ $(i \in I)$ are desired ones for G. It should also be noted that $U_i \setminus W$ $(i \in I)$ are convex sets in the original graph G and hence in the new G as well. Since $U_i \setminus W$ $(i \in I)$ are convex sets in the original graph G, directed paths within $U_i \setminus W$ in the original G are also directed path in the new G. Hence assumption (b) also holds for the new G, I, U_i $(i \in I)$, and r_i $(i \in I)$.

Repeat this process until G becomes empty. Let $W_1, \cdots, W_k$ be the sequence of Ws chosen in the repeated above-mentioned process.

Define for each $i \in I$

$$B_i = \bigcup \{ B_i^{W_\ell} \mid \ell = 1, \cdots, k \}, \tag{2.10}$$

where $B_i^{W_\ell}$ is defined to be B_i^W for $W = W_\ell$. We can easily see that $H_i \equiv (U_i, B_i)$ $(i \in I)$ are desired arborescences with roots r_i $(i \in I)$, one for each corresponding H_i. $\square$

Note that the proof given above leads us to a polynomial algorithm for finding arc-disjoint arborescences that span specified convex sets with roots by using the algorithm in [3].

We can also show the following. Define $I'(v) = \{ i \in I(v) \mid r_i \neq v \}$ for all $v \in V$.

Theorem 2.2: *The following two statements are equivalent to* (a) $($*and* (b)$)$ *in Theorem* 2.1.

(c) *For any vertex subset* $Z \subset V$

$$|\Delta^- Z| \geq |\{ i \in I(Z) \mid r_i \notin Z \}|, \tag{2.11}$$

where $\Delta^- Z$ *denotes the set of arcs* $a \in A$ *such that* $\partial^+ a \notin Z$ *and* $\partial^- a \in Z$.

(d) *There exist spanning trees* $T_i = (U_i, E_i)$ *of* $G[U_i]$ $(i \in I)$ *such that* E_i $(i \in I)$ *are pairwise disjoint and every vertex* $v \in V$ *has in-degree equal to* $|I'(v)|$ *in the union of* T_i $(i \in I)$ $($*as a subgraph* $H = (V, \cup_{i \in I} E_i)$ *of* $G$$)$.

(Proof) We show the implications (c) ⇒ (b) ((a)) ⇒ (d) ⇒ (c).

((c) ⇒ (b)): Let v be any vertex in V. Consider any $Z \subset V$ with $v \in Z$ in (c). Then it follows from (c) (with any such Z) and the max-flow min-cut theorem that (b) for v holds.

((b) ⇒ (d)): This is easy since (a) and (b) are equivalent.

((d) ⇒ (c)): Let Z be any subset of V. Denote by $A_H[Z]$ the set of arcs a in H with $\partial^+ a, \partial^- a \in Z$. Then we have

$$|\Delta^- Z| \geq \sum_{v \in Z} |I'(v)| - |A_H[Z]| \geq |\{ i \in I(Z) \mid r_i \notin Z \}|, \tag{2.12}$$

where the second inequality follows from the fact that $|E_i \cap A_H[Z]| \leq |U_i \cap Z| - 1$ for all $i \in I(Z)$. (Note that $|A_H[Z]| = \sum_{i \in I(Z)} |E_i \cap A_H(Z)| \leq \sum_{i \in I(Z)} |U_i \cap Z| - |I(Z)| = \sum_{v \in Z} |I(v)| - |I(Z)|$.) Hence (2.11) holds. $\square$

It should be noted that because of (d) in Theorem 2.2 a problem of finding minimum-weight arc-disjoint arborescences that span given convex sets with roots can be solved in polynomial time.

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References

[1] K. Bérczi and A. Frank: Variations for Lovász' submodular ideas. Technical Report TR-2008-7, The Egerváry Research Group on Combinatorial Optimization (June 2008).

[2] J. Edmonds: Edge-disjoint branchings. In: *Combinatorial Algorithms* (B. Rustin, ed.) (Academic Press, 1973), pp. 91–96.

[3] N. Kamiyama, N. Katoh, and A. Takizawa: Arc-disjoint in-trees in directed graphs. *Proceedings of the 19th Annual ACM-SIAM Symposium on Discrete Algorithms* (2008), pp. 518–526; also to appear in *Combinatorica*.

[4] A. Schrijver: *Combinatorial Optimization—Polyhedra and Efficiency* (Springer, 2003).