

PACKING DIRECTED CIRCUITS EXACTLY

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ABSTRACT. We give an “excluded minor” and a “structural” characterization of digraphs D that have the property that for every subdigraph H of D , the maximum number of disjoint circuits in H is equal to the minimum cardinality of a set $T \subseteq V(H)$ such that $H \setminus T$ is acyclic.

1. INTRODUCTION

Graphs and digraphs in this paper may have loops and multiple edges. Paths and circuits have no “repeated” vertices, and in digraphs they are directed. A *transversal* in a digraph D is a set of vertices T which intersects every circuit, i.e. $D \setminus T$ is acyclic. A *packing of circuits* (or *packing* for short) is a collection of pairwise (vertex-)disjoint circuits. The cardinality of a minimum transversal is denoted by $\tau(D)$ and the cardinality of a maximum packing is denoted by $\nu(D)$. Clearly $\nu(D) \leq \tau(D)$, and our objective is to study when equality holds. We will show in Section 4 that this is the case for every strongly planar digraph. (A digraph is *strongly planar* if it has a planar drawing such that for every vertex v , the edges with head v form an interval in the cyclic ordering of edges incident with v .) However, in general there is probably no nice characterization of digraphs for which equality holds, and so instead we characterize digraphs such that equality holds for *every subdigraph*. Thus we say that a digraph D *packs* if $\tau(D') = \nu(D')$ for every subdigraph D' of D .

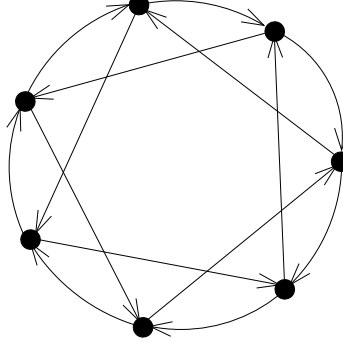
We will give two characterizations: one in terms of excluded minors, and the other will give a structural description of digraphs that pack. We say that an edge e of a digraph D with head v and tail u is *special* if either e is the only edge of D with head v , or it is the only edge of D with tail u , or both. We say that a digraph D is a *minor* of a digraph D' if D can be obtained from a subdigraph of D' by repeatedly contracting special edges. It is easy to see that if a digraph packs, then so do all its minors. Thus digraphs that pack can be characterized by a list of minor-minimal digraphs that do not pack. By an *odd double circuit* we mean the digraph obtained from an undirected circuit of odd length at least three by replacing each edge by a pair of directed edges, one in each direction. The digraph F_7 is defined in Figure 1. The following is our excluded minor characterization.

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FIGURE 1. The digraph F_7 .

Theorem 1.1. *A digraph packs if and only if it has no minor isomorphic to an odd double circuit or F_7 .*

If D is an odd double circuit with k vertices then $\tau(D) = \lceil k/2 \rceil > \nu(D) = \lfloor k/2 \rfloor$. Moreover, $\tau(F_7) = 3 > \nu(F_7) = 2$. Thus odd double circuits and F_7 do not pack and the content of Theorem 1.1 is to prove the converse.

The structural characterization can be stated directly in terms of digraphs, but it is more convenient to rephrase it in terms of bipartite graphs, and therefore we postpone its statement until Section 5. Roughly, the characterization states that a digraph packs if and only if it can be obtained from strongly planar digraphs by means of certain composition operations.

Our main tool in the proof is a characterization of bipartite graphs that have a Pfaffian orientation, found independently by McCuaig [1] and by Robertson, Seymour and the second author [6]. We present the characterization in Section 5. The rest of the paper is organized as follows. In Section 2 we mention three related results. Section 3 reduces the problem to strongly 2-connected digraphs. It is shown in Section 4 that strongly planar digraphs pack. Sections 6-8 show that the property that digraphs pack is preserved under the composition operations of the characterization theorem, thus completing the proof of Theorem 1.1. Finally, Section 9 offers some closing remarks.

2. RELATED RESULTS

In this section we review three related results. The first is a classical theorem of Lucchesi and Younger, of which we only state a corollary [4](Theorem B).

Theorem 2.1. *Let D be a planar digraph and \mathcal{F} be the family of its directed circuits. Then for any set of weights $w : E(D) \rightarrow \mathbb{Z}_+$ we have,*

$$\begin{aligned} & \min\left\{ \sum_{e \in E(D)} w(e)x_e : \sum_{e \in C} x_e \geq 1, \forall C \in \mathcal{F}, x \in \{0, 1\}^{E(D)} \right\} \\ &= \max\left\{ \sum_{C: C \in \mathcal{F}} y_C : \sum_{C: e \in C \in \mathcal{F}} y_C \leq w(e), \forall e \in E(D), y \in \mathbb{Z}_+^{\mathcal{F}} \right\}. \end{aligned} \quad (2.1)$$

Thus, in particular, in a planar digraph the maximum cardinality of a collection of edge-disjoint circuits is equal to the minimum cardinality of a set of edges whose deletion makes the graph acyclic. This relation does not hold for all digraphs, but there is an upper bound on $\tau(D)$ as a function of $\nu(D)$. (A simple construction — splitting each vertex into a “source” and a “sink,” also used in the proof of Corollary 4.1 — shows that the same function serves as an upper bound for both the edge-disjoint as well as vertex-disjoint version of the problem. Note, however, that this construction does not preserve planarity, but it preserves strong planarity.) McCuaig [1] characterized all digraphs D with $\nu(D) \leq 1$; the following follows immediately from his characterization (but there does not seem to be a direct proof).

Theorem 2.2. *For every digraph D , if $\nu(D) \leq 1$, then $\tau(D) \leq 3$.*

In general, Reed, Robertson, Seymour and the second author [5] proved the following.

Theorem 2.3. *There is a function f such that for every digraph D*

$$\tau(D) \leq f(\nu(D)).$$

The function f from the proof of Theorem 2.3, albeit explicit, grows rather fast. The best lower bound of $f(k) \geq \Omega(k \log k)$ was obtained by Noga Alon (unpublished). Finally, the undirected analogue of the problem we study is quite easy. It becomes much harder if we only require that the equality $\nu = \tau$ hold for all *induced* subgraphs. This problem remains open. However, Ding and Zang [2] managed to solve the closely related problem of characterizing graphs for which it is required that a weighted version of the relation $\nu = \tau$ holds. They gave a characterization by means of excluded induced subgraphs, and also gave a structural description of those graphs. We omit the precise statement.

3. STRONG 2-CONNECTIVITY

Let D be a digraph and \mathcal{C} a packing of circuits. We will say that \mathcal{C} *uses a vertex v* if there exists a circuit C in \mathcal{C} with $v \in V(C)$. Consider a digraph D that packs. Then some minimum transversal includes v if and only if $\tau(D \setminus v) = \tau(D) - 1$. As D packs, $\nu(D \setminus v) = \tau(D \setminus v) = \tau(D) - 1 = \nu(D) - 1$. But $\nu(D \setminus v) = \nu(D) - 1$ if and only if every maximum packing uses v . Thus we have shown the following.

Remark 3.1. Let D be a digraph that packs. There exists a minimum transversal of D containing v if and only if every maximum packing of D uses v .

A digraph is *strongly connected* if for every pair of vertices u and v there is a path from u to v . A digraph D is *strongly k -connected* if for every $T \subseteq V(D)$, where $|T| \leq k-1$, the digraph $D \setminus T$ is strongly connected. If D is not strongly connected, then $V(D)$ can be partitioned into non-empty sets X_1, X_2 such that no edge has tail in X_1 and head in X_2 . Let $D_1 := D \setminus X_2$ and $D_2 := D \setminus X_1$. Then D is said to be a *0-sum* of D_1 and D_2 . Since every circuit of D is a circuit of precisely one of D_1 or D_2 , the following is straightforward.

Proposition 3.2. *Let D be the 0-sum of D_1 and D_2 . Then D_1 and D_2 pack if and only if D packs.*

Suppose D is strongly connected, but not strongly 2-connected; thus there is a vertex v such that $D \setminus v$ is not strongly connected. Then there is a partition of $V(D) - \{v\}$ into non-empty sets X_1, X_2 such that all edges with endpoints in both X_1 and X_2 have tail in X_1 and head in X_2 . Let F be the set of all these edges. For $i = 1, 2$ let D_i be the digraph obtained from D by deleting all edges with both endpoints in $X_{3-i} \cup \{v\}$ and identifying all vertices of $X_{3-i} \cup \{v\}$ into a vertex called v . Thus edges of F belong to both D_1 and D_2 ; in D_1 they have head v and in D_2 they have tail v . We say that D is a *1-sum* of D_1 and D_2 .

Let D be a digraph. We denote by $D + uv$ the digraph obtained from D by adding the vertices u, v (if they are not vertices of D) and an edge with tail u and head v . Let us stress that we add the edge even if D already has one or more edges with tail u and head v . We use $D + \{u_1v_1, u_2v_2, \dots, u_kv_k\}$ to denote $D + u_1v_1 + u_2v_2 + \dots + u_kv_k$.

Proposition 3.3. *Let a strongly connected digraph D be the 1-sum of D_1 and D_2 . Then D_1 and D_2 pack if and only if D packs.*

Proof. Since D is strongly connected, the digraphs D_1 and D_2 are minors of D . So if D packs, so do D_1 and D_2 . Conversely, assume that D_1 and D_2 pack. Since every subdigraph of D is either a subdigraph of D_1 or D_2 , or a 0-sum or 1-sum of subdigraphs of D_1 and D_2 , it suffices to show that $\tau(D) = \nu(D)$. Let v, X_1, X_2 , and F be as in the definition of 1-sum. For $i = 1, 2$ let $D'_i := D_i \setminus F$ and let \mathcal{C}_i be a maximum packing of D'_i . Suppose that, for $i = 1, 2$, every maximum packing of D'_i uses the vertex v . It follows from Remark 3.1 that there is a minimum transversal T_i of D'_i using v . Let \mathcal{C} be obtained from the union of $\mathcal{C}_1, \mathcal{C}_2$ by removing the circuit of \mathcal{C}_1 using v . Then \mathcal{C} is a packing of D and $T := T_1 \cup T_2$ is a transversal of D . Moreover, both have cardinality $\tau(D'_1) + \tau(D'_2) - 1$, i.e. $\tau(D) = \nu(D)$. Thus we can assume one of \mathcal{C}_i ($i = 1, 2$), say \mathcal{C}_1 , does not use the vertex v .

For $i = 1, 2$, let F_i be the set of edges f of F such that $\nu(D'_i + f) = \nu(D'_i)$. Consider first the case where $F_1 \cup F_2 = F$. Suppose for a contradiction $\nu(D'_i + F_i) > \nu(D'_i)$ and let \mathcal{F} be a corresponding packing. Clearly \mathcal{F} uses an edge of F_i . Moreover as all edges F of D_i share the endpoint v , \mathcal{F} uses exactly one edge f of F_i . Hence $\nu(D'_i + f) > \nu(D'_i)$, a contradiction. Since (for $i = 1, 2$) $D'_i + F_i$ packs it has a transversal T_i of cardinality $\tau(D'_i)$. As $F_1 \cup F_2 = F$ this implies that $T_1 \cup T_2$ is a transversal of D . Recall that \mathcal{C}_1 does not use v ; thus $\mathcal{C}_1 \cup \mathcal{C}_2$ is a packing of D and $|T_1 \cup T_2| = \tau(D'_1) + \tau(D'_2) = |\mathcal{C}_1 \cup \mathcal{C}_2|$, i.e. $\tau(D) = \nu(D)$.

Thus we may assume there exists $f \in F - F_1 - F_2$. Let \mathcal{C}'_i ($i = 1, 2$) be a maximum packing of $D'_i + f$. Each \mathcal{C}'_i contains a circuit C_i using f . Define \mathcal{C} to be the collection of all circuits of $\mathcal{C}_1, \mathcal{C}_2$ distinct from C_1 and C_2 as well as the circuit $(C_1 \cup C_2) \setminus f$ of D . Let T_i ($i = 1, 2$) be a minimum transversal of D'_i . Then $T := T_1 \cup T_2 \cup \{v\}$ is a transversal of D and \mathcal{C} a packing of D . Moreover, $|T| = \tau(D'_1) + \tau(D'_2) + 1 = |\mathcal{C}|$, i.e. $\tau(D) = \nu(D)$, as desired. \square

4. STRONG PLANARITY

Let us recall that a digraph is *strongly planar* if it has a planar drawing such that for all vertices v , the edges with head v form an interval in the cyclic ordering of edges incident with v determined by the drawing.

Corollary 4.1. *Every strongly planar digraph packs.*

Proof. Let D be a strongly planar digraph with vertex set V and edge set E . We will show that D packs. Since subdigraphs of strongly planar digraphs are strongly planar it suffices to show $\tau(D) = \nu(D)$. Associate to every vertex v a new vertex v' and let V' be the set of all vertices v' . Associate with every edge $e \in E(D)$ with tail u and head v a new edge e' with tail u' and head v . We define a digraph H as follows: the vertex-set of H is $V \cup V'$, and the edge-set of H consists of all the edges e' for $e \in E(D)$ and all the edges of the form vv' , where $v \in V(D)$. Define weights $w : E(H) \rightarrow \mathbb{Z}_+$ as follows: $w(e') = |E(H)|$ for all $e \in E(D)$ and $w(vv') = 1$ for all $v \in V(H)$. It is easy to see that the drawing associated to the strongly planar digraph D can be modified to induce a planar drawing of H . Now equation (2.1) states $\tau(D) = \nu(D)$, as desired. \square

5. BRACES

It will be convenient to reformulate our packing problem about digraphs to one about bipartite graphs. Let G be a bipartite graph with bipartition (A, B) , and let M be a perfect matching in G . We denote by $D(G, M)$ the digraph obtained from G by directing every edge of G from A to B , and contracting every edge of M . When G' is a subgraph of G and $M \cap E(G')$ is a perfect matching of G' we will abbreviate $D(G', M \cap E(G'))$ by $D(G', M)$. It is clear that every digraph is isomorphic to $D(G, M)$ for some bipartite graph G and some perfect matching M . Moreover, the following is straightforward.

Remark 5.1. Let G be a bipartite graph and let M be a perfect matching in G . If G is planar then $D(G, M)$ is strongly planar.

A graph G is k -extendable, where k is an integer, if every matching in G of size at most k can be extended to a perfect matching. A 2-extendable bipartite graph is called a *brace*. The following straightforward relation between k -extendability and strong k -connectivity is very important.

Proposition 5.2. *Let G be a connected bipartite graph, let M be a perfect matching in G , and let $k \geq 1$ be an integer. Then G is k -extendable if and only if $D(G, M)$ is strongly k -connected.*

Let G be a bipartite graph and M a perfect matching in G such that $D(G, M)$ is isomorphic to F_7 . This defines G uniquely up to isomorphism, and the graph so defined is called the *Heawood graph*.

Let G be a bipartite graph, and let e be an edge of G with ends u, v . Consider a new graph obtained from G by replacing e by a path with an even number of vertices and ends u, v and otherwise disjoint from G . Let G' be obtained from G by repeating this operation, possibly for different edges of G . We say that G' is an *even subdivision* of G . The graph G' is clearly bipartite. Now let G, H be bipartite graphs. We say that G *contains* H if G has a subgraph L such that $G \setminus V(L)$ has a perfect matching, and L is isomorphic to an even subdivision of H .

A circuit C in a bipartite graph G is *central* if $G \setminus V(C)$ has a perfect matching. Let G_0 be a bipartite graph, let C be a central circuit of G_0 of length 4, and let G_1, G_2 be subgraphs of G_0 such that $G_1 \cup G_2 = G_0$, $G_1 \cap G_2 = C$, and $V(G_1) - V(G_2) \neq \emptyset \neq V(G_2) - V(G_1)$. Let G be obtained from G_0 by deleting all the edges of C . In this case we say that G is the *4-sum* of G_1 or G_2 along C . This is a slight departure from the definition in [6], but the class of simple graphs obtainable according to our definition is the same, because we allow parallel edges.

Let G_0 be a bipartite graph, let C be a central circuit of G_0 of length 4, and let G_1, G_2, G_3 be three subgraphs of G_0 such that: $G_1 \cup G_2 \cup G_3 = G_0$ and for distinct integers $i, j \in \{1, 2, 3\}$ $G_i \cap G_j = C$ and $V(G_i) - V(G_j) \neq \emptyset$. Let G be obtained from G_0 by deleting all the edges of C . In these circumstances we say that G is a *trism* of G_1, G_2, G_3 along C . We will need the following result.

Theorem 5.3. *Let G be a brace, and let M be a perfect matching in G . Then the following conditions are equivalent.*

- (i) G does not contain $K_{3,3}$,
- (ii) either G is isomorphic to the Heawood graph, or G can be obtained from planar braces by repeatedly applying the trism operation,
- (iii) either G is isomorphic to the Heawood graph, or G can be obtained from planar braces by repeatedly applying the 4-sum operation,
- (iv) $D(G, M)$ has no minor isomorphic to an odd double circuit.

Proof. The equivalence of (i), (ii) and (iii) is the main result of [1] and [6]. Condition (iv) is equivalent to the other three by results of Little [3] and Seymour and Thomassen [7]. See also [1]. \square

We will need the following small variation of Theorem 5.3.

Theorem 5.4. *Let G be a brace, and let M be a perfect matching in G . Then the following conditions are equivalent.*

- (i) G does not contain $K_{3,3}$ or the Heawood graph,
- (ii) G can be obtained from planar braces by repeatedly applying the trism operation,

- (iii) G can be obtained from planar braces by repeatedly applying the 4-sum operation,
- (iv) $D(G, M)$ has no minor isomorphic to an odd double circuit or F_7 .

Proof. This follows from Theorem 5.3 and the fact [6, Theorem 6.7] that if G contains the Heawood graph and is not isomorphic to it, then it contains $K_{3,3}$. \square

We deduce the following information about a minimal counterexample to Theorem 1.1.

Proposition 5.5. *Let G be a bipartite graph and M a perfect matching in G such that the digraph $D := D(G, M)$ has no minor isomorphic to an odd double circuit or F_7 , and every digraph D' with $|V(D')| + |E(D')| < |V(D)| + |E(D)|$ and no minor isomorphic to an odd double circuit or F_7 packs. If $\nu(D) < \tau(D)$, then G is a brace and there exist braces G_1, G_2, G_3 such that G is a trisum of G_1, G_2, G_3 along a circuit C , and each of G_1, G_2, G_3 can be obtained from planar braces by repeatedly applying the trisum operation.*

Proof. It follows from Propositions 3.2 and 3.3 that D is strongly 2-connected. Thus G is a brace by Proposition 5.2. By Corollary 4.1 the digraph D is not strongly planar, and hence G is not planar by Remark 5.1. By Theorem 5.4 the graph G is obtained from planar braces by repeatedly applying the trisum operation. Since G itself is not planar, there is at least one trisum operation involved in the construction of G , and hence G_1, G_2, G_3 and C exist, as desired. \square

In the next three sections we will prove the following result.

Proposition 5.6. *Let G , M , and D be as in Proposition 5.5. Then $\nu(D) = \tau(D)$.*

Proof of Theorem 1.1 (assuming Proposition 5.6). We have already established the “only if” part. To prove the “if” part let D be a digraph with no minor isomorphic to an odd double circuit or F_7 such that every digraph D' with $|V(D')| + |E(D')| < |V(D)| + |E(D)|$ and no minor isomorphic to an odd double circuit or F_7 packs. By Proposition 5.6 we have that $\nu(D) = \tau(D)$, and hence D packs, as desired. \square

We now deduce the structural characterization of digraphs that pack.

Corollary 5.7. *A digraph packs if and only if it can be obtained from strongly 2-connected digraphs that pack by means of 0- and 1-sums. A strongly 2-connected digraph packs if and only if it is isomorphic to $D(G, M)$ for some brace G and some perfect matching M in G , where G is obtained from planar braces by repeatedly applying the trisum operation.*

Proof. The first statement follows from Propositions 3.2 and 3.3. For the second statement let D be a strongly 2-connected digraph. Assume first that D packs, and let G be a bipartite graph and M a perfect matching such that D is isomorphic to $D(G, M)$. By Proposition 5.2 the graph G is a brace. By Theorem 1.1 the digraph D has no minor isomorphic to an odd double circuit or F_7 , and so by Theorem 5.4 G is as desired. The converse implication follows along the same lines. \square

As we alluded to in the Introduction, the second part of Corollary 5.7 can be stated purely in terms of “sums” of digraphs. However, three kinds of sum are needed (see [6]), as opposed to just one. Therefore the formulation we chose is clearer, despite the disadvantage that it involves the transition from a digraph to a bipartite graph.

Finally, we deduce a corollary about packing M -alternating circuits in bipartite graphs. Let G be a bipartite graph, and let M be a perfect matching in G . A circuit C in G is M -alternating if $2|E(C) \cap M| = |E(C)|$. Let $\nu(G, M)$ denote the maximum number of pairwise disjoint M -alternating circuits, and let $\tau(G, M)$ denote the minimum number of edges whose deletion leaves no M -alternating circuit. It is clear that $\nu(G, M) = \nu(D(G, M))$ and $\tau(G, M) = \tau(D(G, M))$. Thus we have the following corollary.

Corollary 5.8. *Let G be a brace, and let M be a perfect matching in G . Then the following three conditions are equivalent.*

- (i) G does not contain $K_{3,3}$ or the Heawood graph,
- (ii) $\tau(G', M') = \nu(G', M')$ for every subgraph G' of G such that $M' = M \cap E(G')$ is a perfect matching in G' , and
- (iii) G can be obtained from planar braces by repeatedly applying the trisum operation.

In fact, the equivalence of (i) and (ii) holds for all bipartite graphs, not just braces. We conclude this section with a lemma that will be needed later. The lemma follows immediately from [6, Theorem 8.2]. We say that a graph is a *cube* if it is isomorphic to the 1-skeleton of the 3-dimensional cube. Thus every cube has 8 vertices and 12 edges.

Lemma 5.9. *Let G be a trisum of G_1, G_2, G_3 along C , where the graphs G_1, G_2, G_3 are obtained from planar braces by repeatedly applying the trisum operation. Then for $i = 1, 2, 3$ we have $|E(G_i)| \geq 12$ with equality if and only if G_i is a cube.*

The remainder of the paper is dedicated to proving Proposition 5.6. Consider D, G, C as in Proposition 5.5, and let k be the number of edges of M with both ends in $V(C)$. As M is a perfect matching of G , $k \in \{0, 1, 2\}$. Proposition 6.2 proves that $k \neq 2$, Proposition 7.2 proves that $k \neq 1$, and finally Proposition 8.2 proves that $k \neq 0$.

6. TRISUM-PART I

Let $D, G, M, G_1, G_2, G_3, C$ be as in Proposition 5.5. For $i = 1, 2, 3$ let M'_i be the set of edges $M \cap E(G_i)$ with at least one end not in $V(C)$, let M_0 be the set of edges of C that are parallel to an edge of M , and let $M_i = M'_i \cup M_0$. We say that M_i is the *imprint* of M on G_i .

Proposition 6.1. *Let a bipartite graph G be a 4-sum of G_1 and G_2 along C , let M be a perfect matching in G such that some two edges of M have both ends in $V(C)$, let $D = D(G, M)$, and for $i = 1, 2$ let M_i be the imprint of M on G_i . If both $D(G_1, M_1)$ and $D(G_2, M_2)$ pack, then $\nu(D) = \tau(D)$.*

Proof. For $i = 1, 2$ let $D_i = D(G_i, M_i)$. Then $|V(D_1) \cap V(D_2)| = 2$; let $V(D_1) \cap V(D_2) = \{u_1, u_2\}$. Moreover, $E(D_1) \cap E(D_2) = \{e_1, e_2\}$, where e_1 has head u_2 and tail u_1 , and e_2 has head u_1 and tail u_2 . For $i = 1, 2$ let $D'_i = D_i \setminus \{e_1, e_2\}$.

Claim 1. For each D'_i ($i = 1, 2$) one of the following holds:

- (1) There exists a maximum packing not using any of u_1 or u_2 . Every minimum transversal does not contain any of u_1 or u_2 .
- (2) For some $k \in \{1, 2\}$ the following holds: all maximum packings use u_k , there exists a maximum packing not using u_{3-k} , and there exists a minimum transversal which contains u_k but not u_{3-k} .
- (3) There exists a maximum packing using both u_1 and u_2 . There exists a minimum transversal using u_1 and a minimum transversal using u_2 . Moreover, either: (a) there is a minimum transversal containing both u_1, u_2 ; or (b) there is a packing of size $\tau(D'_i) - 1$ not using u_1 or u_2 .

Proof of Claim: Observe that for (1)-(3) the statements about transversals (except for the last sentence) follow from the statements about maximum packings and Remark 3.1. Suppose (1) does not hold; then every maximum packing of D'_i uses one of u_1, u_2 . In particular $\nu(D_i) = \nu(D'_i)$. Suppose for a contradiction there exists a maximum packing \mathcal{C}_i of D'_i not using u_1 and a maximum packing \mathcal{C}'_i of D'_i not using u_2 . Remark 3.1 implies that no minimum transversal of D'_i contains u_1 or u_2 . Since $\{e_1, e_2\}$ is the edge-set of a circuit of D_i this implies $\tau(D_i) > \tau(D'_i)$, a contradiction since D_i packs. Thus for some $k \in \{1, 2\}$ every maximum packing of D'_i uses u_k . If (2) does not hold, then all maximum packings use u_{3-k} . If (3)(a) does not hold, no minimum transversal of D'_i uses both u_1 and u_2 . This implies $\tau(D'_i \setminus \{u_1, u_2\}) \geq \tau(D'_i) - 1$. Since D_i packs (3)(b) must hold. \diamond

Claim 2. For $i = 1, 2$, let T_i be a minimum transversal of D'_i and let \mathcal{C}_i be a maximum packing of D'_i . We can assume one of the following holds:

- (a) There exists $k \in \{1, 2\}$ such that \mathcal{C}_1 and \mathcal{C}_2 use u_k but $u_k \notin T_1 \cap T_2$.
- (b) $\{u_1, u_2\} \cap (T_1 \cup T_2) = \emptyset$.

Proof of Claim: Let $T := T_1 \cup T_2$ and let \mathcal{C} be an inclusion-wise maximal packing in $\mathcal{C}_1 \cup \mathcal{C}_2$. If (a) does not hold, then $|T| \leq |\mathcal{C}|$. If (b) does not hold, then $\{u_1, u_2\} \cap T \neq \emptyset$; thus T is a transversal of D . It follows that $\tau(D) = \nu(D)$, as desired. Thus we may assume that (a) or (b) holds. \diamond

We can assume, because of Claim 1 and Claim 2, that D_1, D_2 either both satisfy condition (1) of Claim 1, or they both satisfy condition (3) of Claim 1 and one of D'_1, D'_2 , say D'_1 , satisfies (3)(b). Consider the latter

case first. Let T_1 (resp. T_2) be a minimum transversal of D'_1 (resp. D'_2) using u_1 . Let $T := T_1 \cup T_2$. Let \mathcal{C}_1 be a packing of $D'_1 \setminus \{u_1, u_2\}$ of size $\tau(D'_1) - 1$ and let \mathcal{C}_2 be a maximum packing of D'_2 . Clearly $\mathcal{C} := \mathcal{C}_1 \cup \mathcal{C}_2$ is a packing in D . Since $|T_1 \cup T_2| = \tau(D'_1) + \tau(D'_2) - 1$ and $|\mathcal{C}| = \tau(D'_1) - 1 + \tau(D'_2)$, we have $\tau(D) = \nu(D)$.

Thus we may assume that both D'_1, D'_2 satisfy (1). For $i = 1, 2$, let \mathcal{C}_i be a maximum packing of D_i . Suppose there is $k \in \{1, 2\}$ such that for $i = 1, 2$, $\tau(D'_i + u_k u_{3-k}) = \tau(D'_i)$ and let T_i be the corresponding minimum transversal. Then T_i intersects all $u_{3-k} u_k$ -paths of D_i . Hence $T := T_1 \cup T_2$ is a transversal of D . Moreover, $|T| = \tau(D'_1) + \tau(D'_2) = |\mathcal{C}_1 \cup \mathcal{C}_2|$, i.e. $\tau(D) = \nu(D)$. Thus we can assume there is for $k = 1, 2$ an index $t(k) \in \{1, 2\}$ such that $\tau(D'_{t(k)} + u_k u_{3-k}) > \tau(D'_{t(k)})$. Since D_1, D_2 pack $\nu(D'_{t(k)} + u_k u_{3-k}) > \tau(D'_{t(k)})$; let $\mathcal{F}_{t(k)}$ be the corresponding packing. Some circuit $C_{t(k)}$ of $\mathcal{F}_{t(k)}$ is of the form $P_{t(k)} + u_k u_{3-k}$ where $P_{t(k)}$ is a $u_{3-k} u_k$ -path. For $i = 1, 2$ let T_i be a minimum transversal of D'_i . Note that $T_{t(k)}$ does not intersect $P_{t(k)}$. Observe that we cannot have $t(1) = t(2) = i \in \{1, 2\}$, for otherwise there exist both a $u_1 u_2$ - and $u_2 u_1$ -paths in D'_i which are not intersected by T_i and hence T_i does not intersect all circuits of D'_i , a contradiction. Thus we can assume $t(1) = 1$ and $t(2) = 2$. Let $\mathcal{C} := \mathcal{F}_1 \cup \mathcal{F}_2 \cup \{P_1 \cup P_2\} - \{C_1, C_2\}$. Then \mathcal{C} is a packing of D and $T := T_1 \cup T_2 \cup \{u_1\}$ is a transversal of D . Moreover, $|T| = \tau(D'_1) + \tau(D'_2) + 1 = |\mathcal{C}|$, i.e. $\tau(D) = \nu(D)$, as desired. \square

Proposition 6.2. *Let G, M, D , where $\nu(D) < \tau(D)$, and G_1, G_2, G_3, C be as in Proposition 5.5. Then at most one edge of M has both ends in $V(C)$.*

Proof. Suppose for a contradiction that two edges of M have both ends in $V(C)$. For $i = 1, 2, 3$ let M_i be the imprint of M on G_i . The graphs G_1 and $G_2 \cup G_3$ are obtained from planar braces by repeatedly applying the 4-sum operation, and hence the digraphs $D_1 = D(G_1, M_1)$ and $D_2 = D(G_2 \cup G_3, M_2 \cup M_3)$ have no minor isomorphic to an odd double circuit or F_7 by Theorem 5.4. Thus D_1 and D_2 pack, and hence by Proposition 6.1 $\nu(D) = \tau(D)$, a contradiction. \square

7. TRISUM-PART II

Lemma 7.1. *Let D_1, D_2 be digraphs with $V(D_1) \cap V(D_2) = \{u_1, u_2, u_3\}$ and $E(D_1) \cap E(D_2) = \emptyset$. Let $D = D_1 \cup D_2$, $a \notin V(D)$, $E_1 = \{u_1 u_2, u_1 u_3, u_2 u_3\}$, $E_2 = \{u_2 u_1, u_3 u_1, u_3 u_2\}$, $Z_1 = \{a u_2, u_2 a, u_1 a, a u_3\}$, and $Z_2 = \{a u_2, u_2 a, a u_1, u_3 a\}$, where $a \notin V(D)$. Assume that*

- (a) *if, for $i = 1, 2$, C_i is a circuit of D_i , then $V(C_1) \cap V(C_2) \subseteq \{u_2\}$,*
- (b) *if C is a circuit of D that uses edges of both D_1 and D_2 , then $C = P_1 \cup P_2$ and there exist integers $i, j \in \{1, 2, 3\}$ such that $i < j$ and P_1 is a $u_j u_i$ -path of D_1 and P_2 is a $u_i u_j$ -path of D_2 , and*
- (c) *there exist integers i, j such that $\{i, j\} = \{1, 2\}$, $D_i + E_i$ packs and is strongly 2-connected, and $D_j + Z_j$ packs.*

Then $\tau(D) = \nu(D)$.

Proof. Suppose for a contradiction that $\nu(D) < \tau(D)$.

Claim 1. The digraph D has a packing of size $\nu(D_1) + \nu(D_2) - 1$.

Proof of Claim: Clearly $\nu(D_2 \setminus u_2) \geq \nu(D_2) - 1$, and so the union of any maximum packing of D_1 with any packing of $D_2 \setminus u_2$ of size $\nu(D_2) - 1$ is as desired by (a). This proves Claim 1. \diamond

Claim 2. The digraph D has a transversal of size at most $\tau(D_1) + \tau(D_2) + 1$.

Proof of Claim: By (c) we may assume from the symmetry that $D_1 + E_1$ packs. Clearly $\nu(D_1 + E_1) \leq \nu(D_1) + 1$. Thus $\tau(D_1 + E_1) \leq \tau(D_1) + 1$. Let T_1 be a transversal of $D_1 + E_1$ of size at most $\tau(D_1) + 1$, and let T_2 be a transversal of D_2 of size $\tau(D_2)$. By (b) $T_1 \cup T_2$ is a transversal of D , as required. This proves Claim 2. \diamond

For $i = 1, 2$ let F_i be the set of all edges $f \in E_i$ such that $\nu(D_i + f) = \nu(D_i)$.

Claim 3. For $i = 1, 2$, $\nu(D_i + F_i) = \nu(D_i)$.

Proof of Claim: If $\nu(D_i + F_i) > \nu(D_i)$, then, since every edge of E_i has both ends in $\{u_1, u_2, u_3\}$, we deduce that $\nu(D_i + f) > \nu(D_i)$ for some $f \in F_i$, a contradiction. This proves Claim 3. \diamond

Claim 4. Let $i, j \in \{1, 2, 3\}$ be such that $i < j$, and let D' be a subdigraph of D_1 . If $\nu(D' + u_i u_j) > \nu(D')$, then there exist a maximum packing \mathcal{C} of D' and a path P in D' from u_j to u_i such that every member of \mathcal{C} is disjoint from P .

Proof of Claim: Let \mathcal{C}' be a maximum packing of $D' + u_i u_j$. Since \mathcal{C}' is not a packing of D' , some member of \mathcal{C}' , say C , uses the edge $u_i u_j$. Thus $\mathcal{C}' - \{C\}$ and $\mathcal{C}' \setminus u_i u_j$ satisfy the conclusion of the claim. \diamond

Claim 5. If $D_1 + E_1$ packs and every maximum packing of $D_1 + u_1 u_3$ uses u_2 , then every maximum packing of D_1 uses u_2 .

Proof of Claim: Suppose for a contradiction that every maximum packing of $D_1 + u_1 u_3$ uses u_2 , but some maximum packing of D_1 does not use u_2 . Then $\nu(D_1 + u_1 u_3) > \nu(D_1)$. By Claim 4 applied to $D' = D_1$ there exist a maximum packing \mathcal{C} of D_1 and a path P of D_1 from u_3 to u_1 such that P is disjoint from every member of \mathcal{C} . Let L be a subdigraph of D_1 such that

- (α) L includes P and every member of \mathcal{C} ,
- (β) L includes every member of some maximum packing of D_1 that does not use u_2 , and
- (γ) subject to (α) and (β), $E(L)$ is minimal.

By (α) $\nu(L) = \nu(D_1)$. We claim that $\nu(L + u_1 u_2 + u_2 u_3) > \nu(L)$. To prove this claim suppose for a contradiction that equality holds. Since $D_1 + E_1$ packs we deduce that $\tau(L + u_1 u_2 + u_2 u_3) = \nu(L)$. Let T be a transversal of $L + u_1 u_2 + u_2 u_3$ of size $\nu(L)$. From (β) we deduce that $u_2 \notin T$, but then it follows

that T is a transversal of $L + u_1u_3$, contrary to (α) . This proves that $\nu(L + u_1u_2 + u_2u_3) > \nu(L)$. Let \mathcal{S} be a maximum packing of $L + u_1u_2 + u_2u_3$. We may assume that no member C of \mathcal{S} uses both edges u_1u_2, u_2u_3 , for otherwise $\mathcal{S} \setminus \{C\} \cup \{C + u_1u_3 - u_1u_2 - u_2u_3\}$ is a maximum packing of $D_1 + u_1u_3$ avoiding u_2 , a contradiction. Hence, either $\nu(L + u_1u_2) > \nu(L)$ or $\nu(L + u_2u_3) > \nu(L)$, and so we may assume the former. By Claim 4 applied to $D' = L$ there exists a maximum packing \mathcal{C}' of L and a path P' in L from u_2 to u_1 disjoint from every member of \mathcal{C}' . Since the union of P and all members of \mathcal{C} does not include a path from u_2 to u_1 , there exists an edge $e \in E(P')$ that does not belong to P or any member of \mathcal{C} . Thus $L \setminus e$ satisfies (α) . But $L \setminus e$ includes every member of \mathcal{C}' , and hence it also satisfies (β) , contrary to (γ) . This proves Claim 5. \diamond

Claim 6. Let $i, j \in \{1, 2, 3\}$ with $i < j$. If $u_iu_j \notin F_1$, then $u_ju_i \in F_2$.

Proof of Claim: Suppose for a contradiction that $u_iu_j \notin F_1$ and $u_ju_i \notin F_2$. Let \mathcal{C}_1 be a packing of $D_1 + u_iu_j$ of size $\nu(D_1) + 1$, and let \mathcal{C}_2 be a packing of $D_2 + u_ju_i$ of size $\nu(D_2) + 1$. Then \mathcal{C}_1 includes a circuit C_1 containing u_iu_j , and \mathcal{C}_2 includes a circuit C_2 containing u_ju_i . Let \mathcal{C} be the circuit $(C_1 \setminus u_iu_j) \cup (C_2 \setminus u_ju_i)$. If one of $\mathcal{C}_1 - \{C_1\}, \mathcal{C}_2 - \{C_2\}$ does not use u_2 , then $\mathcal{C} := (\mathcal{C}_1 - \{C_1\}) \cup (\mathcal{C}_2 - \{C_2\}) \cup \{C\}$ is a packing of D of size $\nu(D_1) + \nu(D_2) + 1$ by (a). Then because of Claim 2, $\tau(D) = \nu(D)$ packs, a contradiction. Thus we may assume that both $\mathcal{C}_1 - \{C_1\}, \mathcal{C}_2 - \{C_2\}$ use u_2 for all choices of \mathcal{C}_1 and \mathcal{C}_2 . Thus $i = 1$ and $j = 3$, and every maximum packing of $D_1 + u_1u_3$ or $D_2 + u_3u_1$ uses u_2 . By (c) we may assume that $D_1 + E_1$ and $D_2 + Z_2$ packs. Hence by Claim 5 every maximum packing of D_1 uses u_2 . By Remark 3.1 D_1 has transversal T_1 of size $\nu(D_1)$ with $u_2 \in T_1$, and $D_2 + u_3u_1$ has a transversal T_2 of size $\nu(D_2) + 1$ with $u_2 \in T_2$. By (b) $T_1 \cup T_2$ is a transversal of D of size $\nu(D_1) + \nu(D_2)$. On the other hand, by deleting one of the circuits of \mathcal{C} that contain u_2 we obtain a packing of D of size $\nu(D_1) + \nu(D_2)$. Thus $\nu(D) = \tau(D)$, a contradiction. This proves Claim 6. \diamond

Claim 7. The digraph D has a packing of size $\nu(D_1) + \nu(D_2)$.

Proof of Claim: Suppose not. Then for $i = 1, 2$ every maximum packing of D_i uses u_2 , for otherwise the union of a maximum packing in D_i that does not use u_2 with any maximum packing of D_{3-i} is as desired. By Remark 3.1 the digraph D_i has a transversal T_i of size $\tau(D_i)$ with $u_2 \in T_i$. Let us assume first that $\nu(D_1 + u_1u_3) > \nu(D_1)$. Then $\nu(D_2 + u_3u_1) = \nu(D_2)$ by Claim 6. The graph $D_2 + u_3u_1$ packs (because by (c) $D_2 + E_2$ or $D_2 + Z_2$ packs), and so $\nu(D_2 + u_3u_1 \setminus u_2) = \tau(D_2 + u_3u_1 \setminus u_2)$. If $\nu(D_2 + u_3u_1 \setminus u_2) = \nu(D_2)$, then let \mathcal{C}_1 be a maximum packing in $D_1 + u_1u_3$ and let \mathcal{C}_2 be a maximum packing in $\nu(D_2 + u_3u_1 \setminus u_2)$. Then some circuit of \mathcal{C}_1 uses the edge u_1u_3 (because $\nu(D_1 + u_1u_3) > \nu(D_1)$), and some circuit of \mathcal{C}_2 uses the edge u_3u_1 (because every maximum packing of D_2 uses u_2). Thus \mathcal{C}_1 and \mathcal{C}_2 can be combined as in the proof of Claim 6 to produce the desired packing of D . Thus we may assume that $\nu(D_2 + u_3u_1 \setminus u_2) < \nu(D_2)$. Let T'_2 be a transversal in $D_2 + u_3u_1 \setminus u_2$ of size $\nu(D_2) - 1$; then $T_1 \cup T'_2$ is a transversal in D by (b), and its size is $\nu(D_1) + \nu(D_2) - 1$, contrary to Claim 1. This completes the case when $\nu(D_1 + u_1u_3) > \nu(D_1)$.

Thus we may assume that $\nu(D_1 + u_1u_3) = \nu(D_1)$ and $\nu(D_2 + u_3u_1) = \nu(D_2)$. From the symmetry and (c) we may assume that $D_2 + Z_2$ packs. Since every maximum packing of D_2 uses u_2 , and $\nu(D_2 + u_3u_1) = \nu(D_2)$, we see that $\nu(D_2 + Z_2) = \nu(D_2)$. Since $D_2 + Z_2$ packs, there exists a transversal T_2'' of $D_2 + Z_2$ of size $\tau(D_2)$. Since $T_2'' \cap V(D_2)$ is a transversal of D_2 , we deduce that $a \notin T_2''$, and hence $u_2 \in T_2''$, because T_2'' intersects the circuit of $D_2 + Z_2$ with vertex-set $\{a, u_2\}$. Thus T_2'' is a transversal of $D_2 + u_3u_1$ with $u_2 \in T_2''$, and so $T_1 \cup T_2''$ is a transversal of D by (b). Moreover, $|T_1 \cup T_2''| = \tau(D_1) + \tau(D_2) - 1$, contrary to Claim 1. This completes the proof of Claim 7. \diamond

We are now ready to complete the proof of the lemma. We claim that one of $D_1 + F_1$, $D_2 + F_2$ does not pack. Indeed, if both of them pack, then by Claim 3 the digraph $D_i + F_i$ has a transversal of size $\nu(D_i)$, and the union of those sets is a transversal in D by (b) of size $\nu(D_1) + \nu(D_2)$, contrary to Claim 7. Thus we may assume that $D_2 + F_2$ does not pack.

By (c) the digraph $D_1 + E_1$ packs and is strongly 2-connected, and $D_2 + Z_2$ packs. To motivate the next step, notice that since $D_2 + Z_2$ packs, but $D_2 + F_2$ does not, we have $u_2u_1, u_3u_2 \in F_2$. Since $D_1 + E_1$ packs, so does $D_1 + F_1$, and hence by Claim 3 there exists a transversal T_1 in $D_1 + F_1$ of size $\tau(D_1)$.

We claim that the set T_1 is a transversal in $D_1 + F_1 + u_1u_2$ or $D_1 + F_1 + u_2u_3$. To prove this claim suppose for a contradiction that this is not the case. We deduce that there exist a u_2u_1 -path P_1 and a u_3u_2 -path P_2 in D_1 , both disjoint from T_1 . Since T_1 intersects every circuit of D_1 , it follows that $V(P_1) \cap V(P_2) = \{u_2\}$. Since $D_1 + E_1$ is strongly 2-connected, there exists a path Q in D_1 from $V(P_2) - \{u_2\}$ to $V(P_1) - \{u_2\}$; we may assume that no interior vertex of Q belongs to $V(P_1) \cup V(P_2)$. Let H be the digraph $P_1 \cup P_2 \cup Q + E_1$; then $\nu(H) = 1 < 2 = \tau(H)$, contrary to the fact that $D_1 + E_1$ packs. This proves our claim that T_1 is a transversal in $D_1 + F_1 + u_1u_2$ or $D_1 + F_1 + u_2u_3$.

From the symmetry we may assume that T_1 is a transversal in $D_1 + F_1 + u_1u_2$. Let $F_2' = F_2 - \{u_1u_2\}$. Since $D_2 + Z_2$ packs, so does its minor $D_2 + F_2'$, and so by Claim 3 the digraph $D_2 + F_2'$ has a transversal T_2 of size $\tau(D_2)$. By (b) the set $T_1 \cup T_2$ is a transversal in D , and its size is $\tau(D_1) + \tau(D_2)$, contrary to Claim 7. \square

Proposition 7.2. *Let G, M, D , where $\nu(D) < \tau(D)$, and G_1, G_2, G_3, C be as in Proposition 5.5. Then either none or exactly two edges of M have both ends in $V(C)$.*

Proof. Let A, B denote a bipartition of G . Let v_1, v_2', v_2, v_3 be the vertices of C (in that order), where $v_1, v_2 \in A$. For $i = 1, 2, 3$ let m_i be the edge of M incident with v_i . Suppose for a contradiction that m_2 is the only edge of M with both ends in $V(C)$. We may assume that m_2 is incident with v_2' . Thus m_1, m_3 are distinct and are incident with vertices not on C . We may also assume that $m_1, m_3 \in E(G_1) \cup E(G_2)$. For $i = 1, 2, 3$ let M_i be the imprint of M on G_i (see the paragraph prior to Proposition 6.1 for a definition). Let $J_1 := D(G_1 \cup G_2, M_1 \cup M_2)$, let Q be a cube such that C is a subgraph of Q and otherwise Q is disjoint from

G_3 , and let $J_2 := D(G_3 \cup Q, M'_3)$, where M'_3 is a perfect matching of $G_3 \cup Q$ with $M_3 \subseteq M'_3$ that does not use an edge joining v_1 and v_3 . Such a matching is unique, and it has a unique element, say m_0 , not incident with a vertex of G_3 . Let a denote the vertex of J_2 that results from contracting m_0 , and in both J_1, J_2 let u_1, u_2, u_3 denote the vertices that result from contracting the edges incident with v_1, v_2, v_3 , respectively.

Let D_1 be obtained from J_1 by deleting the edges of C , and let D_2 be obtained from J_2 by deleting the vertex a and edges of $Q \cup C$. We wish to apply Lemma 7.1 to the digraphs D_1 and D_2 . Since u_1 is a source and u_3 is a sink of D_2 , we see immediately that (a) and (b) of that lemma hold. We will show that $i = 1$ and $j = 2$ satisfy (c). Since G_1 and G_2 are braces, so is $G_1 \cup G_2$, and thus J_1 is strongly 2-connected by Proposition 5.2. To show that $D_1 + E_1$ packs we first notice that $D_1 + E_1$ is isomorphic to J_1 . But $G_1 \cup G_2$ is obtained from planar braces by repeatedly applying the trisum operation, and hence J_1 has no odd double circuit or F_7 minor by Theorem 5.4. Moreover, $|V(J_1)| + |E(J_1)| = |E(G_1 \cup G_2)| < |E(G)| = |V(D)| + |E(D)|$ by Lemma 5.9, and hence J_1 (and thus $D_1 + E_1$) pack by the hypothesis of Proposition 5.5. Finally, $D_2 + Z_2$ is a subdigraph of J_2 , and hence it packs, by the argument of this paragraph. Thus $\nu(D) = \tau(D)$ by Proposition 7.1, a contradiction. \square

8. TRISUM-PART III

Let D_1, D_2 be edge-disjoint subdigraphs of a digraph D , let $X \subseteq V(D_1) \cap V(D_2)$, and let C be a circuit of D . We say that C passes from D_1 to D_2 through X if there is no vertex $v \in V(D) - X$ such that the edge of C with head v belongs to D_1 and the edge of C with tail v belongs to D_2 .

Lemma 8.1. *Let D_1 and D_2 be digraphs with $V(D_1) \cap V(D_2) = \{u_1, u_2, u_3, u_4\}$ and $E(D_1) \cap E(D_2) = \emptyset$. Let $D = D_1 \cup D_2$, let $E_1 = \{u_1u_2, u_3u_2, u_3u_4, u_1u_4\}$, and let $E_2 = \{u_2u_1, u_2u_3, u_4u_3, u_4u_1\}$. Assume that*

- (1) *for $i = 1, 2$, $D_i + E_i$ packs,*
- (2) *every circuit of D_1 is disjoint from every circuit of D_2 ,*
- (3) *every circuit of D passes from D_1 to D_2 through $\{u_1, u_3\}$, and it passes from D_2 to D_1 through $\{u_2, u_4\}$.*

Moreover, assume that for every pair $e_1, e_2 \in E_i$ of independent edges one of the following holds:

- (a) $\nu(D_i + e_1 + e_2) \geq \nu(D_i) + 2$,
- (b) $\tau(D_i + e_1) = \tau(D_i)$, or
- (c) $\tau(D_i + e_2) = \tau(D_i)$.

Then $\tau(D) = \nu(D)$.

Proof. Suppose for a contradiction that $\nu(D) < \tau(D)$.

Claim 1. Let $i = 1$ or $i = 2$, and let $F \subseteq E_i$. Then one of the following holds:

- (i) There is an edge $e \in F$ such that $\nu(D_i + e) > \nu(D_i)$,
- (ii) $\tau(D_i + F) = \tau(D_i)$, or
- (iii) there exist independent edges $e_1, e_2 \in F$ such that

$$\nu(D_i) = \nu(D_i + e_1) = \nu(D_i + e_2) < \nu(D_i + e_1 + e_2).$$

Proof of Claim: Suppose (ii) does not hold, i.e. $\tau(D_i + F) > \tau(D_i)$. As $D_i + E_i$ packs, $\nu(D_i + F) > \nu(D_i)$. Now if (i) does not hold then (iii) must hold since if two edges $e_1, e_2 \in F$ appear in the same circuit then e_1, e_2 are independent. \diamond

Claim 2. D has a transversal of size $\nu(D_1) + \nu(D_2) + 1$.

Proof of Claim: If $\tau(D_i + E_i) \leq \tau(D_i) + 1$ for some $i \in \{1, 2\}$, then take the corresponding transversal, and union it with any transversal of D_{3-i} of size $\tau(D_{3-i})$. The resulting set is a transversal in D of size $\nu(D_1) + \nu(D_2) + 1$ by (3), as desired. Thus we may assume that $\tau(D_i + E_i) \geq \tau(D_i) + 2$ for $i = 1, 2$. Since $\nu(D_i + E_i) = \tau(D_i + E_i)$ we may assume that there is a packing of size $\nu(D_1)$ in D_1 and two disjoint paths disjoint from the packing joining u_2 to u_3 and u_4 to u_1 , respectively. Likewise, we may assume that a similar situation occurs in D_2 , but with paths joining u_3 to u_4 and u_1 to u_2 . (If the paths join the other pairs we get a packing of size $\nu(D_1) + \nu(D_2) + 2$, a contradiction, because the union of $\{u_1, u_3\}$, any transversal of D_1 and any transversal of D_2 is a transversal of D of the same size.) Now we use the fact that D_2 satisfies (a), (b) or (c) for the edges u_2u_3 and u_4u_1 . If (a) holds, then we have a packing in D of size $\nu(D_1) + \nu(D_2) + 2$, and so we may assume from the symmetry that (b) holds, where $e_1 = u_2u_3$. Let T_2 be the corresponding transversal. We may also assume that $\nu(D_1 + u_3u_4 + u_1u_2) \leq \nu(D_1) + 1$, for otherwise we produce a packing of D of size $\nu(D_1) + \nu(D_2) + 2$. It follows that $\nu(D_1 + u_3u_4 + u_1u_2 + u_1u_4) \leq \nu(D_1) + 1$, because a packing of $D_1 + u_3u_4 + u_1u_2 + u_1u_4$ that uses u_1u_4 cannot use u_3u_4 or u_1u_2 . Hence $\tau(D_1 + u_3u_4 + u_1u_2 + u_1u_4) = \nu(D_1 + u_3u_4 + u_1u_2 + u_1u_4) \leq \tau(D_1) + 1$. Let T_1 be a corresponding transversal. Then $T_1 \cup T_2$ is a transversal in D of size $\nu(D_1) + \nu(D_2) + 1$ by (3), as desired. \diamond

Let F_i be the set of all edges $e \in E_i$ such that $\tau(D_i + e) > \tau(D_i)$.

Claim 3. The reversal of no edge in F_1 belongs to F_2 .

Proof of Claim: Otherwise we can construct a packing in D of size $\nu(D_1) + \nu(D_2) + 1$, contrary to Claim 2. \diamond

Claim 4. The digraph D has a packing of size $\nu(D_1) + \nu(D_2)$.

Proof of Claim: The union of any maximum packing of D_1 with any maximum packing of D_2 is as desired by (2). \diamond

Claim 5. For some $i \in \{1, 2\}$, F_i includes two independent edges.

Proof of Claim: Suppose for a contradiction that no F_i includes two independent edges. It follows from Claim 3 that there exist adjacent edges $e_1, e_2 \in E_1 - F_1$ and adjacent edges $e_3, e_4 \in E_2 - F_2$ such that e_3, e_4 are the reverses of the edges in $E_1 - \{e_1, e_2\}$. Since $e_1, e_2 \notin F_1$ we deduce from Claim 1 that $\tau(D_1 + e_1 + e_2) = \nu(D_1 + e_1 + e_2) = \nu(D_1)$ and similarly $\tau(D_2 + e_3 + e_4) = \nu(D_2)$. But the union of the corresponding transversals is a transversal in D of size $\nu(D_1) + \nu(D_2)$, contrary to Claim 4. \diamond

Claim 6. At most one of F_1, F_2 includes two independent edges.

Proof of Claim: If both of them do, then (a) holds for those pairs, and we get a packing in D of size at least $\nu(D_1) + \nu(D_2) + 1$, contradicting Claim 2. \diamond

By Claim 5 we may assume that F_2 includes two independent edges. We wish to define a set $F \subseteq E_1 - F_1$. If $E_2 = F_2$, then $F_1 = \emptyset$ by Claim 3, and we put $F = E_1$. Otherwise we proceed as follows. If $F_1 \neq \emptyset$, then it includes a unique edge by Claim 3, Claim 6 and the fact that F_2 includes two independent edges. Let e be the unique member of F_1 . If $F_1 = \emptyset$, then we select $e \in E_1$ such that its reverse does not belong to F_2 . In either case the reverse of e does not belong to F_2 . We put $F = E_1 - \{e\}$. This completes the definition of F . We apply Claim 1 to D_1 and F . Then (i) does not hold, because $F \cap F_1 = \emptyset$. If (ii) holds, then let T_1 be the corresponding transversal, and let T_2 be a transversal of size $\tau(D_2)$ in D_2 if e does not exist, and in D_2 with the reverse of e added otherwise. Then $T_1 \cup T_2$ is a transversal in D by (3) of size $\nu(D_1) + \nu(D_2)$, contrary to Claim 4. Thus (iii) holds. That is, there exist independent edges $e_1, e_2 \in F$ such that $\nu(D_1 + e_1 + e_2) > \nu(D_1)$. Let $e_3, e_4 \in E_2$ be the reverses of e_1, e_2 . Since F_2 includes two independent edges we deduce from the choice of F that $e_3, e_4 \in F_2$. Thus $\nu(D_2 + e_3 + e_4) \geq \nu(D_2) + 2$ by (a). By combining the resulting packings we get a packing in D of size at least $\nu(D_1) + \nu(D_2) + 1$, contrary to Claim 2. \square

Proposition 8.2. *Let G, M, D , where $\nu(D) < \tau(D)$, and G_1, G_2, G_3, C be as in Proposition 5.5. Then at least one edge of M has both ends in $V(C)$.*

Proof. Let A, B denote a bipartition of G . Let u_1, u_2, u_3, u_4 be the vertices of C (in that order), where $u_1, u_3 \in A$ and $u_2, u_4 \in B$. Suppose for a contradiction that no edge of M has both ends in $V(C)$, and let the edges of M incident to vertices of C be $m_1 = u_1u'_1, m_2 = u_2u'_2, m_3 = u_3u'_3, m_4 = u_4u'_4$. For $i = 1, 2, 3, 4$ we will use u_i to also denote the vertex of D that results from contracting m_i . Let Q be a cube such that C is a subgraph of Q , and Q is otherwise disjoint from $G_1 \cup G_2 \cup G_3$. Since G is a brace, $|V(G_i) \setminus \{u_1, \dots, u_4\}|$ is even for $i = 1, 2, 3, 4$. As each of m_1, m_2, m_3, m_4 have exactly one end in C , we may assume (by renumbering G_1, G_2, G_3 and u_1, u_2, u_3, u_4) that $\{m_1, m_2, m_3, m_4\} \subseteq E(G_1)$, or $\{m_3, m_4\} \subseteq E(G_1)$ and $\{m_1, m_2\} \subseteq E(G_2)$. In the former case we may also assume that $|E(G_2)| \leq |E(G_3)|$. If $\{m_1, m_2, m_3, m_4\} \subseteq E(G_1)$ and $|E(G_1)| > 12$, then let $H_1 = G_1$ and $H_2 = G_2 \cup G_3$; otherwise let $H_1 = G_1 \cup G_2$ and $H_2 = G_3$. Thus $|E(H_1)| > 12$ by Lemma 5.9. Then both H_1 and H_2

are obtained from planar braces by repeatedly applying the trisum operation. Let $J_1 = D(H_1, M)$, and let $D_1 = J_1 \setminus E(C)$. Let J_2 be obtained from H_2 by directing every edge from $A \cap V(H_2)$ to $B \cap V(H_2)$, and then contracting every edge of $M \cap E(H_2)$, and let $D_2 = J_2 \setminus E(C)$. Let us notice that u_1, u_3 are sources, and u_2, u_4 are sinks of D_2 . Thus conditions (2) and (3) of Lemma 8.1 hold.

We now prove that condition (1) holds. The graph H_1 is obtained from planar braces by repeatedly applying the 4-sum operation. By Theorem 5.4 the digraph J_1 has no minor isomorphic to an odd double circuit or F_7 . Moreover $|V(J_1)| + |E(J_1)| < |V(D)| + |E(D)|$ by Lemma 5.9, and so J_1 packs by the hypothesis of Proposition 5.5. But J_1 is isomorphic to $D_1 + E_1$, and hence $D_1 + E_1$ packs. To prove that $D_2 + E_2$ packs we first notice that $D_2 + E_2$ is a subdigraph of $D(H_2 \cup Q, M_2)$, where M_2 is a perfect matching of $H_2 \cup Q$ that includes $E(H_2) \cap M$ and no edge with both ends in $V(C)$. But $D(H_2 \cup Q, M_2)$ packs by the hypothesis of Proposition 5.5 and the fact that $|E(H_1)| > 12$. Thus conditions (1)–(3) of Lemma 8.1 hold.

Next we show that for $i = 1, 2$, and for every pair $e_1, e_2 \in E_i$ of independent edges one of (a), (b), (c) holds. We first do so for $i = 2$. It suffices to argue for $e_1 = u_2u_1$ and $e_2 = u_4u_3$. Since $D(H_2 \cup Q, M_2)$ packs by the previous paragraph, we see that $D'_2 = D_2 + \{u_2u_1, u_3u_2, u_4u_3, u_1u_4\}$ also packs. But clearly $\tau(D'_2) > \tau(D_2)$, because u_1, u_3 are sources and u_1, u_4 are sinks of D_2 , and $\{u_1, u_2, u_3, u_4\}$ is the vertex-set of a circuit of D'_2 . If $\nu(D'_2) \geq \nu(D_2) + 2$, then (a) holds. Thus we may assume that $\tau(D'_2) = \tau(D_2) + 1$. Let T be a corresponding transversal of D'_2 . Since $\{u_1, u_2, u_3, u_4\}$ is the vertex-set of a circuit of D'_2 , and $|T| = \nu(D_2) + 1$, we see that $|\{u_1, u_2, u_3, u_4\} \cap T| = 1$. Let $T' = T - \{u_1, u_2, u_3, u_4\}$. If $u_1 \in T$ or $u_2 \in T$, then T' shows that (c) holds and if $u_3 \in T$ or $u_4 \in T$, then T' shows that (b) holds, as desired. This proves that one of (a), (b), (c) holds for $i = 2$.

It remains to show that one of (a), (b), (c) holds for $i = 1$. Let e_1, e_2 be independent edges as in Lemma 8.1; for the purpose of this paragraph we may take advantage of symmetry and assume that $e_1 = u_1u_2$ and $e_2 = u_2u_4$. For $j = 1, 2, 3, 4$ let u_jv_j denote the edges of Q with exactly one end in $V(C)$. Let M_1 be the union of $M \cap E(H_1)$ and two edges of Q , one with ends v_1v_2 and the other with ends v_3v_4 . Let us consider the digraph $D'_1 := D(H_1 \cup Q \setminus E(C), M_1)$. Then D'_1 is isomorphic to the graph $D_1 + \{u_1a, au_2, ab, ba, u_3b, bu_4\}$. If D'_1 packs, then one of (a), (b), (c) holds: clearly $\tau(D'_1) > \tau(D_1)$ because D'_1 has a circuit disjoint from D_1 . If $\nu(D'_1) \geq \nu(D_1) + 2$, then (a) holds; if $\tau(D'_1) = \tau(D_1) + 1$, then let T be a corresponding transversal. If $a \in T$ then $T \cap V(D_1) \cup \{u_1\}$ proves (b). If $b \in T$ then $T \cap V(D_2) \cup \{u_3\}$ proves (c). Thus we may assume that D'_1 does not pack, and so by the hypothesis of Proposition 5.5 we see that $|E(H_2)| \leq |E(Q)|$. Thus H_2 is a cube by Lemma 5.9. In particular, $H_2 = G_3$ and $H_1 = G_1 \cup G_2$. The definition of H_1 and H_2 implies that $\{m_1, m_2, m_3, m_4\} \not\subseteq E(G_1)$ or $|E(G_1)| = 12$.

Let us first assume that $\{m_1, m_2, m_3, m_4\} \subseteq E(G_1)$. Then $|E(G_1)| = 12$, and so G_1 is a cube. Since $|E(G_2)| \leq |E(G_3)|$ and $G_3 = H_2$ is a cube, we deduce that G_1, G_2, G_3 are all cubes. Let a, b (resp. c, d) denote the edges of $M \setminus C$ in G_2 (resp. G_3). Then D is isomorphic to one of the digraphs depicted in Figure 2.

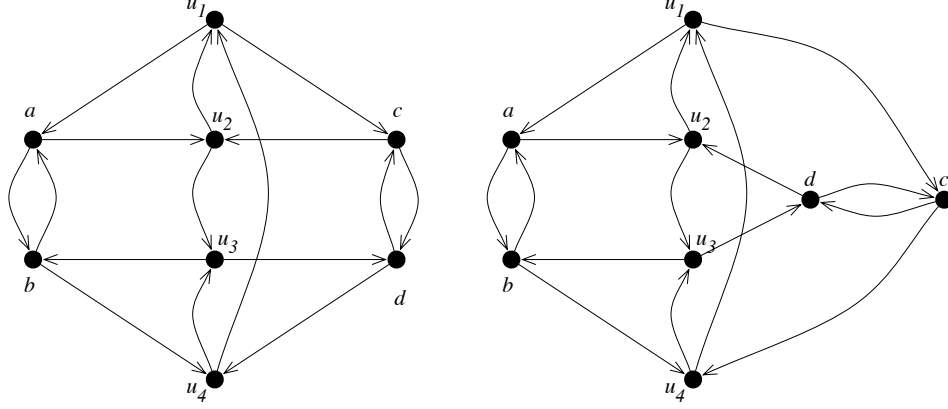


FIGURE 2. Two digraphs.

For both (a) and (b), $\{u_1a, au_2, u_2u_1\}, \{u_3b, bu_4, u_4u_3\}, \{cd, dc\}$ is a packing of circuits and $\{a, u_3, c\}$ is a transversal. In particular, $\nu(D) = 3 = \tau(D)$, a contradiction.

Thus we may assume that $\{m_1, m_2, m_3, m_4\} \not\subseteq E(G_1)$, and so $\{m_3, m_4\} \subseteq E(G_1)$ and $\{m_1, m_2\} \subseteq E(G_2)$. Moreover, $H_1 = G_1 \cup G_2$. For $i = 1, 2$ let L_i be obtained from $G_i \setminus E(C)$ by orienting all the edges of $G_i \setminus E(C)$ from A to B and by contracting all edges of $M \cap E(G_i)$. Then

(*) u_1 is a source and u_2 is a sink of L_1 , and u_3 is a source and u_4 is a sink of L_2 .

Claim 1.

- (1) The digraph L_1 does not have disjoint paths P_1 from u_1 to u_3 and P_2 from u_4 to u_2 .
- (2) The digraph L_2 does not have disjoint paths P_1 from u_3 to u_1 and P_2 from u_2 to u_4 .

Proof of Claim: We may assume that $i = 1$, and suppose for a contradiction that P_1, P_2 exist. For the cube Q we have $V(Q) = \{u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4\}$ and $E(Q) = C \cup \{u_i v_i : i = 1, 2, 3, 4\} \cup \{v_1 v_2, v_2 v_3, v_3 v_4, v_4 v_1\}$. Let $M' = M \cup \{u_1 v_1, u_2 v_2, v_3 v_4\}$. Let Q' be the graph obtained from Q by replacing every edge of C by two parallel edges. Then $D(G_1 \cup Q', M')$ contains as a subdigraph a digraph D' which is obtained from L_1 by adding a new vertex w and edges $u_2 u_1, u_1 u_2, u_3 w, w u_4, w u_1$, and $u_2 w$. But that is a contradiction, because D' has an odd double circuit minor (contract all but one edge of each path comprising L_1) and by Theorem 5.4, Lemma 5.9 and the hypothesis of Proposition 5.5, $D(G_1 \cup Q', M')$ packs, and hence so does D' . \diamond

We now show that one of (a), (b), (c) holds for the pair of edges $u_1 u_4$ and $u_3 u_2$. Indeed, suppose that none of (a), (b), (c) hold. Then $D_1 + u_1 u_4$ has a packing of size $\nu(D_1) + 1$. This packing includes a circuit containing the edge $u_1 u_4$. Hence, D_1 has a packing \mathcal{C} of size $\nu(D_1)$ and a path P_1 from u_4 to u_1 disjoint from every $C \in \mathcal{C}$. Similarly, D_1 has a packing \mathcal{C}' of size $\nu(D_1)$ and a path P_2 from u_2 to u_3 disjoint from

every $C \in \mathcal{C}'$. Since P_1 and P_2 are disjoint from any minimum transversal of D_1 we deduce that their union is acyclic. By $(*)$ we deduce that P_1 can be decomposed into either (α) subpaths P'_1 from u_2 to u_1 of L_2 and P''_1 from u_1 to u_3 of L_1 , or (β) subpaths P'_1 from u_2 to u_4 of L_2 and P''_1 from u_4 to u_3 of L_1 . Similarly, P_2 can be decomposed into either (α') subpaths P'_2 from u_4 to u_2 of L_1 and P''_2 from u_2 to u_1 of L_2 , or (β') subpaths P'_2 from u_4 to u_3 of L_1 and P''_2 from u_3 to u_1 of L_1 . If (α) and (α') occur then the paths P''_1 and P'_2 contradict Claim 1(1). If (β) and (β') occurs then paths P'_1 and P'_2 contradict Claim 1(2). All other cases contradict the fact that $P_1 \cup P_2$ is acyclic.

It remains to show that one of (a), (b), (c) holds for the pair of edges u_1u_2 and u_3u_4 . Suppose it does not. Thus $D_1 + u_3u_4$ has a packing of size $\nu(D_1) + 1$. This packing includes a circuit containing the edge u_3u_4 , and hence D_1 has a packing \mathcal{C} of size $\nu(D_1)$, and a path P from u_4 to u_3 disjoint from every member of \mathcal{C} . It follows from $(*)$ and Claim 1 that P is a subgraph of L_1 . Since \mathcal{C} does not use u_3 or u_4 (because every member of \mathcal{C} is disjoint from P) we deduce that at most one circuit of \mathcal{C} intersects both $E(L_1)$ and $E(L_2)$. Thus either (letting $\nu = \nu(D_1)$ and using $(*)$)

- (A) $\nu(L_1 + u_3u_4) + \nu(L_2) \geq \nu + 1$, or
- (B) $\nu(L_1 + u_2u_1 + u_3u_4) + \nu(L_2 + u_1u_2) \geq \nu + 2$,

where (A) (resp. (B)) occurs when no (resp. exactly one) circuit of \mathcal{C} intersects both $E(L_1)$ and $E(L_2)$. Similarly, either

- (C) $\nu(L_2 + u_1u_2) + \nu(L_1) \geq \nu + 1$, or
- (D) $\nu(L_2 + u_1u_2 + u_4u_3) + \nu(L_1 + u_3u_4) \geq \nu + 2$.

By $(*)$ $\nu(L_1) + \nu(L_2) \leq \nu$. Thus if (A) and (C) hold we deduce that

$$\nu(D_1 + u_1u_2 + u_3u_4) \geq \nu(L_1 + u_3u_4) + \nu(L_2 + u_1u_2) = 2\nu + 2 - \nu(L_1) - \nu(L_2) \geq \nu + 2,$$

where the first inequality follows from $(*)$. It follows that (a) holds, a contradiction. Assume now that (B) and (D) hold. Clearly $\nu(L_2 + u_1u_2 + u_4u_3) \geq \nu(L_2 + u_1u_2)$, $\nu(L_1 + u_2u_1 + u_3u_4) \geq \nu(L_1 + u_3u_4)$ and $\nu(L_1 + u_2u_1 + u_3u_4) + \nu(L_2 + u_1u_2 + u_4u_3) \leq \nu + 2$. Therefore

$$2\nu + 4 \geq \nu(L_1 + u_2u_1 + u_3u_4) + \nu(L_2 + u_1u_2) + \nu(L_2 + u_1u_2 + u_4u_3) + \nu(L_1 + u_3u_4) \geq 2\nu + 4.$$

Thus equality holds throughout, and, in particular, $\nu(L_1 + u_2u_1 + u_3u_4) = \nu(L_1 + u_3u_4)$. Since $\nu(L_2) \geq \nu(L_2 + u_1u_2) - 1$ we have

$$\nu(L_1 + u_3u_4) + \nu(L_2) \geq \nu(L_1 + u_2u_1 + u_3u_4) + \nu(L_2 + u_1u_2) - 1 \geq \nu + 1$$

by (B), and so (A) holds. Thus we have shown that if (B) and (D) hold, then (A) holds as well.

To complete the proof we may assume that either (A) and (D) hold or that (B) and (C) hold. By symmetry we may assume that the former case occurs and that (C) does not hold. We need two claims.

- (E) $\nu(L_2 + u_1u_2) \leq \nu(L_2)$

To prove (E) we subtract the negation of (C) from (A), and use the fact that $\nu(L_1 + u_3u_4) \leq \nu(L_1) + 1$. We find that $\nu(L_2 + u_1u_2) \leq \nu(L_2)$, which is (E).

$$(F) \quad \nu(L_2 + u_4u_3) \leq \nu(L_2)$$

To prove (F) we use the fact that $\nu(L_1 + u_3u_4) + \nu(L_2 + u_4u_3) \leq \nu + 1$. (Otherwise those packings could be combined to produce a packing in D_1 of size $\nu + 1$.) By subtracting this inequality from (A) we obtain (F).

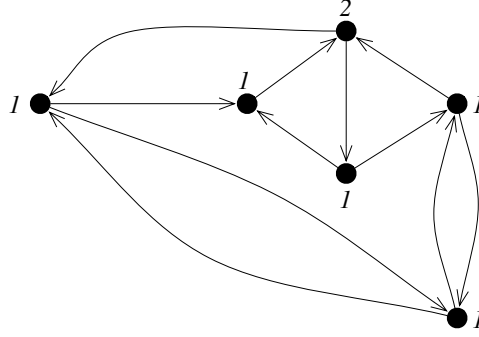
Let $L'_2 = L_2 + \{u_1a, au_2, u_3b, bu_4, ab, ba, u_4u_3\}$. Let Q' be obtained from Q by adding a three-edge path P' joining u_3 and u_4 , and otherwise disjoint from $G \cup Q$. Let M'_2 be a perfect matching of $G_2 \cup Q'$ that includes $M \cap E(G_2)$, two edges of P' , and two edges of $Q \setminus V(C)$: one with ends adjacent to u_1 and u_2 , and the other with ends adjacent to u_3 and u_4 . Thus L'_2 is isomorphic to $D(G_2 \cup Q' \setminus E(C), M'_2)$. The graph $G_2 \cup Q'$ is a subgraph of a brace H in such a way that $H \setminus V(G_2 \cup Q')$ has a perfect matching and H is obtained from planar braces by trisumming. By Theorem 5.4 the digraph L'_2 has no minor isomorphic to an odd double circuit or F_7 . By Lemma 5.9 the digraph L'_2 satisfies $|V(L'_2)| + |E(L'_2)| < |V(D)| + |E(D)|$, and hence L'_2 packs by the hypothesis of Proposition 5.5. We will show that $\tau(L'_2) \geq \nu(L_2) + 2$ and $\nu(L'_2) \leq \nu(L_2) + 1$. This is a contradiction that will prove the proposition.

We first show that $\tau(L'_2) \geq \nu(L_2) + 2$. Indeed, suppose for a contradiction that L'_2 has a transversal T of size at most $\nu(L_2) + 1$. Since $\{b, u_3, u_4\}$ is the vertex-set of a circuit of L'_2 , one of those vertices belongs to T . If $b \in T$, then $T - \{b\}$ is a transversal of $L_2 + u_1u_2 + u_4u_3$ of size $\nu(L_2)$. Thus $\nu(L_2 + u_1u_2 + u_4u_3) + \nu(L_1 + u_3u_4) \leq \nu(L_2) + \nu(L_1 + u_3u_4) \leq \nu + 1$, contrary to (D). If $b \notin T$, then $u_3 \in T$ or $u_4 \in T$, and $a \in T$, because $\{a, b\}$ is the vertex-set of a circuit of L'_2 . Then $T - \{u_3, u_4, a\}$ is a transversal of L_2 by $(*)$ of size $\nu(L_2) - 1$, a contradiction. This proves that $\tau(L'_2) \geq \nu(L_2) + 2$.

Finally, it remains to prove that $\nu(L'_2) \leq \nu(L_2) + 1$. To this end suppose for a contradiction that \mathcal{C} is a packing in L'_2 of size $\nu(L_2) + 2$. Choose a circuit $C \in \mathcal{C}$ such that $b \in V(C)$. If such a choice is not possible choose C with $a \in V(C)$, and if that is not possible choose C arbitrarily. It follows that the packing $\mathcal{C} - \{C\}$ uses at most one of a and u_4 , and hence the packing $\mathcal{C} - \{C\}$ proves that either $\nu(L_2 + u_4u_3) > \nu(L_2)$, or $\nu(L_2 + u_1u_2) > \nu(L_2)$, contrary to (E) and (F). This proves that $\nu(L'_2) \leq \nu(L_2) + 1$, and hence completes the proof of the proposition. \square

9. CONCLUDING REMARKS

Consider a digraph D with weight function $w : V(D) \rightarrow \mathbb{Z}_+$. The weight of a subset $T \subseteq V(D)$ is defined as $\sum_{v \in T} w(v)$. The value of the minimum weight transversal is written $\tau(D, w)$. The cardinality of the largest family \mathcal{C} of circuits with the property that for every $v \in V(D)$ at most $w(v)$ circuits of \mathcal{C} use v , is denoted $\nu(D, w)$. Let $e : V(D) \rightarrow \mathbb{Z}_+$ where $e(v) = 1, \forall v \in V(D)$. Then $\tau(D) = \tau(D, e)$ and $\nu(D) = \nu(D, e)$. Observe that for every digraph D and for all positive weight functions w we have $\tau(D, w) \geq \nu(D, w)$. A natural extension of Theorem 1.1 would be to characterize which are the digraphs

FIGURE 3. Digraph D with $\tau(D, w) > \nu(D, w)$.

D for which $\tau(H, w) = \nu(H, w)$, for every subdigraph H of D and for every weights $w : V(D) \rightarrow Z_+$. This class of digraphs is closed under taking minors, and thus does not contain F_7 or odd double circuits. However, there are other obstructions as is illustrated by the digraph D of Figure 3. Next to each vertex v we indicate the weight $w(v)$. Here we have $3 = \tau(D, w) > \nu(D, w) = 2$, and D does not contain F_7 or an odd double circuit as a minor. In fact many other obstructions can be obtained by a similar construction. A related problem is to study the class of digraphs for which $\tau(D, w) = \nu(D, w)$ for all $w : V(D) \rightarrow Z_+$ but without requiring that the same property hold for every subdigraph. This can be formulated as a hypergraph matching problem where the vertices of the hypergraph are the vertices of the digraph and the edges are the vertex set of circuits of D . There is a long list of obstructions to this property. However the problem has been solved for the special case when D is a tournament [8] or a bipartite tournament [9].

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